

# *Global existence for viscous compressible<sup>(\*)</sup> fluids and their behavior as $t \rightarrow \infty$*

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## 1. Introduction

We consider the system of equations which governs the motion of viscous compressible barotropic fluids; the system is the following

$$(1.1) \quad \begin{aligned} \rho \partial_t u + \rho u \cdot \nabla u - \mu \Delta u - \zeta \nabla(\nabla \cdot u) &= \rho f - \nabla p, \\ \partial_t \rho + \nabla \cdot (\rho u) &= 0, \end{aligned} \quad \text{in } Q_T,$$

where  $Q_T = \Omega \times (0, T)$  with  $\Omega$  being a bounded domain in  $R^3$  and  $T$  being a positive number; moreover,  $\partial_t = \partial/\partial t$ ,  $u = u(t) = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$  is the velocity of the fluid,  $\rho = \rho(t) = \rho(x, t)$  is the density,  $f = f(t) = f(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t))$  is the external force,  $p$  is the pressure, and  $\mu$  and  $\zeta$  are the constant coefficients of viscosity of the fluid. The second equation in (1.1) is the continuity equation.

Many authors have studied the system (1.1). For the initial boundary value problem the results on the existence of solutions are all local in time for large data ([2], [24], [25], [28], [30], and [3], [26], [27] with more general boundary conditions), and global (in time) for small initial data ([16], [17], [31] and [4] with more general boundary conditions but velocity of inflow bounded away from zero). For the one dimensional case, the existence problem can be considered to be completely solved [8], [9], [11], [13] (see also, e.g., [1] for an up-to-date treatment). As far as the Cauchy problem is concerned, we quote [6], [7], [10], [12]. Moreover, the results on the uniqueness are given in [5], [22]. However, in every case the initial density is assumed to be strictly positive.

In this paper we prove existence theorems for some initial boundary value problems using the method developed by the first author in [21].

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<sup>(\*)</sup> This research was partially supported by G.N.A.F.A. of C.N.R. and by M.P.I. 60%.

<sup>(\*\*)</sup> This paper was written during the stay of the author in Dipartimento di Matematica del Politecnico di Milano with the support of C.N.R.

The local and, for small data, global existence theorem is proved in the spaces of less smooth functions than those in [16] and [30]. Then the behavior of the solution as  $t \rightarrow \infty$  is investigated. Finally, a theorem admitting the vacuum state is proved.

The paper is organized as follows :

In section 2 notation, and main results are given.

In section 3 local existence theorem is proved via a generalized Lax-Milgram lemma and a Schauder fixed point theorem.

In section 4 global estimates of certain norms of the solution on  $[0, \infty)$  are derived.

In section 5 the global existence theorem is proved.

In section 6 the behaviour of the global solution as  $t \rightarrow \infty$  is investigated.

In section 7 the existence theorem admitting vacuum at  $t=0$  is established.

## 2. Statements and notation

Let  $\Omega$  be an open bounded set in  $R^3$  with boundary  $\Gamma$ , and  $Q_T = \Omega \times [0, T]$  with  $T$  being a positive number. The motion of a viscous compressible fluid subjected to the external force  $f$ , and independent of temperature, is governed by the system

$$(2.1) \quad \begin{aligned} \rho \partial_t u + \rho u \cdot \nabla u - \mu \Delta u - \zeta \nabla(\nabla \cdot u) &= \rho f - \nabla p, \\ \partial_t \rho + \nabla \cdot (\rho u) &= 0. \end{aligned} \quad \text{in } Q_T,$$

We complete the system (2.1) with the initial and boundary conditions

$$(2.2) \quad \begin{aligned} u(0) &= u_0, \quad \rho(0) = \rho_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Moreover, we consider the following constitutive equation

$$(2.3) \quad p = p(\rho)$$

which expresses how the pressure  $p$  depends on  $\rho$ .

We set

$$\begin{aligned} \partial_i &= \partial / \partial x_i; \quad d_r = d / dr; \quad m_0 = (\text{meas } \Omega)^{-1} \int_{\Omega} \rho_0 dx; \\ (\phi, \psi) &= \sum_{i=1}^3 \int_{\Omega} \phi_i \psi_i dx; \quad ((\phi, \psi)) = \sum_{i=1}^3 \int_{\Omega} \partial_i \phi \partial_i \psi dx; \\ |\phi|^2 &= (\phi, \phi); \quad |\phi|_p = \text{norm in the space } L^p(\Omega) \text{ with } p > 1; \end{aligned}$$

$\|\phi\|_s$  = norm in the Sobolev space  $H^s(\Omega)$  of order  $s$  on  $L^2(\Omega)$ ;

$\|\phi\|_{s,p}$  = norm in the Sobolev space  $H^{s,p}(\Omega)$  of order  $s$  on  $L^p(\Omega)$ ;

$\|\phi\|_{-s}$  = norm in the dual  $H^{-s}(\Omega)$  of  $H_0^s(\Omega)$ ;

( $s$  is a real number). The spaces  $C(\Omega)$ ,  $C^m(\Omega)$ ,  $C_0^m(\Omega)$ ,  $C^\infty(\Omega)$ ,  $H_0^s(\Omega)$ ,  $\dots$ , and their vector-valued analogues are defined as usual.

We assume that the boundary  $\Gamma$  is uniformly of class  $C^k$ , i.e., first, it is possible to choose local coordinates  $(y_1, y_2, y_3)$  in a neighborhood  $B_\xi$  of each point  $\xi \in \Gamma$  such that  $\Gamma \cap B_\xi$  is represented by a function  $y_3 = F(y_1, y_2; \xi)$  of class  $C^k$ ; second, the neighborhoods  $B_\xi$  can be chosen as balls, all of the same size, with respective centers  $\xi$  and bounded by a constant independent of  $\xi$ . We denote by  $\|\cdot\|_{C^k}$  the norm in the class  $C^k$ .

Throughout the paper,  $\varepsilon, \delta, \varepsilon_i$  and  $\delta_i$  ( $i=1, \dots, n$ ) denote arbitrary positive constants, and  $c_\varepsilon, c_\delta, c_{\varepsilon_i}$  and  $c_{\delta_i}$  denote positive constants depending on  $1/\varepsilon, 1/\delta, 1/\varepsilon_i$  and  $1/\delta_i$ , respectively.

