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Growth property of solutions of $-\Delta f=\lambda f$ on noncompact Riemannian manifolds

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Abstract. In this work the nonexistence of nontrivial L^2-solutions of $-\Delta f=\lambda f$ for positive λ on a noncompact two-dimensional Riemannian manifold is considered. We suppose that the manifold is homeomorphic to $\boldsymbol{R}^2$ minus a disk and that its metric approaches rotationally symmetric one near infinity. Completeness or boundary conditions are not required. We claim that if the metric satisfies suitable conditions near infinity, then there is no nontrivial L^2-solution. The obtained result is an extension of one of the previous theorems in the rotationally symmetric case.

§1. Introduction

In 1943, F. Rellich [19] proved that if a domain Ω of $\boldsymbol{R}^n$ includes the outside of some sphere, then, regardless of boundary conditions, any solution of the Helmholtz equation $-\Delta f=\lambda f$ in Ω for a positive constant λ can not be square integrable unless $f\equiv 0$. This result has later been extended to wider types of equations such as the Schrödinger equation $-\Delta f+qf=\lambda f$ ([8], e.t.c.) and more general second-order elliptic equations (e.g., [20], [6]), because those results imply the absence of eigenvalues lying in the continuous spectrum, which is significant in quantum mechanics and quantum scattering theory. One will find in, e.g., [4] and [18] precise histories and meanings of the theory of this kind.

Another problem is presented in connection with the Laplace-Beltrami operator Δ on noncompact Riemannian manifolds. There are many literatures dealing with this kind of problems in the study of the spectrum of $-\Delta$ especially on complete manifolds having nonpositive sectional curvatures. (For example, [2], [3], [7], [15], [16] and [17].) To say nothing of the importance of such a global theory, we like to notice that in some cases, the nonexistence of L^2-solutions results only from the behavior of metric near infinity. The present paper treats such a

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"local" theory without assuming completeness or definiteness of the sign of the curvature. As long as the unique continuation property holds, we have the nonexistence of positive eigenvalues of *any* selfadjoint operator L such that $Lf=\lambda f$ implies $-\Delta f=\lambda f$ at least outside some compact set in the sense of distribution.

Several conditions for guaranteeing the nonexistence of square integrable solutions have been obtained if the manifolds are homeomorphic to $\boldsymbol{R}^n$ minus a ball and rotationally symmetric. This article aims at extending one of the known results to not symmetric manifolds. Before entering into the detailed discussion, however, we like to review typical criteria in the rotationally symmetric case.

THEOREM 1 ([9], cf. also [11]). *Let $\mathcal{M}$ be a two-dimensional Riemannian manifold which admits the local coordinates r, θ, $r_0<r<\infty$, $\theta\in\boldsymbol{S}^1$ by which the metric of $\mathcal{M}$ is represented as*

$$ds^2=dr^2+\rho(r)^2d\theta, \tag{1.1}$$

$\rho(r)$ being a positive function of r. Suppose that
(i) *$\rho(r)$ is absolutely continuous, nondecreasing and $\rho(r)\to\infty$ $(r\to\infty)$,*
(ii) $\displaystyle\int^\infty\frac{dr}{\rho(r)}=\infty.$

Then for any positive constant λ and any nonzero locally square integrable function f which satisfies the equation $-\Delta f=\lambda f$ in the sense of distribution, we can find numbers $C>0$ and $r_1\geq r_0$ such that

$$\int_{r_0<r<R}|f|^2d\mathcal{M}\geq C\int_{r_0}^R\frac{dr}{\rho(r)}$$

holds for any $R\geq r_1$, where $d\mathcal{M}=\rho(r)drd\theta$ and the range of integration on the left indicates $\{(r,\theta)\in\mathcal{M}\,|\,r_0<r<R,\ \theta\in\boldsymbol{S}^1\}$.

THEOREM 2. *Let $\mathcal{M}$ be an n-dimensional $(n\geq 2)$ Riemannian manifold $\{(r,\omega)\,|\,r_0<r<\infty,\omega\in\boldsymbol{S}^{n-1}\}$ and have the metric (1.1) where $d\theta$ is replaced by the line element of $\boldsymbol{S}^{n-1}$. Let λ be an arbitrary positive constant and assume the following conditions:*
(i) *$\rho(r)\in C^2(r_0,\infty)$, $\rho'(r)>0$ and $\rho(r)\to\infty$ $(r\to\infty)$,*
(ii) *$\rho(r)^{-1}\rho'(r)=o(1)$ $(r\to\infty)$,*
(iii) *$\rho'(r)^{-1}\rho''(r)=o(1)$ $(r\to\infty)$,*
(iv) *there exists a positive number α such that*

$$\int^{\infty}\frac{dr}{\rho(r)^{\alpha}}=\infty.$$

Then for any positive number ε *and any nontrivial locally square integrable solution of the equation* $-\Delta f=\lambda f$, *we can find constants* $C>0$ *and* $r_1\geq r_0$ *such that*

$$\int_{r_0<r<R}|f|^2 d\mathcal{M}\geq C\int_{r_0}^{R}\frac{dr}{\rho(r)^{\varepsilon}}$$

hold for any $R\geq r_1$. (Cf. [10] and [11] which treat the Schrödinger-type equation $-\Delta f+q(x)f=\lambda f$.)

Obviously these theorems imply the nonexistence of nonzero L^2-solutions. There is a gap between these theorems, which we have not yet been able to fill. That is, we have not conclusively succeeded in relaxing (ii) of Theorem 1 nor (iii) of Theorem 2. Besides, we do not have any example of L^2-solutions when these conditions are not fulfilled. (If one further removes (ii) or (iv) of Theorem 2, then $\rho=e^{3r}$ and $f=e^{-2(n-1)r}$ form a counterexample.)

Tayoshi's work [21] is a nonsymmetric version of Theorem 2, though not a complete extension. Therefore, it contains a smallness requirement for the curvature. Our purpose is to extend Theorem 1 to asymptotically rotationally symmetric manifolds. Though an additional condition on ρ is needed, nothing will be assumed about the curvature.

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§ 2. Asymptotically rotationally symmetric manifolds

We consider here a two-dimensional manifold $\mathcal{M}$ whose metric is represented in terms of parameters $r\in(r_0,\infty)$ and $\theta\in S^1$ as

$$ds^2=a(r,\theta)dr^2+2b(r,\theta)\rho(r)drd\theta+c(r,\theta)\rho(r)^2d\theta^2.$$

Here $\rho(r)$ and $a(r,\theta)$, $b(r,\theta)$, $c(r,\theta)$ are real-valued functions. They are supposed to satisfy the following assumptions:

ASSUMPTION 1. (i) $\rho(r)$ is a positive nondecreasing absolutely continuous function of r with $\rho'(r)>0$ a.e.,

(ii) $\rho(r)\to\infty\ (r\to\infty)$,

(iii) $\int_{r_0}^{\infty}\rho(r)^{-1}dr=\infty$.

