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A Note on the Fractional Powers of Operators Approximating a Positive Definite Selfadjoint Operator

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Abstract. Let A be an unbounded positive definite selfadjoint operator acting in a Hilbert space X. Suppose that there is a family of bounded positive definite selfadjoint operators $\{A_h : 0<h\leq\bar{h}\}$, where A_h is defined on a closed subspace X_h of X. This note concerns the uniform boundedness and the rate of convergence for the family of fractional powers $\{A_h^s : 0<h\leq\bar{h}\}$ as h tends to 0, with a fixed real exponent s, under certain conditions on the family $\{A_h : 0<h\leq\bar{h}\}$ approximating the limit operator A. The conditions are expressed in terms of the rate of convergence of $A_h^{-1}P_h - A^{-1}$, where P_h is the orthogonal projection form X onto X_h.

Introduction

This note concerns the uniform boundedness and the rate of convergence for a family of fractional powers of bounded positive definite selfadjoint operators $\{A_h : 0<h\leq\bar{h}\}$, where A_h is defined on a Hilbert space X_h which is a closed subspace of a Hilbert space X. Our conclusions are stated under a family of conditions which describe the rate of convergence of $A_h^{-1}P_h - A^{-1}$ in a certain sense, where A is the limit unbounded positive definite selfadjoint operator, P_h is the orthogonal projection from X onto X_h, and the parameter h tends to 0.

The problem comes from the authors' work on the numerical analysis of a finite element method applied to the linear water wave problem. According to their formulation, the linear water wave problem can be represented as the initial value problem for an abstract evolution equation: $\dfrac{d^2\varphi}{dt^2} + A\varphi = \psi$ with values taken in an appropriate Hilbert space X, where A is a realization of a first order pseudo-differential operator which maps the boundary value on Γ_0 of a function, being harmonic in the domain Ω with vanishing normal derivative on the boundary portion

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complementary to Γ_0, to the value of its exterior normal derivative on Γ_0. Here Ω is the water region, and Γ_0 is the boundary portion of Ω corresponding to the water surface at rest.

Using a standard technique of the finite element method, we have full discrete approximation problems, for example: $\frac{\varphi_{n+1}-2\varphi_n+\varphi_{n-1}}{\tau^2}+A_h\varphi_n=\psi_n$, where φ_n and ψ_n are elements of the finite dimensional subspace X_h of X, and A_h is a non-negative selfadjoint operator acting on X_h. We have treated a family of approximate problems including the above example. In order to derive error estimates for the solutions of the approximate problems, the result of this note has been freely used.

The proof of the result of this note is based on integral representation formulas of $A^{-s}\varphi$ and $A_h^{-s}P_h\varphi$ for $\varphi\in X$ and Heinz's inequality. The aim of the note is to show the proof concretely, independently on the full context of the authors' original problem in numerical analysis. The authors feel that the result itself is interesting from the point of view of abstract study. As far as the authors know, there is no appropriate literature from which we can quote the present result.

The organization of the paper is as follows. In § 1, the setting of the problem and the result are stated. In § 2, the background of the study is explained. Finally § 3 is devoted to the proof of the result.

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§ 1. Setting of the problem and conclusions

Let A be a positive definite selfadjoint operator acting in a Hilbert space X. Let $\{X_h\}$ be a family of closed subspaces of X for $h\in(0,\bar{h}]$ with $\bar{h}<\infty$. Suppose there is a positive definite bounded selfadjoint operator A_h acting on the space X_h, which itself is considered to be a Hilbert space with the inner product induced from the original Hilbert space X.

We assume that the spectrum of A_h are uniformly bounded below by a positive constant $\underline{\alpha}$ with the property that $A\geq\underline{\alpha}$. Assume further that there is a positively valued bounded function $\varepsilon(h)$ with the following properties $(\varepsilon-0)$, $(\varepsilon-1)$ and $(\varepsilon-2)$.