Now we state our results.

**THEOREM 1.** *Let  $\Omega$  be a bounded domain in  $R^3$  with a boundary  $\Gamma$  uniformly of class  $C^3$ . Let  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$  and  $\rho_0 \in H^2(\Omega)$ , such that  $0 < \alpha \leq \rho_0 \leq \beta < \infty$  ( $\alpha, \beta$  are positive constants),  $\|u_0\|_2 + \|\nabla \rho_0\|_1 \leq r_0$ ,  $r_0 > 0$  being given,  $f \in L^2(0, T; H^1(\Omega))$ ,  $\partial_t f \in L^2(Q_T)$  and  $p \in C_{loc}^2((0, \infty))$ .*

*Then there exist a  $\bar{T} = \bar{T}(r_0, m_0) > 0$ ,  $\bar{T} \leq T$  and positive constants  $\alpha(\bar{T}, r_0, m_0)$ ,  $\beta(\bar{T}, r_0, m_0)$  such that there exists a solution  $(u, \rho)$  of (2.1)–(2.3) on  $(0, \bar{T})$  satisfying*

$$\begin{aligned} (2.4) \quad & u \in L^2(0, \bar{T}; H^3(\Omega)); \quad \partial_t u \in L^2(0, \bar{T}; H_0^1(\Omega)); \\ & \rho \in L^\infty(0, \bar{T}; H^2(\Omega)); \quad \partial_t \rho \in L^\infty(0, \bar{T}; H^1(\Omega)); \\ & 0 < \alpha(\bar{T}, r_0, m_0) \leq \rho \leq \beta(\bar{T}, r_0, m_0) < \infty. \end{aligned}$$

Moreover, the solution  $(u, \rho)$  is unique.

**THEOREM 2.** *Let  $\Omega$  be a bounded domain in  $R^3$  with a boundary  $\Gamma$  uniformly of class  $C^3$ . Let  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $\rho_0 \in H^{1,q}(\Omega)$  for some  $q$ ,  $q \in (3, 6]$ , such that  $0 < \alpha \leq \rho_0 \leq \beta < \infty$  ( $\alpha, \beta$  are positive constants),  $\|u_0\|_2 + \|\nabla \rho_0\|_q \leq r_0$ ,  $r_0 > 0$  being given,  $f \in L^2(0, T; H^1(\Omega))$ ,  $\partial_t f \in L^2(Q_T)$  and  $p \in C_{loc}^2((0, \infty))$ .*

*Then there are a  $\bar{T} = \bar{T}(r_0, m_0) > 0$ ,  $\bar{T} \leq T$  and positive constants  $\alpha(\bar{T}, r_0, m_0)$ ,  $\beta(\bar{T}, r_0, m_0)$  such that there exists a solution of (2.1)–(2.3) on  $(0, \bar{T})$  satisfying*

$$\begin{aligned}
u &\in L^2(0, \bar{T}; H^{2,q}(\Omega)); \quad \partial_t u \in L^2(0, \bar{T}; H_0^1(\Omega)); \\
\rho &\in L^\infty(0, \bar{T}; H^{1,q}(\Omega)) \cap L^\infty(Q_{\bar{T}}); \quad \partial_t \rho \in L^\infty(0, \bar{T}; L^q(\Omega)); \\
0 &< \alpha(\bar{T}, r_0, m_0) \leq \rho \leq \beta(\bar{T}, r_0, m_0) < \infty.
\end{aligned}$$

Moreover, the solution  $(u, \rho)$  is unique.

**THEOREM 3.** *Let  $\Omega, u_0, \rho_0, q$  and  $p$  be as in Theorem 2. Besides, assume that  $f \in L^\infty(0, \infty; H^1(\Omega))$ ,  $\partial_t f \in L^\infty(0, \infty; L^2(\Omega))$ ,  $dp/d\rho > 0$  on  $(0, \infty)$  and  $r_0 + \sup_{0 \leq t < \infty} \{\|f(t)\|_1, |\partial_t f(t)|_2\}$  is sufficiently small. Then there exists a (unique) solution  $(u, \rho)$  of the problem (2.1)–(2.3) on  $(0, \infty)$  such that*

$$\begin{aligned}
u &\in L^\infty(0, \infty; H^2(\Omega)); \quad \partial_t u \in L_{\text{loc}}^2(0, \infty; H_0^1(\Omega)) \cap L^\infty(0, \infty; L^2(\Omega)); \\
\rho &\in L^\infty(0, \infty; H^{1,q}(\Omega)) \cap L^\infty(Q_\infty); \quad \partial_t \rho \in L^\infty(0, \infty; L^q(\Omega));
\end{aligned}$$

and

$$0 < \alpha(r_0, m_0) \leq \rho \leq \beta(r_0, m_0) < \infty.$$

**THEOREM 4.** *Let  $\Omega, u_0, \rho_0$  and  $p$  satisfy the assumptions of Theorem 2 with  $r_0$  sufficiently small,  $dp/d\rho > 0$  on  $(0, \infty)$ , and let  $f = -\nabla g$ , where  $g \in H^{2,3/2}(\Omega)$  and  $\|g\|_{H^{2,3/2}(\Omega)}$  is small enough.*

*Then the global solution  $(u(t), \rho(t))$  from Theorem 3 converges to an equilibrium state  $(0, \bar{\rho})$  as  $t \rightarrow \infty$  in  $L^2(\Omega)$  and a. e. in  $\Omega$ , where  $\bar{\rho} \in H^{2,3/2}(\Omega)$  is a (unique) solution of the equations*

$$\nabla p(\bar{\rho}) = \bar{\rho} f \quad \text{in } \Omega,$$

$$\int_\Omega \bar{\rho} dx = \int_\Omega \rho_0 dx.$$

**THEOREM 5.** *Let  $0 \leq \rho_0 \leq \beta$ ,  $\rho_0 \in H^2(\Omega)$ ,  $u_0, f$  satisfy the assumptions of Theorem 1 and  $p \in C_{\text{loc}}^2([0, \infty))$ . Further, we assume that  $(\Delta u_0 + \nabla(\nabla \cdot u_0) + \nabla p(\rho_0))/\sqrt{\rho_0}$  belongs to  $L^2(\Omega)$ .*