DEFINITION 1. We set

$$t(r)=\exp\left(-\int_{r_0}^{r}\frac{ds}{\rho(s)}\right)$$

and sometimes use this t as a variable which takes place of r without changing the letters designating the functions.

DEFINITION 2. For each positive number m, the quantity $h(r; m)$ is the one which satisfies

$$\int_{r}^{r+h(r;m)}\frac{ds}{\rho(s)}=mt(r),$$

and $\varphi(r; m)$ is defind as

$$\varphi(r; m)=\operatorname*{ess\ inf}_{r\leq s\leq r+h(r;m)}\rho(s)^2\rho'(s).$$

ASSUMPTION 2. For every $m>0$, one has

$$\int^{\infty}\varphi(r; m)\rho(r+h(r; m))^{-1}dr=\infty.$$

REMARK. This assumption is rather complicated and looks not so easy to be verified. But we shall see in the examples at the end of this article that if $\rho(r)t(r)$ is bounded and $\rho'(r)\leq 1$, and moreover $\rho(r)^2\rho'(r)$ is nondecreasing or nonincreasing, then Assumption 2 will be satisfied.

ASSUMPTION 3. $\rho'(r)^{-1}t(r)=o(1)\ (r\to\infty)$.

ASSUMPTION 4. The functions a, b and c are of class C^1 and $a>0$, $c>0$, $ac-b^2>0$.

DEFINITION 3. $g=\sqrt{ac-b^2}$, $A=a/g$, $B=b/g$, $C=c/g$.

ASSUMPTION 5. (i) $A\to 1$, $B\to 0$, $C\to 1$, as $r\to\infty$ uniformly in θ.
(ii) There are positive numbers k, $l\,(l<2)$ and $r_1\geq r_0$ such that

$$g(r,\theta)\geq k,\quad g_r/g\geq -l\rho'/\rho$$

for $r\geq r_1$, $\theta\in S^1$. Furthermore,

$$bg_\theta/(\rho' g^2)=o(1),\quad tg_\theta/(\rho' g^2)=o(1)$$

hold as $r\to\infty$ uniformly in θ. (The subscripts stand for the derivatives.)

DEFINITION 4. A function $f(t,\theta)$ is said to satisfy the condition of Definition 4 in a region if there exists a positive continuous nondecreasing function $\psi(x)$ $(x>0)$ corresponding to the region which fulfills $\int_{\to 0} x^{-1}\psi(x)dx<\infty$ and f satisfies

$$|f(t_1,\theta_1)-f(t_2,\theta_2)|\leq\psi(\sqrt{t_1^2+t_2^2-2t_1t_2\cos(\theta_1-\theta_2)})$$

for any two points (t_1,θ_1) and (t_2,θ_2) in that region.

REMARK. The square root indicates the distance in the t,θ-plane. This condition is a generalization of the uniform Hölder continuity, the latter being the particular case where $\psi(x)=Kx^\alpha$ with some constants $K>0$ and $0<\alpha<1$.

ASSUMPTION 6. As functions of t and θ, the functions $\rho t^{-1}A_r$, $\rho t^{-1}B_r$, $\rho t^{-2}C_r$, $t^{-1}A_\theta$, $t^{-2}B_\theta$ and $t^{-1}C_\theta$ have the limit values at $t=0$ (i.e., $r=\infty$) uniformly in θ and satisfy the condition of Definition 4 in the neighborhood of $t=0$.

Our purpose is to prove the following theorem:

THEOREM 3. *Let* Δ *be the Laplace-Beltrami operator on a two-demensional Riemannain manifold* $\mathcal{M}$ *which satisfies Assumptions* 1-6. *Then for any positive constant* λ *and any nonzero lecally square integrable solution of* $-\Delta f=\lambda f$ *we can find numbers* $C>0$ *and* $r_1\geq r_0$ *such that*

$$\int_{r_0<r<R}|f|^2 d\mathcal{M}\geq C\int_{r_0}^{R}\frac{dr}{\rho(r)} \tag{2.1}$$

holds for every $R\geq r_1$.

This theorem is obtained by combining the following two theorems. The first one is on an estimate for solutions in terms of the so-called isothermal coordinates, i.e., the ones which satisfy (2.2) below. (They give a conformal mapping from $\mathcal{M}$ to a region of R^2. See, e.g., [22]). The second is on the existence of an appropriate system of isothermal coodinates.

THEOREM 4. *Let a two-dimensional Riemannian manifold* $\mathcal{M}$ *admit an isothermal system of coordinates* (u, v), $u_0<u<\infty$, $v\in \boldsymbol{S}^1$ *by which its metric is represented with a positive function* $\tau(u, v)$ *as*

$$ds^2=\tau(u, v)(du^2+dv^2). \tag{2.2}$$

Suppose moreover that $\tau(u, v)$ *is absolutely continuous with respect to* u *for a.e.* v *and of class* C^1 *with respect to* v *for a.e.* u. *Furthermore, we assume that*

$$\varphi(u)=\operatorname*{ess\,inf}_{v\in S^1}\frac{\partial}{\partial u}\tau(u, v)$$

satisfies $\varphi(u)\geq 0$ *and*

$$\int^{\infty}\varphi(u)du=\infty.$$

Then for every nontrivial locally square integrable solution of $-\Delta f=\lambda f$, $\lambda>0$, *we can find numbers* $C>0$ *and* $u_1\geq u_0$ *such that*

$$\int_{u_0<u<U}|f|^2 d\mathcal{M}\geq CU \tag{2.3}$$

holds for every $U\geq u_1$, *where* $d\mathcal{M}$ *is the volume element of* $\mathcal{M}$.

THEOREM 5. *If a two-dimensional Riemannian manifold* $\mathcal{M}$ *satisfies Assumptions* 1-6, *then there exist a number* r_1, *a function* $u(r, \theta)$ *and a multi-valued function* $v(r, \theta)$ *defined for* $r\geq r_1$, $\theta\in \boldsymbol{S}^1$ *which are of class* C^1 *and satisfy*

$$\begin{cases} v_r=Bu_r-A\rho^{-1}u_\theta, \\ v_\theta=C\rho u_r-Bu_\theta. \end{cases}$$

Here (a) *For each fixed* θ, *the function* $u(r, \theta)$ *is monotone increasing with* r, *and* $u(r, \theta)\to\infty$ *as* $r\to\infty$. *Moreover,* $u_r(r, \theta)$ *is absolutely continuous with respect to* r. *Meanwhile* v_θ *is single-valued,* $v_\theta>0$ *and the value of* $v(r, \theta)$ *is determined up to the difference of* $2k\pi$ $(k=0, \pm 1, \pm 2, \cdots)$.
(b) u *and* v *form a system of isothermal coordinates with*

$$ds^2=\tau(u, v)(du^2+dv^2),$$

$$\tau=\frac{g}{Cu_r^2-2B\rho^{-1}u_r u_\theta+A\rho^{-2}u_\theta^2}.$$

(c) *The function*

$$\varphi(u)=\operatorname*{ess\,inf}_{v\in S^1}\frac{\partial}{\partial u}\tau(u,v)$$

satisfies $\int_{u_0}^{\infty}\varphi(u)du=\infty$.

