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Invariants of equivalence classes of plats

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Abstract. We study equivalence classes of plats in R^3. Some equivalence classes of integral matrices are assigned to equivalence classes of plats. From the matrices we obtain numerical invariants of equivalence classes of plats.

1. Introduction.

By connecting strings in pairs on the top and the bottom of a braid (Figure 1), we obtain a plat (Figure 2).

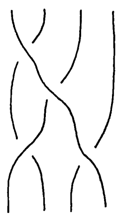

Figure 1

Figure 2

If it comes from a braid with 2n-strands, it is called a 2n-plat. Two plats are said to be equivalent if there is a homeomorphism h which carries upper and lower halves of R^3 to themselves and a plat to the other plat (Figure 3).

One of the main problems on plats is to decide when two plats are equivalent. The problem has been studied through considering Heegaard splittings of 2-fold brached covering spaces along plats [See 2, 8, 9, 10, 11]. Our method does not depend on 2-fold branched coverings but seems to relate to infinite cyclic coverings. In this note we assign a matrix to a plat and show that if two plats are equivalent then assigned matrices are equivalent in our sense. Furthermore we produce a numerical invariant

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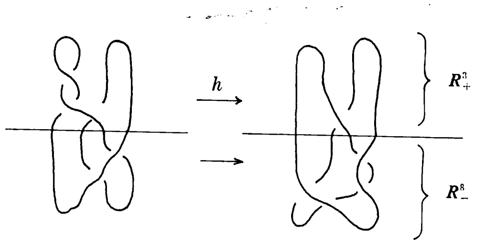

Figure 3

from the matrices which is similar to the invariant obtained by Lustig-Moriah [8, 9]. Generally it needs a large amount of calculation to obtain the invariant. But it is quite elementary and can be done by an electric computer.

Our invariant distinguishes two plats which are equivalent as links. The equivalence classes of 6-plats correspond to the equivalence classes of Heegaard splittings of associated 3-manifolds via 2-fold branched covering spaces. So our invariant can be used to distinguish Heegaard splittings of 3-manifolds which are homeomorphic.

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2. Definitions.

Let $R^3_+=\{(x, y, z)\in R^3|z\geqq 0\}$, $R^3_-=\{(x, y, z)\in R^3|z\leqq 0\}$ and $\rho: R^3_+\to R^3_-$ be defined by $\rho(x, y, z)=(x, y, -z)$. Let $A=A_1\cup A_2\cup\cdots\cup A_n$ be a union of arcs which are properly embedded in R^3_+ as shown in Figure 4.

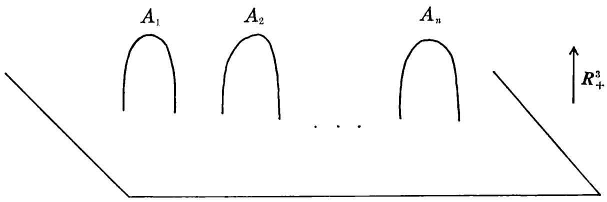

Figure 4

It is assumed that the arcs are unlinked and unknotted. Let $A'=A'_1\cup A'_2\cup\cdots A'_n$ be defined by $A'_i=\rho(A_i)$ for any $i=1, \cdots, n$. Then A' is the union of the arcs in R^3_-.

DEFINITION 1. Let B_{2n} be the braid group of 2n-strands. Here in this note we regard an element of B_{2n} as an isotopy class an orientation preserving homeomorphism $\phi: (\partial R^3_+, \partial A) \to (\partial R^3_+, \partial A)$.

DEFINITION 2. Let $\phi: (\partial R^3_+, \partial A) \to (\partial R^3_+, \partial A) = (\partial R^3_-, \partial A)$ be a homeomorphism. The identification space $(R^3_+, A) \cup_\phi (R^3_-, A')$ can be identified with R^3 containing $L = A \cup_\phi A'$ as a link. In this situation we call $(R^3_+, A) \cup_\phi (R^3_-, A')$ a plat representation of the link L. We also call an isotopy class $[\phi]$ a plat (or 2n-plat). Hence we call an element of B_{2n} a plat in this note.