$(\varepsilon-0)$ $$\sup_{0<h\leq\bar{h}}\varepsilon(h)=\bar{\varepsilon}<\infty.$$

$(\varepsilon-1)$ $$\lim_{h\to 0}\varepsilon(h)=0.$$

$(\varepsilon-2)$ $$\begin{cases}\text{There exists a positive number } \alpha\\ \text{independent of } h\in(0,\bar{h}]\\ \text{satisfying } \|A_h\|\leq\dfrac{\alpha}{\varepsilon(h)}\,.\end{cases}$$

We denote the orthogonal projection from X onto X_h by P_h. Let s be a real number. In this paper, we use the fractional powers of operators A^s and A_h^s, which are defined through the operational calculus formulas with the aid of the resolution of the identity corresponding to the self-adjoint operators A, and A_h, respectively. We say that condition $(A_{\varepsilon,s})$ holds, if there exists a constant C independent of h such that for any $h\in(0,\bar{h}]$ and $\varphi\in D(A^s)$ the following inequality satisfies:

$(A_{\varepsilon,s})$ $$\|(A_h^{-1}P_h-A^{-1})\varphi\|\leq C\varepsilon(h)^{1+s}\|A^s\varphi\|.$$

We say that condition (ε_s) holds if both $(A_{\varepsilon,0})$ and $(A_{\varepsilon,s})$ hold.

Further we consider the following conditions $(\tilde{A}_{\varepsilon,s})$ for $s\geq 0$, and (B_s) for $s\in\boldsymbol{R}$.

$(\tilde{A}_{\varepsilon,s})$ $$\|(A_h^{-s}P_h-A^{-s})\varphi\|\leq C\varepsilon(h)^s\|\varphi\| \quad \text{for } \varphi\in X.$$

(B_s) $$\|A_h^sP_h\varphi\|\leq C\|A^s\varphi\| \quad \text{for } \varphi\in D(A^s).$$

In the above two conditions the constant C may depend on s, but does not depend on $h\in(0,\bar{h}]$.

The following are the conclusions of this note.

THEOREM 1. *Condition $(A_{\varepsilon,0})$ implies condition $(\tilde{A}_{\varepsilon,s})$ for $s\in[0,1]$ with the constant C independent of $s\in[0,1]$.*

THEOREM 2. *If $s\geq 0$, then condition (ε_s) implies condition $(A_{\varepsilon,\sigma})$ for $\sigma\in[0,s]$ with the constant C independent of $\sigma\in[0,s]$.*

THEOREM 3. *If $s\geq 0$, then condition (ε_s) implies condition (B_σ) for $\sigma\in[0,1+s]$ with the constant C independent of $\sigma\in[0,1+s]$.*

THEOREM 4. *If* $-1\leq s\leq 0$, *then condition* $(A_{\varepsilon,s})$ *implies condition* (B_σ) *for* $\sigma\in[s,0]$ *with the constant* C *independent of* $\sigma\in[s,0]$.

§ 2. Background of the study

Our companion paper Matsuki and Ushijima [2] treats the fully discrete approximation of the following conservative second order linear evolution equation with values in X.

$$(E)\qquad \begin{cases} \dfrac{d^2\varphi}{dt^2}+A\varphi=\psi,\ t>0, \\ \varphi(0)=\varphi^1,\ \dfrac{d\varphi}{dt}(0)=\varphi^0. \end{cases}$$

A family of approximate problem $(E_{h,\tau})$, with the parameter $h\in(0,\bar{h}]$ and the time mesh $\tau\in(0,\infty)$, are obtained from (E) by replacing X and A with X_h and A_h, respectively, and by discretizing the time variable through Newmark's method. In order to give error estimates for the solution of $(E_{h,\tau})$, we have used freely the results mentioned in the above Theorems. The second author already used the partial results of Theorems in his previous works (for example, [4], [5]).

The framework of the present setting of $(E_{h,\tau})$ corresponds to the finite element approximation of the evolution linear water wave problem. Some of earlier results of the authors were given in [4], [5] and [6]. The selfadjoint operator A in the linear water wave problem is closely connected with the Steklov eigenvalue problem for the harmonic functions in a domain. In [6], we have shown that condition $(A_{\varepsilon,s})$ with $\varepsilon(h)=h$ holds for $s\in[-1/2,\ 1/2]$ if the problem is suitably well set.