*Then there exists a  $\bar{T} = \bar{T}(r_0, m_0) > 0$  and a positive constant  $\beta_1(\bar{T}, r_0, m_0)$  such that there exists a solution  $u(x, t), \rho(x, t)$  of (2.1)–(2.3) satisfying*

$$\begin{aligned}
u &\in L^2(0, \bar{T}; H^3(\Omega)); \quad \partial_t u \in L^2(0, \bar{T}; H_0^1(\Omega)); \quad \sqrt{\rho} \partial_t u \in L^\infty(0, \bar{T}; L^2(\Omega)); \\
\rho &\in L^\infty(0, \bar{T}; H^2(\Omega)); \quad \partial_t \rho \in L^\infty(0, \bar{T}; H^1(\Omega)); \quad 0 \leq \rho \leq \beta_1(\bar{T}, r_0, m_0).
\end{aligned}$$

Moreover, the solution  $(u, \rho)$  is unique.

We note that the solvability of (1.1) with non-negative initial density

has been considered in [20] and [23] in the one-dimensional case. However, it is difficult to compare the results of these papers with ours. In [20], a free boundary problem is investigated assuming  $\rho=0$  on the interface of the barotropic gas and the vacuum. In [23], the existence of a weak solution of (1.1) is considered with  $\rho_0 \geq 0$ , but the regularity of the solution is not investigated.

### 3. Proofs of Theorem 1, 2

PROOF OF THEOREM 1. Throughout this paper from now on, for simplicity, we assume  $\mu=\zeta=\beta=1$ . First, we consider the following auxiliary problem.

Let

$$F = \{\phi | \phi \in L^2(0, T; H^2(\Omega)), \partial_t \phi \in L^2(Q_T), \phi = 0 \text{ on } \Gamma\}$$

with the natural norm};

and let  $\Phi$  be a closure of the space

$$\{\phi | \phi \in L^2(0, T; H^2(\Omega)), \partial_t \phi \in L^2(0, T; H^2(\Omega)), \phi = 0 \text{ on } \Gamma\}$$

in the norm

$$\|\phi\|_\Phi = \|\phi\|_F + \|\phi(0)\|_1$$

and consider the following problem:

*Find  $(\rho, u)$  that satisfies*

$$(3.1) \quad \rho \partial_t u - \Delta u - \nabla(\nabla \cdot u) = -\rho \bar{u} \cdot \nabla \bar{u} + \rho f - \nabla p, \quad \text{a. e. in } Q_T,$$

$$(3.2) \quad \partial_t \rho + \nabla \cdot (\rho \bar{u}) = 0,$$

$$u = 0 \text{ on } \Gamma, \quad u(0) = u_0, \quad \rho(0) = \rho_0, \quad 0 < \alpha \leq \rho_0 \leq 1,$$

$$\text{such that } u \in F, \quad \rho \in L^\infty(0, T; H^2(\Omega)), \quad \partial_t \rho \in L^\infty(0, T; H^{1,q}(\Omega)),$$

$$0 < \alpha_1 \leq \rho \leq \beta_1 \quad (\alpha_1, \beta_1 \text{ are suitable positive constants}).$$

In (3.1), (3.2),  $\bar{u} \in L^2(0, T; H^3(\Omega))$ ,  $\partial_t \bar{u} \in L^2(0, T; H^1(\Omega))$ ,  $\bar{u} = 0$  on  $\Gamma$ , and  $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ ,  $\rho_0 \in H^2(\Omega)$  are given functions and  $q \in (3, 6]$ . For simplicity, we assume  $q=6$ .

First, the existence of a unique solution  $\rho$  of the continuity equation follows from the method of characteristics.

We notice that this method requires that  $\bar{u} \in C^0(0, T; C^1(\Omega))$  and  $\rho_0 \in C^1(\Omega)$ . But the estimates below hold under the above assumptions on  $\bar{u}$ , and  $\rho_0$ . So we use, for simplicity, a formal approach. The correct pro-

cedure is to consider a regularization of  $\bar{u}$ , and  $\rho_0$ , and then to pass to limit.

Now, let  $y(\tau, x, t)$  be the solution of

$$\frac{dy}{d\tau} = \bar{u}(y, \tau), \quad y(t, x, t) = x;$$

then, the explicit formula for  $\rho$  is

$$(3.3) \quad \rho(x, t) = \rho_0(y(0, x, t)) \exp\left(-\int_0^t \nabla \cdot \bar{u}(y(\tau, x, t), \tau) d\tau\right)$$

From (3.3) it follows

$$(3.4) \quad 0 < \alpha \exp\left(-\int_0^t |\nabla \cdot \bar{u}|_\infty d\tau\right) \leq \rho(x, t) \leq \exp\left(\int_0^t |\nabla \cdot \bar{u}|_\infty d\tau\right).$$

Now applying the gradient operator  $\nabla$  to (3.2), we have

$$(3.5) \quad \partial_t \nabla \rho + \bar{u} \cdot \nabla(\nabla \rho) + \nabla \bar{u} \cdot \nabla \rho + \rho \nabla \nabla \cdot \bar{u} + \nabla \rho \nabla \cdot \bar{u} = 0.$$

Multiplying (3.5) by  $|\nabla \rho|^{p-2} \nabla \rho$  with  $2 \leq p \leq 6$  and integrating over  $\Omega$ , we get

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} |\nabla \rho|_p^p + \int_\Omega \left( \frac{1}{p} (\bar{u} \cdot \nabla(|\nabla \rho|^p) + \rho |\nabla \rho|^{p-2} \nabla(\nabla \cdot \bar{u}) \cdot \nabla \rho \right. \\ \left. + |\nabla \rho|^p \nabla \cdot \bar{u} + |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \bar{u} \cdot \nabla \rho \right) dx = 0. \end{aligned}$$