§3. Proof of Theorem 4

Let $\|\ \|$ and $(\ ,\ )$ are the norm and the inner product of $L^2(S^1;dv)$ and regard the function $f(u,v)$ as an $L^2(S^1;dv)$-valued function $f(u,\cdot)$ (shortly $f(u)$ or f). Since $-\Delta f=\lambda f$ is

$$f_{uu}+f_{vv}+\lambda\tau f=0,$$

$f(u,v)$ belongs to $H^2_{\mathrm{loc}}((u_0,\infty)\times S^1)$ provided f is locally square integrable (cf. [1]).

Now set

$$\begin{aligned}F(u)&=\|f_u\|^2+(f_{vv},f)+\lambda(\tau f,f)\\&\quad(=\|f_u\|^2-\|f_v\|^2+\lambda(\tau f,f)).\end{aligned}$$

This is a real-valued absolutely continuous function of u. Let us denote d/du by a prime in the sequel.

LEMMA 1. $$F'(u)\geq\lambda\varphi(u)\|f\|^2\quad a.e.$$

PROOF.
$$\begin{aligned}F'(u)&=2\mathrm{Re}(f_{uu}+f_{vv}+\lambda\tau f,f_u)+\lambda(\tau_u f,f)\\&=\lambda(\tau_u f,f)\\&\geq\lambda\varphi(u)\|f\|^2.\end{aligned}$$

(Justification of the differentiation under $(\ ,\ )$ is given in [12].)

LEMMA 2. *There exists a number $u_1\geq u_0$ such that $F(u_1)>0$.*

PROOF. Let I be an interval $\subset(u_0,\infty)$ where $f(u,\cdot)\neq 0$. Set $g(u)=\log\|f\|^2$ for $u\in I$. Differentiation and the Schwarz inequality yield

$$\begin{aligned}g''(u)&=2\left\{\frac{\mathrm{Re}(f_{uu},f)+\|f_u\|^2}{\|f\|^2}-\frac{2[\mathrm{Re}(f_u,f)]^2}{\|f\|^4}\right\}\\&\geq-\frac{2}{\|f\|^2}\{\|f_u\|^2+(f_{vv},f)+\lambda(\tau f,f)\}\end{aligned}$$

$$=-2e^{-g(u)}F(u). \tag{3.1}$$

If we suppose, contrary to the statement, that

$$F(u)\leq 0 \quad \text{for a.e. } u\geq u_0 \tag{3.2}$$

then

$$\begin{aligned}-F(u)&=-F(r)+\int_u^r F'(s)ds\\ &\geq\lambda\int_u^r \varphi(s)\|f(s)\|^2 ds\end{aligned}$$

in virtue of Lemma 1. The left-hand side is independent of r, while the right-hand side is an increasing function of r. Hence we can let $r\to\infty$ to have the inequality

$$-F(u)\geq\lambda\int_u^\infty \varphi(s)\|f(s)\|^2 ds. \tag{3.3}$$

Now, (3.1) implies in particular

$$g''(u)\geq 0 \quad \text{for } u\in I$$

by the assumption (3.2). Hence $g(u)$ must be bounded from below by a straight line so that it can not tend to $-\infty$ at any finite point. That means

$$f(u,\cdot)\neq 0 \quad \text{everywhere in } (u_0,\infty)$$

and (3.1) holds almost everywhere. Moreover, we can find a constant $K>0$ such that

$$g(s)-g(u)\geq -K(s-u) \quad \text{for } u_0<u<s<\infty. \tag{3.4}$$

Therefore, from (3.1), (3.3) and (3.4) it follows that

$$\begin{aligned}g''(u)&\geq 2\lambda\int_u^\infty \varphi(s)e^{g(s)-g(u)}ds\\ &\geq 2\lambda\int_u^\infty \varphi(s)e^{-K(s-u)}ds.\end{aligned}$$

Hence

$$g'(U)\geq g'(u_0)+2\lambda\int_{u_0}^U\int_u^\infty \varphi(s)e^{-K(s-u)}dsdu.$$

But we have

$$\int_{u_0}^{\infty}\int_{u}^{\infty}\varphi(s)e^{-K(s-u)}dsdu=\int_{u_0}^{\infty}\varphi(s)e^{-K(s-u_0)}\int_{u_0}^{s}e^{K(u-u_0)}duds$$
$$=\frac{1}{K}\int_{u_0}^{\infty}\varphi(s)(1-e^{-K(s-u_0)})ds$$
$$\geq\frac{1-e^{-K}}{K}\int_{u_0+1}^{\infty}\varphi(s)ds$$
$$=\infty$$

by dint of the condition $\int_{u_0}^{\infty}\varphi(s)ds=\infty$ which is assumed in Theorem 4. This shows that $g'(u)\to\infty$ and hence $\|f(u)\|^2=e^{g(u)}\to\infty$ as $u\to\infty$ which is impossible because $\int_{u_0}^{\infty}\varphi(s)\|f(s)\|^2ds<\infty$ by (3.3) while $\int_{u_0}^{\infty}\varphi(s)ds$ should be ∞. Lemma 2 is established.

LEMMA 3. *There exist numbers $C>0$ and $u_2\geq u_1$ such that for every $U\geq u_2+1$ one has*

$$\int_{u_2}^{U}(\tau f,f)du\geq C\int_{u_2+1}^{U-1}F(u)du.$$

PROOF. Let $u_3\geq u_1$, $U\geq u_3+2$ be arbitrary and $\sigma(u)=\sigma_{u3,U}(u)$ be a C^2-function which satisfies (i) $0\leq\sigma(u)\leq 1$ $(u_1\leq u<\infty)$, (ii) $\sigma(u)=0$ for $u\leq u_3$ and $u\geq U$, (iii) $\sigma(u)=1$ for $u_3+1\leq u\leq U-1$, (iv) the parts of its graph over the interval (u_3, u_3+1) and $(U-1, U)$ do not change their shape with u_3 and U so that $\mu\equiv\sup_{u_2<u<\infty}|\sigma''(u)|$ does not depend on u_3 and U.