§ 3. Proof of the results

PROPOSITION 1. *Let* $D=A^{-1}$ *and let* $D_h=A_h^{-1}P_h$. *Then for* $s\in(0,1)$ *it holds that*

$$(1)\qquad A^{-s}\varphi=\frac{\sin(\pi s)}{\pi}\int_0^\infty \nu^{s-1}D(\nu+D)^{-1}\varphi\,d\nu \quad \textit{for}\ \varphi\in X,$$

and

$$(2)\qquad A_h^{-s}P_h\varphi=\frac{\sin(\pi s)}{\pi}\int_0^\infty \nu^{s-1}D_h(\nu+D_h)^{-1}\varphi\,d\nu \quad \textit{for}\ \varphi\in X.$$

PROOF. The theory of fractional powers of operator tells us that the right hand side of (1) coincides with $D^s\varphi$ (see e.g. Krein [1]). Hence if one admits the equality $A^{-s}=D^s$, there is nothing to do as for the proof of the validity of (1). For the sake of selfcontainedness, however, the following discussion is added, which assures directly the validity of (1), where A^{-s} is the bounded operator defined through an operational calculus formula.

Let $\{E(\lambda);\ \lambda\in \boldsymbol{R}\}$ be the resolution of the identity corresponding to the selfadjoint operator A. It is to be noted that $E(\lambda)=0$ for $\lambda<\underline{\alpha}$. As is well known, for any $f(\lambda)\in C((0,\infty))\cap L^\infty([\underline{\alpha},\infty))$ we can define uniquely a bounded operator $f(A)$ through the formula:

$$f(A)=\int_{\underline{\alpha}-0}^{\infty} f(\lambda)dE(\lambda).$$

More precisely, $f(A)$ is determined in the following weak form for any $\varphi, \psi\in X$,

$$\begin{aligned}(f(A)\varphi,\psi)&=\int_{\underline{\alpha}-0}^{\infty} f(\lambda)d(E(\lambda)\varphi,\psi)\\ &=\lim_{\varepsilon\downarrow 0,\bar{\alpha}\uparrow\infty}\int_{\underline{\alpha}-\varepsilon}^{\bar{\alpha}} f(\lambda)d(E(\lambda)\varphi,\psi).\end{aligned}$$

As an example of the above operational calculus, we have $f(A)=A^{-s}$ for $f(\lambda)=\lambda^{-s}$ with $s\geq 0$. For $(\lambda,\nu)\in(0,\infty)\times(0,\infty)$, and $s\in(0,1)$, let

$$f(\lambda,\nu)=\nu^{s-1}\lambda^{-1}(\nu+\lambda^{-1})^{-1}.$$

Since, as a function of $\lambda, f(\lambda,\nu)\in C((0,\infty))\cap L^\infty([\alpha,\infty))$ for a fixed parameter $\nu\in(0,\infty)$, we have

$$f(A,\nu)=\nu^{s-1}D(\nu+D)^{-1},\quad \nu>0.$$

For the moment, we admit the validity of the following equality (3).

$$\int_0^\infty f(A,\nu)d\nu=\int_{\underline{\alpha}-0}^{\infty}\left(\int_0^\infty f(\lambda,\nu)d\nu\right)dE(\lambda). \tag{3}$$

Let $F\varphi$ be the right-hand side of (1). Then we have

$$\begin{aligned}F\varphi&=\frac{\sin(\pi s)}{\pi}\int_0^\infty f(A,\nu)\varphi\, d\nu\\ &=\frac{\sin(\pi s)}{\pi}\int_{\underline{\alpha}-0}^{\infty}\left(\int_0^\infty f(\lambda,\nu)d\nu\right)dE(\lambda)\varphi\end{aligned}$$

$$=\int_{a-0}^{\infty} \lambda^{-s}\, dE(\lambda)\varphi$$
$$=A^{-s}\varphi,$$

since it holds that

$$\int_0^{\infty} f(\lambda, \nu)d\nu=\lambda^{-s}\int_0^{\infty} \tau^{s-1}(\tau+1)^{-1}\, d\tau$$
$$=\lambda^{-s}B(s, 1-s)$$
$$=\lambda^{-s}\frac{\pi}{\sin(\pi s)}\,.$$

