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Operators of class $(S)_{+}^{1}$, Altman's condition and the complementarity problem

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Abstract. We study the complementarity problem associated to a closed convex cone in a reflexive Banach space and to an operator of the form $T(x)=T_1(x)-T_2(x)$, where T_1 is of class $(S)_{+}^{1}$ and T_2 is a compact operator, generally diferant from zero. Several interesting conclusions are obtained.

1. Introduction.

Let $(E, \| \ \|)$ be a Banach space and let $\boldsymbol{K}\subset E$ be a closed convex cone. If E^* is the topological dual of E, we consider the dual pair $\langle E, E^*\rangle$ defined by the natural duality and we denote by $\boldsymbol{K}^*$ the dual cone of $\boldsymbol{K}$, that is, $\boldsymbol{K}^*=\{y\in E^* \mid \langle x, y\rangle \geqq 0;\ \forall x\in \boldsymbol{K}\}$.

Given a mapping $f: \boldsymbol{K}\to E^*$, the general complementarity problem associated to f and $\boldsymbol{K}$ is:

$$\text{C. P.}(f, \boldsymbol{K}): \left\| \begin{array}{l} \text{find } x_0\in \boldsymbol{K} \text{ such that} \\ f(x_0)\in \boldsymbol{K}^* \text{ and } \langle x_0, f(x_0)\rangle = 0 \end{array} \right.$$

It is well known that this problem has numerous applications in such areas as: Optimization, Game Theory and Economics [4], [10-11], [15], [17-23], [28-29].

Since it has interesting and important applications in Mechanics (Elasticity, Hydrodynamics, Free Boundary Problems) and Engineering, it is important to study this problem in infinite dimensional spaces.

In this paper, we consider the case when E is a reflexive Banach space and $f(x)=T_1(x)-T_2(x)$, with T_1 and T_2 satisfying special conditions. Such a case seems to be frequently used in practical problems. We will suppose that T_1 is bounded and satisfies condition $(S)_{+}^{1}$ and T_2 is (ws)-compact (see Definitions 2 and 5).

In [15] we studied in Hilbert spaces some complementarity problems

Key words: Nonlinear complementarity problem, reflexive Banach spaces, operators of class $(S)_{+}^{1}$.

with bounded solutions set.

Now, in this paper we are interested to know when the problem C.P.$(T_1-T_2, \boldsymbol{K})$ has a solution in a set of the form $B_K(0,r)=\{x\in \boldsymbol{K} \mid \|x\| \leq r\}$, fact which is very important for some numerical methods or for some practical problems.

At the same time, we show the importance of operators of class $(S)_+$ in the study of this problem. We note that this class of operators was much studied in nonlinear analysis [5-9].

A generalization of Altman's condition is also essential for our results. We finish this paper with some consequences of the principal results.

2. Preliminaries.

Let $(E, \| \ \|)$ be a Banach space. We denote by E^* the topological dual of E and by $\langle E, E^*\rangle$ the natural pairing. In this paper the "weak topology" on E or E^* refers respectively to the topology defined by the canonical duality between E and E^*.

We denote by (w)-lim the limit with respeect to the weak topology.

We say that a Banach space $(E, \| \ \|)$ is locally uniformly convex if for every $\varepsilon>0$ and x with $\|x\|=1$ there exists $\delta(\varepsilon, x)>0$ such that the inequality $\|x-y\| \geq \varepsilon$ implies $\|x+y\| \leq 2(1-\delta(\varepsilon, x))$ for every $y\in E$ with $\|y\|=1$.

Certainly, every uniformly convex Banach space is locally uniformly convex and reflexive. *We say that a Banach space $(E, \| \ \|)$ is Kadeç if for each sequence $\{x_n\}_{n\in N}$ in E which converges weakly to x_* with $\lim_{n\to\infty}\|x_n\| = \|x_*\|$ we have $\lim_{n\to\infty}\|x_n-x_*\|=0$.*

Each L^p space $(1<p<\infty)$ has this property as does $l_1(S)$ and any locally uniformly convex Banach space.

A Banach space $(E, \| \ \|)$ is said to be strictly convex if for every $x, y \in E$ with $x\neq y$, $\|x\|=\|y\|=1$ we have that $\|\lambda x+(1-\lambda)y\|<1$, for every $\lambda\in$ $(0,1)$. Every locally convex Banach space is strictly convex.

Let $\Omega\subset R^n$ be an open set and $1<p<\infty$. The Sobolev space $W^{m,p}(\Omega)$ is the space of u in $L^p(\Omega)$ (with respect to Lebesgue n-measure) with all the derivatives $D^\alpha u$ also in $L^p(\Omega)$ for all distribution derivatives of order $\leq m$. The norm of this space is obtained by injecting it into the product of L^p-spaces, one for each derivative by jet-mapping $u\to\{D^\alpha u \mid |\alpha| \leq m\}$.

The space $W_0^{m,p}(\Omega)$ is the closed subspace of $W^{m,p}(\Omega)$ obtained by taking closure of the subspace of testing functions with compact support in Ω.

We note that the Sobolev space $W_0^{m,p}(\Omega)$, $(1<p<\infty)$ are locally uniformly convex (since it is uniformly convex) and reflexive.

Several classical results establish interesting relations between reflexivity and strictly convexity. We recall two of these results.

If $(E, \|\ \|)$ *is a reflexive Banach space then for every* $a>1$ *there exists a norm* $\|\ \|_a$ *on* E *such that:*

i) $(E, \|\ \|)_a)$ *and* $(E^*, \|\ \|_a^*)$ *are strictly convex* (*where* $\|\ \|_a^*$) *is the dual norm of* $\|\ \|_a$),

ii) $(1/a)\|\ \|_a \leqq \|\ \| \leqq a\|\ \|_a$,

iii) $(1/a)\|\ \|_a^* \leqq \|\ \|^* \leqq \|\ \|_a^*$.

