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Logarithmic compactifications of the generalized Jacobian variety

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Introduction

Let X be a connected complete curve over an algebraically closed field k, which is reduced and has at most ordinary double points. T. Oda and C. S. Seshadri [O-S], using geometric invariant theory, constructed compactifications of its generalized Jacobian variety Pic_X^0, whose points stand for torsion-free $\mathcal{O}_X$-modules of rank one on each component of X. In their construction, each polarization on X gives rise to a compactified generalized Jacobian variety as well as a convex polyhedral decomposition of the cohomology group $H^1(\Gamma(X), \mathbf{Q})$ of the dual graph $\Gamma(X)$ of X, in terms of which they described the structure of the compactified generalized Jacobian variety.

In this paper, we give a simple construction of compactifications of the generalized Jacobian variety Pic_X^0 of X with smooth irreducible components from the viewpoint of schemes with "logarithmic structures" founded by J.-M. Fontaine and L. Illusie. Replacing the multiplicative group $\mathcal{O}_X^*$ by the abelian sheaf M_X^{gp} associated to the logarithmic structure M_X on X (1.1), we compactify the toric part of the generalized Jacobian variety Pic_X^0, and obtain compactifications of it, whose points stand for elements of the first cohomology group of the sheaf M_X^{gp}. In our construction, each admissible convex polyhedral decomposition of $H^1(\Gamma(X), \mathbf{Q})$ in the sense of 4.1 gives a compactification of the generalized Jacobian variety. In the logarithmic construction, we assume that each irreducible component of X has no self-intersection. This is because we cannot define, as a Zariski sheaf, a natural log. str., i.e. a log. str. of semi-stable type (2.6) on a curve with self-intersection. Our method is very simple and gives cohomological interpretation for the compactifications. Our result is an attempt at the compactification by "logarithmic" method. We hope to apply this method to schemes of higher dimension.

In Section 1, we recall the notion of the logarithmic structure founded by J.-M. Fontaine and L. Illusie, and sheaf theory on the Zariski site on the category of fine saturated log. schemes. In Section 2, we give two short exact sequences of cohomology groups for a semi-stable curve X with log. str., one of which allows us to compactify naturally the toric part of the generalized Jacobian variety J_X in order to get a compactified generalized Jacobian variety. The other implies that the compactified generalized Jacobian variety is the quotient, by a discrete group, of the fiber bundle associated to the torsor J_X over the Jacobian variety of the normalization of X. In Section 3, from the viewpoint of log. schemes, we construct compactifications of algebraic tori, which represent subsheaves of a finite direct sum of the logarithmic multiplicative group (2.12). In Section 4, we define the compactified generalized Jacobian variety of a semi-stable curve for each admissible convex polyhedral decomposition of $H^1(\Gamma(X), \mathbf{Q})$, and prove our main theorem.

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CONVENTION. In this paper, a ring (resp. a monoid) means a commutative ring (resp. monoid) having a unit element 1. A homomorphism of rings (resp. monoids) is assumed to preserve 1. A subring (resp. submonoid) is assumed to contain 1 of the ring (resp. monoid). For a ring (resp. monoid) P, we denote by P^* the group of invertible elements of P. All schemes are assumed to be locally noetherian. If $\mathcal{C}$ denotes a category, $\mathcal{C}(X, Y)$ denotes the set of morphisms from X to Y.

1. Logarithmic structures of Fontaine-Illusie

We recall elementary definitions on monoids. See [Ka 1] and [Ka 2, § 5].

For a monoid P, we denote by P^{gp} the group $\{ab^{-1}; a, b \in P\}$ ($a_1b_2^{-1} = a_2b_2^{-1}$ if and only if $a_1b_2c = a_2b_1c$ for some $c \in P$). A monoid P is said to be *interal* if $ab = ac$ implies $b = c$ in P, i.e., if $P \to P^{gp}$ is injective. A monoid P is said to be *fine* if P is finitely generated and integral. A monoid P is said to be *saturated* if P is integral and has the following property (when regarded as a submonoid of P^{gp}): If $a^n \in P$ for some $n \geq 1$, then $a \in P$. A monoid P is said to be *fine and saturated* (fs) if P is finitely generated and saturated.

We recall the notion of "logarithmic structures" founded by J.-M. Fontaine and L. Illusie. See [Ka 1], [Ka 2].

DEFINITION 1.1. (1) A *pre-logarithmic structure* on a scheme X is a sheaf of monoids M on the Zariski site X_{Zar} endowed with a monoid homomorphism $\alpha : M \to \mathcal{O}_X$ with respect to the multiplication on $\mathcal{O}_X$. It is denoted by (M, α), or simply by M.

(2) A pre-log. str. (M, α) on a scheme X is called a *logarithmic structure* on X if $\alpha^{-1}(\mathcal{O}_X^*)$ is isomorphic to $\mathcal{O}_X^*$ via α. Let us identify $\alpha^{-1}(\mathcal{O}_X^*)$ with $\mathcal{O}_X^*$ by α. A scheme X endowed with a log. str. (M, α) is called a *logarithmic scheme*, denoted by (X, M, α), (X, M) or simply $\underline{X}$. For a log. scheme $\underline{X}$, we denote by X the underlying scheme of $\underline{X}$.

(3) A morphism $(X, M) \to (Y, N)$ of schemes with pre-log. str.'s is defined to be a pair of a morphism of schemes $f : X \to Y$ and a homomorphism $h : f^{-1}(N) \to M$ such that the diagram

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{h} & M \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{O}_Y) & \xrightarrow{f} & \mathcal{O}_X \end{array}$$

is commutative. (We use the notation f^{-1}, not f^*, for the inverse image of a sheaf, for we shall make a special use of the notation f^*, cf. 1.4.) A morphism $(X, M) \to (Y, N)$ of schemes with pre-log. str.'s is said to be a *morphism* of log. schemes if their pre-log. str.'s are log. str.'s.

Example 1.2. (1) For any scheme X, we call the log. str. $M = \mathcal{O}_X^* \subset \mathcal{O}_X$ the trivial log. str. on X.

(2) A standard example is (X, M) where X is a regular scheme with a fixed reduced divisor D with simple normal crossings and M is the log. str. on X defined as

$$M = \{g \in \mathcal{O}_X;\ g \text{ is invertible outside } D\} \subset \mathcal{O}_X.$$

1.3. *Log. str.'s associated to pre-log. str.'s.* For a pre-log. str. (M, α) on X, we define *its associated log. str.* M^a to be the push out of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & M \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids on X_{Zar}, endowed with

$$M^a \to \mathcal{O}_X;\ (a, b) \mapsto \alpha(a)b \quad (a \in M, b \in \mathcal{O}_X^*).$$

Then M^a is universal for homomorphisms of pre-log. str.'s from M to log. str.'s on X.

1.4. *Inverse images of log. str.* For a morphism of schemes $f: Y \to X$ and for a log. str. (M, α) on X, we define the inverse image $(f^*M, f^*\alpha)$ of (M, α) by f to be the log. str. on Y associated to the pre-log. str. $(f^{-1}M, f^{-1}\alpha)$. Here $f^{-1}(M)$ denotes the sheaf-theoretic inverse image of M, and $f^{-1}\alpha$ is the composite map $f^{-1}(M) \to f^{-1}(\mathcal{O}_X) \to \mathcal{O}_Y$.

Example 1.5. Let P be a monoid, X a scheme, and assume that we are given a homomorphism $\iota: P \to \Gamma(X, \mathcal{O}_X)$, or equivalently $P_X \to \mathcal{O}_X$ where P_X denotes the constant sheaf on X corresponding to P. By abuse of notation, (X, P) denotes the scheme X with the log. str. associated to the pre-log. str. $P_X \to \mathcal{O}_X$. In particular, if A is a ring, $X = \operatorname{Spec} A[P]$, and ι is the inclusion homomorphism $P \to A[P]$, the log. str. is called the *canonical log. str.*

DEFINITION 1.6. (1) A log. scheme (X, M) is called an *fs log. scheme* if it satisfies the following condition:

(FS) There exist an open covering (U_i) of X, fs monoids P_i, and homomorphisms $h_i: P_i \to \mathcal{O}_{U_i}$ (P_i is regarded here as a constant sheaf) such that $M|_{U_i}$ is isomorphic to the log. str. associated to the pre-log. str. (P_i, h_i) for each i.

(2) For an fs log. scheme (X, M), a *chart* of M is called a homomorphism $P_X \to M$ for an fs monoid P which induces $(P_X)^a \cong M$.

In this paper we are mainly concerned with fs log. schemes.

1.7. *Finite inverse limits.* We can show that the category of fs log. schemes has finite inverse limits. The category (fs/$\underline{S}$) of fs log. schemes over an fs log. scheme $\underline{S} = (S, M_S)$ has finite inverse limits.

Let $\underline{X} = (X, M_X)$ be an fs log. scheme over $\underline{S}$. We denote by ε the covariant functor $\underline{S}' = (S', M_{S'}) \mapsto S'$ from (fs/$\underline{S}$) to the category (Sch/S) of S-schemes. In general, this functor ε does not preserve fiber products, i.e., the following diagram may not be commutative:

$$\begin{array}{ccc} \mathrm{fs}/\underline{S} & \xrightarrow{\varepsilon} & \mathrm{Sch}/S \\ \underline{X}\times_{\underline{S}}-\downarrow & & \downarrow X\times_S- \\ \mathrm{fs}/\underline{S} & \xrightarrow{\varepsilon} & \mathrm{Sch}/S. \end{array}$$

We give two sufficient conditions for the above diagram to be commutative:

(1) M_X is the inverse image of the log. str. of M_S by the structure morphism $f: X \to S$ of the underlying schemes.

(2) M_X is of semi-stable type over S. (See 2.6 for the definition.)

In the first case, the assertion follows from

$$(\mathrm{fs}/\underline{S})((Y, M_Y),\ (X, f^*M_S)) \cong (\mathrm{Sch}/S)(Y, X),$$

for any fs log. scheme (Y, M_Y) over (S, M_S). (cf. [Ka 1, (1.4)]) The assertion in the other case follows from the following lemma.

LEMMA 1.8. *Let P be an* fs *monoid and $a \in P$. Let Q be the push-out $P \oplus_N N^n$ defined by $N \to P$ $(1 \mapsto a)$ and the diagonal homomorphism $N \to N^n$. Then the canonical homomorphism $f: P \to Q$ is universally saturated, i.e., for any* fs *monoid P', and any homomorphism $g: P \to P'$, the push-out $Q \oplus_P P'$ of $Q \xleftarrow{f} P \xrightarrow{g} P'$ in the category of monoids is fine and saturated.*

PROOF. We have only to prove this lemma when P (resp. Q) is the monoid N (resp. N^n) with respect to addition and a is the element $1 \in N$. We denote by $a+b$ the product of a and $b \in P'$, and by 0 the unit element of P'. Since $-^{\mathrm{gp}}$ is the left adjoint functor of the inclusion functor from the category of abelian groups to that of monoids, this functor preserves direct limits. Therefore $(Q \oplus_P P')^{\mathrm{gp}}$ is isomorphic to the quotient of $Q^{\mathrm{gp}} \oplus P'^{\mathrm{gp}}$ by the submodule generated by $((1, 1, \cdots, 1), 0) - ((0, 0, \cdots, 0), g(1))$. It follows from this that for any $q \in (Q \oplus_P P')^{\mathrm{gp}}$, there exist unique elements $(m_1, m_2, \cdots, m_n) \in N^n$ and $p \in P'^{\mathrm{gp}}$ such that q is the class of $((m_1, m_2, \cdots, m_n), p)$ and that $m_i = 0$ for some i. Since $Q' = Q \oplus_P P'$ injects into Q'^{gp} [Ka 1, (4.1)(iv)], an element $((m_1, \cdots, m_n), p)$ of Q'^{gp} belongs to the submonoid Q' if and only if $p \in P'$. It follows from this that Q' is saturated. □

Let (X, M_X) be an fs log. scheme. We now prove that for any point $x \in X$, there exists a locally closed subset of X containing x on which the quotient $M_X/\mathcal{O}_X^*$ is a constant sheaf. This result leads us to computing the cohomology of log. str.'s.