Bearing in mind

$$\int_\Omega (\bar{u} \cdot \nabla(|\nabla \rho|^p) + |\nabla \rho|^p \nabla \cdot \bar{u}) dx = 0,$$

we find

$$\frac{d}{dt} |\nabla \rho|_p \leq c(|\nabla \rho|_p |\nabla \bar{u}|_\infty + |\rho|_\infty |\nabla(\nabla \cdot \bar{u})|_p),$$

whence

$$(3.6) \quad |\nabla \rho|_p \leq \exp\left(c \left(\int_0^t |\nabla \bar{u}|_\infty d\tau\right)\right) (|\nabla \rho(0)|_p + \int_0^t (|\rho|_\infty |\nabla(\nabla \cdot \bar{u})|_p) d\tau).$$

Now applying to (3.2) the operators  $\partial_i \partial_j$ , multiplying the result by  $\partial_i \partial_j \rho$  and integrating over  $\Omega$ , after some calculations, we obtain

$$\frac{d}{dt} (\sum_{i,j=1}^3 |\partial_i \partial_j \rho|_2^2) \leq c(|\nabla \bar{u}|_\infty + 1) \sum_{i,j=1}^3 |\partial_i \partial_j \rho|_2^2 + c \|\bar{u}\|_3^2 (|\rho|_\infty^2 + |\nabla \rho|_6^2),$$

whence

$$(3.7) \quad \|\rho\|_2^2 \leq c \exp \left( c \int_0^t (|\nabla \bar{u}|_\infty + 1) d\tau \right) (\|\rho_0\|_2^2 + \int_0^t (\|\rho\|_\infty^2 + \|\nabla \rho\|_2^2) \|\bar{u}\|_2^2 d\tau).$$

Now we prove the existence of a solution of (3.1). We let

$$\begin{aligned} E(u, \phi) &= \int_0^T \{(\rho \partial_t u - \Delta u - \nabla(\nabla \cdot u), \partial_t \phi - k(\Delta \phi + \nabla(\nabla \cdot \phi)))\} dt - \\ &\quad - (u(0), \Delta \phi(0) + \nabla(\nabla \cdot \phi(0))), \\ L(\phi) &= - \int_0^T (\rho \bar{u} \cdot \nabla \bar{u} - \rho f + \nabla p, \partial_t \phi - k(\Delta \phi + \nabla(\nabla \cdot \phi))) dt - \\ &\quad - (u_0, \Delta \phi(0) + \nabla(\nabla \cdot \phi(0))), \end{aligned}$$

where  $k = (\sup_{Q_T} \rho)^{-1}$ .

First,  $L(\phi)$  is a linear continuous form on  $\Phi$  with respect to the norm  $\|\phi\|_\Phi$ . Moreover, bearing in mind that  $|\Delta \phi - \nabla(\nabla \cdot \phi)|_2 \geq c \|\phi\|_2$ , we have

$$\begin{aligned} E(\phi, \phi) &= \int_0^T (|\sqrt{\rho} \partial_t \phi|_2^2 + k |\Delta \phi + \nabla(\nabla \cdot \phi)|_2^2 - k(\rho \partial_t \phi, \Delta \phi + \nabla(\nabla \cdot \phi))) dt + \\ &\quad + \frac{1}{2} (|\nabla \phi(T)|_2^2 + |\nabla \phi(0)|_2^2) + \frac{1}{2} |\nabla \cdot \phi(T)|_2^2 + \frac{1}{2} |\nabla \cdot \phi(0)|_2^2 \geq \\ &\geq \int_0^T \{|\sqrt{\rho} \partial_t \phi|_2^2 + k |\Delta \phi + \nabla(\nabla \cdot \phi)|_2^2 - \frac{3}{4} |\sqrt{\rho} \partial_t \phi|_2^2 - \frac{k}{3} |\Delta \phi + \nabla(\nabla \cdot \phi)|_2^2 + \\ &\quad + \frac{1}{2} (|\nabla \phi(T)|_2^2 + |\nabla \phi(0)|_2^2) + \frac{1}{2} (|\nabla \cdot \phi(T)|_2^2 + |\nabla \cdot \phi(0)|_2^2)\} dt \\ &\geq c \|\phi\|_\Phi^2. \end{aligned}$$

Then there exists a  $u \in F$  such that

$$(3.8) \quad E(u, \phi) = L(\phi)$$

is satisfied for every  $\phi \in \Phi$  (see [29] p. 208).

Now let  $\tilde{\phi}$  be the solution of the problem

$$\begin{aligned} \partial_t \tilde{\phi} - k(\Delta \tilde{\phi} + \nabla(\nabla \cdot \tilde{\phi})) &= 0 && \text{in } Q_T, \\ \tilde{\phi}(0) &= h(x) && \text{in } \Omega, \\ \tilde{\phi} &= 0 && \text{on } \Gamma \times (0, T) \end{aligned}$$

with  $h(x)$  smooth enough,  $h=0$  on  $\Gamma$ . Replacing in (3.8)  $\phi$  by  $\tilde{\phi}$ , we have

$$(u(0), \Delta h + \nabla(\nabla \cdot h)) = (u_0, \Delta h + \nabla(\nabla \cdot h)),$$

which implies  $u(0)=u_0$ .

Let now  $\bar{\phi}$  be a solution of the following problem

$$\begin{aligned} \partial_t \bar{\phi} - k(\Delta \bar{\phi} - \nabla(\nabla \cdot \bar{\phi})) &= g(x, t) & \text{in } Q_T, \\ \bar{\phi}(0) &= 0 & \text{in } \Omega, \\ \bar{\phi} &= 0 & \text{on } \Gamma \times (0, T) \end{aligned}$$

with  $g$  smooth enough. Replacing  $\phi$  with  $\bar{\phi}$  in (3.8), we have

$$\int_0^T (\rho \partial_t u - \Delta u - \nabla(\nabla \cdot u) + \rho \bar{u} \cdot \nabla \bar{u} + \nabla p - \rho f, g) dt = 0.$$

This implies that  $(u, \rho)$  satisfies a.e. in  $Q_T$  the following system

$$(3.9) \quad \rho \partial_t u - \Delta u - \nabla(\nabla \cdot u) = -\rho \bar{u} \cdot \nabla \bar{u} + \rho f - \nabla p,$$

$$(3.10) \quad \partial_t \rho + \nabla \cdot (\rho \bar{u}) = 0.$$

Now we prove more regularity for  $u$ . To avoid tedious calculations and notation, we work directly with the derivatives with respect to  $t$  of  $u$  instead of its differential quotients.