This is Brezis-Crandall-Pazy's theorem [24, p. 177].

Also, as a consequence of some classical results proved by Lindenstraus, Asplund and Troyanski we have the following result [12].

If $(E, \|\ \|)$ *is a reflexive Banach space, then there exists on* E *an equivalent norm* $\|\ \|_1$ *such that* $(E, \|\ \|_1)$ *and* $(E^*, \|\ \|_1^*)$ *are locally uniformly convex. Moreover the norms* $\|\ \|_1$ *and* $\|\ \|_1^*$ *are Fréchet differentiable.*

Let $(E, \|\ \|)$ be a Banach space and let $\boldsymbol{K} \subset E$ be a closed convex cone, that is $\boldsymbol{K}+\boldsymbol{K} \subseteqq \boldsymbol{K}$, $\lambda \boldsymbol{K} \subseteqq \boldsymbol{K}$, for every $\lambda \in R_+$ and $\boldsymbol{K}$ is closed. We suppose given two mappings $T_1, T_2 : \boldsymbol{K} \to E^*$.

We will study in this paper the following complementarity problem:

$$\text{C. P.}(T_1 - T_2, \boldsymbol{K}) : \left\| \begin{array}{l} \text{find } x_0 \in \boldsymbol{K} \text{ such that } T_1(x_0) - T_2(x_0) \in \boldsymbol{K}^* \\ \text{and } \langle x_0, T_1(x_0) - T_2(x_0) \rangle = 0 \end{array} \right.$$

Examples. We obtain interesting examples of such problem considering the complementarity problem associated to the Von Kárman operator or to its compact perturbations.

Let $(E, \langle\ ,\ \rangle$ be a Hilbert space. In practical problems E is a closed subspace of a particular Sobolov space. Suppose defined a closed convex cone $\boldsymbol{K} \subset E$. Consider $T_1(x) = x - \lambda L(x)$, where $\lambda \in \boldsymbol{R}_+$ and L is a linear self-adjoint compact operator and $T_2(x) = -L_2(x)$ or $T_2(x) = -C(x) - R(x)$, where L_2, C and R are nonlinear compact operators with some physical interpretation [19]. The operator $T_1 - T_2$ is in fact the Von Kárman operator or a compact perturtation of this operator. The problem $CP(T_1 - T_2, \boldsymbol{K})$ is in this case the mathematical model used to study the existence of the post-critical equilibrium state of a thin elastic plate [19], [20].

We say that $\boldsymbol{K}$ is locally compact if there exists a fundamental system $\mathcal{U}$ of neighborhoods of zero such that for every $U \in \mathcal{U}$, $U \cap \boldsymbol{K}$ is compact.

From Klee's theorem [1], we know that $\boldsymbol{K}$ is locally compact, if and only if, $\boldsymbol{K}$ has a compact base, that is there exists a convex compact set $B \subset \boldsymbol{K}$ such that $0 \notin B$ and for every $x \in \boldsymbol{K} \setminus \}0\}$ there exists a unique $\lambda \in$

$\boldsymbol{R}_* \backslash \{0\}$ and a unique $b \in B$ such that $x = \lambda b$.

We say that an operator (not necessary linear) $T : \boldsymbol{K} \rightarrow E^*$ is bounded if $T(B)$ is bounded for every bounded set $B \in \boldsymbol{K}$.

About the bilinear form $\langle , \rangle$ we remark the following fact which will be used in this paper.

If $\{x_n\}_{n \in N} \subset E$ is convergent to x_* and $\{y_n\}_{n \ni N} \subset E^*$ is weakly convergent to y_* (in particular if $\{y\}_{n \in N}$ is convergent to y_*) then $\lim_{n \to \infty} \langle x_n, y_n \rangle = \langle x_*, y_* \rangle$. To show this we use the decomposition : $\langle x_n, y_n \rangle = \langle x_n - x_*, y_n - y_* \rangle + \langle x_*, y_n \rangle + \langle x_n, y_* \rangle - 2\langle x_*, y_* \rangle$ and the fact that $\{y_n\}$ is bounded.

3. Results

A. The case of locally compact cones.

We use in this section the following classical result.

THEOREM 1 [Hartman-Stampacchia]. *Let E be a Banach space, E^* its topological dual and let C be a compact convex set in E.*

If $f : C \rightarrow E^*$ is a continuous mapping, then there exists $x_* \in C$ such that $\langle x - x_*, f(x_*) \rangle \geqq 0$, for every $x \in C$.

PROOF. This variant of the classical Hartman-Stampacchia theorem is a particular case of Theorm 2 proved in [16]. ■

Let $(E, \| \|)$ be a Banach space, $\boldsymbol{K} \subset E$ a pointed closed convex cone and suppose given two mappings $T_1, T_2 : \boldsymbol{K} - E^*$. We introduce now a condition which is essential for this paper.

DEFINITION 1. We say that T_2 satisfies Altman's condition with respect to T_1 for $r > 0$ (with respect to $\boldsymbol{K}$) if for every $x \in \boldsymbol{K}$ with $\|x\| = r$ we have $\langle x, T_2(x) \rangle \leqq \langle x, T_1(x) \rangle$.

REMARK 1. If E is a Hilbert space and $T_1(x) = x$ for every $x \in E$. then we obtain from Definition 1 the classical Altman's condition for T_2. This condition is an essential assumption in several known fixed point theorems [2], [31].