LEMMA 1.9. *Let* (X, M_X) *be an* fs *log. scheme. For any point* $x \in X$, *there exist a neighborhood* U *of* x *and an* fs *monoid* P *and a chart* $h: P_U \to M_X|_U$ *such that the composite of* $h: P \to M_{X,x}$ *with the canonical homomorphism* $M_{X,x} \to M_{X,x}/\mathcal{O}^*_{X,x}$ *is an isomorphism.*

PROOF. Let P be the fs monoid $M_x/\mathcal{O}^*_x$ and ρ the canonical homomorphism $M_x \to M_x/\mathcal{O}^*_x$. Since P is a torsion-free fs monoid, P^{gp} is isomorphic to Z^r. We define σ to be a section of ρ^{gp}, i.e., $\rho^{gp} \circ \sigma = \mathrm{id}$. We verify $(\rho^{gp})^{-1}(M_x/\mathcal{O}^*_x) = M_x$ by the following lemma, so $\sigma^{-1}(M_x) = P$. Therefore this lemma follows from the Zariski version of [Ka 1, (2.10)]. □

LEMMA 1.10. *Let* P *be an integral monoid and* $A \subset P$ *a subgroup. Then*

$$(\rho^{gp})^{-1}(P/A) = P,$$

where $\rho: P \to P/A$ *is the canonical homomorphism.*

The proof is elementary, and is left to the reader.

From now on, a log. str. (resp. a log. scheme) means an fs log. str. (resp. an fs log. scheme) throughout this paper.

PROPOSITION 1.11. *Let* (T, M_T) *be a noetherian log. scheme, i.e., the underlying scheme* T *is noetherian. Put* $\mathcal{F} := M_T/\mathcal{O}^*_T$. *Then, there exists a family* $(U_i)_{i=1,\dots,n}$ *of subschemes of* T *such that the following conditions are satisfied*:

(1) $T = \coprod_{i=1}^n U_i$ *(as sets)*.

(2) $U_i \subset \bigcup_{j \geq i} U_j$ *is an open immersion for each* i.

(3) $\mathcal{F}|_{U_i}$ *is a constant sheaf for each* i.

PROOF. It is sufficient to show that for any maximal point η of T, there exists a neighborhood U of η with $\mathcal{F}|_U$ constant, for we apply the above claim to $T \setminus U$ and are allowed to repeat the argument because T is noetherian. Take a neighborhood U of η and a chart $h: P_U \to M_X|_U$ of $(U, M_X|_U)$ as in 1.9. We may assume that U is irreducible. Since η is the generic point of U and $P/h^{-1}(\mathcal{O}^*_y)$ is isomorphic to $M_y/\mathcal{O}^*_y$, we have $P \cong M_y/\mathcal{O}^*_y$ for all $y \in U$. Hence $P \cong (M_X/\mathcal{O}^*_X)|_U$. □

We define the Zariski topology on the category of fs log. schemes, and state some results similar to those in the classical case. We will use the following notation: $\mathcal{C}$ is the category of fs log. schemes over a log. scheme $\underline{S}=(S, M_S)$, and $h_{\underline{X}}$ is the contravariant functor $\underline{Y}\mapsto\mathcal{C}(\underline{Y}, \underline{X})$ from $\mathcal{C}$ to the category of sets.

DEFINITION 1.12. A morphism $(f, h):(X, M)\to(Y, N)$ is said to be an *open immersion* if it satisfies: (a) $f: X\to Y$ is an open immersion as a morphism of schemes and (b) $f^*N\to M$ is an isomorphism. A *covering* of an object (U, M_U) of $\mathcal{C}$ is a family $((f_\lambda, h_\lambda):(U_\lambda, M_\lambda)\to(U, M_U))_{\lambda\in\Lambda}$ of morphisms of log. schemes such that:

(1) the morphism f_λ is an open immersion for each λ.

(2) $U=\cup_\lambda U_\lambda$ as topological spaces.

We denote by $\mathcal{C}_{\mathrm{Zar}}$ the big site on $\underline{S}$ defined by these coverings. This Grothendieck topology is called the Zariski topology.

PROPOSITION 1.13. *The Zariski topology on $\mathcal{C}$ is coarser than the canonical topology, i.e. every representable presheaf is a sheaf on $\mathcal{C}_{\mathrm{Zar}}$.*

PROOF. One can verify immediately the gluing lemma of morphisms, which proves the proposition. □

REMARK. By replacing "an open immersion" in (1.12) (a) by "étale" (resp. "flat"), we obtain the étale (resp. flat) topology on $\mathcal{C}$. Then we can also prove that the canonical topology on $\mathcal{C}$ is finer than the étale (resp. flat) topology.

DEFINITION 1.14. (1) A natural transformation $f:\mathfrak{U}\to\mathfrak{F}$ of presheaves on $\mathcal{C}_{\mathrm{Zar}}$ is called an *open immersion* if it is a monomorphism in the category of presheaves on $\mathcal{C}_{\mathrm{Zar}}$ and satisfies that for any $\underline{X}\in\mathrm{Ob}(\mathcal{C})$ and any morphism $h_{\underline{X}}\to\mathfrak{F}$, $h_{\underline{X}}\times_{\mathfrak{F}}\mathfrak{U}$ is representable and the morphism $\mathrm{pr}_1: h_X\times_{\mathfrak{F}}\mathfrak{U}\to h_X$ of log. schemes is an open immersion in the sense of 1.12.

(2) Let $\mathfrak{F}$ be a presheaf on $\mathcal{C}_{\mathrm{Zar}}$. A family $(\mathfrak{F}_i)_{i\in I}$ of subpresheaves $\mathfrak{F}_i$ (resp. open subpresheaves) is a *covering* (resp. an *open covering*) if $\mathfrak{F}(X)=\cup_{i\in I}\mathfrak{F}_i(X)$ for any log. scheme X such that the underlying scheme of X is isomorphic to Spec K for some field K.

PROPOSITION 1.15. *Let $\mathfrak{F}$ be a presheaf on $\mathcal{C}_{\mathrm{Zar}}$, and $(\mathfrak{F}_i)_{i\in I}$ a family of subpresheaves of $\mathfrak{F}$. We assume the following:*

(1) *$\mathfrak{F}_i$ is represented by $\underline{X}_i\in\mathrm{Ob}(\mathcal{C})$ for all i;*

(2) *for any $i, j\in I$, the inclusion $\mathfrak{F}_i\cap\mathfrak{F}_j\hookrightarrow\mathfrak{F}_i$ is an open immersion. Then there exists a representable sheaf $\mathfrak{G}$ with open immersion $\mathfrak{F}_i\to\mathfrak{G}$ on*

C_{Zar} *such that*:

(1) $(\mathfrak{F}_i)_{i\in I}$ *is an open covering of* $\mathfrak{G}$;

(2) *the following diagram is commutative*;

$$\begin{array}{ccc} \mathfrak{F}_j\cap\mathfrak{F}_i & \longrightarrow & \mathfrak{F}_i \\ \downarrow & & \downarrow \\ \mathfrak{F}_j & \longrightarrow & \mathfrak{G} \end{array}$$

(3) $\mathfrak{G}$ *is isomorphic to the sheaf associated to* $\cup_i\mathfrak{F}_i$.

In particular, by (3) $\mathfrak{G}$ *is uniquely determined up to isomorphism.*

PROOF. One can prove the gluing lemma as in the case of schemes. This proposition, as in the classical case, follows from the gluing lemma. □

PROPOSITION 1.16. *Let* $\mathfrak{X}$ *be a sheaf on* C_{Zar}, $\sim$ *an equivalence relation on* $\mathfrak{X}$, *and* $(\mathfrak{X}_i)_i$ *a covering of* $\mathfrak{X}$ *such that for each* i, $\mathfrak{X}_i$ *is representable and such that* $\mathfrak{X}$ *is the sheaf associated to* $\cup_i\mathfrak{X}_i$. *Suppose that the equivalence relation* $\sim$ *satisfies the following*:

(1) $\mathfrak{X}_i\cap\varphi^{-1}(\varphi(\mathfrak{X}_j))\hookrightarrow\mathfrak{X}$ *is an open immersion for each* i *and* j, *where* φ *is the quotient morphism* $\mathfrak{X}\rightarrow\mathfrak{X}/\sim$;

(2) *The restriction of* φ *to* $\mathfrak{X}_i$ *is a monomorphism for each* i.

Then $\mathfrak{X}/\sim$ *is representable and* $(\mathfrak{X}_i\rightarrow\mathfrak{X}/\sim)$ *is an open covering of* $\mathfrak{X}/\sim$.

PROOF. We show that $(\varphi(\mathfrak{X}_i))_i$ satisfies the conditions (1) and (2) in 1.15. By the assumption (2), $\varphi(\mathfrak{X}_i)$ is representable. The condition (2) follows from the assumptions and the commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_i\cap\varphi^{-1}(\varphi(\mathfrak{X}_j)) & \longrightarrow & \mathfrak{X}_i \\ \downarrow & & \downarrow \\ \varphi(\mathfrak{X}_i)\cap\varphi(\mathfrak{X}_j) & \longrightarrow & \varphi(\mathfrak{X}_i). \end{array}$$

It is immediate to verify that $\mathfrak{X}/\sim$ is the sheaf associated to $\cup_i\varphi(\mathfrak{X}_i)$. □

COROLLARY 1.17. *Let* $\mathfrak{X}$ *be a representable sheaf on* C_{Zar}, *and* $(\mathfrak{X}_i)_i$ *an open covering of* $\mathfrak{X}$. *Let* H *be an abelian group acting* $\mathfrak{X}$ *in such a way that*:

(1) *For any* $h\in H\backslash\{0\}$ *and any* i, $h(\mathfrak{X}_i)\cap\mathfrak{X}_i=\emptyset$;

(2) *For any* $h\in H$ *vnd any* i, $h(\mathfrak{X}_i)=\mathfrak{X}_j$ *for some* j.

Then the quotient $\mathfrak{X}/H$ by the action of H is representable and $(\mathfrak{X}_i \to \mathfrak{X}/H)$ is an open covering of $\mathfrak{X}/H$.

PROOF. Let us denote by φ the quotient morphism $\mathfrak{X} \to \mathfrak{X}/\mathrm{H}$. Since $\mathfrak{X}_i \cap \varphi^{-1}(\mathfrak{X}_j)$ is equal to $\cup_{h \in H}(\mathfrak{X}_i \cap h(\mathfrak{X}_j))$, the corollary follows from the assumptions and 1.16. $\square$

To end this section, we recall the definition of torsors on C_{Zar}. First, we define a torsor as follows.

DEFINITION 1.18. Let $\mathfrak{S}$ be a sheaf on C_{Zar} and $\mathfrak{G}$ a sheaf of groups.

(1) A *right $\mathfrak{G}$-sheaf* over $\mathfrak{S}$ is a sheaf $\mathfrak{X}$ endowed with a right action u of $\mathfrak{G}$ on $\mathfrak{X}$ and a morphism $\mathfrak{p}: \mathfrak{X} \to \mathfrak{S}$ such that $\mathfrak{p}(u(x, g)) = \mathfrak{p}(x)$ for any $S' \in \mathrm{Ob}(C)$, $x \in \mathfrak{X}(S')$, $g \in \mathfrak{G}(S')$. We write a right $\mathfrak{G}$-sheaf $(\mathfrak{X}, u, \mathfrak{p})$ simply by $\mathfrak{X}$.

(2) A morphism $\mathfrak{X} \to \mathfrak{Y}$ of right $\mathfrak{G}$-sheaves is an $\mathfrak{S}$-morphism of sheaves compatible with the action of $\mathfrak{G}$.