Next we give definitions of equivalence of plats. Two definitions will be considered. One is required to be orientation preserving while the other is not.

DEFINITION 3. Two plats $[\phi]$ and $[\psi]$ are said to be (orientation preservingly) equivalent if there is an orientation preserving homeomorphism $h: R^3_+ \cup_\phi R^3_- \to R^3_+ \cup_\psi R^3_-$ which satisfies $h(R^3_\pm) = R^3_\pm$, $h(A) = A$ and $h(A') = A'$. In particular if h preserves orientations of A and A' it is said to be orientation preservingly equivalent.

Next we define a subgroup K_{2n} of B_{2n}.

DEFINITION 4. An element $[\phi]$ of B_{2n} belongs to K_{2n} if and only if there is a homeomorphism $\hat{\phi}: (R^3_+, A) \to (R^3_+, A)$ such that $\hat{\phi}|\partial R^3_+ = \phi$.

Under the definitions we obtain the following lemma.

LEMMA 1. *Two plats $[\phi], [\psi] \in B_{2n}$ are equivalent if and only if there are $g_1, g_2 \in K_{2n}$ which satisfy $\phi = g_2 \circ \psi \circ g_1$.*

The proof of the lemma is due to standard argument (See, for example, Birman [3, Theorem 1]) and we omit it. Note that if $[\phi]$ and $[\psi]$ are orientation preservingly equivalent then g_1 and g_2 can be taken to preserve the orientation of A.

3. Jacobian matrices.

Any plat $\phi: (\partial R^3_+, \partial A) \to (\partial R^3_+, \partial A)$ induces an isomorphism $\phi_*: \pi_1(\partial R^3_+ - \partial A) \to \pi_1(\partial R^3_+ - \partial A)$. The group $\pi_1(\partial R^3_+ - \partial A)$ is identified with a free group of rank $2n$ by taking $a_1, \cdots, a_n, b_1, \cdots, b_n$ in Figure 5 as a generating system.

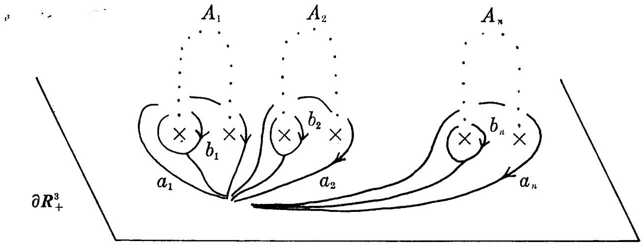

Figure 5

Then $\phi : (\partial R^3_+, \partial A) \to (\partial R^3_+, \partial A)$ induces an isomorphism $\phi_* : F(a_1, \cdots, a_n, b_1, \cdots, b_n) \to F(a_1, \cdots a_n, b_1, \cdots, b_n)$. Let G denotes an infinite cyclic group Z when we work in oriented case and a cyclic group Z_2 of order 2 in general case. The group G will be presented as $G=\langle t|\phi\rangle$ for oriented case while $G=\langle t|t^2=1\rangle$ for general case. Let $\alpha : F=F(a_1, \cdots, a_n, b_1, \cdots, b_n) \to G$ be a homomorphism defined by $\alpha(a_i)=1, \alpha(b_i)=t$ for $i=1, \cdots, n$. We denote $^- : ZG \to ZG$ an involution defined by $\overline{(\sum n_i t^i)}=\sum n_i t^{-i}$. Hereafter we work on orientation preserving equivalence and simply call it equivalence. But it will be easy to modify statements to adapt to non orientation preserving case.