Then, integration by parts and the equation $f_{uu}+f_{vv}+\lambda\tau f=0$ show

$$\frac{1}{2}\int_{u_3}^{U}\sigma''\|f\|^2du=\int_{u_3}^{U}\sigma\cdot\frac{1}{2}\frac{d^2}{du^2}\|f\|^2du$$
$$=\int_{u_3}^{U}\sigma\{\mathrm{Re}(f_{uu},f)+\|f_u\|^2\}du$$
$$=\int_{u_3}^{U}\sigma\{\|f_u\|^2-(f_{vv},f)-\lambda(\tau f,f)\}du.$$

Hence

$$\int_{u_3}^{U}\Big(\Big(\frac{\mu}{2}+2\sigma\lambda\tau\Big)f,f\Big)du\geq\int_{u_3}^{U}\sigma\{\|f_u\|^2-(f_{vv},f)+\lambda(\tau f,f)\}du$$
$$\geq\int_{u_3}^{U}\sigma(u)F(u)du,$$

where we used the relation $(f_{vv}, f) = -\|f_v\|^2 \leq 0$. But, because of the inequality $\inf_v \tau(u, v) \geq \int_{u_0}^{u} \varphi(s)ds + \text{const.} \to \infty$ $(u \to \infty)$ we can find positive numbers k and $u_2 \geq u_1$ for which

$$\mu \leq 2k\tau(u, v) \quad \text{for} \quad u \geq u_2, \ v \in S^1$$

holds. Accordingly, for every $U \geq u_2 + 2$ one sees

$$\begin{aligned}(2\lambda + k)\int_{u_2}^{U} (\tau f, f)du &\geq \int_{u_2}^{U}\left(\left(\frac{\mu}{2} + 2\sigma\lambda\tau\right)f, f\right)du \\ &\geq \int_{u_2}^{U} \sigma F du \\ &\geq \int_{u_2+1}^{U-1} F du\end{aligned}$$

which proves Lemma 3.

Now we turn to the proof of Theorem 4. From Lemmas 1 and 2, we have $F(u) \geq C > 0$ for $u \geq u_1$. Hence for every $U \geq u_2 + 2$, Lemma 3 shows

$$\begin{aligned}\int_{u_0 < u < U} |f|^2 d\mathcal{M} &= \int_{u_0}^{U}\int_{S^1} |f(u, v)|^2 \tau(u, v) dv du \\ &\geq \int_{u_2}^{U} (\tau f, f) du \\ &\geq C(U - u_2 - 2) \\ &\geq CU,\end{aligned}$$

where the same letter C was used to designate several different numbers. Theorem 4 is thus proved by considering $u_2 + 2$ as u_1.

§ 4. Proofs of Theorem 5 and Theorem 3

We begin with Theorem 5. Since $dr = -\rho^{-1}dt$, the metric is expressed in terms of t and θ in such a way that

$$ds^2 = \rho^2 t^{-2}(a dt^2 - 2bt dt d\theta + ct^2 d\theta^2). \tag{4.1}$$

First, we intend to show the existence of the solution of

$$\Delta u = \frac{t}{\rho^2 g}\{(Ctu_t + Bu_\theta)_t + (Bu_t + At^{-1}u_\theta)_\theta\} = 0 \quad (t \neq 0) \tag{4.2}$$

which has the singularity like $-\log t$ at $t=0$. Note that we can think of t and θ as the polar coordinates of a plane and (4.2) as an equation considered in the neighborhood of the origin.

In order to prove the existence, we will refer to the Hartman-Winter theorem which is one of the improved versions of the theory by A. Korn and L. Lichtenstein on the solvability of the Beltrami equations ([13], [14]). Before doing that, we had better express (4.2) in terms of the Cartesian coordinates through

$$x=t\cos\theta,\quad y=t\sin\theta$$

to have

$$ds^2=\rho^2t^{-2}(\alpha dx^2-2\beta dxdy+\gamma dy^2)$$

where

$$\begin{aligned}\alpha&=a\cos^2\theta+2b\cos\theta\sin\theta+c\sin^2\theta,\\ \beta&=(c-a)\cos\theta\sin\theta+b(\cos^2\theta-\sin^2\theta),\\ \gamma&=a\sin^2\theta-2b\cos\theta\sin\theta+c\cos^2\theta.\end{aligned}$$

Then $\alpha\gamma-\beta^2=ac-b^2$ and

$$\Delta=\frac{t^2}{\rho^2 g}\left\{\frac{\partial}{\partial x}\left(\tilde{C}\frac{\partial}{\partial x}+\tilde{B}\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left(\tilde{B}\frac{\partial}{\partial x}+\tilde{A}\frac{\partial}{\partial y}\right)\right\},\tag{4.3}$$

in which $\tilde{A}=\alpha/g$, $\tilde{B}=\beta/g$, $\tilde{C}=\gamma/g$.

Now we set

$$u=-\log t+\xi(t,\theta)=\int_{r_0}^{r}\rho(s)^{-1}ds+\xi(r,\theta)\tag{4.4}$$

and want to establish the existence of and estimates for $\xi(r,\theta)$. The equation $\Delta u=0$ therefore reads $\Delta\xi=\Delta(\log t)$. Applying the expression (4.3) to $\Delta\xi$ and the middle member of (4.2) to $\log t$ in place of u, we obtain

$$(\tilde{C}\xi_x+\tilde{B}\xi_y)_x+(\tilde{B}\xi_x+\tilde{A}\xi_y)_y=t^{-1}C_t+t^{-2}B_\theta.\tag{4.5}$$

We are now in a position to quote the following theorem given by Hartman and Wintner [5].

THEOREM (a) *Suppose that four functions* $A_1(x,y)$, $B_1(x,y)$, $B_2(x,y)$

and $C_1(x, y)$ *fulfill* $A_1C_1-(B_1+B_2)^2/4>0$ *and, as functions of the polar coordinates* (t, θ), *satisfy the condition of Definition 4 near* $t=0$. *Suppose also that* $D(x, y)$ *and* $E(x, y)$ *are continuous functions and* $\phi(x, y)$ *is a function of class* C^2. *Then there exists a number* R_1 *depending only on* ψ*'s of Definition 4 corresponding to* A_1, B_1, B_2, C_1 *and on the bounds of* $|A_1|, |B_1|, |B_2|, |C_1|, |D|$ *and* $|E|$ *such that for any* $R\leq R_1$, *the equation*

$$(C_1\xi_x+B_1\xi_y)_x+(B_2\xi_x+A_1\xi_y)_y+D\xi=E$$

has a C^1*-solution* in* $x^2+y^2\leq R^2$ *which satisfies* $\xi=\phi$ *on* $x^2+y^2=R^2$.
(b) *If* A_1, B_1, B_2, C_1 *are of class* C^1 *and their partial derivatives together with* D, E *and the second order derivatives of* ϕ *satisfy the condition of Definition 4, then the solution is of class* C^2.