Hence we have (1). The validity of (3) comes from the fact that it holds for any $\varphi\in X$,

$$\int_0^{\infty}\left(\int_{a-0}^{\infty} f(\lambda, \nu)\, d(E(\lambda)\varphi, \varphi)\right) d\nu$$
(4)
$$=\int_{a-0}^{\infty}\left(\int_0^{\infty} f(\lambda, \nu)d\nu\right)d(E(\lambda)\varphi, \varphi).$$

Fubini's Theorem establishes the validity of (4) since $f(\lambda, \nu)$ is positive on $(0, \infty)\times(0, \infty)$ and since the right hand side of (4) equals

$$\frac{\pi}{\sin(\pi s)}\left\| A^{-s/2}\varphi\right\|^2<\infty.$$

Substituting

$$(X, A, D, \varphi)=(X_h, A_h, A_h^{-1}, P_h\varphi)$$

into (1), we have

$$A_h^{-1}P_h\varphi=\frac{\sin(\pi s)}{\pi}\int_0^{\infty} \nu^{s-1}A_h^{-1}(\nu+A_h^{-1})^{-1}\, P_h\varphi\, d\nu \tag{5}$$
$$\text{for } \varphi\in X.$$

Hence (2) follows from the fact that

$$(A_h^{-1})(\nu+A_h^{-1})^{-1}P_h\varphi=D_h(\nu+D_h)^{-1}\varphi \quad \text{for } \varphi\in X. \tag{6}$$

To prove (6), let $\phi=(\nu+D_h)^{-1}\varphi$. Then we have $(\nu+A_h^{-1}P_h)\phi=\varphi$. This implies $(\nu+A_h^{-1})P_h\phi=P_h\varphi$. Hence we obtain

$$P_h\psi=(\nu+A_h^{-1})^{-1}P_h\varphi.$$

Therefore it holds that

$$D_h(\nu+D_h)^{-1}\varphi=D_h\psi=A_h^{-1}P_h\psi=A_h^{-1}(\nu+A_h^{-1})^{-1}P_h\varphi.$$

∎

PROOF OF THEOREM 1: It suffices to consider the case $s\in(0,1)$, for $(\tilde{A}_{\varepsilon,0})$ holds trivially, and $(\tilde{A}_{\varepsilon,1})=(A_{\varepsilon,0})$.

Let $D=A^{-1}$, and let $D_h=A_h^{-1}P_h$. For $\varphi\in X$, by Proposition 1, it holds that

$$\begin{aligned}&A^{-s}\varphi-A_h^{-s}P_h\varphi\\ &=\frac{\sin(\pi s)}{\pi}\int_0^\infty \lambda^{s-1}\{D(\lambda+D)^{-1}-D_h(\lambda+D_h)^{-1}\}\varphi\,d\lambda \\ &=\frac{\sin(\pi s)}{\pi}\int_0^\infty \lambda^{s}\{(\lambda+D_h)^{-1}-(\lambda+D)^{-1}\}\varphi\,d\lambda.\end{aligned}\tag{7}$$

We have

$$\begin{aligned}&\left\|\int_0^{\varepsilon(h)} \lambda^{s}\{(\lambda+D_h)^{-1}-(\lambda+D)^{-1}\}\varphi\,d\lambda\right\|\\ &\le 2\int_0^{\varepsilon(h)}\lambda^s\lambda^{-1}\,d\lambda\|\varphi\|=2\frac{\varepsilon(h)^s}{s}\|\varphi\|.\end{aligned}\tag{8}$$