First we multiply (3.9) by  $\partial_t u$ , integrate over  $\Omega$ , and obtain

$$(3.11) \quad |\sqrt{\rho} \partial_t u|_2^2 + \frac{1}{2} \frac{d}{dt} |\nabla u|_2^2 + \frac{1}{2} \frac{d}{dt} |\nabla \cdot u|_2^2 = (-\rho \bar{u} \cdot \nabla \bar{u} + \rho f - \nabla p, \partial_t u).$$

Then, integrating (3.11) with respect to  $t$ , we get

$$\begin{aligned} (3.12) \quad & \frac{1}{2} |\nabla u(t)|_2^2 + \frac{1}{2} |\nabla \cdot u(t)|_2^2 + \frac{2}{3} \int_0^t |\sqrt{\rho} \partial_\tau u|_2^2 d\tau \leq \\ & \leq \frac{1}{2} |\nabla u(0)|_2^2 + \frac{1}{2} |\nabla \cdot u(0)|_2^2 + \int_0^t (p, \partial_t \nabla \cdot u(\tau)) d\tau + \\ & + c \sup_{0 \leq \tau \leq t} |\rho|_\infty \int_0^t (|\bar{u}|_6^2 |\nabla \bar{u}|_3^2 + |f|_2^2) d\tau. \end{aligned}$$

Now we differentiate (3.9) with respect to  $t$  so that we get

$$\begin{aligned} (3.13) \quad & \partial_t \rho \partial_t u + \rho \partial_{tt}^2 u - \Delta \partial_t u - \nabla(\nabla \cdot \partial_t u) = -\partial_t \rho \bar{u} \cdot \nabla \bar{u} - \\ & - \rho \partial_t \bar{u} \cdot \nabla \bar{u} - \rho \bar{u} \cdot \nabla \partial_t \bar{u} - \nabla \partial_t p + \partial_t \rho f + \rho \partial_t f. \end{aligned}$$

Multiplying (3.13) by  $\partial_t u$ , integrating over  $\Omega$ , and bearing in mind the continuity equation, we obtain



$$\begin{aligned}
(3.14) \quad & \frac{1}{2} d_t |\sqrt{\rho} \partial_t u|_2^2 + \frac{1}{2} \int_{\Omega} \partial_t \rho |\partial_t u|^2 dx + |\nabla \partial_t u|_2^2 + |\nabla \cdot \partial_t u|_2^2 \leq \\
& \leq |\rho|_{\infty} |\bar{u}|_{\infty} |\nabla \bar{u}|_2 |\nabla \bar{u}|_3 |\partial_t u|_6 + |\rho|_{\infty} |\bar{u}|_6 |\bar{u}|_6 |\nabla(\nabla \bar{u})|_2 |\partial_t u|_6 + \\
& + |\rho|_{\infty} |\bar{u}|_{\infty} |\bar{u}|_6 |\nabla \bar{u}|_3 |\nabla \partial_t u|_2 + |\sqrt{\rho}|_{\infty} |\sqrt{\rho} \partial_t \bar{u}|_2 |\nabla \bar{u}|_3 |\partial_t u|_6 + \\
& + |\sqrt{\rho}|_{\infty} |\bar{u}|_{\infty} |\nabla \partial_t \bar{u}|_2 |\sqrt{\rho} \partial_t u|_2 + |\partial_t p|_2 |\nabla \partial_t u|_2 + \\
& + |\partial_t \rho|_2 |f|_3 |\partial_t u|_6 + |\sqrt{\rho}|_{\infty} |\partial_t f|_2 |\sqrt{\rho} \partial_t u|_2.
\end{aligned}$$

Integrating (3.14) with respect to  $t$ , and bearing in mind the continuity equation and  $|\nabla u|_3^2 \leq c \|u\|_1 |\Delta u|_2$  (see [21]), we find

$$\begin{aligned}
(3.15) \quad & \frac{1}{2} |\sqrt{\rho(t)} \partial_t u(t)|_2^2 + \int_0^t |\nabla \partial_t u|_2^2 d\tau \leq \\
& \leq \frac{1}{2} |\sqrt{\rho(0)} \partial_t u(0)|_2^2 + c \int_0^t |\sqrt{\rho}|_{\infty} |\bar{u}|_{\infty} |\sqrt{\rho} \partial_t u|_2 |\nabla \partial_t \bar{u}|_2 d\tau + \\
& + c \int_0^t |\rho|_{\infty} (|\bar{u}|_{\infty} |\nabla \bar{u}|_2^{3/2} |\Delta \bar{u}|_2^{1/2} |\nabla \partial_t u|_2 + |\nabla \bar{u}|_2^2 |\Delta \bar{u}|_2 |\nabla \partial_t u|_2) d\tau + \\
& + c \int_0^t |\sqrt{\rho}|_{\infty} (|\sqrt{\rho} \partial_t \bar{u}|_2 |\nabla \bar{u}|_2^{1/2} |\Delta \bar{u}|_2^{1/2} |\nabla \partial_t u|_2 + |\bar{u}|_{\infty} |\nabla \partial_t \bar{u}|_2 |\sqrt{\rho} \partial_t u|_2) d\tau + \\
& + \int_0^t (|\partial_t p|_2 + c |\partial_t \rho|_2 |f|_3) |\nabla \partial_t u|_2 d\tau + \int_0^t |\sqrt{\rho}|_{\infty} |\partial_t f|_2 |\sqrt{\rho} \partial_t u|_2 d\tau + \\
& \leq \frac{1}{2} |\sqrt{\rho(0)} \partial_t u(0)|_2^2 + c_{\delta} \left[ \int_0^t |\sqrt{\rho} \bar{u}|_{\infty}^2 |\sqrt{\rho} \partial_t u|_2^2 d\tau + \right. \\
& + \sup_{0 \leq \tau \leq t} |\rho|_{\infty}^2 \int_0^t (|\bar{u}|_{\infty}^2 |\nabla \bar{u}|_2^3 |\Delta \bar{u}|_2 + |\nabla \bar{u}|_2^4 |\Delta \bar{u}|_2^2) d\tau + \\
& + \sup_{0 \leq \tau \leq t} |\rho|_{\infty} \int_0^t (|\sqrt{\rho} \partial_t \bar{u}|_2^2 |\nabla \bar{u}|_2 |\Delta \bar{u}|_2 + |\partial_t f|_2^2) d\tau + \\
& \left. + \int_0^t (|\partial_t \rho|_2^2 |f|_3^2 + |\partial_t p|_2^2) d\tau \right] + \delta \int_0^t (6 |\nabla \partial_t u|_2^2 + |\sqrt{\rho} \partial_t u|_2^2) d\tau + \delta \int_0^t |\nabla \partial_t \bar{u}|_2^2 d\tau.
\end{aligned}$$