We note that $\tilde{A}=A\cos^2\theta+2B\cos\theta\sin\theta+C\sin^2\theta$, etc. Also, for an arbitrary smooth function φ, we have $\varphi_x=\cos\theta\cdot\varphi_t-t^{-1}\sin\theta\cdot\varphi_\theta$, $\varphi_y=\sin\theta\cdot\varphi_t+t^{-1}\cos\theta\cdot\varphi_\theta$ and $\varphi_t=-\rho t^{-1}\varphi_r$. Therefore, it is easy to see from Assumption 6 that (4.5) satisfies the conditions in (b) of the Hartman-Wintner theorem with an arbitrary ϕ. Thus we are able to find a C^2-solution ξ to (4.5) in a neighborhood of $t=0$, say, $t\leq t_1$. Hence we have a solution $u=-\log t+\xi$ of (4.2) whose difference from $-\log t$ is of class C^2 in $t\leq t_1$.

Consider an arbitrary rectifiable Jordan curve Γ. If it is included in $t\leq t_1$ and does not surround nor pass through $t=0$, then from the equation (4.2) we have

$$-\iint_{\text{interior of }\Gamma}\{(Ctu_t+Bu_\theta)_t+(Bu_t+At^{-1}u_\theta)_\theta\}dtd\theta=0$$

and Green's theorem indicates that the value of the contour integral

$$\int_\Gamma(Bu_t+At^{-1}u_\theta)dt-(Ctu_t+Bu_\theta)d\theta \tag{4.6}$$

is zero. If Γ encloses $t=0$, the value of (4.6) is not zero. But such a value does not depend on Γ so long as its orientation is unchanged. We can evaluate this value by taking as Γ the counterclockwise oriented circle $\{t=\delta\}$ and making $\delta\to 0$. Namely:

$$(4.6)=\int_0^{2\pi}\{-C(\delta, \theta)\cdot\delta\cdot(-\delta^{-1}+\xi_t)-B(\delta, \theta)\cdot\xi_\theta\}d\theta$$

* The definition is described in [5]. But here we need only the case (b).

$$=\int_0^{2\pi} C(\delta,\theta)d\theta-\int_0^{2\pi}\{C(\delta,\theta)\cdot\delta\cdot\xi_t+B(\delta,\theta)\cdot\xi_\theta\}d\theta,$$

and because $\xi\in C^2$ in $t<t_1$, we have $|\xi_t|\leq$const., $|\xi_\theta|\leq$const. t. Using them together with the fact $C(t,\theta)\to 1$ $(t\to 0)$ uniformly in θ, we obtain

$$(4.6)\to 2\pi \quad \text{as} \quad \delta\to 0.$$

Thus (4.6) has the value 2π for such Γ. Let $v=v(t,\theta)$ be defined by

$$v(t,\theta)=\int_{(t_2,\theta_2)}^{(t,\theta)}(Bu_t+At^{-1}u_\theta)dt-(Ctu_t+Bu_\theta)d\theta$$

for $t<t_1$ and $\theta\in \boldsymbol{S}^1$, where (t_2,θ_2) is an arbitrarily chosen fixed point and the path of integral is taken at will within $0<t\leq t_1$. Clearly v is a multi-valued function whose values are determined up to the difference of $2n\pi$ $(n=0,\pm 1,\pm 2,\cdots)$. As is easily seen, u and v satisfy

$$\begin{cases} v_t=Bu_t+At^{-1}u_\theta, \\ v_\theta=-Ctu_t-Bu_\theta, \end{cases} \quad \text{for } t<t_1,\ \theta\in \boldsymbol{S}^1. \tag{4.7}$$

We can also write down these relations in terms of r and θ using $\partial/\partial t=-\rho t^{-1}\partial/\partial r$ to have

$$\begin{cases} v_r=Bu_r-A\rho^{-1}u_\theta, \\ v_\theta=C\rho u_r-Bu_\theta, \end{cases} \quad \text{for } r>{}^\exists r_1,\ \theta\in \boldsymbol{S}^1. \tag{4.8}$$

This implies

$$\begin{aligned} du^2+dv^2&=(Cu_r^2-2B\rho^{-1}u_ru_\theta+A\rho^{-2}u_\theta^2)(Adr^2+2B\rho drd\theta+C\rho^2d\theta^2)\\ &=\frac{Cu_r^2-2B\rho^{-1}u_ru_\theta+A\rho^{-2}u_\theta^2}{g}ds^2. \end{aligned}$$

That is,

$$ds^2=\tau(du^2+dv^2), \tag{4.9}$$

$$\tau=\frac{g}{Cu_r^2-2B\rho^{-1}u_ru_\theta+A\rho^{-2}u_\theta^2}. \tag{4.10}$$

All kinds of estimates appearing in the sequel are derived from the boundedness of ξ and its derivatives up to the second order. At first we have

$$|\xi_r|\le m\rho^{-1}t,\quad |\xi_\theta|\le mt,$$
$$|\xi_{rr}|\le m\rho^{-2}t(1+\rho'+t),\quad |\xi_{r\theta}|\le m\rho^{-1}t,\quad |\xi_{\theta\theta}|\le mt, \tag{4.11}$$

here m is some constant and $t<t_1$. From (4.4), (4.8) and the boundedness of B, we have

$$u_r=\rho^{-1}(1+\rho\xi_r)\ge\rho^{-1}(1-mt)>0,$$
$$v_\theta=C(1+\rho\xi_r)-B\xi_\theta\ge C(1-mt)-m|B|t>0$$

for sufficiently small t, or in other words, large* r, say, $r>{}^{\exists}r_1$.

Let J be the Jacobian

$$J=\begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix}=C\rho u_r^2-2Bu_ru_\theta+A\rho^{-1}u_\theta^2. \tag{4.12}$$

Then

$$\begin{aligned} J&=C\rho^{-1}(1+\rho\xi_r)^2-2B\rho^{-1}(1+\rho\xi_r)\xi_\theta+A\rho^{-1}\xi_\theta^2\\ &\ge C\rho^{-1}(1+o(1))\\ &>0 \qquad \text{for } r>{}^{\exists}r_1. \end{aligned} \tag{4.13}$$

From these relations we conclude that the curve $\{u(r,\theta)=U\}$ for each constant U is a Jordan curve which surrounds $r=0$ and the function v gives a C^1-mapping from this curve onto S^1. Moreover, the curve corresponding to a larger U encloses those to smaller U's so that u and v form an isothermal coordinate system over a part of $\mathcal{M}$.