(3) A right $\mathfrak{G}$-torsor over $\mathfrak{S}$ is a $\mathfrak{G}$-sheaf $(\mathfrak{X}, u, \mathfrak{p})$ over $\mathfrak{S}$ satisfying the following condition: there exists an epimorphism $f: \mathfrak{T} \to \mathfrak{S}$ in the category of sheaves on C_{Zar} such that $\mathfrak{T} \times_{\mathfrak{S}} \mathfrak{X}$ is a trivial $\mathfrak{G}$-sheaf over $\mathfrak{T}$, i.e.

$$\begin{array}{ccc} \mathfrak{T} \times \mathfrak{G} & \cong & \mathfrak{T} \times_{\mathfrak{S}} \mathfrak{X} \\ \mathrm{pr}_1 \searrow & & \swarrow \mathfrak{p}_{\mathfrak{T}} \\ & \mathfrak{T}. & \end{array}$$

One can verify fundamental properties on $\mathfrak{G}$-torsors. The proofs are essentially the same as those in the classical case. See [D-G, III, § 4, $n^\circ 1$]. Let $(\mathfrak{X}, u, \mathfrak{p})$ be a right $\mathfrak{G}$-torsor over $\mathfrak{S}$. For a left $\mathfrak{G}$-sheaf $\mathfrak{F}$, we can define the $\mathfrak{G}$-sheaf $\mathfrak{X} \vee^{\mathfrak{G}} \mathfrak{F}$ associated to $\mathfrak{F}$ as in the classical case. (*loc. cit.*)

PROPOSITION 1.19. *Let $\mathfrak{S}$ be a log. scheme over S, $(\mathfrak{X}, u, \mathfrak{p})$ a right $\mathfrak{G}$-torsor over $\mathfrak{S}$, and $\mathfrak{F}$ a representable $\mathfrak{G}$-sheaf. Then $\mathfrak{X} \vee^{\mathfrak{G}} \mathfrak{F}$ is represented by an S-log. scheme.*

PROOF. Let $(\mathfrak{S}_i)$ be an open covering of $\mathfrak{S}$ such that $\mathfrak{S}_i \times_{\mathfrak{S}} \mathfrak{X}$ is trivial for each i. We get an open covering $(\mathfrak{S}_i \times_{\mathfrak{S}} (\mathfrak{X} \vee^{\mathfrak{G}} \mathfrak{F}))_i$ of $\mathfrak{X} \vee^{\mathfrak{G}} \mathfrak{F}$. Since $\mathfrak{S}_i \times_{\mathfrak{S}} (\mathfrak{X} \vee^{\mathfrak{G}} \mathfrak{F})$ is isomorphic to $\mathfrak{S}_i \times \mathfrak{F}$ for each i, $\mathfrak{X} \vee^{\mathfrak{G}} \mathfrak{F}$ is representable by 1.15. $\square$

2. Fundamental exact sequences

The aim of this section is to prove the commutative diagram in 2.13 and the short exact sequence in 2.19. The former is important in our construction of the toroidal compactification of the generalized Jacobian variety, and the latter implies that the compactified generalized Jacobian variety is the quotient of the toroidal compactification above by a discrete group. In Section 4, we will define the compactified generalized Jacobian variety for each admissible convex polyhedral decomposition, in the sense of 4.1, of the first Q-coefficient cohomology of the dual graph of a semi-stable curve.

First we prove two key propositions which are useful in the proof of 2.10.

PROPOSITION 2.1. *Let $f: X \to Y$ be a proper surjective morphism of schemes with geometrically connected fibers, and $\mathfrak{F}$ an abelian sheaf on Y. Suppose that there exists a partition $(U_i)_{i=1,\dots,n}$ of Y into a finite number of subschemes satisfying the following:*

(1) *$U_i \hookrightarrow \bigcup_{j\geq i} U_j$ is an open immersion for each i;*

(2) *$\mathfrak{F}|_{U_i}$ is a constant sheaf for each i.*

Then the sheaf $\mathfrak{F}$ is isomorphic to $f_* f^{-1}\mathfrak{F}$.

PROOF. The proof will be carried out in three steps.

Step 1. Let $\mathfrak{F}$ be the constant sheaf P_Y corresponding to an abelian group P. Since $f^{-1}\mathfrak{F}$ is isomorphic to the constant sheaf P_X on X corresponding to P, we have:

$$\begin{aligned}(f_* f^{-1}\mathfrak{F})_y &\cong (f_* P_X)_y \\ &\cong f'_*(g'^{-1}P_X)_y = \Gamma(X', P_{X'}) \quad \text{for each } y \in Y,\end{aligned}$$

where X' is the fiber product $X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y}$ with g' the first projection and f' the second projection. By [EGA, IV (4.5.7)] and the assumption on f, X' is connected, so $\Gamma(X', P_{X'}) \cong P$. Hence we have proved the proposition when $\mathfrak{F}$ is a constant sheaf.

Step 2. Let $j: V \hookrightarrow Y$ be an open immersion, P an abelian group, $\mathfrak{F}$ the sheaf $j_! P_V$ we obtain by extending the constant sheaf P_V by zero outside V. Let $\mathfrak{G}$ be the constant sheaf P_Y on Y, i the closed immersion $Z = Y \setminus V \to Y$, $\mathfrak{G}_V$ the sheaf $j_! j^{-1}\mathfrak{G} = \mathfrak{F}$ and $\mathfrak{G}_Z$ the sheaf $i_* i^{-1}\mathfrak{G}$. Then we can verify that there exists a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 \to & \mathfrak{G}_V & \to & \mathfrak{G} & \to & \mathfrak{G}_Z & \to 0 \\ & \downarrow a & & \downarrow b & & \downarrow c & \\ 0 \to & f_*f^{-1}\mathfrak{G}_V & \to & f_*f^{-1}\mathfrak{G} & \to & f_*f^{-1}\mathfrak{G}_Z & \to 0. \end{array}$$

By Step 1, the above morphisms b and c are isomorphisms. Hence a is an isomorphism, and this completes Step 2.

Step 3. Let $\mathfrak{F}$ be a sheaf satisfying the hypotheses in the proposition. We prove the proposition by induction on the number n of subschemes U_i. Let V (resp. Z) denote U_1 (resp. $\bigcup_{i\geq 2} U_i$). Replacing $\mathfrak{G}$ by $\mathfrak{F}$ in Step 2, we have the commutative diagram with exact rows. The morphism a (resp. c) is an isomorphism by Step 2 (resp. the induction hypothesis). We conclude that b is an isomorphism. □

PROPOSITION 2.2. *Let $f: X \to Y$ be a universally open surjective morphism of schemes with geometrically irreducible fibers, and $\mathfrak{F}$ an abelian sheaf on X. Assume that, for any point $x \in X$, there exist a neighborhood U of $f(x) \in Y$ and a partition (V_i) of U such that $\mathfrak{F}|_{f^{-1}(U)}$ satisfies the assumptions* (1) *and* (2) *in* 2.1 *with respect to $(f^{-1}(V_i))$. Then, for any point $x \in X$, there exists a neighborhood U of $f(x) \in Y$ such that, for each $p \geq 1$ and each neighborhood $V \subset U$ of $f(x)$,*

$$H^p(X_{(V)}, \mathfrak{F}|_{X_{(V)}}) = 0,$$

where $X_{(V)} = X \times_Y V$.

PROOF. We will prove the proposition in two steps.

Step 1. Let $\mathfrak{F}$ be a constant sheaf on X. We will give the proof by induction on the dimension of Y and the number of irreducible components of Y. If Y is irreducible, then, by the assumption on f, X is also irreducible ([EGA, IV(4.5.7)]). Thus we have

$$H^p(X, \mathfrak{F}) = 0 \quad \text{for each } p \geq 1.$$

If $\dim Y = 0$, it follows from the assumption on f that X is a direct sum of irreducible components of X. It is easily checked that the proposition is true in this case. Let n be the dimension of Y, m the number of irreducible components of Y, x a point of X, and $y = f(x)$.

The question is local on Y, so we may assume that y belongs to each irreducible component Y_i of Y and to each irreducible component of $Y_i \cap Y_j$ for any i, j. We denote by Z (resp. U) the closed subscheme $\bigcup_{i<j}(Y_i \cap Y_j)$ (resp. the open subscheme $Y \backslash Z$). We have an exact sequence

$$0 \to \mathfrak{F}_{X(U)} \to \mathfrak{F} \to \mathfrak{F}_{X(Z)} \to 0,$$

where $X_{(U)} = X \times_Y U$, $X_{(Z)} = X \times_Y Z$. The dimension of Z is less than that of Y. Then, by the induction hypothesis, there exists a neighborhood U_y of y such that, for any integer $p \geq 1$ and any open set $V \subset U_y$ containing y,

$$H^p(X_{(Z \cap V)}, \mathfrak{F}|_{X(Z \cap V)}) = 0.$$

Since $\mathfrak{F}$ is constant and $X_{(Z \cap V)}$ is conected, we have

$$(1) \qquad H^p(X_{(V)}, \mathfrak{F}_{X(U)}|_{X(V)}) \cong H^p(X_{(V)}, \mathfrak{F}|_{X(V)})$$

for any neighborhood $V \subset U_y$ of y, and any $p \geq 1$. Let U_i be the open subset $Y \backslash (\bigcup_{j \neq i} Y_j)$ and Z_i the closed subset $Y \backslash U_i$. Since the number of irreducible components of Z_i is less than that of Y, by the induction hypothesis, in a fashion similar to the above argument, there exists a neighborhood V_i of y such that, for any neighborhood $V_i' \subset V_i$ and any $p \geq 1$,

$$(2) \qquad H^p(X_{(V_i')}, \mathfrak{F}_{X(U_i)}|_{X(V_i')}) \cong H^p(X_{(V_i')}, \mathfrak{F}|_{X(V_i')}).$$

Now let $U_0 = (\bigcap_i V_i) \cap U_y$. Because $U = \coprod_{i \geq 1} U_i$ as topological spaces, we have a commutative diagram

$$\begin{array}{ccc}
H^p(X_{(V_0)}, (\mathfrak{F}|_{X(V_0)})_{X(U \cap V_0)}) & \to & H^p(X_{(V_0)}, \mathfrak{F}|_{X(V_0)}) \\
\downarrow & & \| \\
\bigoplus_{i \geq 1} H^p(X_{(V_0)}, (\mathfrak{F}|_{X(V_0)})_{X(U_i \cap V_0)}) & \to & H^p(X_{(V_0)}, \mathfrak{F}|_{X(V_0)}) \\
\downarrow & & \| \\
H^p(X_{(V_0)}, (\mathfrak{F}|_{X(V_0)})_{X(U_i \cap V_0)}) & \to & H^p(X_{(V_0)}, \mathfrak{F}|_{X(V_0)})
\end{array}$$

for any neighborhood $V_0 \subset U_0$ of y and any $p \geq 1$. By (1) (resp. (2)), the first (resp. third) horizontal arrow is an isomorphism, hence the second horizontal arrow is also an isomorphism. Then we can verify that

$$H^p(X_{(V_0)}, (\mathfrak{F}|_{X(V_0)})_{X(U_i \cap V_0)}) = 0 \quad \text{for each } p \geq 1.$$

Therefore $H^p(X_{(V_0)}, \mathfrak{F}|_{X_{(V_0)}})=0$ for each $p\geq 1$.