Next we define Jacobian matrix $J\phi_*$ which is associated with a plat ϕ.

$$J\phi_*=\begin{pmatrix} \left(\dfrac{\partial\phi(a_i)}{\partial a_j}\right) i, j=1, \cdots, n & \left(\dfrac{\partial\phi(a_i)}{\partial b_j}\right) i, j=1, \cdots, n \\ \left(\dfrac{\partial\phi(b_i)}{\partial a_j}\right) i, j=1, \cdots, n & \left(\dfrac{\partial\phi(b_j)}{\partial b_j}\right) i, j=1, \cdots, n \end{pmatrix}^{\alpha}.$$

Here a matrix $(c_{ij})^{\alpha}$ denotes $(\alpha(c_{ij}))$. The symbol $\dfrac{\partial}{\partial x}$ denotes free derivative of Fox [5]. We denote four submatrices of $J\phi_*$ by A, B, C and D as follows:

$$J\phi_*=\begin{pmatrix} C & A \\ D & B \end{pmatrix}$$

where the size of the four matrices are $n\times n$. The submatrix $\begin{pmatrix} A \\ B \end{pmatrix}$ of $J\phi_*$ is important for our purpose. We denote it by $R\phi_*$. Namely $R\phi_*=\begin{pmatrix} A \\ B \end{pmatrix}$.

4. The main results.

THEOREM 1. *Suppose that two plats* $[\phi], [\psi]\in B_{2n}$ *are equivalent. Let* $R\phi_*=\begin{pmatrix} A \\ B \end{pmatrix}$ *and* $R\psi_*=\begin{pmatrix} A' \\ B' \end{pmatrix}$. *Then there are* $n\times n$ *unimodular matrices* U, G *and an* $n\times n$ *matrix* W *which satisfy*

$$\begin{pmatrix} U & 0 \\ W & *U^{-1} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} G=\begin{pmatrix} A' \\ B' \end{pmatrix}$$

where, for $U=(u_{ij})$, $*U$ *denotes* ${}^t\bar{U}$, *that is* $*U=(\overline{u_{ji}})$.

To prove the theorem the following is a key.

LEMMA 2. *If* ϕ *belongs to* K_{2n} *then* $J\phi_*$ *has a form* $\begin{pmatrix} U & 0 \\ W & *U^{-1} \end{pmatrix}$ *where* U *and* W *are as in Theorem* 1.

PROOF. The idea of the proof is similar to that of Fukuhara-Kanno [6, Lemma 4]. We set

$$J\phi_*=\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

where U_{ij} are $n\times n$ matrices. It suffices to show that $U_{12}=0$ and $\overline{U_{11}}\,{}^tU_{22}=E_n$.

Let $i:\pi_1(\partial R^3_+-\partial A)\to\pi_1(R^3_+-A)$ be a homomorphism induced from the inclusion map. Then ker i is the normal closure of $a_1,\cdots,a_n$. Since ϕ extends to a self-homeomorphism of (R^3_+, A), $\phi_*(a_i)$ belongs to ker i. Hence $\phi_*(a_i)$ is a product of conjugates of $a_1,\cdots,a_n$ and can be presented as $\phi_*(a_i)=\prod_{k=1}^m g_k a_{i_k}^{\epsilon_k} g_k^{-1}(\epsilon_k=\pm 1)$. Since $\left(\dfrac{\partial g_k a_{i_k}^{\epsilon_k} g_k^{-1}}{\partial b_j}\right)^\alpha=(1-g_k a_{i_k}^{\epsilon_k} g_k^{-1})^\alpha\left(\dfrac{\partial g_k}{\partial b_j}\right)^\alpha=0$, we have

$$\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^\alpha=0 \qquad (i,j=1,\cdots,n). \tag{1}$$