What remains is to show (c) of the theorem. To this end, we enumerate the estimates near $r=\infty$ of the derivatives which follow from (4.11) and Assumptions 4-6. (Note that $t=o(\rho')$ by Assumption 3.)

$$\begin{aligned} &u_r=\rho^{-1}(1+\rho\xi_r)=O(\rho^{-1}),\\ &u_\theta=\xi_\theta=O(t),\\ &v_r=B\rho^{-1}(1+\rho\xi_r)-A\rho^{-1}\xi_\theta=O(\rho^{-1}(|B|+t))=o(\rho^{-1}),\\ &v_\theta=C(1+\rho\xi_r)-B\xi_\theta=C+o(1),\\ &u_{rr}=-\rho^{-2}\rho'+\xi_{rr}=-\rho^{-2}\rho'(1+O(\rho'^{-1}t))=-\rho^{-2}\rho'(1+o(1)),\\ &u_{r\theta}=\xi_{r\theta}=O(\rho^{-1}t),\\ &u_{\theta\theta}=\xi_{\theta\theta}=O(t), \end{aligned}$$

* From now on, we denote the bounds of "sufficiently large r" by the same letter r_1. It is not confusing if we replace old r_1 by a larger new one.

$$v_{rr}=Bu_{rr}+B_r u_r-A\rho^{-1}u_{r\theta}+A\rho^{-2}\rho' u_\theta-A_r\rho^{-1}u_\theta=o(\rho^{-2}\rho'),$$
$$v_{r\theta}=Bu_{r\theta}+B_\theta u_r-A\rho^{-1}u_{\theta\theta}-A_\theta\rho^{-1}u_\theta=O(\rho^{-1}t),$$
$$v_{\theta\theta}=C\rho u_{r\theta}+C_\theta\rho u_r-Bu_{\theta\theta}-B_\theta u_\theta=O(t).$$

In order to calculate $\partial/\partial u$, we first note the following chain rule. Let $f(r,\theta)$ be an arbitrary function for a while. Then

$$\begin{cases} u_r f_u+v_r f_v=f_r, \\ u_\theta f_u+v_\theta f_v=f_\theta, \end{cases}$$

and Cramér's formula shows

$$f_u=J^{-1}\begin{vmatrix} f_r & f_\theta \\ v_r & v_\theta \end{vmatrix}$$

J being the Jacobian (4.12). Since $\tau=J^{-1}\rho g$ according to (4.10), we have

$$\tau_u=J^{-2}\begin{vmatrix} (\rho g)_r & (\rho g)_\theta \\ v_r & v_\theta \end{vmatrix}-\rho g J^{-3}\begin{vmatrix} J_r & J_\theta \\ v_r & v_\theta \end{vmatrix}. \tag{4.14}$$

Applying the estimates enumerated above to

$$\begin{vmatrix} J_r & J_\theta \\ v_r & v_\theta \end{vmatrix}=v_\theta(u_{rr}v_\theta+u_r v_{r\theta}-u_{r\theta}v_r-u_\theta v_{rr})-v_r(u_{r\theta}v_\theta+u_r v_{\theta\theta}-u_{\theta\theta}v_r-u_\theta v_{r\theta}),$$

we have

$$\begin{aligned}\text{the right-hand side}&=(C+o(1))\{-C\rho^{-2}\rho'+o(\rho^{-2}\rho')+O(\rho^{-2}t)\}+o(\rho^{-2}) \\ &=-C^2\rho^{-2}\rho'+o(\rho^{-2}\rho')\end{aligned} \tag{4.15}$$

because of Assumption 3.

The first term on the right-hand side of (4.14) is estimated as

$$\begin{aligned}J^{-2}\rho' g\{C(1+o(1))+C\rho\rho'^{-1}g_r/g+\rho'^{-1}(O(B)+O(t))g_\theta/g\} \\ \geq J^{-2}\rho' g C(1-l) \quad \text{for } r\geq{}^\exists r_1\end{aligned}$$

on account of Assumption 4 with a slight change in taking the constant l within the range $0<l<2$. Using (4.13), (4.14) and (4.15) we obtain

$$\begin{aligned}\tau_u&\geq C^{-1}(1+o(1))g(1-l)\rho^2\rho'+C^{-1}(1+o(1))g\rho^2\rho' \\ &\geq(2-l)C^{-1}g\rho^2\rho' \quad \text{for } r\geq{}^\exists r_1\end{aligned}$$

by changing l again but keeping that $2-l>0$. However, since $C\to 1$

and $g(u, v)$ is bounded from below, we eventually conclude that there is a constant $K>0$ for which

$$\tau_u \geq K\rho^2\rho' \quad \text{for a.e. } r \geq {}^{\exists}r_1 \tag{4.16}$$

holds.

Now, let m be the number which appeared in (4.11). Temporarily we fix an $r \geq r_1$ and put

$$\delta = h(r; m\pi)$$

h being the function in Definition 2. Furthermore, let $\max_{0\leq\theta\leq 2\pi} u(r, \theta)$ be attained at $\theta=\theta_0$. Then it follows from $u_r=\rho^{-1}+\xi_r$, $t'=-\rho^{-1}t$ and $|\xi_r| \leq m\rho^{-1}t$ that

$$\begin{aligned} u(r+\delta, \theta_0) &= u(r, \theta_0) + \int_r^{r+\delta} \rho(s)^{-1}ds + \int_r^{r+\delta} \xi_r(s, \theta_0)ds \\ &\geq u(r, \theta_0) + m\pi t(r) - m\int_r^{r+\delta} \rho(s)^{-1}t(s)ds \\ &= u(r, \theta_0) + m(\pi-1)t(r) + mt(r+\delta). \end{aligned}$$

On the other hand, (4.11) shows

$$\begin{aligned} u(r+\delta, \theta) - u(r+\delta, \theta_0) &= \int_{\theta_0}^{\theta} \xi_\theta(r+\delta, \omega)d\omega \\ &\geq -m|\theta-\theta_0|t(r+\delta) \\ &\geq -m\pi t(r+\delta). \end{aligned}$$

Consequently

$$\begin{aligned} u(r+\delta, \theta) &\geq u(r, \theta_0) + m(\pi-1)\{t(r) - t(r+\delta)\} \\ &\geq u(r, \theta_0) \end{aligned}$$

since $t(r)$ is nonincreasing. Set $U=u(r, \theta_0)=\max_\theta u(r, \theta)$. Then

$$\min_{0\leq\theta\leq 2\pi} u(r+\delta, \theta) \geq U$$

and we have at last come to know that the contour $\{u(r, \theta)=U\}$ lies between the circles of radii r and $r+h(r; m\pi)$. This fact together with (4.16) shows that the function $\varphi(U)=\operatorname{ess\,inf}_{v\in S^1} \tau_u(U, v)$ satisfies

$$\varphi(U) \geq \operatorname*{ess\,inf}_{r\leq s\leq r+h(r;m\pi)} K\rho^2(s)\rho'(s) = K\varphi(r; m\pi)$$

hence by virtue of Assumption 2, we are led to

$$\int^{\infty}\varphi(U)dU\geq K\int^{\infty}\varphi(r;\, m\pi)\rho(r+h(r;\, m\pi))^{-1}(1+o(1))dr$$
$$=\infty.$$

Thus Theorem 5 is proved.