THEOREM 2. *If $\boldsymbol{K} \subset E$ is a locally compact convex cone in a Banach space E and $T_1, T_2 : \boldsymbol{K} \rightarrow E^*$ two mappings and the following assumptions are satisfied:*

1°) *T_1 and T_2 are continuous,*

2°) *T_2 satisfies Altman's condition with respect to T_1 for some $r>0$, then the problem C.P.$(T_1-T_2, \boldsymbol{K})$ has a solution x_* with $\|x_*\| \leqq r$.*

PROOF. Since $\boldsymbol{K}$ is locally compact the set $\boldsymbol{K}_r=\{x\in \boldsymbol{K} \mid \|x\|\leqq r\}$ is convex and compact in E. By Theorem 1 [Hartman-Stampacchia] there exists an element $x^*\in \boldsymbol{K}_r$ such that $(\alpha_1): \langle x-x_*, T_1(x_*)-T_2(x_*)\geqq 0$, for all $x\in \boldsymbol{K}_r$.

We show now that x_* is a solution of the problem $C.P.(T_1-T_2, \boldsymbol{K})$. Indeed, about x_* we have two possible cases:

i) $\|x_*\|<r$. Then for every $x\in \boldsymbol{K}$ there exists $\lambda\in(0,1)$ such that $u=\lambda x+(1-\lambda)x_*\in \boldsymbol{K}_r$. Using the element u in the inequality (α_1) we get $\langle x-x_*, T_1(x_*)-T_2(x_*)\rangle\geqq 0$, for all $x\in \boldsymbol{K}$, which is equivalent to the fact that x_* solves the problem $C.P.(T_1-T_2, \boldsymbol{K})$.

ii) $\|x_*\|=r$. In this case from (α_1) we have $\langle 0-x_*, T_1(x_*)-T_2(x_*)\rangle \geqq 0$, that is, $\langle x^*, T_1(x_*)-T_2(x_*)\rangle\leqq 0$ or $\langle x_*, T_1(x_*)\rangle\leqq\langle x_*, T_2(x_*)\rangle$, and using assumption 2°) we obtain $(\alpha_2): \langle x_*, T_1(x_*)-T_2(x_*)\rangle=0$.

The proof is finished if we show that $T_1(x_*)-T_2(x_*)\in T^*$.

Indeed, from (α_1) and (α_2) we have $\langle x, T_1(x_*)-T_2(x_*)\rangle\geqq 0$; for all $x\in \boldsymbol{K}_r$.

Scaling leads to $\langle x, T_1(x_*)-T_2(x_*)\rangle\geqq 0$, for $x\in \boldsymbol{K}$, that is, we have $T_1(x_*)-T_2(x_*)\in \boldsymbol{K}^*$. ■

COROLLARY. *Let K be a locally compact convex cone in a Banach space. If $T: \boldsymbol{K}\to E^*$ is a continuous mapping and for some $r>0$ we have $\langle x, T(x)\rangle\geqq 0$, for all $x\in \boldsymbol{K}$ with $\|x\|=r$ then the problem C.P.$(T, \boldsymbol{K})$ has a solution x_* with $\|x_*\|\leqq r$.*

PROOF. We consider $T_1=T$, $T_2=0$ and we apply Theorem 2. ■

B. The case of Galerkin cones.

In this section we extend the principal result of section A to general Galerkin cones. First, we need to consider some results about duality mappings.

Let $(E, \|\ \|)$ be a Banach space. We denote by $(E^*, \|\ \|_*)$ the dual of E, where $\|\ \|_*$ is the dual norm of $\|\ \|$ and by $\langle E, E\rangle$ the duality defined by E and E^*. We say that a continuous and strictly increasing function $\phi: R_+\to R_+$ is a weight if $\phi(0)=0$ and $\lim_{r\to\infty}\phi(r)=+\infty$. Given a weight ϕ, a duality mapping on E associated to ϕ is a mapping $J: E\to 2^{E^*}$ such that $J(x)=\{x^*\in E^* \mid \langle x, x^*\rangle=\|x\|\|x\|_* \text{ and } \|x^*\|_*=\phi(\|x\|)\}$.

A consequence of Hahn-Banach theorem is the fact that for every $x\in E$ we have that $J(x)$ is nonempty [12].

Examples. 1°). If E is a Hilbert space and $\phi(r)=r$, for every $r\in R_+$ then the duality mapping associated to ϕ is $J(x)=x$, for every $x\in E$.

2°) If $E=L^p(\Omega)$, $\|u\|=\left(\int_\Omega |u|^p dx\right)^{1/p}=\|u\|_{L^p(\Omega)}$, $\phi(r)=r^{p-1}$ then $J(u)=|u|^{p-2}\cdot u$.

3°) If $E=W_0^{1,p}(\Omega)$, $\|u\|=\left(\sum_{i=1}^n \|D_i u\|_{L^p(\Omega)}^p\right)^{1/p}$, $\phi(r)=r^{p-1}$ then

$$J(u)=-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

The domain Ω is supposed to be bounded [8].

A duality mapping is a monotone operator and it is strictly monotone if E is strictly convex.

If $(E, \|\ \|)$ is a reflexive Banach space with $(E^*, \|\ \|_*)$ strictly convex then a duality mapping associated to a weight function ϕ is a demicontinuous point-to-point mapping.

Another important classical result about duality mapping is the following.

If $(E, \|\ \|)$ is a Banach space then a duality mapping on D is a point-to-point mapping and norm continuous if and only if the norm of E is Fréchet differentiable [9], [12].

DEFINITION 2. Let D be a subset of E. We say that a mapping $T: D\to E^*$ satisfies condition $(S)_+^1$ if for any sequence $\{x_n\}_{n\in N}\subset D$ with $(w)\text{-}\lim_{n\to\infty} x_n=x_*$, $(w)\text{-}\lim_{n\to\infty} T(x_n)=u\in E^*$ and $\limsup_{n\to\infty}\langle x_n, T(x_n)\rangle\leqq\langle x_n, u\rangle$ we have that $\{x_n\}_{n\in N}$ has a subsequence norm convergent to x^*.