Step 2. Let $\mathfrak{F}$ be a sheaf on X satisfying the hypotheses of the proposition, i.e. there exists a partition $(U_i)_{i=1,\cdots,n}$ of Y satisfying the conditions (1) and (2) in 2.1 with respect to $(f^{-1}(U_i))_i$. We will give the proof by induction on n. If $n=1$, the assertion follows from Step 1. If $n\geq 2$, we consider the long exact sequence

$$\begin{aligned}\cdots\to H^{p-1}(X, \mathfrak{F}_{X_{(Z_1)}})&\to H^p(X, \mathfrak{F}_{X_{(U_1)}})\to H^p(X, \mathfrak{F})\\ &\to H^p(X, \mathfrak{F}_{X_{(Z_1)}})\to\cdots,\end{aligned}$$

where $Z_1=X\backslash U_1$. The proof will be completed by the induction hypothesis and the following lemma. □

LEMMA 2.3. *Let X, Y, f be as in 2.2, U an open subset of Y, j the open immersion $X_{(U)}=X\times_Y U\to X$, and $\mathfrak{F}$ a constant sheaf on $X_{(U)}$. Then, for any $x\in X$, there exists a neighborhood V of $y=f(x)$ such that, for any neighborhood $V'\subset V$ of y and $p\geq 1$,*

$$H^p(X_{(V')}, (j_!\mathfrak{F})|_{X_{(V')}})=0.$$

PROOF. Let P be the abelian group to which $\mathfrak{F}$ corresponds, and $\mathfrak{G}$ the constant sheaf on X corresponding to P. By the proof of Step 1 in 2.2, both $R^pf_*\mathfrak{G}=0$ and $R^pf_*\mathfrak{G}_{X_{(Z)}}=R^pf_{(Z)*}\mathfrak{G}|_{X_{(Z)}}=0$ for each $p\geq 1$, where Z is the complement of U. Since $X_{(Z)}$ is connected for a sufficiently small neighborhood of y, the homomorphism $R^0f_*\mathfrak{G}\to R^0f_*\mathfrak{G}_{X_{(Z)}}$ is surjective. □

COROLLARY 2.4. *Let $X, Y, f, \mathfrak{F}$ be as in 2.2. Then we have*

$$R^pf_*\mathfrak{F}=0 \quad \textit{for each } p\geq 1.$$

COROLLARY 2.5. *Let $X, Y, f, \mathfrak{F}$ be as in 2.2. Then we have*

$$H^p(Y, f_*\mathfrak{F})\cong H^p(X, \mathfrak{F}) \quad \textit{for each } p\geq 0.$$

Next we prove some fundamental exact sequences which are important in the compactification of the generalized Jabobian variety.

We fix the following notation throughout this section: Let k be a field, S the affine scheme Spec k, and $\underline{S}$ the log. scheme (S, M_S) with the log. str. defined by $N\to k(1\mapsto 0)$. Let us denote by X a reduced proper geometrically connected curve over S such that:

(1) all the irreducible components of X are smooth and geometrically irreducible;

(2) the singular points of X are at most k-rational ordinary double points.

We denote by $\{Q_j\}_{j\in J}$ (resp. $\{X_i\}_{i\in I}$) the set of singular points of X (resp. irreducible components of X). Let p be the normalization $\tilde{X}=\coprod_{i\in I}X_i\to X$ of X.

We associate to X, as usual, a connected graph $\Gamma(X)=\{I, J\}$ with I as the set of vertices and with J as the set of edges. $j\in J$ is an edge joining vertices i and i' if and only if Q_j is a transversal intersection of the irreducible components X_i and $X_{i'}$. We assign and fix an arbitrary orientation to $\Gamma(X)$. Corresponding to the orientation, we name two points on the normalization $\tilde{X}$ of X lying above Q_j as Q_j^+ and Q_j^-, where Q_j^+ is on $\tilde{X}_i$ and Q_j^- is on $\tilde{X}_{i'}$, if j is an edge from i to i'.

We consider a natural log. str. on X.

DEFINITION 2.6. A log. str. on X is said to be *of semi-stable type* over $\underline{S}$ if it satisfies the following conditions:

(1) For any smooth point $x\in X$, there exists a neighborhood U of x such that $M_X|_U\cong$ the inverse image log. str. of S;

(2) For any double point $Q_j\in X$, there exist an affine open neighborhood U of Q_j and a chart (α: $N\oplus N\to M_U$, β: $N\to M_S$, γ: $N\to N\oplus N$) of the morphism $(U, M_U)\to(S, N)$ such that (i) γ is the diagonal homomorphism, and (ii) the composite of α with $M_U\to\mathcal{O}_X$ maps $(1, 0)$ (resp. $(0, 1)$) to a local equation x_i of X_i (resp. $x_{i'}$ of $X_{i'}$) if j is an edge from i to i' in $\Gamma(X)$. See [Ka 1, (2.9)] for the definition of charts of morphisms.

We say that $\underline{X}=(X, M_X)$ is a curve of semi-stable type over $\underline{S}$ if M_X is of semi-stable type over $\underline{S}$.

2.7. REMARK. The above curve X has a log. str. of semi-stable type over $\underline{S}$. Let U_j be a neighborhood of Q_j as in 2.6 (2), and V the open subset $X\backslash\{Q_j; j\in J\}$. It is verified that for any log. str. M on U_j of semi-stable type over S as in 2.6, $M|_{V\cap U_j}$ is canonically isomorphic to the inverse image of log. str. M_S. This proves the above assertion. cf. [K-N, § 1].

In the rest of this section, let $\underline{X}=(X, M_X)$ be a curve of semi-stable type over $\underline{S}$, and f the structure morphism $X\to S$ of underlying schemes. We have the following commutative diagram with exact rows.

$$\begin{array}{lccccccc}
(1) & 0 \to & \mathcal{O}_X^* & \xrightarrow{a} & p_*\mathcal{O}_{\tilde{X}}^* & \xrightarrow{b} & \bigoplus_{j\in J} \iota_{Q_j *}\mathcal{O}_{S_j}^* & \to 0 \\
& & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\
(2) & 0 \to & f^*M_S^{gp} & \xrightarrow{c} & p_*((fp)^*M_S^{gp}) & \xrightarrow{d} & \bigoplus_{j\in J} \iota_{Q_j *}M_{S_j}^{gp} & \to 0.
\end{array}$$

where the sheaf $f^*M_S^{gp}$ (resp. $(fp)^*M_S^{gp}$) is the group associated to the inverse image log. str. of M_S by f (resp. fp), the log. scheme (Q_j, M_{S_j}) is isomorphic to $\underline{S}$, and ι_{Q_j} is the closed immersion $Q_j \to X$. The columns of homomorphisms are defined by the inclusion $\mathcal{O}_X^* \to f^*M_S$ (1.1). The homomorphisms a and c are defined by $p : \tilde{X} \to X$. The homomorphism b is the map defined as follows: (i) At points of $X \backslash \{Q_j\}$, the homomorphism b sends every element of the stalk to 1, (ii) at a point Q_j with an edge $j \in J$ from i to i', b sends an element $u = (u^+, u^-)$ of the stalk $(p_*\mathcal{O}_{\tilde{X}}^*)_{Q_j} = \mathcal{O}_{\tilde{X},Q_j^+}^* \times \mathcal{O}_{\tilde{X},Q_j^-}^*$ to $u^+(Q_j^+)/u^-(Q_j^-)$. Here $u^+(x^+)$ etc. denote the evaluation of the sections u^+ etc. at those points, i.e. the image in the residue field. The definition of d is the same as that of b. It is immediate to verify that this diagram is commutative and has exact rows. We note that $f^*M_S^{gp}$ (resp. $p_*(((fp)^*M_S)^{gp})$ is isomorphic to the direct sum of $\mathcal{O}_X^*$ and the constant sheaf $\boldsymbol{Z}$ (resp. the direct sum of $p_*\mathcal{O}_{\tilde{X}}^*$ and the direct images of the constant sheaves $\boldsymbol{Z}$ on X_i).

We give a relative version of the above diagram. For an $\underline{S}$-log. scheme $\underline{S}'$, let us denote by (Y, M_Y) the fiber product $\underline{X} \times_{\underline{S}} \underline{S}'$, and by $\tilde{Y}$ the fiber product $\tilde{X} \times_S S'$. For each Q_j, we denote by S_j' (resp. $S_j'^{\pm}$) the fiber product $Q_j \times_X Y$ (resp. $Q_j^{\pm} \times_{\tilde{X}} \tilde{Y}$) in (Sch/$S$). The second projection $Y \to S'$ (resp. $\tilde{Y} \to Y$, resp. $S_j' \to Y$) is denoted by $f_{S'}$ (resp. $p_{S'}$, resp. ι_{Q_j}). Replacing X (resp. S, resp. $\tilde{X}$, resp. S_j) in the diagram above, we have the following proposition.

PROPOSITION 2.8. *Let the notation be as in 2.7. For any $\underline{S}$-log. scheme $\underline{S}'$, we have a commutative diagram with exact rows*

$$\begin{array}{lccccccc}
(1) & 0 \to & \mathcal{O}_Y^* & \xrightarrow{a} & p_{S'*}\mathcal{O}_{\tilde{Y}}^* & \xrightarrow{b} & \bigoplus_{j\in J} \iota_{Q_j *}\mathcal{O}_{S_j'}^* & \to 0 \\
& & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\
(2) & 0 \to & f_{S'}^*M_{S'}^{gp} & \xrightarrow{c} & p_{S'*}((f_{S'}p_{S'})^*M_{S'}^{gp}) & \xrightarrow{d} & \bigoplus_{j\in J} \iota_{Q_j *}M_{S_j'}^{gp} & \to 0.
\end{array}$$

PROOF. The definitions of homomorphisms in the diagram are the same as those in 2.7. We consider the short exact sequence

$$0\to f_{S'}^{-1}(M_{S'}^{gp}/\mathcal{O}_{S'}^{*})\to p_{S'*}(f_{S'}p_{S'})^{-1}(M_{S'}^{gp}/\mathcal{O}_{S'}^{*})\to \bigoplus_{j\in J}\iota_{Q_j*}(M_{S'_j}^{gp}/\mathcal{O}_{S'_j}^{*})\to 0.$$

It follows from [Ka 1, (1.4.1)] that the group on the left (resp. right) hand side is the cokernel of α (resp. γ). Since $p_{S'}$ is integral, $R^1p_{S'*}\mathcal{O}_{\tilde{Y}}^{*}=0$ ([EGA, IV(21.8.1)]). This implies that the middle group is the cokernel of β. Using homological algebra, we have only to show that (1) is an exact sequence. Let $y\in Y$, $y^{\pm}\in\tilde{Y}$ over y, and $t=f_{S'}(y)$. We reduce the proof to the exactness of the following sequence.

$$0\to 1+\mathfrak{m}_{Y,y}\to(1+\mathfrak{m}_{Y,y^+})\times(1+\mathfrak{m}_{Y,y^-})\to 1+\mathfrak{m}_{T,t}\to 0,$$

where $\mathfrak{m}_{Y,y}$ etc. are the maximal ideal of $\mathcal{O}_{Y,y}$ etc. This follows from the fact that X has at most ordinary double points and that $S'\to S$ is flat. □

We recall the Picard functor and define the logarithmic Picard fuuctor as sheaves on the big Zariski site on the category (fs/$\underline{S}$) of fs log. schemes over $\underline{S}$ (1.12).

DEFINITION 2.9. $\mathrm{Pic}_{\underline{X}/\underline{S}}$ (resp. $P_{\underline{X}/\underline{S}}^{\log}$, resp. $\mathrm{Pic}_{\underline{X}/\underline{S}}^{\log}$) on $\mathcal{C}_{\mathrm{Zar}}$ is the sheaf associated to the presheaf

$$\underline{S}'\mapsto H^1_{\mathrm{Zar}}(X_{S'},\mathcal{O}_{X_{S'}}^{*})$$
$$(\text{resp. } \underline{S}'\mapsto H^1_{\mathrm{Zar}}(X_{S'},f_{S'}^{*}M_{S'}^{gp}),$$
$$\text{resp. } \underline{S}'\mapsto H^1_{\mathrm{Zar}}(X_{S'},M_{X_{S'}}^{gp})),$$

where $(X_{S'},M_{X_{S'}})$ is the fiber product $\underline{X}\times_{\underline{S}}\underline{S}'$ in (fs/$\underline{S}$) and $f_{S'}$ is the second projection $X_{S'}\to S'$ of underlying schemes.