Let us turn to the proof of Theorem 3. Suppose that the solution of $-\Delta f=\lambda f$ does not vanish identically on $\{(r,\theta)\in\mathcal{M}|r>\alpha\}$ for any α. By dint of Theorem 5 there is an isothermal system of coordinates at least over a part of $\mathcal{M}$ which is favorable to Theorem 4. From the assumption above, f is not identically zero in that part, hence f satisfies (2.3). The inequality (2.1) is now clear if we substitute the correspondence $U\leftrightarrow R$ and the relation $U=\int_{r_0}^{R}dr/\rho(r)+o(1)$. Thus we have only to show the unique continuation property of f all over $\mathcal{M}$, because if so, $f\not\equiv 0$ should imply what we supposed above. To this end, set $\sigma=\int_{r_0}^{r}ds/\rho(s)$. This gives a C^1-diffeomorphism between (r,θ) and (σ,θ). Hence, recalling that a, b, $c\in C^1$ and $ac-b^2>0$ everywhere, we have the property of Definition 4, this time the Lipschitz condition, for A, B and C uniformly in every compact set K of δ, θ-plane. Moreover, since the Beltrami differential equations (4.8) are written as

$$v_{\sigma}=Bu_{\sigma}-Au_{\theta},\quad v_{\theta}=Cu_{\sigma}-Bu_{\theta},$$

the Hartman-Wintner theorem again indicates that there exists a positive number R_0 depending only on K such that we can find C^1-solutions u, v in each disk $\subset K$ of radius R_0. In terms of u and v the equation $-\Delta f=\lambda f$ has the representation $f_{uu}+f_{vv}+\lambda\tau(u,v)f=0$ in that disk with a continuous τ. It follows, e.g., from [1] and [18] (p. 226) that f has the unique continuation property in the sense that if a locally L^2 solution vanishes on some open set in the disk, then $f\equiv 0$ throughout the disk. From this fact the unique continuation property on the whole $\mathcal{M}$ is clear, because the δ, θ-plane is covered by the disks of such a property.

§ 5. Examples of $\rho(r)$

Our assumptions on $\rho(r)$ are rather indirect. The following examples will offer criteria which are easier to verify.

Example 1. If $\rho(r)t(r)$ is bounded and $\rho'(r)\leq 1$ a.e., and if $\rho^2\rho'$ is a nonderceasing or nonincreasing function for sufficiently large r, then Assumption 2 is fulfilled. To see this, let us suppose $\rho(r)t(r)\leq C/2$. Then for an arbitrary positive number m, we can find a number r_1 such that for $r\geq r_1$ we have

$$\rho(r)t(r)\leq C(1-mt(r)). \tag{5.1}$$

Hence

$$\int_r^{r+mC}\frac{ds}{\rho(s)}\geq mC\rho(r+mC)^{-1}\geq mC\rho(r)^{-1}(1+mC\rho(r)^{-1})^{-1}$$

because $\rho'(r)\leq 1$. But the relation (5.1) implies $C\geq\rho(r)t(r)(1+mC\rho(r)^{-1})$ and hence

$$\int_r^{r+mC}\frac{ds}{\rho(s)}\geq mt(r)$$

which means

$$h(r;m)\leq mC \quad \text{for } r\geq r_1.$$

If $\rho^2\rho'$ is nondecreasing, we have $\varphi(r;m)=\rho(r)^2\rho'(r)\geq\text{const.}>0$ so that

$$\int_{r_1}^{\infty}\varphi(r;m)\rho(r+h(r;m))^{-1}dr\geq\text{const.}\int_{r_1}^{\infty}\rho(r+mC)^{-1}dr=\infty.$$

While if $\rho^2\rho'$ is nonincreasing, we obtain

$$\varphi(r;m)\geq\rho(r+mC)^2\rho'(r+mC)$$

which implies

$$\int_{r_1}^{\infty}\varphi(r;m)\rho(r+h(r;m))^{-1}dr\geq\int_{r_1}^{\infty}\rho(r+mC)\rho'(r+mC)dr=\infty.$$

Example 2. Consider

$$\rho(r)=\rho_0(r)-\rho_0'(r)(1-k(r))\sin r \qquad (r\geq r_0)$$

where $\rho_0(r)$ is a positive function with absolutely continuous derivative. Moreover, let $k(r)$ be absolutely continuous. We assume (i) $\rho_0(r)\to\infty$, (ii) $0\leq\rho_0'(r)\leq 1$, (iii) $0<k(r)\leq 1$, (iv) $k(r)^{-1}k'(r)\to 0$, (v) $\rho_0'(r)k(r)$ is nonincreasing, (vi) $\rho_0'(r)k(r)\exp\left(\int_{r_0}^{r}[\rho_0(s)+1]^{-1}ds\right)\to\infty$, (vii) $\int_{r_0}^{\infty}\rho_0(r)\rho_0'(r)k(r)dr=$

∞, (viii) $\rho_0'(r)^{-1}\rho_0''(r)k(r)^{-1}\to 0$, where the limits are considered when $r\to\infty$. With these conditions we can show that $\rho(r)$ satisfies Assumptions 2 and 3. In fact, one has $\rho(r)\leq\rho_0(r)+1$ and

$$\rho'(r)=\rho_0'(r)\{1-(1-k(r))\cos r+k'(r)\sin r-\rho_0'(r)^{-1}\rho_0''(r)(1-k(r))\sin r\},$$
$$t(r)=\exp\left(-\int_{r_0}^{r}\rho(s)^{-1}ds\right)\leq\exp\left(-\int_{r_0}^{r}[\rho_0(s)+1]^{-1}ds\right) \tag{5.2}$$

which imply

$$0<\rho'(r)^{-1}t(r)\leq\frac{\text{const.}}{\rho_0'(r)k(r)\exp\left(\int_{r_0}^{r}[\rho_0(s)+1]^{-1}ds\right)} \qquad \text{(for large } r\text{)},$$

hence $\rho'(r)^{-1}t(r)\to 0$ $(r\to\infty)$ (Assumption 3) is shown. Moreover, (5.2) yields

$$t(r)\leq\exp\left(-\int_{r_0}^{r}\frac{\rho_0'(s)}{\rho_0(s)+1}ds\right)=\frac{\rho_0(r_0)+1}{\rho_0(r)+1}.$$

Therefore, by putting $c=2(\rho_0(r_0)+1)m$ where m is an arbitrary positive number, we have

$$\begin{aligned}\int_{r}^{r+c}\rho(s)^{-1}ds&\geq c(\rho_0(r+c)+1)^{-1}\\&=c[\rho_0(r)+c\rho_0'(r+\theta c)+1]^{-1} \quad (0<{}^{\exists}\theta<1)\\&\geq\frac{c}{2(\rho_0(r)+1)}\\&\geq mt(r)\end{aligned}$$

for large r. Hence $h(r;m)\leq c$. On the other hand, from $\rho'(r)\geq$const. $\rho_0'(r)k(r)$ and $\rho_0(r+c)-c-1\leq\rho(r)\leq\rho_0(r)+1$ it follows that

$$\begin{aligned}&\varphi(r;m)\rho(r+h(r;m))^{-1}\\&\quad\geq\rho(r+c)^{-1}\rho(r)^2\operatorname*{ess\,inf}_{r\leq s\leq r+c}\rho'(s)\\&\quad\geq(\rho_0(r+c)+1)^{-1}(\rho_0(r+c)-c-1)^2\cdot\text{const.}\,\rho_0'(r+c)k(r+c)\\&\quad\geq\text{const.}\,\rho_0(r+c)\rho_0'(r+c)k(r+c)\end{aligned}$$

for large r, where we used (i) and (v). Thus the required inequality in Assumption 2 is obtained.