PROPOSITION 3. *Let $(E, \|\ \|)$ be a Banach space which is Kaceç and such that E^* is strictly convex.*

If J is a duality mapping on E associated to a weight ϕ then J satisfies $(S)_+^1$.

PROOF. Since E^* is strictly convex we have that J is a potni-to-point mapping.

Consider a sequence $\{x_n\}_{n\in N}\subset E$ such that $(w)\text{-}\lim_{n\to\infty} x_n=x_*$, $(w)\text{-}\lim_{n\to\infty} J(x_n)=u$ and $\lim_{n\to 8}\sup\langle x_n, J(x_n)\rangle\leqq\langle x_*, u\rangle$.

From the definition of J we have

$$\langle x_n - x_*, J(x_n) - J(x_*)\rangle$$
$$= \langle x_n, J(x_n)\rangle - \langle x_*, J(x_n)\rangle - \langle x_n, J(x_*)\rangle + \langle x_*, J(x_*)\rangle$$
$$= [\phi(\|x_n\|) - \phi(\|x_*\|)]\cdot[\|x_n\| - \|x_*\|] + [\|J(x_*)\|_* \cdot \|x_n\| - \langle x_n, J(x_*)\rangle]$$
$$+ [\|J(x_n)\|_* \|x_*\| - \langle x_*, J(x_n)\rangle],$$

that is

$\beta_1)$: $\langle x_n - x_*, J(x_n) - J(x_n)\rangle \geqq [\phi(\|x_n\|) - \phi(\|x_*\|)]\cdot[\|x_n\| - \|x_*\|] \geqq 0$, which implies $0 \leqq \liminf_{n\to\infty} [\phi(\|x_n\|) - \phi(\|x_*\|)][\|x_n\| - \|x_*\|] \leqq \limsup_{n\to\infty} [\phi(\|x_n\|) - \phi(\|x_*\|)]$ $[\|x_n\| - \|x_*\|] \leqq \limsup_{n\to\infty} \langle x_n, J(x_n)\rangle - \lim_{n\to\infty} \langle x_*, J(x_n)\rangle - \lim_{n\to\infty} \langle x_n - x_*, J(x_*)\rangle \leqq \langle x_*, u\rangle$ $- \langle x_*, u\rangle = 0$. That is we have

$\beta_2)$: $\lim_{n\to\infty} [\phi(\|x_n\|) - \phi(\|x_*\|)][\|x_n\| - \|x_*\|] = 0$.

We show now that $\beta_2)$ implies that $\{\|x_n\|\}_{n\in N}$ is convergent to $\|x_*\|$. To prove that $\{\|x_n\|\}_{n\in N}$ is convergent to $\|x_*\|$ we show that every subsequence of $\{\|x_n\|\}_{n\in N}$ has a subsequence convergent to $\|x_*\|$. [24]

Indeed, let $\{\|x_{n_k}\|\}_{k\in N}$. The sequence $\{x_{n_k}\}_{k\in N}$ is bounded since $\{x_{n_k}\}$ is weakly convergent to x_*. Hence $\{\|x_{n_k}\|\}_{k\in N}$ has a convergent subsequence. We denote this last subsequence by $\{\|x_i\|\}_{i\in N}$. The sequence $\{\|x_i\|\}_{i\in N}$ must be convergent to $\|x_*\|$.

Indeed, if we suppose the contrary we have $\lim_{i\to\infty} \|x_i\| = \|x_*\| + c$, with $c \neq 0$.

The mapping ϕ being continuous we have $\lim_{i\to\infty} \phi(\|x_i\|) = \phi(\|x_*\| + c)$ and $\phi(\|x_*\| + c) \neq \phi(\|x_*\|)$ since ϕ is strictly increasing. So, $\lim_{i\to\infty} [\phi(\|x_i\|) - \Phi\|x_*\|)]$ $= \alpha \neq 0$ and hence $\lim_{i\to\infty} [\phi(\|x_i\|) - \phi(\|x_*\|)][\|x_i\| - \|x_*\|] = \alpha c \neq 0$, which is a contradiction to $\beta_2)$.

Hence $\{x_n\}_{n\in N}$ is weakly convergent to x_*, $\{\|x_n\|\}_{n\in N}$ is convergent to $\|x_*\|$ and since E is Kadeç we obtain that $\{x_n\}_{n\in N}$ is convergent to x_* and finally we have that J satisfies $(S)_+^1$. ■

In nonlinear analysis is used a condition similar to condition $(S)_+^1$ denonoted by $(S)_+$ [5-8]. We recall the definition of this condition.

DEFINITION 3 [5-8]. A mapping $T: E \to E^*$ is said to satisfy condition $(S)_+$ if for any sequence $\{x_n\}_{n\in N} \subset E$ which converges weakly to x_* in E and for which $\limsup_{n\to\infty} \langle x_n - x_*, T(x_n)\rangle \leqq 0$, we have the norm convergence of $\{x_n\}_{n\in N}$ to x_*.

If $(E, \| \|)$ is Banach space which is Kadeç and E^* is strictly convex then a duality mapping associated to $\phi(r)=r$, for every $r\in R_+$ satisfies condition $(S)_+$ [8], but the primary interest of definition of condition $(S)_+$ is the fact that one can verify this property under suitable concrete hypotheses for the maps of a Sobolev space $W_0^{m,p}(\Omega)$ into conjugate space $W_0^{-m,p'}(\Omega)$ (where $p'=(p/p-1)$) obtained from an elliptic operator in generalized divergence form $T(u)=\sum_{\alpha\leqq m}(-1)^{|\alpha|}D^\alpha T_\alpha(x,u,\cdots,D^m u)$.