This implies $U_{12}=0$

Let us consider the covering space of $\partial R^3_+ - \partial A$, denoted by $\widetilde{\partial R^3_+ - \partial A}$, which is associated with $\alpha : \pi_1(\partial R^3_+ - \partial A) = F(a_1, \cdots, a_n, b_1, \cdots, b_n) \to G$. Let $\pi : \widetilde{\partial R^3_+ - \partial A} \to \partial R^3_+ - \partial A$ denote the covering map. We choose a sufficiently large disk D in ∂R^3_+ which satisfies $D \supset \partial A$. Let $X = \partial R^3_+ - D$. We can assume without loss of generality that $\phi | X = id$ for any $\phi \in B_{2n}$. Take a base point p_0 of $\partial R^3_+ - \partial A$ such that $p_0 \in X$. Let $\widetilde{p_0}$ be a base point of $\widetilde{\partial R^3_+ - \partial A}$ such that $\widetilde{p_0} \in \pi^{-1}(p_0)$. For a_i and b_i in $\pi_1(\partial R^3_+ - \partial A)$ we choose their liftings $\widetilde{a_i}$ and $\widetilde{b_i}$ with respect to π where we assume that they have $\widetilde{p_0}$ as their starting points. Note that $\widetilde{a_i}$ is still a loop because $\alpha(a_i)=1$ while $\widetilde{b_i}$ is not. Set $\tilde{X} = \pi^{-1}(X)$. We can regard $\widetilde{a_i}$ and $\widetilde{b_i}$ as elements of $H_1(\widetilde{\partial R^3_+ - \partial A})$ and $H_1(\widetilde{\partial R^3_+ - \partial A}, \tilde{X})$ respectively. Now let us consider the intersection pairing

$$\langle\ ,\ \rangle : H_1(\widetilde{\partial R^3_+ - \partial A}) \otimes H_1(\widetilde{\partial R^3_+ - \partial A}, \tilde{X}) \to ZG$$

which is defined by $\langle x, y\rangle = \sum_{g \in G} g(gx, y)$ where

$$(\ ,\ ) : H_1(\widetilde{\partial R^3_+ - \partial A}) \otimes H_1(\widetilde{\partial R^3_+ - \partial A}, \tilde{X}) \to Z$$

denotes the ordinary intersection pairing. Then it follows immediately that the pairing $\langle\ ,\ \rangle$ has the properties:

$$\text{(2)} \qquad \begin{gathered} \langle x+x', y\rangle = \langle x, y\rangle + \langle x', y\rangle,\ \langle x, y+y'\rangle = \langle x, y\rangle + \langle x, y'\rangle, \\ \langle gx, y\rangle = g^{-1}\langle x, y\rangle \quad \text{and} \quad \langle x, gy\rangle = g\langle x, y\rangle. \end{gathered}$$

It is also obvious that

$$\text{(3)} \qquad \langle \widetilde{a_i}, \widetilde{b_j}\rangle = \delta_{ij} \in ZG \quad \text{and} \quad \langle \widetilde{a_i}, \widetilde{a_j}\rangle = 0.$$

We recall the formula of free differential calculus. Let ω be an element of $\pi_1(\partial R^3_+ - \partial A)$ and $\tilde{\omega}$ be its lifting. Let us represent ω as a word in $F(a_1, \cdots, a_n, b_1, \cdots, b_n)$ like $\omega = \omega(a_1, \cdots, a_n, b_1, \cdots, b_n)$. When we regard $\tilde{\omega}$ as a 1-chain on $\widetilde{R^3_+ - \partial A}$ we obtain

$$\text{(4)} \qquad \tilde{\omega} = \sum_{i=1}^{n} \left(\frac{\partial \omega}{\partial a_i}\right)^{\alpha} \widetilde{a_i} + \sum_{i=1}^{n} \left(\frac{\partial \omega}{\partial b_i}\right)^{\alpha} \widetilde{b_i}.$$

From (1) and (4) it follows that

$$\tilde{\phi}(\widetilde{a_i})=\widetilde{\phi(a_i)}=\sum_{j=1}^{n}\left(\frac{\partial\phi(a_i)}{\partial a_j}\right)^{\alpha}\widetilde{a_j}+\sum_{j=1}^{n}\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^{\alpha}\widetilde{b_j}=\sum_{j=1}^{n}\left(\frac{\partial\phi(a_i)}{\partial a_j}\right)^{\alpha}\widetilde{a_j}$$

and

$$\tilde{\phi}(\widetilde{b_i})=\widetilde{\phi(b_i)}=\sum_{j=1}^{n}\left(\frac{\partial\phi(b_i)}{\partial a_j}\right)^{\alpha}\widetilde{a_j}+\sum_{j=1}^{n}\left(\frac{\partial\phi(b_i)}{\partial b_j}\right)^{\alpha}\widetilde{b_j}.$$