Let C_0 be the constant in condition $(A_{\varepsilon,0})$. By the second resolvent equation and condition $(A_{\varepsilon,0})$ we have

$$\begin{aligned}&\left\|\int_{\varepsilon(h)}^\infty \lambda^{s}\{(\lambda+D_h)^{-1}-(\lambda+D)^{-1}\}\varphi\,d\lambda\right\|\\ &=\left\|\int_{\varepsilon(h)}^\infty \lambda^{s}(\lambda+D_h)^{-1}(D-D_h)(\lambda+D)^{-1}\varphi\,d\lambda\right\|\\ &\le\int_{\varepsilon(h)}^\infty \lambda^s\lambda^{-1}C_0\,\varepsilon(h)\lambda^{-1}\|\varphi\|d\lambda\\ &=C_0\,\varepsilon(h)\int_{\varepsilon(h)}^\infty\lambda^{s-2}\,d\lambda\|\varphi\|\\ &=C_0\,\varepsilon(h)\frac{\varepsilon(h)^{s-1}}{1-s}\|\varphi\|.\end{aligned}\tag{9}$$

Therefore equality (7) and estimations (8) and (9) imply

$$\|A^{-s}\varphi - A_h^{-s}P_h\varphi\| \leq \tilde{C}_s\, \varepsilon(h)^s \|\varphi\| \quad \text{for } \varphi \in X$$

with

$$\tilde{C}_s = \frac{\sin(\pi s)}{\pi}\left(\frac{2}{s} + \frac{C_0}{1-s}\right) \leq 2 + C_0. \tag{10}$$

∎

Hereafter we use the constant $\tilde{C}_s$ defined by (10) for $s \in (0, 1)$, together with $\tilde{C}_0 = 1$ and $\tilde{C}_1 = C_0$.

PROPOSITION 2. *Condition* $(A_{\varepsilon,0})$ *with* $C = C_0$ *implies condition* (B_s) *with* $C = \alpha^s\tilde{C}_s + 1$ *for* $s \in [0, 1]$.

PROOF: Since we have for $\varphi \in D(A^s)$

$$A_h^s P_h \varphi = A_h^s P_h (A^{-s} - A_h^{-s} P_h) A^s \varphi + P_h A^s \varphi,$$

it holds that

$$\begin{aligned}
\|A_h^s P_h \varphi\| &\leq \|A_h^s\| \, \|(A^{-s} - A_h^{-s} P_h) A^s \varphi\| + \|A^s \varphi\| \\
&\leq (\text{by Theorem 1}) \\
&\leq \alpha^s \varepsilon(h)^{-s} \tilde{C}_s \varepsilon(h)^s \|A^s \varphi\| + \|A^s \varphi\| \\
&= (\alpha^s \tilde{C}_s + 1) \|A^s \varphi\|.
\end{aligned}$$

∎

To prove Theorem 2 we quote a theorem concerning Heinz's inequality from the book of Krein [1] as the following Lemma 1.

LEMMA 1. *Let* A, *and* B, *be positive selfadjoint operators acting on Hilbert spaces* X, *and* Y, *respectively. Suppose there is a bounded operator* T *from* X *into* Y *and that* M *is a constant greater than or equal to* $\|T\|$. *Suppose also that* $TD(A) \subset D(B)$ *and that there is a constant* N *such that*

$$\|BTx\| \leq N\|Ax\| \quad \text{for all } x \in D(A).$$

Then we have for $t \in [0, 1]$

$$TD(A^t) \subset D(B^t)$$

and

$$\|B^t Tx\| \leq M^{1-t} N^t \|A^t x\| \quad \textit{for all } x \in D(A^t).$$

For the proof of Lemma 1, see Theorem 7.1 of Chapter 1 of [1] and the remark after its proof.

PROOF OF THEOREM 2: In Lemma 1, set

$$(X, Y, A, B, T, M, N, t) = \left(X, X, \varepsilon(h)^s A^s, I, \frac{A_h^{-1} P_h - A^{-1}}{\varepsilon(h)}, C_0, C_s, \frac{\sigma}{s}\right),$$

where C_0, and C_s, stand for the constant C in condition $(A_{\varepsilon,0})$, and $(A_{\varepsilon,s})$, respectively. It holds condition $(A_{\varepsilon,\sigma})$ with $C = C_0^{1-\sigma/s} C_s^{\sigma/s} \leq \max(C_0, C_s)$. Then we have the conclusion. It is to be noted that $(\varepsilon(h)^s A^s)^t = \varepsilon(h)^\sigma A^\sigma$ for $t = \dfrac{\sigma}{s}$. ∎

PROOF OF THEOREM 3: Let $n = [s]$. The proof is given through an induction argument on n.