We note that operator T given by the differential expression written above makes sense as a mapping of $W_0^{m,p}(\alpha)$ into its conjugate space $W^{-m,p'}(\Omega)$.

In [4] is proved that if some assumptions are satisfied then the operator $T=\sum_{\alpha\leqq m}(-1)^{|\alpha|}D^\alpha T_\alpha(x,u,\cdots,D^m u)$ satisfies $(S)_+$. Other details about condition $(S)_+$ we find in [7], [6], [8], [9]. The next result shows that condition $(S)_+$ is satisfied for another operators which are not duality mappings. We say that $f: E\to E^*$ is strongly ρ-monotone if there exists a continuous strictly increasing function $\rho: R_+\to R_+$ such that $\rho(0)=0$ and $\langle x-y, f(x)-f(y)\rangle\geqq\rho(\|x-y\|)$, for all $x,y\in E$.

PROPOSITION 4. *Each strongly ρ-monotone mapping $T: E\to E^*$ satisfies condition $(S)_+$.*

PROOF. Let $\{x_n\}_{n\in N}$ be a sequence weakly convergent to x_* in E and such that $\limsup_{n\to\infty}\langle x_n-x_*, T(x_n)\rangle\leqq 0$. Since

$$\rho(\|x_n-x_*\|)\leqq\langle x_n-x_*, T(x_n)\rangle-T(x_*)\rangle$$
$$=\langle x_n-x_*, T(x_*)\rangle-\langle x_n-x_*, T(x_*)\rangle$$

we obtain

$$0\leqq\liminf_{n\to\infty}\rho(\|x_n-x_*\|)\leqq\limsup_{n\to\infty}\rho(\|x_n-x_*\|)$$
$$\leqq\limsup_{n\to\infty}\langle x_n-x_*, T(x_n)\rangle-\lim_{n\to\infty}\langle x_n-x_*, T(x_*)\rangle\leqq 0,$$

which implies $\lim_{n\to\infty}\rho(\|x_n-x_*\|)=0$ and since ρ is strictly increasing and continuous we can show that $\|x_n-x_*\|$ is convergent to 0. ■.

REMARK 2. We remark that the class of operators satisfying condition $(S)_+$ is inveriant under compact perturbations that is, if $T_1: E\to E^*$ satisfies $(S)_+$ and $T_2: E\to E^*$ is compact then T_1+T_2 satisfies $(S)_+$.

In particular when $E=E^*$ is a Hilbert space then the operators of class $(S)_+$ contains Leray-Schauder operators, that is, the operators of the form $T=I-G$, where G is a compact operator.

PROPOSITION 5. *If a mapping* $T: E\to E^*$ *satisfies condition* $(S)_+$ *then it satisfies* $(S)_+^1$.

PROOF. Let $\{x_n\}_{n\in N}$ be a weakly convergent sequence to x_* in E, such that $\{T(x_n)\}_{n\in N}$ is weakly convergent to $u\in E^*$ and $\limsup_{n\to\infty}\langle x_n, T(x_n)\rangle\leqq\langle x_*, u\rangle$.

We have $\langle x_n-x_*, T(x_n)\rangle=\langle x_n, T(x_n)\rangle-\langle x_*, T(x_n)\rangle$ which imylies, $\limsup_{n\to\infty}\langle x_n-x_*, T(x_n)\rangle=\limsup_{n\to\infty}\langle x_n, T(x_n)\rangle-\lim_{n\to\infty}\langle x_*, T(x_n)\rangle\leqq\langle x_*,u\rangle-\langle x_*,u\rangle=0$. Since $(S)_+$ is satisfied for T we have that $\{x_n\}_{n\in N}$ is norm convergent to x_*. ■

In conclusion, the class of mappings satisfying condition $(S)_+^1$ is sufficiently large. Now, we will introduce the concept of Galerkin cone.

Let $(E, \|\ \|)$ be a Banach space and let $\boldsymbol{K}\subset E$ be a closed convex cone.

DEFINITION 4. We say that $\boldsymbol{K}$ is a Galerkin cone if there exists a countable family of convex subcones $\{\boldsymbol{K}_n\}_{n\in N}$ of $\boldsymbol{K}$ such that:

1°) $\boldsymbol{K}_n$ is locally compact for every $n\in N$,

2°) $\boldsymbol{K}_n\subseteqq\boldsymbol{K}_m$ whenever $n\leqq m$,

3°) $\boldsymbol{K}=\overline{\bigcup_{n\in N}\boldsymbol{K}_n}$.

We denote a Galerkin cone by $\boldsymbol{K}(\boldsymbol{K}_n)_{n\in N}$. In practical problems we obtain Galerkin cones, by approximation by the finite element method or if $\boldsymbol{K}$ has a Schauder base.

For example in our paper [20] is proved that if E is an infinite-dimensional separable Banach lattice with the Krein-Milman property such that $\boldsymbol{K}=\{x\in E|x\geqq 0\}$ has a bounded base B then $\boldsymbol{K}$ has a Schauder base $\{b_n\}_{n\in N}$ where b_n is an extreme point for B.

Let $(E, \|\ \|)$ and $(F, \|\ \|)$ be two Banach spaces and let $T: D\to F$ be a mapping, where D is a subset of E.

DEFINITION 5. We say that T is (ws)-compact operator if for every weakly convergent sequence $\{x_n\}_{n\in N}$ in D, we have that $\{T(x_n)\}_{n\in N}$ has a subsequence norm convergent in F.

As examples of (ws)-compact operators we have compact operators and strongly continuous operators. (For strongly continuous operator see [32] and [25].

THEOREM 4. *Let* $(E, \|\ \|)$ *be a reflexive Banach space and let* $\boldsymbol{K}(\boldsymbol{K}_n)_{n\in N}$

be a Galerkin cone in E.