Hence

$$\begin{aligned}\delta_{ij}&=\langle\tilde{\phi}(\widetilde{a_i}),\tilde{\phi}(\widetilde{b_j})\rangle\\&=\left\langle\sum_{k=1}^{n}\left(\frac{\partial\phi(a_i)}{\partial a_k}\right)^{\alpha}\widetilde{a_k},\sum_{k=1}^{n}\left(\frac{\partial\phi(b_j)}{\partial a_k}\right)^{\alpha}\widetilde{a_k}+\sum_{k=1}^{n}\left(\frac{\partial\phi(b_j)}{\partial b_k}\right)^{\alpha}\widetilde{b_k}\right\rangle\\&=\sum_{k=1}^{n}\overline{\left(\frac{\partial\phi(a_i)}{\partial a_k}\right)^{\alpha}}\left(\frac{\partial\phi(b_j)}{\partial b_k}\right)^{\alpha}.\end{aligned}$$

This implies $\overline{U_{11}}{}^tU_{22}=E_n$ completing the proof.

Proof of Theorem 1. From Lemma 1 and 2 there are $n\times n$ unimodular matrices U, G and an $n\times n$ matrix W and H which satisfy

$$\begin{pmatrix}U&0\\W&{}^*U^{-1}\end{pmatrix}\begin{pmatrix}C&A\\D&B\end{pmatrix}\begin{pmatrix}{}^*G^{-1}&0\\H&G\end{pmatrix}=\begin{pmatrix}C'&A'\\D'&B'\end{pmatrix}.$$

Comparing $(1,2)$ and $(2,2)$-components of the both sides of the above equation we obtain Theorem 1.

We consider equivalence relation between matrices like $R\phi_*$.

Definition 5. Two matrices $R\phi_*=\begin{pmatrix}A\\B\end{pmatrix}$ and $R\psi_*=\begin{pmatrix}A'\\B'\end{pmatrix}$ are said to be equivalent if and only if there are $n\times n$ unimodular matrices U, G and an $n\times n$ matrix W which satisfy

$$\begin{pmatrix}U&0\\W&{}^*U^{-1}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}G=\begin{pmatrix}A'\\B'\end{pmatrix}.$$

In term of this equivalence we can restate Theorem 1 as two plats ϕ and ψ are equivalent only if $R\phi_*$ and $R\psi_*$ are equivalent.

We choose a nice representative from an equivalence class of $R\phi_*$.

Theorem 2. *Let ϕ be a plat and let $R\phi_*=\begin{pmatrix}A\\B\end{pmatrix}$. Then $\begin{pmatrix}A\\B\end{pmatrix}$ is* *equivalent to a matrix which has a form*

$$\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \\ 1 & * \\ 0 & B_{22} \end{pmatrix}$$

where A_{22} and B_{22} are $(n-1)\times(n-1)$ matrices while A and B are $n\times n$ matrices.

PROOF. From the definition of free differential calculus and the fact that $\phi(a_1\cdots a_{i-1})^\alpha=1$ we have

$$\left(\frac{\partial\phi(a_1\cdots a_n)}{\partial b_j}\right)^\alpha=\sum_{i=1}^n\phi(a_1\cdots a_{i-1})^\alpha\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^\alpha=\sum_{i=1}^n\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^\alpha.$$

On the other hand, since $\phi(a_1\cdots a_n)=a_1\cdots a_n$, we have

$$\left(\frac{\partial\phi(a_1\cdots a_n)}{\partial b_j}\right)^\alpha=\left(\frac{\partial a_1\cdots a_n}{\partial b_j}\right)^\alpha=0.$$

The two equations above imply

$$\sum_{i=1}^n\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^\alpha=0 \tag{5}$$

for any $j=1,\cdots,n$. Applying the fundamental theorem of free differential calculus we have

$$\begin{aligned} t-1=(\phi(b_i)-1)^\alpha&=\sum_{j=1}^n\left(\frac{\partial\phi(b_i)}{\partial b_j}\right)^\alpha(b_j-1)^\alpha+\sum_{j=1}^n\left(\frac{\partial\phi(b_i)}{\partial a_j}\right)^\alpha(a_j-1)^\alpha \\ &=(t-1)\sum_{j=1}^u\left(\frac{\partial\phi(b_i)}{\partial b_j}\right)^\alpha. \end{aligned}$$