Let $n = 0$. Then we have by Proposition 2,

$$\|A_h^s P_h \varphi\| \leq C_s \|A^s \varphi\| \quad \text{for } \varphi \in D(A^s). \tag{11}$$

For $\varphi \in D(A^{s+1})$, we have

$$A_h^{s+1} P_h \varphi = A_h^{s+1} P_h (A^{-1} - A_h^{-1} P_h) A\varphi + A_h^s P_h A\varphi.$$

Therefore it holds that

$$\|A_h^{s+1} P_h \varphi\| \leq \|A_h^{s+1}\| \, \|(A^{-1} - A_h^{-1} P_h) A\varphi\| + \|A_h^s P_h A\varphi\|.$$

Conditions $(\varepsilon - 2)$ and $(A_{\varepsilon,s})$ with $C = C'_s$, and inequality (11) yield

$$\|A_h^{s+1} P_h \varphi\| \leq \alpha^{s+1} \varepsilon(h)^{-s-1} C'_s \varepsilon(h)^{1+s} \|A^s A\varphi\| + C_s \|A^s A\varphi\|.$$

Hence we have

$$\|A_h^{s+1} P_h \varphi\| \leq (\alpha^{1+s} C'_s + C_s) \|A^{s+1} \varphi\| \quad \text{for } \varphi \in D(A^{s+1}). \tag{12}$$

Namely we have (B_{s+1}) with $C_{s+1} = C_s + \alpha^{s+1} C'_s$. The validity of (B_σ) for $\sigma \in [0, s+1]$ with $C = \max(1, C_{s+1})$ follows from Lemma 1.

Now we assume the validity of Theorem 3 for s with $[s] = n$. Suppose $[s] = n + 1$. By Theorem 2 condition (ε_s) implies condition $(A_{\varepsilon,s-1})$. Hence the induction hypothesis implies the estimate (11). The preceding argument from the estimate (11) to the estimate (12) for the case of $n = 0$

is also valid for the case of general n. Therefore we have (B_{s+1}) with

$$C_{s+1}=\alpha^{s+1}C'_s+\alpha^s C'_{s-1}+\cdots+\alpha^{s-[s]+1}C'_{s-[s]}+\alpha^{s-[s]}\tilde{C}_{s-[s]}+1,$$

where C'_{s-i} denotes the constant in condition $(A_{\varepsilon,s-i})$ for $i=0, 1, \cdots, [s]$. The validity of (B_σ) for $\sigma\in[0, s+1]$ with the constant C independent of σ follows from Lemma 1. ■

Before entering the proof of Theorem 4, for the sake of clarity we restate conditions $(A_{\varepsilon,-s})$ and (B_{-s}) for $s\geq 0$.

$(A_{\varepsilon,-s})$ $$\|(A_h^{-1}P_h-A^{-1})\varphi\|\leq C\varepsilon(h)^{1-s}\|A^{-s}\varphi\|, \quad \varphi\in X.$$

(B_{-s}) $$\|A_h^{-s}P_h\varphi\|\leq C\|A^{-s}\varphi\|, \quad \varphi\in X.$$

Auxiliary we introduce the following condition (C_{-s}) for $s\geq 0$.

(C_{-s}) $$\|(A_h^{-s}P_h-A^{-s})\varphi\|\leq C\|A^{-s}\varphi\|, \quad \varphi\in X.$$

In the above 3 conditions, C is a constant independent of h, but possibly dependent on the specified condition. Due to the triangle inequality, conditions (B_{-s}) and (C_{-s}) are mutually equivalent.

PROPOSITION 3. *For any $s\in[0, 1]$, condition $(A_{\varepsilon,-s})$ implies condition (B_{-s}).*

PROOF: It suffices to show the validity of (C_{-s}) under condition $(A_{\varepsilon,-s})$. Let $D=A^{-1}$, and let $D_h=A_h^{-1}P_h$. Since the case of $s=0$ or $s=1$ is easy to see, we assume that $0<s<1$.