It is clear that this definition is functorial with respect to X. The underlying scheme of $\underline{X}\times_{\underline{S}}\underline{S}'$ is isomorphic to the fiber product $X\times_S S'$ because $\underline{X}$ is a curve of semi-stable type over $\underline{S}$ (1.7). By definition and the above, we have:

$$\mathrm{Pic}_{\underline{X}/\underline{S}}(S')=H^0(S',R^1(f_{S'})_*\mathcal{O}_{X_{S'}}^{*});$$
$$P_{\underline{X}/\underline{S}}^{\log}(S')=H^0(S',R^1(f_{S'})_*((f_{S'}^{*}M_{S'})^{gp}));$$
$$\mathrm{Pic}_{\underline{X}/\underline{S}}^{\log}(S')=H^0(S',R^1(f_{S'})_*(M_{X_{S'}}^{gp})).$$

It is immediate to verify that $\mathrm{Pic}_{\underline{X}/\underline{S}}$ is represented by the usual Picard scheme $\mathrm{Pic}_{X/S}$ with the inverse image log. str. of M_S. Let us denote by $\mathrm{Pic}_{X/S}^0$ (resp. $\mathrm{Pic}_{\underline{X}/\underline{S}}^0$) the connected component of the origin of $\mathrm{Pic}_{X/S}$ (resp.

with the inverse image log. str. of M_S), called the generalized Jacobian variety of X over S.

THEOREM 2.10. *Let $\underline{S}$ be a log. scheme* (Spec k, $k^*\oplus N$) *and $\underline{X}$ a curve of semi-stable type over $\underline{S}$. Suppose that the underlying scheme X is smooth over S. Then $P^{\log}_{\underline{X}/\underline{S}}$ is isomorphic to* $\mathrm{Pic}_{X/S}$.

PROOF. We consider a short exact sequence

$$0\to\mathcal{O}_Y^*\xrightarrow{\alpha}(f_{S'}^*M_{S'})^{\mathrm{gp}}\to f_{S'}^{-1}(M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^*)\to 0.$$

Here the notation is an in 2.7. We have only to show that $R^1(f_{S'})_*(\alpha)$ is an isomorphism. By 2.4 and 1.11, $R^1f_{S'*}(f_{S'}^{-1}(M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^*))=0$. It follows Lemma 2.11 (3) that $R^1f_{S'*}(\alpha)$ is injective. □

LEMMA 2.11. *Let the notation be as in* 2.7. *Then we have*:

(1) $M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^*\cong(f_{S'})_*f_{S'}^{-1}(M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^*)\cong(f_{S'})_*(f_{S'}^*M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^*)$;

(2) $M_{S'}^{\mathrm{gp}}\cong(f_{S'})_*f_{S'}^*M_{S'}^{\mathrm{gp}}$;

(3) $0\to\mathcal{O}_{S'}^*\to f_{S'*}f_{S'}^*M_{S'}^{\mathrm{gp}}\to f_{S'*}f_{S'}^{-1}(M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^*)\to 0$ *is exact.*

PROOF. The question being local on S', we may assume that S' is noetherian. (1) follows from 1.11 and 2.1. Apply (1) to the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0\to & \mathcal{O}_{S'}^* & \to & M_{S'}^{\mathrm{gp}} & \to & M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^* & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0\to & f_{S'*}\mathcal{O}_Y^* & \to & (f_{S'})_*f_{S'}^*M_{S'}^{\mathrm{gp}} & \to & (f_{S'})_*f_{S'}^{-1}(M_{S'}^{\mathrm{gp}}/\mathcal{O}_{S'}^*) & \to & R^1(f_{S'})_*\mathcal{O}_Y^* \end{array}$$

Since $\mathcal{O}_{S'}^*\cong(f_{S'})_*\mathcal{O}_Y^*$, we verify (2) and (3), using (1) and the five lemma. □

2.12. Now we define an abelian sheaf $\boldsymbol{G}_m$ (resp. $\boldsymbol{G}_m^{\log}$) on $\mathcal{C}_{\mathrm{Zar}}$, called the multiplicative group (resp. the log. multiplicative group), as follows: For $(X, M_X)\in\mathrm{Ob}(\mathcal{C})$, $\boldsymbol{G}_m(X, M_X)$ (resp. $\boldsymbol{G}_m^{\log}(X, M_X)$) is defined as the group $\Gamma(X, \mathcal{O}_X^*)$ (resp. the group $\Gamma(X, M_X^{\mathrm{gp}})$) of global rections of $\mathcal{O}_X^*$ (resp. M_X^{gp}). The restriction maps of $\boldsymbol{G}_m$ and $\boldsymbol{G}_m^{\log}$ are defined in a natural way. For a free abelian group A of finite rank, we define an abelian sheaf $A\otimes\boldsymbol{G}_m$ (resp. $A\otimes\boldsymbol{G}_m^{\log}$) on $\mathcal{C}_{\mathrm{Zar}}$ by $(X, M_X)\mapsto A\otimes_{\boldsymbol{Z}}\Gamma(X, \mathcal{O}_X^*)$ (resp. $(X, M_X)\mapsto A\otimes_{\boldsymbol{Z}}\Gamma(X, M_X^{\mathrm{gp}})$). By the definition of log. str., $\boldsymbol{G}_m$ is a subgroup of $\boldsymbol{G}_m^{\log}$, so $A\otimes\boldsymbol{G}_m$ acts on $A\otimes\boldsymbol{G}_m^{\log}$ by translation.

COROLLARY 2.13. *There exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \to & H^1(\Gamma(X), \boldsymbol{Z})\otimes \boldsymbol{G}_m & \to & \mathrm{Pic}_{\underline{X}/\underline{S}} & \to & \bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^1(\Gamma(X), \boldsymbol{Z})\otimes \boldsymbol{G}_m^{\log} & \to & P_{\underline{X}/\underline{S}}^{\log} & \to & \bigoplus_{i\in I}P_{\underline{X}_i/\underline{S}}^{\log} & \to 0, \end{array}$$

where $\underline{X}_i$ is the irreducible component X_i of X with the inverse image of the log. str. M_S. Moreover, the right vertical arrow is an isomorphism and the left square is a push-out.

PROOF. We are done by the exact sequences (1) and (2) in 2.8, as well as 2.10, 2.11 (2) and the fact that $H^1(Y, p_{S'*}\mathcal{O}_{\tilde{Y}}^*)=H^1(\tilde{Y},\mathcal{O}_{\tilde{Y}}^*)$ [EGA, IV(21.8.2)]. □

DEFINITION 2.14. By 2.13, we define a homomorphism

$$d\colon P_{\underline{X}/\underline{S}}^{\log}\to \bigoplus_{i\in I}\boldsymbol{Z}$$

by the zero homomorphism from $H^1(\Gamma(X), \boldsymbol{Z})\otimes\boldsymbol{G}_m^{\log}$ to $\bigoplus_{i\in I}\boldsymbol{Z}$ and the composite of the degree map $\bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}\to\bigoplus_{i\in I}\boldsymbol{Z}$ and the surjection $\mathrm{Pic}_{\underline{X}/\underline{S}}\to\bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}$. We denote by $\tilde{J}_{\underline{X}/\underline{S}}^{\log}$ the kernel of the homomorphism d, and by $\bigoplus_{i\in I}\tilde{J}_{\underline{X}_i/\underline{S}}^{\log}$ the image of $\bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}^0$ under the vertical homomorphism on the right hand side in 2.13 above.

COROLLARY 2.15. *There exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \to & H^1(\Gamma(X), \boldsymbol{Z})\otimes \boldsymbol{G}_m & \to & \mathrm{Pic}_{\underline{X}/\underline{S}}^0 & \to & \bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}^0 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^1(\Gamma(X), \boldsymbol{Z})\otimes \boldsymbol{G}_m^{\log} & \to & \tilde{J}_{\underline{X}/\underline{S}}^{\log} & \to & \bigoplus_{i\in I}\tilde{J}_{\underline{X}_i/\underline{S}}^{\log} & \to 0, \end{array}$$

where $\bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}^0$ (resp. $\mathrm{Pic}_{\underline{X}/\underline{S}}^0$) is the kernel of the degree map $\bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}\to\bigoplus_{i\in I}\boldsymbol{Z}$ (resp. the inverse image of $\bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}^0$ under the homomorphism $\mathrm{Pic}_{\underline{X}/\underline{S}}\to\bigoplus_{i\in I}\mathrm{Pic}_{\underline{X}_i/\underline{S}}$).

PROOF. This follows from 2.13 and the above definitions. □

To conclude this section, we study the relationship between $P_{\underline{X}/\underline{S}}^{\log}$ and $\mathrm{Pic}_{\underline{X}/\underline{S}}^{\log}$, and define the sheaf $J_{\underline{X}/\underline{S}}^{\log}$.

PROPOSITION 2.16. *Let $\underline{X}$ be a curve of semi-stable type over $\underline{S}$, $\underline{S}'=$*

$(S', M_{S'})$ an $\underline{S}$-log. scheme, and $\underline{Y}=(Y, M_Y)$ the fiber product $\underline{X}\times_{\underline{S}}\underline{S}'$. Let the notation be as in 2.7. Then we have a short exact sequence

$$0\to f_{S'}^* M_{S'}^{gp}\to M_Y^{gp}\to \bigoplus_{j\in J}\iota_{Q_j*}Z\to 0,$$

where Z denotes the constant sheaf on S'_j.

PROOF. If $S'=S$, we have a short exact sequence

$$(1)\qquad 0\to(f^*M_S)^{gp}/\mathcal{O}_X^*\to M_X^{gp}/\mathcal{O}_X^*\xrightarrow{\alpha}\bigoplus_{j\in J}\iota_{Q_j*}Z\to 0,$$

where α is the homomorphism defined as follows. Since M_X is of semi-stable type over $\underline{S}$, the quotient $M_{X,Q_j}^{gp}/\mathcal{O}_{X,Q_j}^*$ is isomorphic to $Z\oplus Z$ by $[\xi_i]\mapsto(1,0)$ and $[\xi_{i'}]\mapsto(0,1)$ if j is an edge from i to i' in $\Gamma(X)$ and the image of ξ_i (resp. $\xi_{i'}$) of M_{X,Q_j} in $\mathcal{O}_X$ is a local equation of X_i (resp. $X_{i'}$). We define α at Q_j by $M_{X,Q_j}^{gp}/\mathcal{O}_{X,Q_j}^*\cong Z\oplus Z\to Z((a,b)\mapsto a-b)$. If φ_X is the first projection $Y\to X$ of underlying schemes, the inverse image of (1) by φ_X is also an exact sequence. Thus we have

$$(2)\qquad 0\to((f\varphi_X)^*M_S)^{gp}\to(\varphi_X^*M_X)^{gp}\to\bigoplus_{j\in J}\iota_{Q_j*}Z\to 0.$$

Since $\underline{Y}$ is the fiber product $\underline{X}\times_{\underline{S}}\underline{S}'$, the square

$$(3)\qquad \begin{array}{ccc} (f\varphi_X)^*M_S^{gp} & \longrightarrow & \varphi_X^*M_X^{gp} \\ \downarrow & & \downarrow \\ f_{S'}^*M_{S'}^{gp} & \longrightarrow & M_Y^{gp} \end{array}$$

is a push-out. By (2) and (3), we have the short exact sequence in the proposition. □

COROLLARY 2.17. *We have an exact sequence*

$$\bigoplus_{j\in J}Z\xrightarrow{\delta}P_{\underline{X}/\underline{S}}^{\log}\to\mathrm{Pic}_{\underline{X}/\underline{S}}^{\log}\to 0,$$

where δ is induced by the connecting homomorphism $H^0(\bigoplus_{j\in J}\iota_{Q_j*}Z)\to H^1(f_{S'}^*M_{S'}^{gp})$.