Since ZG is a domain, this implies

$$\sum_{j=1}^n\left(\frac{\partial\phi(b_i)}{\partial b_j}\right)^\alpha=1. \tag{6}$$

Similarly we have

$$0=(\phi(a_i)-1)^\alpha=\sum_{j=1}^n\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^\alpha(b_j-1)^\alpha+\sum_{j=1}^n\left(\frac{\partial\phi(a_i)}{\partial a_j}\right)^\alpha(a_j-1)^\alpha$$

$$=(t-1)\sum_{j=1}^{n}\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^{\alpha}.$$

This implies

$$\sum_{j=1}^{n}\left(\frac{\partial\phi(a_i)}{\partial b_j}\right)^{\alpha}=0. \tag{7}$$

Due to (5), (6) and (7), if we multiply $\begin{pmatrix} A \\ B \end{pmatrix}$ by

$$\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
 & & & \cdots & & & & & & \cdots & & \\
 & & & \cdots & & & & & & \cdots & & \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\
 & & & \cdots & & & & & & \cdots & & \\
 & & & \cdots & & & & & & \cdots & & \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}$$

from the left and by

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 0 & \cdots & 0 & 0 \\
 & & & \cdots & & \\
 & & & \cdots & & \\
1 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}$$

from the right, we obtain a matrix which has a form

$$\begin{pmatrix}
0 & 0 \\
0 & A_{22} \\
1 & * \\
0 & B_{22}
\end{pmatrix}$$

where A_{22} and B_{22} are $(n-1)\times(n-1)$ matrices. This completes the proof of Theorem 2.

We call a matrix which has the form as in Theorem 2 a reduced form of $R\phi_*$.

THEOREM 3. *Let ϕ and ψ be plat representations of knots. Suppose* *that ϕ and ψ are equivalent. Let*

$$R\phi_*\sim\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \\ 1 & * \\ 0 & B_{22} \end{pmatrix},\quad R\psi_*\sim\begin{pmatrix} 0 & 0 \\ 0 & A'_{22} \\ 1 & * \\ 0 & B'_{22} \end{pmatrix}.$$

Then there are unimodular matrices U and G and a matrix W which satisfy

(i) $$A'_{22}=UA_{22}G$$
(ii) $$B'_{22}=WAG+{}^*U^{-1}B_{22}G.$$

PROOF. Let

$$\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \\ 1 & * \\ 0 & B_{22} \end{pmatrix}$$

and

$$\begin{pmatrix} A' \\ B' \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & A'_{22} \\ 1 & * \\ 0 & B'_{22} \end{pmatrix}$$

be equivalent. Then there are matrices $U=\begin{pmatrix} u_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, $G=\begin{pmatrix} g_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ and $W=\begin{pmatrix} w_1 & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$ such that

$$\begin{pmatrix} U & 0 \\ W & {}^*U^{-1} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} A' \\ B' \end{pmatrix}G.$$

Then it follows

(8) $$UA=A'G$$

(9) $$WA+{}^*U^{-1}B=B'G \quad \text{or} \quad {}^*UWA+B={}^*UB'G.$$

The equation (8) means

$$\begin{pmatrix} u_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & A'_{22} \end{pmatrix}\begin{pmatrix} g_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \tag{10}$$

From this we obtain $U_{12}A_{22}=0$ and $A'_{22}G_{21}=0$. Let us recall that $\det A_{22}$ and $\det A'_{22}$ are essentially Alexander polynomials of the knots. So $\det A_{22}\neq 0$ and $\det A'_{22}\neq 0$. This implies that $U_{12}=0$ and $G_{21}=0$. From (10) we also obtain $U_{22}A_{22}=A'_{22}G_{22}=0$ which implies (i) of Theorem 3. We have $*U=\begin{pmatrix} *u_{11} & *U_{21} \\ 0 & *U_{22} \end{pmatrix}$ and $G=\begin{pmatrix} g_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix}$.