For $\varphi\in X$, by Proposition 1 and a calculation similar to the proof of Theorem 1, it holds that

$$\begin{aligned} &A^{-s}\varphi-A_h^{-s}P_h\varphi \\ &=\frac{\sin(\pi s)}{\pi}\int_0^\infty \lambda^{s-1}\{D(\lambda+D)^{-1}-D_h(\lambda+D_h)^{-1}\}\varphi\, d\lambda \qquad (13)\\ &=\frac{\sin(\pi s)}{\pi}\int_0^\infty \lambda^s(\lambda+D_h)^{-1}(D-D_h)(\lambda+D)^{-1}\varphi\, d\lambda. \end{aligned}$$

By condition $(\varepsilon-2)$ and condition $(A_{\varepsilon,-s})$, we have

$$\left\|\int_0^{\varepsilon(h)} \lambda^s(\lambda+D_h)^{-1}(D-D_h)(\lambda+D)^{-1}\varphi\, d\lambda\right\|$$

$$
(14)\quad
\begin{aligned}
&\leq \int_0^{\varepsilon(h)} \lambda^s \cdot \|A_h\| \cdot C_{-s}\, \varepsilon(h)^{1-s} \cdot \|D^s(\lambda+D)^{-1}\varphi\| d\lambda \\
&\leq \alpha\varepsilon(h)^{-1} C_{-s}\, \varepsilon(h)^{1-s} \int_0^{\varepsilon(h)} \lambda^{s-1}\, d\lambda \|A^{-s}\varphi\| \\
&\leq \alpha C_{-s}\, \varepsilon(h)^{-s} \frac{\varepsilon(h)^s}{s} \left\| A^{-s}\varphi \right\| = \alpha C_{-s} \frac{1}{s} \left\| A^{-s}\varphi \right\| .
\end{aligned}
$$

Analogously to (14), it holds that

$$
(15)\quad
\begin{aligned}
&\left\| \int_{\varepsilon(h)}^{\infty} \lambda^s (\lambda+D_h)^{-1}(D-D_h)(\lambda+D)^{-1}\varphi\, d\lambda \right\| \\
&\leq \int_{\varepsilon(h)}^{\infty} \lambda^s \cdot \lambda^{-1} \cdot C_{-\varepsilon}\, \varepsilon(h)^{1-s} \cdot \lambda^{-1} \cdot \|A^{-s}\varphi\| d\lambda \\
&= C_{-s}\, \varepsilon(h)^{1-s} \frac{\varepsilon(h)^{s-1}}{1-s} \|A^{-s}\varphi\| \\
&= C_{-s} \frac{1}{1-s} \|A^{-s}\varphi\|.
\end{aligned}
$$

From (13), (14) and (15), we obtain

$$\|A^{-s}\varphi - A_h^{-s} P_h \varphi\| \leq C_s' \|A^{-s}\varphi\| \quad \text{for } \varphi \in X$$

with

$$C_s' = \frac{\sin(\pi s)}{\pi} \left(\frac{\alpha}{s} + \frac{1}{1-s} \right) C_{-s} \leq (\alpha+1) C_{-s} .$$

∎

PROPOSITION 4. *For any* $s \in [0, 1]$, *condition* (B_{-s}) *implies condition* $(B_{-\sigma})$ *for* $\sigma \in [0, s]$ *with the constant* C *independent of* $\sigma \in [0, s]$.

PROOF: Set in Lemma 1

$$(X, Y, A, B, T, M, N, t) = \left(X, X_h, A^{-s}, A_h^{-s}, P_h, 1, C_{-s}, \frac{\sigma}{s} \right),$$

where C_{-s} stands for constant C in (B_{-s}). Then we have for $\sigma \in [0, s]$

$$\|A_h^{-\sigma} P_h \varphi\| \leq C_{-s}^{\sigma/s} \|A^{-\sigma}\varphi\|, \ \varphi \in X.$$

Replacing $C_{-s}^{\sigma/s}$ with $\max(1, C_{-s})$, we obtain the desired conclusion. ∎

PROOF OF THEOREM 4: Propositions 3 and 4 imply the conclusion of Theorem 4. ∎

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