REMARK. It can readily be understood that $\rho_0(r)=r^{\alpha}$ $(0<\alpha\leq 1)$ and

$\rho_0(r)=\log r$ meet the requirements (i), (ii), (iii) and $\rho_0'(r)\exp\left(\int_{r_0}^{r}[\rho_0(s)+1]^{-1}ds\right)$ $\to\infty$ and $\rho_0'(r)^{-1}\rho_0''(r)\to 0$. Moreover, if $\rho_0(r)$ fulfills these conditions, the particular choice $k(r)=1$, i.e., $\rho(r)=\rho_0(r)$ is valid. The choice $\rho_0(r)=r^{\alpha}$, $k(r)=r^{-\beta}$ $(0<\alpha\leq 1,\ 0<\beta\leq 1,\ r_0\geq 1)$ gives an example of ρ which contains the sine term.

Example 3. To illustrate the requested decreasing order of a, b and c, let $\rho(r)=r$. In this case $t=r^{-1}$ and of course ρ satisfies all the conditions required. Choose for example $a=1-r^{-\alpha}\cos\theta$, $b=r^{-\alpha}\sin\theta$, $c=1+r^{-\alpha}\cos\theta$ where the exponent α is taken larger than 2. Then $g=\sqrt{1-r^{-2\alpha}}$ and A, B, C are very close to a, b, c respectively as well as their derivatives. Therefore, $t^{-2}A_r\sim\alpha t^{\alpha-1}\cos\theta$, $t^{-2}B_r\sim-\alpha t^{\alpha-1}\sin\theta$, $t^{-3}C_r\sim-\alpha t^{\alpha-2}\cos\theta$, $t^{-1}A_\theta\sim-t^{\alpha-1}\sin\theta$, $t^{-2}B_\theta\sim-t^{\alpha-2}\cos\theta$ and $t^{-1}C_\theta\sim t^{\alpha-1}\sin\theta$. Thus we see Assumption 6 be satisfied with e.g. the Hölder condition. Assumption 5 is obvious.

References

[1] Agmon, S., Lectures on Elliptic Boundary Value Problems, van Nostrand, Princeton 1965.

[2] Donnelly, H., Spectral geometry for certain noncompact Riemannian manifolds, Math. Z. **169** (1979), 63-76.

[3] Donnelly, H., Eigenvalues embedded in the continuum for negatively curved manifolds, Michigan Math. J. **28** (1981), 53-62.

[4] Eastham, M. S. P. and H. Kalf, Schrödinger-type Operators with Continuous Spectra, Res. Notes in Math. No. 65, Pitman, Boston-London-Melbourn, 1982.

[5] Hartman, P. and A. Wintner, On uniform Dini condition in the theory of linear partial differential equations of elliptic type, Amer. J. Math. **77** (1955), 329-354.

[6] Ikebe, T. and J. Uchiyama, On the asymptotic behavior of eigenfunctions of second-order elliptic operators, J. Math. Kyoto Univ. **11** (1971), 425-448.

[7] Karp, L., Noncompact Riemannian manifolds with purely continuous spectrum, Michigan Math. J. **31** (1984), 339-347.

[8] Kato, T., Growth properties of solutions of the reduced wave equation with a variable coefficient, Comm. Pure Appl. Math. **12** (1959), 403-425.

[9] Konno, R., Estimate for eigenfunctions of the Laplace-Beltrami operator on surfaces of revolution, Memoirs Inst. Sci. Tech. Meiji Univ. **11** (1972), 65-71 (Japanese).

[10] Konno, R., Asymptotic behavior of solutions of the equation $-\Delta u+qu=\lambda u$ on spherically symmetric Riemannian manifolds, Memoirs Inst. Sci. Tech. Meiji Univ. **13** (1974), 1-8 (Japanese).

[11] Konno, R., Asymptotic behavior of eigenfunctions corresponding to positive eigenvalues of the Schrödinger-type operator on a spherically symmetric Riemannian manifold, Technical Rep. School Eng. Meiji Univ. **54** (1988), 1-15.

[12] Konno, R., Iiyama, Y. and A. Moriya, The Schrödinger-type equation on higher

dimensional Riemannian manifolds, Memoirs Inst. Sci. Tech. Meiji Univ. **31** (1993).
[13] Korn, A., Zwei Anwendungen der Methode der sukzessiven Annäherungen, Schwarz Festschrift (1914), 215-229.
[14] Lichtenstein, L., Zur Theorie der konformen Abbildung. Konforme Abbildung nicht-analytischer singularitätenfreier Flächenstücke auf ebene Gebiete, Bull. Int. de l'Acad. Sci. Cracovie, Ser. A (1916), 192-217.
[15] McKean, H. P., An upper bound to the spectrum of Δ on a manifold of negative curvature, J. Difierential Geom. **4** (1970), 359-366.
[16] Pinsky, M. A., The spectrum of the Laplacian on a manifold of negative curvature. I, J. Differential Geom. **13** (1978), 87-91.
[17] Pinsky, M. A., Spectrum of the Laplacian on a manifold of negative curvature. II, J. Differenital Geom. **14** (1979), 609-620.
[18] Reed, M. and B. Simon, Method of Modern Mathematical Physics; IV Analysis of Operators, Academic Press, New York and London, 1978.
[19] Rellich, F., Über das asymptotische Verhalten der Lösungen von $\Delta u+\lambda u=0$ unendlichen Gebieten, J'ber. Deutsch. Math. Verein. **53** (1943), 57-65.
[20] Roze, S. N., On the spectrum of a second-order elliptic operators, Mat. Sb. **80** (112) (1969), 195-209 (Russian)=Math. USSR Sb., 183-197 (English translation).
[21] Tayoshi, T., The continuous spectra of elliptic differential operators in unbounded domains, J. Math. Kyoto Univ. **27** (1987), 731-781.
[22] Encyclopedic Dictionary of Mathematics, by Math. Soc. Japan, MIT Press, Cambridge, Massachusetts, London, 1977.

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