PROOF. The corollary follows from the cohomology long exact sequence applied to the short exact sequence in 2.16. □

2.18. We define the degree map $\mathrm{Pic}^{\log}_{\underline{X}/\underline{S}} \to \boldsymbol{Z}$ and a sheaf $J^{\log}_{\underline{X}/\underline{S}}$ whose subsheaves give compactifications of the generalized Jacobian variety. We identify the cochain complex of the graph $\Gamma(X)$ with the chain complex [O-S, I. 4]. Let ∂ be the boundary map from the group $C_1(\Gamma(X), \boldsymbol{Z})$ of 1-chains to the group $C_0(\Gamma(X), \boldsymbol{Z})$ of 0-chains. Then ∂ is the composite of the homomorphim δ in 2.17 with the homomorphism d in 2.14. Consider the following diagram.

$$\begin{array}{ccccccc} & \bigoplus_{j\in J}\boldsymbol{Z} & \xrightarrow{\delta} & P^{\log}_{\underline{X}/\underline{S}} & \to & \mathrm{Pic}^{\log}_{\underline{X}/\underline{S}} & \to 0 \\ & \downarrow & \searrow\partial & \downarrow d & & \downarrow \mathrm{degree} & \\ 0\to & \mathrm{Im}\,\partial & \to & \bigoplus_{i\in I}\boldsymbol{Z} & \xrightarrow{\mathrm{sum}} & \boldsymbol{Z} & \to 0. \end{array}$$

The composite of ∂ with the sum map is trivial. Hence there exists a unique homomorphism $\mathrm{Pic}^{\log}_{\underline{X}/\underline{S}} \to \boldsymbol{Z}$ making the above diagram commutative. We call it the *degree map*. Let us denote by $J^{\log}_{\underline{X}/\underline{S}}$ the kernel of the degree map. Applying the snake lemma to the above diagram, we conclude the following theorem:

THEOREM 2.19. *Let $\underline{X}$ be a curve of semi-stable type over $\underline{S}$. Let us denote by $\tilde{J}^{\log}_{\underline{X}/\underline{S}}$ the kernel of the homomorphism $d: P^{\log}_{\underline{X}/\underline{S}} \to \bigoplus_{i\in I}\boldsymbol{Z}$ (2.14). Then we have a short exact sequence*

$$0 \to H_1(\Gamma(X), \boldsymbol{Z}) \to \tilde{J}^{\log}_{\underline{X}/\underline{S}} \to J^{\log}_{\underline{X}/\underline{S}} \to 0.$$

PROOF. We have only to prove that the homomorphism $H_1(\Gamma(X), \boldsymbol{Z}) \to \tilde{J}^{\log}_{\underline{X}/\underline{S}}$ is injective. It follows from 2.1 and the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0\to & f_{S'}^{*}M_{S'}^{\mathrm{gp}}/\mathcal{O}_Y^{*} & \to & M_Y^{\mathrm{gp}}/\mathcal{O}_Y^{*} & \to & \bigoplus_{j\in J}\iota_{Q_j *}\boldsymbol{Z} \\ & \| & & \downarrow\alpha & & \downarrow\beta \\ 0\leftarrow & f_{S'}^{*}M_{S'}^{\mathrm{gp}}/\mathcal{O}_Y^{*} & \to & p_{S'*}((f_{S'}p_{S'})^{*}M_{S'}^{\mathrm{gp}}/\mathcal{O}_Y^{*} & \to & \bigoplus_{j\in J}\iota_{Q_j *}M^{\mathrm{gp}}_{S'_j} \end{array}$$

Here α is defined by means of the map $M_X^{\mathrm{gp}}/\mathcal{O}_X^{*} \to p_*(fp)_*M_S^{\mathrm{gp}}/\mathcal{O}_X^{*}$ while β is defined to be the composite $\boldsymbol{Z} \to M_S^{\mathrm{gp}} (1 \mapsto (1, 1) \in \mathcal{O}_S^{*} \oplus \boldsymbol{Z})$ with $M_S^{\mathrm{gp}} \to M_{S'}^{\mathrm{gp}}$. □

3. Compactifications of the tori

In this section, we recall facts on convex polyhedral cones and polyhedra in $\boldsymbol{R}^n$ ([Od 1], [KKMS]), and give compactifications of tori which play an important role in compactifying the generalized Jacobian variety. We use the following notation: Let N be a finitely generated free abelian group, $N_{\boldsymbol{Q}}$ the $\boldsymbol{Q}$-vector space $N\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, N' the dual abelian group $\mathrm{Hom}_{\mathrm{Ab}}(N, \boldsymbol{Z})$, $N'_{\boldsymbol{Q}}$ the dual vector space of $N_{\boldsymbol{Q}}$.

We say that a subset $\sigma\subset N$ (resp. $\sigma\subset N_{\boldsymbol{Q}}$) is a *convex polyhedral cone* in N (resp. $N_{\boldsymbol{Q}}$) if $\rho={}'_{\boldsymbol{R}}\cap N$ (resp. $\sigma=\sigma'_{\boldsymbol{R}}\cap N_{\boldsymbol{Q}}$) for some convex rational polyhedral cone $\sigma'_{\boldsymbol{R}}$ ([KKMS, § 1]). We regard convex polyhedral cones in N (resp. in $N_{\boldsymbol{Q}}$) as submonoids of N (resp. $N_{\boldsymbol{Q}}$) with respect to addition. By [KKM, § 1] and simple computation, one concludes that the set of convex polyhedral cones is the set of fs submonoids P of N with N/P^{gp} torsion-free. If σ is a convex polyhedral cone in N (resp. $N_{\boldsymbol{Q}}$), we denote by $\sigma^{\vee}$ the set of $r\in N'$ (resp. $N'_{\boldsymbol{Q}}$) such that $\langle a, r\rangle\geq 0$ for all $a\in\sigma$. Here $\langle\ ,\ \rangle: N\times N'\to\boldsymbol{Z}$ (resp. $N_{\boldsymbol{Q}}\times N'\to\boldsymbol{Q}$) is the duality pairing. $\sigma^{\vee}$ is said to be the *positive dual* of σ in N (resp. in N). For a subset τ of N , we denote by $\tau^{\perp}$ the set

$$\{r\in N'\,;\langle a, r\rangle=0 \quad \text{for any } a\in\tau\}.$$

A subset τ of σ is called a *face* of σ if $\tau=\{a\in\sigma; \langle a, r\rangle=0\}$ for some $r\in\sigma^{\vee}$. The faces of σ correspond to "prime ideals" of σ as a monoid as follows.

DEFINITION 3.1. Let P be a monoid. A subset I of a monoid P is said to be an *ideal* if $IP\subset I$. An ideal $\mathfrak{p}$ of P is *prime* $P\backslash\mathfrak{p}$ is a submonoid of P. For a submonoid S of P, we denote by $S^{-1}P$ the monoid $\{s^{-1}a; a\in P, s\in S\}$ where $s_1^{-1}a_1=s_2^{-1}a_2$ if and only if $ts_1a_2=ts_2a_1$ for some $t\in S$. In particular, for a prime ideal $\mathfrak{p}$ of P, we define $P_{\mathfrak{p}}=S^{-1}P$, where $S=P\backslash\mathfrak{p}$. See [Ka 2] and [Od 2, I, § 5].

PROPOSITION 3.2. *Let σ be a convex polyhedral cone in N. Then the set of faces of σ is equal to the set of submonoids $\tau\subset\sigma$ such that $\sigma\backslash\tau$ is a prime ideal. The map $\tau\mapsto\sigma^{\vee}\cap\tau^{\perp}$ is an order reversing bijection between faces of σ and those of $\sigma^{\vee}$. Moreover, for a face τ of σ, the positive dual $\tau^{\vee}$ is isomorphic to $(\sigma^{\vee})_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal $\sigma\backslash\tau$.*

PROOF. This is proved by straightforward computation. See [Od 2, I, § 5]. □

In the rest of this section, let $\underline{S}=(S, M_S)$ be the log. scheme (Spec k, $k^*\oplus N$) defined by $1\in N\mapsto 0\in k$, π the global section $1\in N\subset\Gamma(S, M_S)$, and

$\mathcal{C}$ the category of fs log. schemes over $\underline{S}$. We now define subsheaves of a finite direct sum of $\boldsymbol{G}_m^{\log}$ (2.12), which play an important role in compactifying tori. The log. construction which follows is inspired by the example in [Mum, 6].

Let N_0 (resp. N_1) be the free abelian group of rank 1 (resp. of rank r) with basis e_0 (resp. $e_1, \cdots, e_r$), N_0' (resp. N_1') the dual group of N_0 (resp. N_1) with the dual basis f_0 (resp. $f_1, \cdots, f_r$). Let N be the direct sum $N_0 \oplus N_1$ and N' the dual group of N. First we have a pairing

$$(N \otimes \boldsymbol{G}_m^{\log}) \times N' \longrightarrow \boldsymbol{G}_m^{\log}$$

defined by $((s_0, s_1) \otimes x, (a_0, a_1)) \mapsto \pi^{\langle s_0, a_0 \rangle} x^{\langle s_1, a_1 \rangle}$ for $(s_0, s_1) \in N_0 \oplus N_1 = N$ and $(a_0, a_1) \in N_0' \oplus N_1' = N'$, where $\langle\ ,\ \rangle : N_1 \times N_1'$ (resp. $N_0 \times N_0'$) $\to \boldsymbol{Z}$ is the duality pairing. We denote this pairing by $\langle\ ,\ \rangle$, too. For a bounded convex polyhedron Δ in $N_1 \otimes \boldsymbol{Q} = N_{1\boldsymbol{Q}}$, we denote by $\Delta^{\sim}$ the polyhedral cone $\{\lambda(e_0+\xi);\ \xi \in \Delta,\ \lambda \in \boldsymbol{Q}_{\geq 0}\}$. Using this pairing, for $(X, M_X) \in \mathrm{Ob}(\mathcal{C})$ and a bounded convex polyhedron Δ, we define $T_\Delta^{\log}(X, M_X)$ to be the set

$$\{x \in N_1 \otimes \boldsymbol{G}_m^{\log}(X, M_X);\ \langle e_0 \otimes \pi_X + x, a \rangle \in \Gamma(X, M_X) \subset \Gamma(X, M_X^{\mathrm{gp}}) \quad \text{for all } a \in (\Delta^{\sim})^{\vee} \cap N'\},$$

where π_X is the image of π by the structure morphism $\underline{X} \to \underline{S}$. By the functoriality of $\langle\ ,\ \rangle$, the sheaf $T_\Delta^{\log}$ of sets is well-defined. We can verify that for bounded convex polyhedra Δ and Δ' in $N_{1\boldsymbol{Q}}$, $T_{\Delta \cap \Delta'}^{\log}$, is the intersection of $T_\Delta^{\log}$ and $T_{\Delta'}^{\log}$ [Od 1, Appendix A.1].

PROPOSITION 3.3. *Let the notation be as above. The sheaf $T_\Delta^{\log}$ is represented by the log. affine scheme* $(\mathrm{Spec}\, k[(\Delta^{\sim})^{\vee} \cap N']/(f_0), (\Delta^{\sim})^{\vee} \cap N')$ *over* (S, M_S).

PROOF. Let $\underline{X}_\Delta := (\mathrm{Spec}\, k[(\Delta^{\sim})^{\vee} \cap N']/(f_0), (\Delta^{\sim})^{\vee} \cap N')$. Let e_i (resp. f_i) be the basis of N (resp. the dual basis of N') as above. For any morphism $(g, g^M) : \underline{S}' \to \underline{X}_\Delta$, we have a global section $\Sigma e_i \otimes g^M(f_i) \in N_1 \otimes \boldsymbol{G}_m^{\log}(\underline{S}')$. Since g^M is a homomorphism $M_{X_\Delta} \to M_{S'}$, any element $\Sigma a_i f_i \in (\Delta^{\sim})^{\vee} \cap N'$ $(a_i \in \boldsymbol{Z})$ maps to $\Sigma a_i e_i \otimes g^M(f_i) \in M_{S'}$. Hence $\Sigma e_i \otimes g^M(f_i) \in T_\Delta^{\log}(\underline{S}')$. Conversely, give $\Sigma s_j \otimes x_j \in N_1 \otimes \Gamma(S', M_{S'}^{\mathrm{gp}})$, we have a homomorphism $(\Delta^{\sim})^{\vee} \cap N' \to \Gamma(S', M_{S'})$ defined by $a_0 f_0 + a \mapsto \pi^{a_0} \prod x_j^{\langle s_j, a \rangle}$, where $a \in N_1'$ and $a_0 \in \boldsymbol{Z}$. This homomorphism gives a unique $\underline{S}$-morphism $\underline{S}' \to \underline{X}_\Delta$. $\square$

We study functors $T_\Delta^{\log}$ more closely. Our results are essentially due to [Mum, 6]. Let A be a complete discrete valuation ring with quotient field K, $\tilde{S}=\operatorname{Spec} A$, and η (resp. s) the generic point (resp. the closed point) of $\tilde{S}$. Let $\tilde{G}=G_{m,A}^{\oplus r}$.