Note that U_{22} and G_{22} are also unimodular because U and G are unimodular. Now we calculate the both sides of (9):

$$\begin{aligned} *UWA+B&=\begin{pmatrix} *u_{11} & *U_{21} \\ 0 & *U_{22} \end{pmatrix}\begin{pmatrix} w_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}+\begin{pmatrix} 1 & * \\ 0 & B_{22} \end{pmatrix} \\ &=\begin{pmatrix} 1 & *u_{11}W_{12}A_{22}+*U_{21}W_{22}A_{22}+* \\ 0 & *U_{22}W_{22}A_{22}+B_{22} \end{pmatrix} \end{aligned} \tag{11}$$

and

$$\begin{aligned} *UB'G&=\begin{pmatrix} *u_{11} & *U_{21} \\ 0 & *U_{22} \end{pmatrix}\begin{pmatrix} 1 & B'_{12} \\ 0 & B'_{22} \end{pmatrix}\begin{pmatrix} g_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix} \\ &=\begin{pmatrix} *u_{11}g_{11} & *u_{11}G_{12}+*u_{11}B'_{12}G_{22}+*U_{21}B'_{22}G_{22} \\ 0 & *U_{22}B'_{22}G_{22} \end{pmatrix} \end{aligned} \tag{12}$$

Comparing the $(2,2)$-components of (11) and (12) we have

$$*U_{22}W_{22}A_{22}+B_{22}=*U_{22}B'_{22}G_{22}. \tag{13}$$

Since U_{22} and G_{22} are unimodular we multiply U_{22}^{-1} from the left and G_{22}^{-1} from the right to both sides of (13). Then we have

$$*U_{22}W_{22}A_{22}+B_{22}=*U_{22}B'_{22}G_{22}.$$

This implies (ii) of Theorem 3 completing the proof.

REMARK 1. The first equation (i) of Theorem 3 shows $A'=UAG^{-1}$. This means the matrices A and A' are similar. Let us substitute G in the second equation (ii) of Theorem 3 by $A'^{-1}UA$. Then we have

$$*W+BA^{-1}=*UB'A'^{-1}U. \tag{14}$$

The entries of the matrix BA^{-1} are elements of the quotient field, say

F, of ZG. Then BA^{-1} can be regarded as a bilinear form from $ZG\times ZG\rightarrow F/ZG$. So is $B'A'^{-1}$. The equation (14) means the bilinear form $BA^{-1}: ZG\times ZG\rightarrow F/ZG$ and $B'A'^{-1}: ZG\times ZG\rightarrow F/ZG$ are isomorphic. Hence we proved that an isomorphism class of BA^{-1} is an invariant of the equivalence class of $\begin{pmatrix} A \\ B \end{pmatrix}$.

If we substitute t by -1 then Theorem 3 holds for non orientable case.

We need the following lemma to show the equivalence classes of our matrices give rise to a numerical invariant.

LEMMA 3. *Let* $\begin{pmatrix} A \\ B \end{pmatrix}$ *and* $\begin{pmatrix} A' \\ B' \end{pmatrix}$ *are equivalent. Then the following holds*:

$$\gcd_{i,j=1,\dots,n}(a_{ij})=u\gcd_{i,j=1,\dots,n}(a'_{ij})$$

and

$$\det B\equiv v\det B' \mod \gcd_{i,j=1,\dots,n}(a_{ij})$$

where u *and* v *are units of* ZG.

Lemma 3 can be easily obtained from (8) and (9) of the proof of Theorem 3. Using the lemma we have a corollary.

COROLLARY. *For a matrix* $\begin{pmatrix} A \\ B \end{pmatrix}$, $\gcd_{i,j=1,\dots,n}(a_{ij})$ *and* $\det B \bmod \gcd_{i,j=1,\dots,n}(a_{ij})$ *are, np to multiplication of units, invariants of an equivalence class of* $\begin{pmatrix} A \\ B \end{pmatrix}$.

REMARK 2. Corollary above holds for non orientation preserving case if we substitute t by -1.