Suppose given two continuous mappings $T_1, T_2 : \boldsymbol{K} \to E^*$.

If the following assumptions are satisfied:

1°) *T_1 is bounded and satisfies condition $(S)^1_+$ with respect to $\boldsymbol{K}$.*

2°) *T_2 is a (ws)-compact operator,*

3°) *T_2 satisfies Altman's condition with respect to T_1 for an $r>0$ with respect to $\boldsymbol{K}$,*

then the Problem C.P.$(T_1-T_2, \boldsymbol{K})$ has a solution x_* such that $\|x_*\| \leqq r$.

PROOF. Let x_n be a solution to the complementarity problem C.P. $(T_1-T_2, \boldsymbol{K}_n)$ obtained by Theorem 2, with $\|x_n\| \leqq r$:

$$(T_1-T_2)(x_n) \in \boldsymbol{K}_n^*, \qquad \langle x_n, (T_1-T_2)(x_n)\rangle = 0\,. \tag{1}$$

Since $\{x_n\}_n$ and $\{T_1(x_n)\}_n$ are bounded, there exists a subsequence $\{x_{\phi(n)}\}_n$ ($\phi : N \to N$, strictly monotone) such that

$$x_{\phi(n)} \rightharpoonup x_* \quad \text{and} \quad T_1(x_{\phi(n)}) \rightharpoonup u \tag{2}$$

for some $x_* \in \boldsymbol{K}$ and $u \in E^*$. By the (ws)-compactness of T_2 there exists a further subsequence $\{x_{\psi(n)}\}_n$ of $\{x_{\phi(n)}\}_n$ for which

$$T_2(x_{\psi(n)}) \text{ is norm convergent to a } v \in E^*\,. \tag{3}$$

Then $x_* \in \boldsymbol{K}$ and $(T_1-T_2)(x_{\psi(n)}) \rightharpoonup u-v$. Moreover, we can show that $u-v \in \boldsymbol{K}^*$. In fact, let $x \in \boldsymbol{K}_{\psi(n)}$, $m \geqq n$. Then $\langle x, (T_1-T_2)(x_{\psi(m)})\rangle \geqq 0$ holds since $(T_1-T_2)(x_{\psi(m)}) \in \boldsymbol{K}^*_{\psi(m)} \subset \boldsymbol{K}^*_{\psi(n)}$ by definition. Letting $m \to \infty$, we obtain $\langle x, u-v\rangle \geqq 0$ for any $x \in \boldsymbol{K}_{\psi(n)}$. Since $\cup_n \boldsymbol{K}_{\psi(n)}$ is dense in $\boldsymbol{K}$, this proves $u-v \in \boldsymbol{K}^*$.

From these and the equations (1), (2) and (3), we obtain

$$\langle x_{\psi(n)}, T_1(x_{\psi(n)})\rangle = \langle x_{\psi(n)}, (T_1-T_2)(x_{\psi(n)})\rangle + \langle x_{\psi(n)}, T_2(x_{\psi(n)})\rangle$$
$$\longmapsto \langle x_*, v\rangle \leq \langle x_*, u\rangle\,. \qquad (x_* \in \boldsymbol{K}, u-v \in \boldsymbol{K}^*)$$

This yields $x_{\psi(n)} \to x_*$ since T_1 satisfies $(S)^1_+$. Therefore $(T_1-T_2)(x_{\psi(n)})$ is norm convergent to $(T_1-T_2)(x_*) = u-v \in \boldsymbol{K}^*$, hence the second equality in (1) gives

$$\langle x_*, (T_1-T_2)(x_*)\rangle = 0\,.$$

Thus we have shown that x_* is a solution to the problem C.P. $(T_1-T_2, \boldsymbol{K})$. ■

(In the proof above, "$\rightharpoonup$" denotes the weak—or weak*—convergence, while "→" means the norm convergence.)

In the next result conditition $(S)_+^1$ is replaced by other conditions.

THEOREM 5. *Let* $(E, \| \ \|)$ *be a reflexive Banach space and let* $\boldsymbol{K}(\boldsymbol{K}_n)_{n\in N}$ *be a Galerkin cone in* E.

Suppose given two continuous mappings $T_1, T_2 : \boldsymbol{K} \to E^*$. *If the following assumption are satisfied:*

1°) T_2 *satisfies Altman's condition with respect to* T_1 *for an* $r>o$,

2°) $T_1 - T_2$ *is sequentially weak-to-weak continuous,*

3°) *If* $\{x_n\}_{n\in N} \subset \boldsymbol{K}$, $(w)\text{-}\lim x_n = x_0$ *and* $\langle x_n, T_1(x_n) - T_2(x_n)\rangle = 0$ *for every* n, *we have* $\langle x_0, T_1(x_0) - T_2(x_0)\rangle \leqq 0$,

then the problem $C.P.(T_1 - T_2, \boldsymbol{K})$ *has a solution* x_* *with* $\|x_*\| \leq r$.

PROOF. From Theorem 2 we have that for every $n\in N$ the problem $C.P.(T_1 - T_2, \boldsymbol{K}_n)$ has a solution x_n with $\|x_n\| \leqq r$.

Since E is reflexive the sequence $\{x_n\}_{n\in N}$ has a subsequence $\{x_{n_k}\}_{k\in N}$ weakly convergent to an element $x_* \in \boldsymbol{K}$. We have $\|x_*\| \leqq r$.

We denote the sequence $\{x_{n_k}\}_{k\in N}$ again by $\{x_n\}_{n\in N}$. Since for every $n\in N$, we have that $(T_1 - T_2)(x_n) \in \boldsymbol{K}_n^*$, we deduce, since $\boldsymbol{K}$ is a Galerkin cone, the inequality $\langle x, (T_1 - T_2)(x_m)\rangle \geqq 0$ for every $x\in \boldsymbol{K}_n$ and every $m \geqq n$, which implies by assumption 2, $\langle x, (T_1 - T_2)(x_*)\rangle \geqq 0$, for every $x \in \boldsymbol{K}_n$.