PROPOSITION 3.4. *There exists a one-to-one correspondence among:*

(I) *normal affine schemes U of finite type over A, such that $U_\eta=\tilde{G}_\eta$, $U_s\neq 0$, and such that $\tilde{G}$ acts on U extending the translation action of $\tilde{G}_\eta$ on U_η.*

(II) *nonempty closed, bounded polyhedra $\Delta\subset N_{1Q}$*

and

(III) *$\underline{S}$-log. schemes* $(\operatorname{Spec} k[P]/(f_0), P)$ *such that P is an* fs *submonoid of N' with* (a) *π goes to f_0 and* (b) $N_0'+P=N'(=P^{gp})$.

PROOF. It follows from [Mum, (6.1)] that there exists a bijection between (I) and (II). We give a one-to-one correspondence between (II) and (III). Given Δ, define by 3.3 the log. scheme representing $T_\Delta^{\log}$. Then we have a map from (II) to (III). Conversely, if P is a submonoid of $N'=N_0'+N_1'$ with the element f_0 and $N_0'+P=N'$, define Δ to be the image of $P_Q^\vee\cap(e_0+N_{1Q})$ under the second projection $N_Q\to N_{1Q}$. We leave it to the reader to check that these set up mutually inverse maps between the sets (II) and (III) of the proposition. □

PROPOSITION 3.5. *Let the notation be as in* 3.4.

(1) *There is a one-to-one correspondence between:*

(I) *orbits Z of $\tilde{G}_s$ on the closed fibre $(U_\Delta)_s$ of U_Δ,*

and

(II) *faces $\sigma\prec\Delta$.*

Moreover, $\dim Z_\sigma+\dim\sigma=\operatorname{rank} N_1$, *where Z_σ denotes the orbit corresponding to a face $\sigma\prec\Delta$.*

(2) *Let τ, σ be faces of Δ. Then σ contains τ if and only if the closure of Z_τ contains Z_σ.*

(3) *$(U_\Delta)_s$ is a finite union of orbits, hence each component contains a unique open orbit, and the set of components is in one-to-one correspondence with the vertices of Δ.*

(4) *If Δ_1, Δ_2 are polyhedra, then U_{Δ_1} is an open subset of U_{Δ_2} if and only if Δ_1 is a face of Δ_2.*

PROOF. See [Mum, (6.3)] for (1) and (2), [Mum, (6.4)] for (3), and [Mum, (6.5)] for (4). □

PROPOSITION 3.6. *If Δ, Δ' are polyhedra in N_{1Q}, then we have:*

(1) *$\Delta\subset\Delta'$ if and only if $T_\Delta^{\log}\subset T_{\Delta'}^{\log}$.*

(2) *$\Delta\prec\Delta'$ if and only if $T_\Delta^{\log}\subset T_{\Delta'}^{\log}$ is an open immersion.*

PROOF. (1) If $T_\Delta^{\log}\subset T_{\Delta'}^{\log}$, then by computing the image of the identity of $\underline{X}_\Delta$ under the inclusion $T_\Delta^{\log}\subset T_{\Delta'}^{\log}$, we conclude that Δ is contained in Δ'. The converse is trivial.

(2) If the inclusion map $T_\Delta^{\log}\subset T_{\Delta'}^{\log}$ is an open immersion, so is the morphism of the underlying schemes. Thus by (1) and 3.5, (4), Δ is a face of Δ'. Conversely, if Δ is a face of Δ', then the polyhedral cone $\Delta^\sim$ is a face of $\Delta'^\sim$. It follows from 3.2 and the following lemma that the morphism $\underline{X}_\Delta\hookrightarrow\underline{X}_{\Delta'}$ defined by $\Delta\hookrightarrow\Delta'$ is an open immersion. □

LEMMA 3.7. *Let P be an* fs *monoid, and $\mathfrak{p}$ a prime ideal of P. Then the morphism* $(\operatorname{Spec}\boldsymbol{Z}[P_\mathfrak{p}], P_\mathfrak{p})\to(\operatorname{Spec}\boldsymbol{Z}[P], P)$ *defined by $P\hookrightarrow P_\mathfrak{p}$ is an open immersion in the sense of* 1.12.

PROOF. Since $P\backslash\mathfrak{p}$ is finitely generated, the above morphism of underlying schemes is an open immersion. Then we have only to show that $(\operatorname{Spec}\boldsymbol{Z}[P_\mathfrak{p}], P_\mathfrak{p})\to(\operatorname{Spec}\boldsymbol{Z}[P_\mathfrak{p}], P)$ is an isomorphism. We consider the "exact" commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & P^a & \longrightarrow & P^a/\mathcal{O}_X^* & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X^* & \longrightarrow & (P_\mathfrak{p})^a & \longrightarrow & (P_\mathfrak{p})^a/\mathcal{O}_X^* & \longrightarrow & 0 \end{array}$$

where $X=\operatorname{Spec}\boldsymbol{Z}[P_\mathfrak{p}]$ and P^a (resp. $(P_\mathfrak{p})^a$) is the log. str. associated to P (resp. $P_\mathfrak{p}$). To prove the above statement, we have only to verify that the third vertical arrow is an isomorphism. The homomorphism is surjective, because the image of $P\backslash\mathfrak{p}$ is contained in $\mathcal{O}_X^*$. Since both P and $P_\mathfrak{p}$ and integral and the homomorphism of groups associated to them is an isomorphism, the above homomorphism is injective. □

DEFINITION 3.8. A set Σ of bounded convex polyhedra in N_Q is called a *convex polyhedral decomposition* of N_Q if it satisfying the following:

(a) For each $\Delta\in\Sigma$, any face τ of Δ is in Σ;

(b) For each Δ, $\Delta'\in\Sigma$ with $\Delta\cap\Delta'\neq\emptyset$, $\Delta\cap\Delta'$ is a face of Δ.

Moreover if $\cup_{\Delta\in\Sigma}\Delta=N_Q$, then Σ is said to be a *complete convex polyhedral decomposition* of N_Q. A convex polyhedral decomposition Σ of N_Q is *locally finite* if given $\Delta\in\Sigma$, the number of $\Delta'\in\Sigma$ such that $\Delta\cap\Delta'\neq\emptyset$ is finite.

COROLLARY 3.9. *For any convex polyhedral decomposition Σ of N_{1Q}, the sheaf associated to* $\bigcup_{\Delta\in\Sigma} T_{\Delta}^{\log} : \mathcal{C}_{\mathrm{Zar}} \to \mathrm{Sets}$ *is representable. By abuse of notation, it is denoted by $T_{\Sigma}^{\log}$.*

PROOF. It follows from 1.15 and 3.6. □

Now we study the irreducible components of $T_{\Delta}^{\log}$, and $T_{\Sigma}^{\log}$.

PROPOSITION 3.10. *Let Δ be a bounded convex polyhedron in N_{1Q}, and v a vertex of Δ. Then the irreducible component Z_v of the underlying scheme of $T_{\Delta}^{\log}$ (3.5, (3)) is the closed subscheme* $\mathrm{Spec}\, k[(\Delta^{\sim})^{\vee}\cap N']/(I_v) \hookrightarrow \mathrm{Spec}\, k[(\Delta^{\sim})^{\vee}\cap N']/(f_0)$. *Here I_v is the prime ideal of $(\Delta^{\sim})^{\vee}\cap N'$ corresponding to the face $v^{\sim}$ of $\Delta^{\sim}$* (3.2).

PROOF. See the proof of [Mum, 6.3]. □

Fix an element $\xi_0 \in N_{1Q}$. We define a Q-linear map $\rho_{\xi_0} : N_Q = N_{0Q} \oplus N_{1Q} \to N_{1Q}$ by $a_0 f_0 + \xi \mapsto a_0(-\xi_0) + \xi$, and $\iota_{\xi_0} : N'_{1Q} \hookrightarrow N'_Q$ by $b \mapsto -\langle \xi_0, b\rangle f_0 + b$. Then we have a commutative diagram

$$\begin{array}{ccccc} N_Q & \times & N'_Q & \longrightarrow & Q \\ \rho_{\xi_0}\downarrow & & \downarrow \iota_{\xi_0} & & \| \\ N_{1Q} & \times & N'_{1Q} & \longrightarrow & Q \end{array}$$

The "integral part" of the above diagram is

$$\begin{array}{ccccc} N & \times & N' & \longrightarrow & Z \\ \rho_{\xi_0}\downarrow & & \uparrow \iota_{\xi_0} & & \| \\ \tilde{N}_1 & \times & \tilde{N}'_1 & \longrightarrow & Z \end{array}$$

Here $\tilde{N}_1 = \rho_{\xi_0}(N) = N_1 + Z\xi_0$, and $\tilde{N}'_1 = \{f \in N'_{1Q};\ \langle \tilde{N}_1, f\rangle \subset Z\}$. For any polyhedron Δ with the vertex $\{\xi_0\}$, the monoid $\Delta_{\xi_0}^{\vee} \cap \tilde{N}'_1$ is isomorphic to $((\Delta^{\sim})^{\vee}\cap N')\backslash I_{\xi_0}$ by ι_{ξ_0} where Δ_{ξ_0} is the polyhedral cone $\rho_{\xi_0}(\Delta^{\sim})$, and I_{ξ_0} is the prime ideal of $(\Delta^{\sim})^{\vee}\cap N'$ corresponding to the face $\{\xi_0\}^{\sim}$ of $\Delta^{\sim}$. Therefore the k-algebra $k[(\Delta^{\sim})^{\vee}\cap N']/(I_v)$ is isomorphic to $k[\Delta_{\xi_0}^{\vee}\cap \tilde{N}'_1]$.

PROPOSITION 3.11. *Let $\xi_0, \rho_{\xi_0}, \iota_{\xi_0}$ be as above, and let Σ be a convex polyhedral decomposition of N_{1Q} containing $\{\xi_0\}$, and $Z_{\{\xi_0\}}$ the irreducible component of $T_{\Sigma}^{\log}$ corresponding to $\{\xi_0\}$. Then*:

(1) *$Z_{\{\xi_0\}}$ is $\tilde{G}_s$-invariant, and $(T_{\Delta}^{\log})_{\{\xi_0\}\prec\Delta\in\Sigma}$ is a covering of $Z_{\{\xi_0\}}$*;

(2) *$Z_{\{\xi_0\}}$ is proper over S if and only if Σ satisfies the following*

conditions:

(a) *The number of* $\Delta \in \Sigma$ *containing* $\{\xi_0\}$ *is finite.*

(b) $\rho_{\xi_0}(\bigcup_{\{\xi_0\}\prec\Delta\in\Sigma}\Delta)=N_{1Q}$.

In particular, if Σ *is a locally finite convex polyhedral decomposition of* N_{1Q}, *then every irreducible component of* $T_{\Sigma}^{\log}$ *is proper over* k

PROOF. (1) follows from 3.5, (2) and 3.6. By 3.10 and [KKMS, § 2, Theorem 7], we can verify (2) immediately. □

4. Construction

In this section, we construct compactifications of the generalized Jacobian variety of a reduced connected complete semi-stable curve over a field. Given an admissible convex polyhedral decomposition of $H^1(\Gamma(X), \boldsymbol{Q})$ below in the sense of 4.1, we compactify the generalized Jacobian variety.

We use the following notation in Section 2. Let k be a field, S the affine scheme Spec k, and $\underline{S}$ the log. scheme (S, M_S) with the log. str. defined by $N\to k$ $(1\mapsto 0)$. Let us denote by X a reduced proper geometrically connected curve over S such that:

(1) all the irreducible components of X are smooth and geometrically irreducible;

(2) the singular points of X are at most k-rational ordinary double points.