Now we sum (3.12) and (3.15), and for suitable  $\delta$  ( $\delta \leq 1/12$ ), we obtain

$$\begin{aligned}
& \frac{1}{2} |\sqrt{\rho} \partial_t u|_2^2 + \frac{1}{2} |\nabla u(t)|_2^2 + \frac{1}{2} |\nabla \cdot u(t)|_2^2 + \frac{1}{3} \int_0^t |\sqrt{\rho} \partial_\tau u|_2^2 d\tau + \frac{1}{4} \int_0^t |\nabla \partial_t u|_2^2 dt \leq \\
& \leq \frac{1}{2} |\sqrt{\rho(0)} \partial_t u(0)|_2^2 + \frac{1}{2} |\nabla u(0)|_2^2 + \frac{1}{2} |\nabla \cdot u(0)|_2^2 + \\
(3.16) \quad & + \int_0^t (p, \partial_\tau \nabla \cdot u(\tau)) d\tau + c \sup_{0 \leq \tau \leq t} |\rho|_\infty \int_0^t (|\bar{u}|_6^2 |\nabla \bar{u}|_3^2 + |f|_3^2) d\tau + \\
& + c_\delta \left[ \int_0^t |\sqrt{\rho} \bar{u}|_\infty^2 |\sqrt{\rho} \partial_\tau u|_2^2 d\tau + \sup_{0 \leq \tau \leq t} |\rho|_\infty^2 \int_0^t (|\bar{u}|_\infty^2 |\nabla \bar{u}|_2^3 |\Delta \bar{u}|_2 + \right. \\
& + |\nabla \bar{u}|^4 |\Delta \bar{u}|_2^2) d\tau + \sup_{0 \leq \tau \leq t} |\rho|_\infty \int_0^t (|\sqrt{\rho} \partial_\tau \bar{u}|_2^2 |\nabla \bar{u}|_2 |\Delta \bar{u}'|_2 + \\
& \left. + |\partial_\tau f|_2^2) d\tau + \int_0^t (|\partial_\tau \rho|_2^2 (|f|_3^2 + |\partial p / \partial \rho|_2^2) d\tau \right] + \delta \int_0^t |\nabla \partial_\tau \bar{u}|_2^2 d\tau .
\end{aligned}$$

The estimate (3.16) implies

$$\begin{aligned}
(3.17) \quad & \sqrt{\rho} \partial_t u \in L^\infty(0, T; L^2(\Omega)) ; \quad \partial_t u \in L^2(0, T; H_0^1(\Omega)) ; \\
& u \in L^\infty(0, T; H_0^1(\Omega)) .
\end{aligned}$$

Now we consider the system

$$(3.18) \quad -\Delta u - \nabla(\nabla \cdot u) = -\rho \partial_t u - \rho \bar{u} \cdot \nabla \bar{u} - \nabla p + \rho f \equiv v .$$

This is a system with a strongly elliptic left-hand side. By a classical result on elliptic systems (see, e.g., [14]) there exists a positive constant  $c$  such that

$$(3.19) \quad \|u\|_3 \leq c \|\Delta u + \nabla(\nabla \cdot u)\|_1 .$$

Thanks to (3.11) and (3.17), we have

$$(3.20) \quad v \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) .$$

The relations (3.19) and (3.20) imply

$$u \in L^2(0, T; H^3(\Omega)) .$$

Since  $v \in L^\infty(0, T; L^2(\Omega))$ , a similar consideration as above leads to

$$u \in L^\infty(0, T; H^2(\Omega)) .$$

Now the existence and uniqueness of the solution to the system (3.1), (3.2) or (3.9), (3.10) enables us to define the map  $u = G\bar{u}$  given by the

composition of  $g : \bar{u} \rightarrow \rho$  and  $d : (\rho, \bar{u}) \rightarrow u$ . The fixed point of  $G$  is the solution of the system (2.1), (2.2).

Let us consider the set

$$B = \{ \phi \mid \sup (\| \phi \|_{L^2(\zeta_0, \bar{T}; H^3(\Omega))} ; \| \phi \|_{L^\infty(\zeta_0, \bar{T}; H^2(\Omega))} ; \| \partial_t \phi \|_{L^\infty(\zeta_0, \bar{T}; L^2(\Omega))} ; \| \partial_t \phi \|_{L^2(\zeta_0, \bar{T}; H^1(\Omega))}) \leq r \}$$

with  $r^2 = \bar{c} e^{\|u_0\|_1} (|\Delta u_0|_2^2 + |u_0 \cdot \nabla u_0|_2^2 + |f(0)|_2^2 + |\nabla p(\rho_0)|_2^2) / \alpha$ , where  $\bar{c} \geq 1$  is a suitable constant.

It is clear that  $B$  is a compact set in  $L^2(Q_{\bar{T}})$ . As we are going to use a fixed point theorem, we have to show that  $GB \subseteq B$ , and  $G$  is continuous in  $B$  with respect to the norm in  $L^2(Q_{\bar{T}})$ .