Now, since for every $x\in \boldsymbol{K}_n$, we have that $\langle x, (T_1 - T_2)(x_*)\rangle \geqq 0$, using again that $\boldsymbol{K}$ is a Galerkin cone we obtain that $\langle x, (T_1 - T_2)(x_*)\rangle \geqq 0$, for all $x\in \boldsymbol{K}$, that is $T_1(x_*) - T_2(x_*) \in \boldsymbol{K}^*$. The proof is finshed if we show that $\langle x_*, T_1(x_*) - T_2(x_*)\rangle = 0$.

Indeed, because $\langle x_n, T_1(x_n) - T_2(x_n)\rangle = 0$, for every $n\in N$ and $\{x_n\}$ is weakly convergent to x_* we obtain from assumption 3°) $\langle x_*, T_1(x_*) - T_2(x_*)\rangle \leqq 0$ and since $T_1(x_*) - T_2(x_*) \in \boldsymbol{K}^*$ we have $\langle x_*, T_1(x_*) - T_2(x_*)\rangle = 0$. ■

REMARK 3. We give now a condition which implies that $T_1 - T_2$ is sequentially weak-to-weak continuous. If $T\in L(E, E^*)$ we denote by T^* the adjoint of T.

We say that $f : \boldsymbol{K} \to E^*$ is Gâteaux differentiable along the convex cone $\boldsymbol{K} \subset E$ if the function f has a linear Gâteaux differential $f'(x) \in L(E, E^*)$ at every $x \in \boldsymbol{K}$.

We suppose E to be a reflexive Banach space and $T_1, T_2 : \boldsymbol{K} \to E^*$ two mappings.

We can show that if $T_1 - T_2$ is Gâteaux differentiable along $\boldsymbol{K}$ and for each $y^* \in E^*$ and every bounded sequence $\{x_n\}_{n\in N} \subset \boldsymbol{K}$ there exists a subsequence $\{x_{n_k}\}_{k\in N}$ such that $\bigcup_{k\in N} [(T_1 - T_2)'(x_{n_k})]^*(y^*)$ is strongly precompact then $T_1 - T_2$ is sequentially weak-to-weak continuous on $\boldsymbol{K}$. (For the proof see Lemma 4 in [25]).

4. Consequences of principal results and comments.

COROLLARY 1. *Let $(E, \|\ \|)$ be a reflexive Banach space and let $\boldsymbol{K}(\boldsymbol{K}_n)_{n\in N}$ be a Galerkin cone in E. Suppose given two continuous mappings $T_1, T_2 : \boldsymbol{K} \to E^*$.*

If the following assumptions are satisfied:

1°) T_1 is bounded, and strongly ρ-monotone

2°) T_2 is (ws)-compact and satisfies Altman's condition with respect to T_1 for an $r>0$ with respect to $\boldsymbol{K}$,

then the problem C.P.$(T_1 - T_2, \boldsymbol{K})$ has a solution x_* such that $\|x_*\| \leqq r$.

PROOF. Consequence of Proposition 4 and Theorem 4. ■

COROLLARY 2. *Let $E = H$ be a Hilbert space and let $\boldsymbol{K}(\boldsymbol{K}_n)_{n\in N}$ be a Galerkin cone in H.*

If $T : \boldsymbol{K} \to H$ is continuous, (ws)-compact and there exists $r>0$ such that for every $x \in \boldsymbol{K}$ with $\|x\| = r$ we have $\langle x, T(x)\rangle \leqq \|x\|^2$, then the problem C.P.$(I - T, \boldsymbol{K})$ has a solution x_* such that $\|x_*\| \leqq x$.

PROOF. Consequence of Theorem 4. ■

REMARK 4. We remark that Altman's condition "$\langle x, T(x)\rangle \leqq \|x\|^2$, for every $x \in \boldsymbol{K}$ with $\|x\| = r$" is satisfied if $T : \boldsymbol{K} \to H$ monotone decreasing on rays.

We say that $T : \boldsymbol{K} \to H$ is monotone decreasing on rays with respect to $\boldsymbol{K}$ if there exists $t_0 > 0$ such that for every $x \in \boldsymbol{K}$ and every s, t such that $s \geqq t \geqq t_0$ we have $\langle x, T(tx)\rangle \geqq \langle x, T(sx)\rangle$.

A variant of this notion is used in [31] but with respect to a vector space.

If T is continuous and monotone decreasing on rays the Altman's condition is satisfied for T. Indeed, since T is monotone decreasing on rays then for every $\lambda > 0$ and $x \in \boldsymbol{K}$, $t \to \langle \lambda x, T(t\lambda x)\rangle$ is monotone decreasing for $t \geqq t_0$, which is equivalent to say that $t \to \langle x, T(t\lambda x)\rangle$ is is monotone decreasing for $t \geqq t_0$.

Therefore we see that $t \to \langle x, T(tx)\rangle$ is monotone decreasing for $t > 0$ (let $\lambda \downarrow 0$).

Thus, together with the continuity of T, we have that $\langle x, T(tx) \leqq \langle x, T(0)\rangle$ (let $t \downarrow 0$), which implies that $\langle x, T(x)\rangle \leqq \|x\|^2$, for x with $\|x\| \geqq$

$\|T(0)\|$.

COROLLARY 3. *Let H be a Hilbert space and let $\boldsymbol{K}(\boldsymbol{K}_n)_{n\in N}$ be a Galerkin cone in H.*

If $T:\boldsymbol{K}\to H$ is continuous, (ws)-compact and monotone decreasing on rays with respect to $\boldsymbol{K}$, then the problem C.P.$(I-T,\boldsymbol{K})$ has a solution.