Let $\Gamma(X)$ be the dual graph of X and $\underline{X}=(X, M_X)$ a curve of semi-stable type over $\underline{S}$ (2.6). Before stating our main theorem, we discuss an action of $H=H_1(\Gamma(X), \boldsymbol{Z})$ on the set of bounded convex polyhedra $H^1(\Gamma(X), \boldsymbol{Q})$.

Identifying the cochain complex of the graph $\Gamma(X)$ with the chain complex [O-S, I.4], we regard H as a subgroup of $H^1(\Gamma(X), \boldsymbol{Z})$ by the composite of the inclusion $H\hookrightarrow C_1(\Gamma(X), \boldsymbol{Z})$ with the projection $C_1(\Gamma(X), \boldsymbol{Z})\to H^1(\Gamma(X), \boldsymbol{Z})$. Moreover, H is regarded as a subgroup of $H^1(\Gamma(X), \boldsymbol{Q})$ by the canonical homomorphism $H^1(\Gamma(X), \boldsymbol{Z})\to H^1(\Gamma(X), \boldsymbol{Q})$. Hence H acts naturally on the set of bounded convex polyhedra in $H^1(\Gamma(X), \boldsymbol{Q})$ by translation.

DEFINITION 4.1. Let Σ be a complete convex polyhedral decomposition of $H^1(\Gamma(X), \boldsymbol{Q})$. We say that Σ is *admissible* if Σ satisfies the following conditions:

(ST) The convex polyhedral decomposition Σ is stable under the above action of $H=H_1(\Gamma(X), \boldsymbol{Z})$;

(DC) For any bounded convex polyhedron Δ in Σ and for any $a\in H\backslash\{0\}$,

$$\Delta\cap(a+\Delta)=\varnothing.$$

Example 4.2. Let X be a curve as above. If the rank of $H^1(\Gamma(X), \boldsymbol{Z})<$ the girth of $\Gamma(X)$, i.e. the minimum length of cycles in $\Gamma(X)$, then any non-degenerate Namikawa decomposition of $H^1(\Gamma(X), \boldsymbol{Q})$ [O-S, I] is admissible.

We now state our main theorem.

THEOREM 4.3. *Let* $\underline{S}$ *be the log. scheme* $(\mathrm{Spec}\, k, k^*\oplus N)$ *defined by* $N\to k(1\mapsto 0)$, $\underline{X}$ *a curve of semi-stable type over* $\underline{S}$ (2.6), $\Gamma(X)$ *the dual graph of* X, *and* Σ *a complete locally finite polyhedral decomposition of* $H^1(\Gamma(X), \boldsymbol{Q})$ (3.8). *Suppose that* Σ *is admissible in the sense of* (4.1). *Then there exists a subsheaf* $J^{\log}_{\underline{X}/\underline{S},\Sigma}$ *of sets contaiued in* $J^{\log}_{\underline{X},\underline{S}}$ (2.18) *satisfying the following*:

(1) $J^{\log}_{\underline{X}/\underline{S},\Sigma}$ *is represented by an S-log. scheme* J_Σ *whose underlying scheme is proper over* S;

(2) *Each irreducible component of* J_Σ *canonically contains the generalized Jacobian variety* $\mathrm{Pic}^0_{X/S}$ (*before* 2.10) *as an open dense subscheme*;

(3) *The generalized Jacobian variety* $\mathrm{Pic}^0_{X/S}$ *with the inverse image log. str. of* M_S *acts on* $J^{\log}_{\underline{X}/\underline{S},\Sigma}$ *extending the translation action of* $\mathrm{Pic}^0_{X/S}$.

REMARK. The above compactifications of $\mathrm{Pic}^0_{X/S}$ are independent of orientations of $\Gamma(X)$ fixed in 2.

Before proving the theorem, we show some preliminary results.

Let G be a semi-abelian variety over S, i.e. an extension of an abelian variety A by an algebraic torus T. Let $\underline{G}$ (resp. $\underline{A}$, resp. $\underline{T}$) be the sheaf on $(\mathrm{fs}/\underline{S})_{\mathrm{Zar}}$ represented by the scheme G (resp. A, resp. T) with the inverse image log. str. of M_S. We assume that T is split over k and that the sequence

$$0\to\underline{T}\to\underline{G}\to\underline{A}\to 0$$

is exact as sheaves on $(\mathrm{fs}/\underline{S})_{\mathrm{Zar}}$. The algebraic torus T is canonically isomorphic to $\mathrm{Par}(T)\otimes\boldsymbol{G}_m=\underline{\mathrm{Hom}}(\mathrm{Char}(T), \boldsymbol{G}_m)$ where $\mathrm{Char}(T)$ is the character group of T and $\mathrm{Par}(T)=\mathrm{Hom}(\mathrm{Char}(T), \boldsymbol{Z})$ is the group of one-parameter subgroup of T. We define the sheaf $T^{\log}$ to be the set of homomorphisms from $\mathrm{Char}(T)$ to $\boldsymbol{G}_m^{\log}$ (2.12). It is clear that $T^{\log}$ is isomorphic to $\mathrm{Par}(T)\otimes\boldsymbol{G}_m^{\log}$. The sheaf $\tilde{G}^{\log}$ is defined to be the push-out of

$$\begin{array}{ccc} \underline{T} & \longrightarrow & \underline{G} \\ \downarrow & & \\ T^{\log} & & \end{array}$$

in the category of abelian sheaves on $(\mathrm{fs}/\underline{S})_{\mathrm{Zar}}$. Here the homomorphism $\underline{T}\to T^{\log}$ is defined by the inclusion map $\boldsymbol{G}_m\to\boldsymbol{G}_m^{\log}$. Fix a locally finite complete convex polyhedral decomposition Σ in $\mathrm{Par}(T)\otimes\boldsymbol{Q}$. By the notation and results in § 3, $T_\Sigma^{\log}:=(\mathrm{Par}(T)\otimes\boldsymbol{G}_m^{\log})_\Sigma$ is representable. We can regard $\underline{G}$ as a $\underline{T}$-torsor over $\underline{A}$. Then, by the definition of $\tilde{G}^{\log}$ and 1.19, the sheaf $\tilde{G}_\Sigma^{\log}:=\underline{G}\vee^{\underline{T}}T_\Sigma^{\log}$ is a subsheaf of $\tilde{G}^{\log}$ and is representable.

PROPOSITION 4.4. *Let the notation and assumptions be as above. Then every irreducible component of the underlying scheme of* $\tilde{G}_\Sigma^{\log}$ *is proper over* S.

PROOF. It follows from 3.11, (2) that any irreducible component of $T_\Sigma^{\log}$ is proper over S. Thus every irreducible component of $\tilde{G}_\Sigma^{\log}$ is proper over A. Because A is proper over S, we have completed the proof. □

PROPOSITION 4.5. *Let the notation and assumptions be as in* 4.4. *Assume that an abelian group* H *acts on* $\tilde{G}_\Sigma^{\log}$ *in such a way that*:

(1) *for any* $\Delta\in\Sigma$ *and* $h\in H$, *the image of* $\tilde{G}_\Delta^{\log}$ *under* h *is* $\tilde{G}_{\Delta'}^{\log}$ *for some* $\Delta'\in\Sigma$ *and* $\tilde{G}_\Delta^{\log}\cap\tilde{G}_{\Delta'}^{\log}=\varnothing$ *if* $h\neq 0$;

(2) *the action of* H *induces an action on* Σ *by* (1) *and the cardinarity of* Σ/H *is finite.*

Then, the quotient $G_\Sigma^{\log}$ *of* $\tilde{G}_\Sigma^{\log}$ *by* H *is representable by a log. scheme whose underlying scheme is proper over* S.

PROOF. It follows from 1.17 and the assumption (1) that $G_\Sigma^{\log}$ is representable. By the above assumptions, $G_\Sigma^{\log}$ has a finite unmber of irreducible components, each of which is isomorphic to some irreducible component of $\tilde{G}_\Sigma^{\log}$. Thus, using 4.4, we are done. □

PROOF OF 4.3. We define $\tilde{J}_{\underline{X}/\underline{S},\Sigma}^{\log}$ to be the fiber bundle $J_{\underline{X}/\underline{S}}\vee^{\underline{T}}T_\Sigma^{\log}$ (3.9) and $J_{\underline{X}/\underline{S},\Sigma}^{\log}$ to be the quotient of $\tilde{J}_{\underline{X}/\underline{S},\Sigma}^{\log}$ by the action of H (2.19).

To verify that $\tilde{J}^{\log}_{\underline{X}/\underline{S},\Sigma}$ is stable under the action of H. We have only to prove that for any $\Delta\in\Sigma$ and any $h\in H=H_1(\Gamma(X),\boldsymbol{Z})$, we have

(a) $$\tilde{J}^{\log}_{\underline{X}/\underline{S},\Delta}+h=\tilde{J}^{\log}_{\underline{X}/\underline{S},\Delta-h},$$

for the sheaf $J^{\log}_{\underline{X}/\underline{S},\Sigma}$ has the property (1) by 1.19, 3.9, 4.5 and the assumption (DC) on Σ. We define a homomorphism $\iota: H\hookrightarrow H^1(\Gamma(X),\boldsymbol{Z})\otimes \boldsymbol{G}_m^{\log}$ by $h\mapsto h\otimes\pi_{S'}$ for any $\underline{S}$-log. scheme and any $h\in H$, where $\pi_{S'}$ is the image of the global section $\pi=(1,1)$ of $M_S=k^*\oplus N$ under the structure morphism of $\underline{S}$. Thus H acts on $H^1(\Gamma(X),\boldsymbol{Z})\otimes\boldsymbol{G}_m^{\log}$ by translation. We can verify that this action is compatible with the action of H on Σ, i.e. for any $\Delta\in\Sigma$ and any $h\in H$,

$$T^{\log}_{\Delta}+\iota(h)=T^{\log}_{\Delta-\iota(h)}.$$

By 2.19, H acts on $\tilde{J}^{\log}_{\underline{X}/\underline{S}}$ by translation. On the other hand, through the composite of the above homomorphism ι with the homomorphism $H^1(\Gamma(X),\boldsymbol{Z})\otimes\boldsymbol{G}_m^{\log}\hookrightarrow\tilde{J}^{\log}_{\underline{X}/\underline{S}}$, the group H also acts on $\tilde{J}^{\log}_{\underline{X}/\underline{S}}$. Then, by the exact sequence in the proof of 2.19, the former action of H is the same as the latter modulo $\mathrm{Pic}^0_{\underline{X}/\underline{S}}$ (before 2.10), i.e. the following diagram is commutative.

$$\begin{array}{ccccc} H & \rightarrow & \tilde{J}^{\log}_{\underline{X}/\underline{S}} & & \\ & & & \searrow & \\ \iota\downarrow & & & & \tilde{J}^{\log}_{\underline{X}/\underline{S}}/\mathrm{Pic}^0_{\underline{X}/\underline{S}} \\ & & & \nearrow & \\ H^1(\Gamma(X),\boldsymbol{Z})\otimes\boldsymbol{G}_m^{\log} & \rightarrow & \tilde{J}^{\log}_{\underline{X}/\underline{S}} & & \end{array}$$

Thus, by the assumption (ST) on Σ, the action of H on $\tilde{J}^{\log}_{\underline{X}/\underline{S}}$ defines an action on $\tilde{J}^{\log}_{\underline{X}/\underline{S},\Sigma}$ which satisfies (a). It follows from 3.5 that $J^{\log}_{\underline{X}/\underline{S},\Sigma}$ has the property (2). The composite of $J_{\underline{X}/\underline{S}}\rightarrow\tilde{J}^{\log}_{\underline{X}/\underline{S},\Sigma}$ (2.15) with $\tilde{J}^{\log}_{\underline{X}/\underline{S},\Sigma}\rightarrow J^{\log}_{\underline{X}/\underline{S},\Sigma}$ (2.19) gives the action of $J_{\underline{X}/\underline{S}}$ on $J^{\log}_{\underline{X}/\underline{S},\Sigma}$ by translation. □

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