Next we prove that  $GB \subseteq B$  for suitable  $\bar{T}$ . In fact, assuming  $\bar{u} \in B$ , from (3.4), (3.6) and (3.7), we have

$$(3.21) \quad \begin{aligned} \alpha e^{-c r \sqrt{t}} &\leq \rho(x, t) \leq e^{c r \sqrt{t}}, \\ |\nabla \rho(t)|_p &\leq e^{c r \sqrt{t}} (|\nabla \rho_0|_p + e^{c r \sqrt{t}} r t^{1/2}) \quad (2 \leq p \leq 6), \\ \|\rho(t)\|_2 &\leq c e^{c(r+1)\sqrt{t}} (\|\rho_0\|_2 + t e^{c r \sqrt{t}} (1 + |\nabla \rho_0|_2 + r t^{1/2} e^{c r \sqrt{t}})). \end{aligned}$$

Therefore from (3.15) and (3.21), it follows the inequality

$$(3.22) \quad \begin{aligned} &\frac{1}{2} |\sqrt{\rho(t)} \partial_t u(t)|_2^2 + \frac{1}{2} \int_0^t |\nabla \partial_\tau u|_2^2 d\tau \leq \\ &\leq \frac{1}{2} |\sqrt{\rho(0)} \partial_t u(0)|_2^2 + c_\delta \left[ t r^2 e^{c r \sqrt{t}} \sup_{0 \leq \tau \leq t} |\sqrt{\rho} \partial_\tau u|_2^2 + t(r^4 + r^6) e^{c r \sqrt{t}} \right] + \\ &+ \int_0^t |\partial_\tau f|_2^2 d\tau e^{c r \sqrt{t}} + \sup_{0 \leq \tau \leq t} |\partial_\tau \rho|_2^2 \int_0^t (|f|_3^2 + |\partial p / \partial \rho|_\infty^2) d\tau + \\ &+ \delta \int_0^t |\nabla \partial_\tau u|_2^2 d\tau + \delta t \sup_{0 \leq \tau \leq t} |\sqrt{\rho} \partial_\tau u|_2^2 + \delta r^2. \end{aligned}$$

Bearing in mind (3.6), and taking  $\delta$  and  $\bar{T}$  suitably, we derive from (3.22)

$$(3.23) \quad \|\sqrt{\rho} \partial_t u\|_{L^\infty(\zeta_0, \bar{T}; L^2(\Omega))}^2 + \|\partial_t u\|_{L^2(\zeta_0, \bar{T}; H^1(\Omega))}^2 \leq \frac{1}{c} r^2 e^{-\|u_0\|_1}$$

for some constant  $c > 0$ . Now from (3.20), with help of (3.7), (3.18), (3.19), and the inequality  $|\phi|_3^2 \leq c |\phi|_2 |\nabla \phi|_2$ ,  $\forall \phi \in H_0^1(\Omega)$ , we find

$$\begin{aligned}
\int_0^{\bar{T}} \|u\|_3^2 dt &\leq 3c \sup_{0 \leq t \leq \bar{T}} (|\nabla \rho|^2 |\partial_t u|_2) \left( \int_0^{\bar{T}} \|\partial_t u\|_1^2 dt \right)^{1/2} \bar{T}^{1/2} + \\
&+ 3c \sup_{0 \leq t \leq \bar{T}} |\rho|_\infty^2 \int_0^{\bar{T}} |\nabla \partial_t u|_2^2 dt + 9c \int_0^{\bar{T}} (|\nabla \rho|^2 (|\bar{u}|_\infty^2 |\nabla \bar{u}|_2^2 + |f|_2^2) + \\
&+ |\rho|_\infty^2 (|\nabla \bar{u}|_4^2 + |\bar{u}|_\infty^2 |\nabla(\nabla \bar{u})|_2^2 + |\nabla f|_2^2) \\
&+ |\partial^2 p / \partial \rho^2|_\infty^2 |\nabla \rho|_4^4 + |\partial p / \partial \rho|_\infty^2 |\nabla(\nabla \rho)|_2^2) dt \leq \\
&\leq r^2
\end{aligned}$$

for suitable  $\bar{T}$  (we denote with the same letter, eventual, different suitable  $\bar{T}$ ). Whence (3.23) and (3.24) imply

$$(3.25) \quad GB \subseteq B.$$

Now we prove the continuity of  $G$  is  $L^2(Q_T)$ . First we observe that if  $\{\bar{u}^n\} \subseteq B$ , then there exists a subsequence (denoted again by  $\{\bar{u}^n\}$ ) such that as  $n \rightarrow \infty$ ,  $\bar{u}^n \rightarrow \bar{u}$  strongly in  $L^2(0, \bar{T}; H^2(\Omega))$ , weakly  $*$  in  $L^\infty(0, T; H^2(\Omega))$ , and  $\partial_t \bar{u}^n \rightarrow \partial_t \bar{u}$  weakly in  $L^2(0, \bar{T}; H^1(\Omega))$ . Let  $\rho^n$  and  $\rho$  be the solutions of

$$\partial_t \rho^n + \nabla \cdot (\rho^{n-n} u^n) = 0 \quad \text{with } \rho^n(0) = \rho_0,$$

and

$$\partial_t \rho + \nabla \cdot (\rho \bar{u}) = 0 \quad \text{with } \rho(0) = \rho_0,$$

respectively.

Now  $\tilde{\rho}^n = \rho^n - \rho$  satisfies

$$(3.26) \quad \partial_t \tilde{\rho}^n + \bar{u}^n \cdot \nabla \tilde{\rho}^n + (\bar{u}^n - \bar{u}) \cdot \nabla \rho + \tilde{\rho}^n \nabla \cdot \bar{u}^n + \rho \nabla \cdot (\bar{u}^n - \bar{u}) = 0$$

with  $\tilde{\rho}^n(0) = 0$ . Multiplying (3.26) by  $\tilde{\rho}^n$ , integrating over  $Q_T$ , and applying Gronwall lemma, we have

$$|\tilde{\rho}^n|_2^2 \leq \exp(cr\bar{T}) \int_0^{\bar{T}} (|\bar{u} - \bar{u}^n|_2 \cdot |\nabla \rho|_2^2 + |\rho \nabla \cdot (\bar{u}^n - \bar{u})|_2^2) dt.$$

This implies that  $\rho^n \rightarrow \rho$  strongly in  $L^\infty(0, \bar{T}; L^2(\Omega))$ .