PROOF. Consequence of Corollary 2 since T satisfies Altman's condition. ■

The next result is a fixed point theorem similar to a fixed point theorem proved by Shinbrot in [31, Theorem 5], but our theorem is with respect to a cone.

Shinbrot's theorem is proved by a long proof and supposing the weak continuity, or by another method using the topological degree. Our result is a fixed point theorem on a Galerkin cone and it is a consequence of the Complementarity Theory. We say that $T:\boldsymbol{K}\to\boldsymbol{K}$ is completely continuous if T is continuous and maps every bounded subset into a relatively compact set.

THEOREM 6. *Let H be a Hilbert space and let $\boldsymbol{K}(\boldsymbol{K}_n)_{n\in N}$ be a Galerkin cone in H. If $T:\boldsymbol{K}\to\boldsymbol{K}$ is completely continuous and monotone decreasing on rays with respect to $\boldsymbol{K}$ then T has a fixed point.*

PROOF. Consequence of Corollary 3 and of the fact that in this case the complementarity problem C. P. $(I-T,\boldsymbol{K})$ is equivalent to the existence of a fixed point for T on $\boldsymbol{K}$ [16]. ■

We consider now the case $E=R^n$ endowed with euclidean structure.

COROLLARY 4. *Let $\boldsymbol{K}\subset R^n$ be a closed pointed convex cone, $G:\boldsymbol{K}\to R^n$ a continuous mapping and $b\in R^n$ an arbitrary element. If there exists $r>0$ such that*

β_3) $\langle x, G(x)-b\rangle\geqq 0$, *for every $x\in\boldsymbol{K}$ with $\|x\|=r$,*
then the problem C. P. $(G-b,\boldsymbol{K})$ has a solution x_* such that $\|x_*\|\leqq r$.

PROOF. We consider $T_1(x)=x$, $T_2(x)=x-[G(x)-b]$ for every $x\in\boldsymbol{K}$ and we apply Theorem 2. ■

REMARK 5. Condition (β_3) is satisfied for an $r>0$ if there exists a constant $a>0$ such that

β_4) $\langle x, G(x)\rangle\geqq a\|x\|^2$, for every $x\in\boldsymbol{K}$.

Indeed, in this case if we choose $r>0$ such that $\|b\|\leqq ar$, we obtain for every $x\in \boldsymbol{K}$ with $\|x\|=r$, $\langle x, G(x)\rangle \geqq a\|x\|^2=ar^2\geqq r\|b\|\geqq\langle x, b\rangle$.

Condition (β_4) is studied in [25] when G is linear.

Finally, we remark that condition (β_3) is also satisfied if

$$\lim_{\|x\|\to\infty}\frac{\langle x, G(x)\rangle}{\|x\|}=+\infty .$$

Corollary 4 is true on locally compact cones in Hilbert space and it is interesting to compare this result to the results obtained in [15] and [29].

In the general case, operator T_2 satisfies Altman's condition with respect to T_1 if there exists $\gamma>0$ such that $\langle x, T_1(x)\rangle\geqq\gamma\|x\|^{\rho}$. with $\rho>2$ and if T_2 is linear and continuous.

Indeed, in this case we choose $r>0$ such that $\|T_2\|\leqq\gamma r^{\rho-2}$.

The principal results of this paper can be used to develop a new computation method to approximate solutions of the problem $C.P.(T_1-T_2, \boldsymbol{K})$ in R^n (endowed with euclidean structure) using the global optimization.

Let $\boldsymbol{K}\subset \boldsymbol{R}^n$ be a closed convex cone. We can show that the problem $C.P.(T_1-T_2, \boldsymbol{K})$ is equivalent to the following fixed point problem:

$$(F.P.):\quad\left\|\begin{array}{l}\text{find } x_*\in \boldsymbol{K} \text{ such that } \phi(x_*)=x_* \text{ where}\\ \phi(x)=P_K[x-\tau(T_1(x)-T_2(x))]\\ P_K \text{ is the orthogonal projection on } \boldsymbol{K} \text{ and}\\ \tau\in R_+\backslash\{0\}.\end{array}\right.$$

Consider the function ψ defined by $\psi(x)=\|x-\phi(x)\|$.

If we know that the problem $C.P.(T_1-T_2, \boldsymbol{K})$ has a solution x_* such that $\|x_*\|\leqq r$, then to approximate a solution of this problem is equivalent to approximate a global minimum x_* of ψ on a box containing the set $\{x\in \boldsymbol{K}\mid \|x\|\leqq r\}$ knowing that $\psi(x_*)=0$.

This method is different from the method studied in [30].

As a global optimization method we can use the cubic algorithm [13], [14] or its generalization for σ-Hölder continuous functions.

The operator P_K can be computed by the algorithm proposed in [3].

Note. When $T: E\to E^*$ is continuous and E is a reflexive Banach space then the (ws)-compactness of T is equivalent to the complete continuity of T. This fact is not true if E is not reflexive.

Comments. It seems to be interesting to apply our results when T_2

is a compact operator and $T_1(x)=x-f(x)$, with $f(x)$ a φ-contraction. We recall this notion, well known in the fixed point theory. We say that $f: E\to E^*$ is a φ-contraction if there is a mapping $\varphi: R_+\to R_+$ satisfying:

i) $\|f(x)-f(g)\| \leqq \varphi(\|x-y\|)$, for all $x, y\in E$,

ii) $\varphi(t)<t$, for all $t\in R_+\backslash\{0\{$.

In this case, we can show that if φ is continuous then the operator $T_1(x)=x-f(x)$ satisfies condition $(S)_+$.

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