5. Examples.

We would like to give very simple examples. Though these examples can be shown to be inequivalent using Heegaard splittings of covering spaces [See 1, 2, 11], we show it somewhat directly.

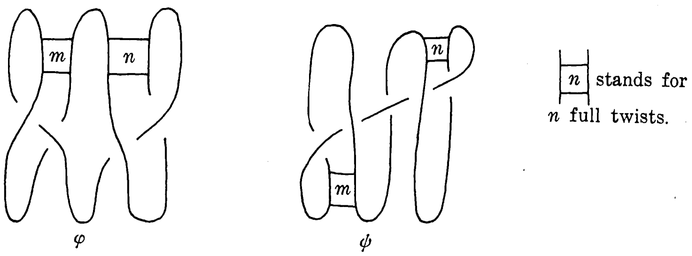

Figure 6

Let consider the following two plats ϕ and ψ (Figure 6).

Here we denote $\gcd(4m-1, 4n-1)$ by d and suppose that d is neither 1 nor 3. To simplify calculation we substitute t with -1. Then

$$(R\phi_*)^{t=-1}=\begin{pmatrix} A \\ B \end{pmatrix}^{t=-1} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-4m & 0 \\ 0 & 0 & 1-4n \\ 1 & * & * \\ 0 & 1-2m & 0 \\ 0 & -1+4m & 1+4n \end{pmatrix}$$

and

$$(R\psi_*)^{t=-1}=\begin{pmatrix} A' \\ B' \end{pmatrix}^{t=-1} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-4m & 0 \\ 0 & 0 & 1-4n \\ 1 & * & * \\ 0 & 1+4m & 0 \\ 0 & -1+4m & 1+4n \end{pmatrix}.$$

We have $\det B'_{22}-\det B_{22}=(1+4n)(1+4m-1+2m)=6m(4n-1+2)\equiv 3 \bmod d$ and $\gcd_{i,j=1,\dots,n}(a_{ij})=d$. Since d is neither 1 nor 3, $3\not\equiv 0 \bmod d$. Thus $\det B_{22}\not\equiv\det B'_{22} \bmod \gcd_{i,j=1,\dots,n}(a_{ij})$. This implies that $\begin{pmatrix} A \\ B \end{pmatrix}$ and $\begin{pmatrix} A' \\ B' \end{pmatrix}$ are not equivalent. In conclusion we can say that ϕ and ψ are not equivalent by Theorem 1. Note that these two plats are equivalent as knots.

References

[1] Birman, J. S. and H. M. Hilden, On the mapping class groups of closed surfaces as covering space, Ann. Math. Studies **66** (1971), 81-115.

[2] Birman, J. S. and H. M. Hilden, Heegaard splittings of branched coverings of S^3,

Trans. Amer. Math. Soc. **213** (1975), 315-352.
[3] Birman, J. S., On the equivalence of Heegaard splittings of closed, orientable 3-manifolds, Ann. Math. Studies **84** (1975), 137-164.
[4] Birman, J. S., On the stable equivalence of plat representations of knots and links, Canad. J. Math. **28** (1976), 264-290.
[5] Fox, R. H., Free differential calculus. I., Ann. of Math. **57** (1953), 547-560.
[6] Fukuhara, S. and J. Kanno, Extended Alexander matrices of 3-manifolds I, Tokyo J. Math. **8** (1985), 107-120.
[7] Hilden, H. M., Generators for two groups related to the braid group, Pacific J. Math. **59** (1975), 475-486.
[8] Lustig, M. and Y. Moriah, Nielsen equivalence in Fuchsian groups and Seifert fibered spaces, Topology **30** (1991), 191-204.
[9] Lustig, M. and Y. Moriah, Generalized Montesinos knots, tunnels and N-torsion, Math. Ann. **295** (1993), 167-189.
[10] Montesinos, J. M., Minimal plat representations of prime knots and links are not unique, Canad. J. Math. **28** (1976), 161-167.
[11] Montesinos, J. M. and C. Safont, On the Birman invariants of Heegaard splittings, Pacific J. Math. **132** (1988), 113-142.

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