Now let  $u^n$  and  $u$  be the solutions of

$$(3.27) \quad \rho^n \partial_t u^n - \Delta u^n - \nabla(\nabla \cdot u^n) = -\rho^n \bar{u}^n \cdot \nabla \bar{u}^n + \rho^n f - \nabla p(\rho^n)$$

with  $u^n(0) = u_0$ , and

$$(3.28) \quad \rho \partial_t u - \Delta u - \nabla(\nabla \cdot u) = -\rho \bar{u} \cdot \nabla \bar{u} + \rho f - \nabla p(\rho)$$

with  $u(0) = u_0$ , respectively.

Subtracting (3.27) and (3.28), we obtain

$$\begin{aligned}
(3.29) \quad & \rho^n \partial_t u^n - \rho \partial_t u - \Delta(u^n - u) - \nabla(\nabla \cdot (u^n - u)) = \\
& = -\rho^n \bar{u}^n \cdot \nabla \bar{u}^n + \rho \bar{u} \cdot \nabla \bar{u} + \bar{\rho}^n f - \nabla p(\rho^n) + \nabla p(\rho).
\end{aligned}$$

Denoting  $U^n = u^n - u$ ,  $\bar{U}^n = \bar{u}^n - \bar{u}$ , from (3.29), we have

$$\begin{aligned}
(3.30) \quad & \rho^n \partial_t U^n - \Delta U^n - \nabla(\nabla \cdot U^n) = -\bar{\rho}^n \partial_t u - \rho^n \bar{U}^n \cdot \nabla \bar{u}^n - \\
& - \bar{\rho}^n \bar{u} \cdot \nabla \bar{u}^n - \rho \bar{u} \cdot \nabla \bar{U}^n + \bar{\rho}^n f - \nabla p(\rho^n) + \nabla p(\rho).
\end{aligned}$$

Multiplying (3.30) by  $\partial_t U^n$  and integrating over  $Q_T$ , we find

$$\begin{aligned}
& \int_0^T |\sqrt{\rho^n} \partial_t U^n|_2^2 dt + \frac{1}{2} |\nabla U^n|_2^2 + \frac{1}{2} |\nabla \cdot U^n|_2^2 = \int_0^T (-\bar{\rho}^n \partial_t u - \rho^n \bar{U}^n \cdot \nabla \bar{u}^n - \\
& - \bar{\rho}^n \bar{u} \cdot \nabla \bar{u}^n - \rho \bar{u} \cdot \nabla \bar{U}^n + \bar{\rho}^n f - \nabla p(\rho^n) + \nabla p(\rho), \partial_t U^n) dt.
\end{aligned}$$

Thanks to

$$\begin{aligned}
& \bar{\rho}^n \longrightarrow 0 \quad \text{strongly in } L^\infty(0, \bar{T}; L^2(\Omega)), \\
& \bar{U}^n \longrightarrow 0 \quad \text{strongly in } L^2(0, \bar{T}; H_0^1(\Omega)),
\end{aligned}$$

it is a routine matter to prove that  $U^n \rightarrow 0$  strongly in  $L^2(Q_T)$ . Consequently, the map  $G$  is continuous in  $L^2(Q_T)$ , and the existence of a local solution is completely proved.

Now we proceed to prove the uniqueness of the solution by the same procedure as that used for the continuity of  $G$ . Let  $(\rho, u)$  and  $(\bar{\rho}, \bar{u})$  be two solutions of (2.1) and (2.2) and let  $\bar{\rho} = \rho - \bar{\rho}$  and  $\bar{u} = u - \bar{u}$ . Then  $\bar{\rho}$  and  $\bar{u}$ , respectively, satisfy the equations (3.26) with  $\bar{\rho}^n = \bar{\rho}$ ,  $\rho^n = \rho$ ,  $\rho = \bar{\rho}$  and (3.30) with  $U^n = \bar{u}$ ,  $u^n = u$ ,  $u = \bar{u}$ ,  $\bar{\rho}^n = \bar{\rho}$ ,  $\rho^n = \rho$ ,  $\rho = \bar{\rho}$ . Therefore it is easy to derive the inequality

$$\begin{aligned}
(3.31) \quad & d_t(|\sqrt{\rho} \bar{u}|_2^2 + |\bar{\rho}|_2^2) + |\nabla \bar{u}|_2^2 + |\nabla \cdot \bar{u}|_2^2 \\
& \leq \omega(t)(|\sqrt{\rho} \bar{u}|_2^2 + |\bar{\rho}|_2^2) + \delta c |\nabla \bar{u}|_2^2 + \delta |\nabla \cdot \bar{u}|_2^2,
\end{aligned}$$

where

$$\begin{aligned}
\omega(t) = & |\nabla \bar{u}|_\infty + c_\delta(|\bar{u} \cdot \nabla \bar{u}|_3^2 + \|\partial p(\cdot)/\partial \rho\|_{c_{loc}^0}^2 + |f|_3^2 + \\
& + |\partial_t \bar{u}|_3^2 + |\nabla \bar{\rho}|_3^2 + |\nabla \cdot u|_\infty + |\bar{\rho}|_\infty^2).
\end{aligned}$$

Now, integrating the differential inequality (3.31), we get ( $\delta \leq 1/(2c)$ )

$$\begin{aligned}
& |\sqrt{\rho(t)} \bar{u}(t)|_2^2 + |\bar{\rho}(t)|_2^2 = 0 \quad \text{a. e. in } (0, \bar{T}), \\
& \int_0^T (|\nabla \bar{u}|_2^2 + |\nabla \cdot \bar{u}|_2^2) d\tau = 0
\end{aligned}$$

Hence

$$\tilde{u}=0, \quad \tilde{\rho}=0 \quad \text{a. e. in } Q_{\bar{T}}.$$

Theorem 1 is completely proved.

PROOF OF THEOREM 2. Let

$$B = \{ \phi \mid \sup ( \|\phi\|_{L^2(0,T;H^2,q(\Omega))} ; \|\sqrt{\rho}\partial_t\phi\|_{L^\infty(0,T;L^2(\Omega))} ; \\ \|\phi\|_{L^\infty(0,T;H^2(\Omega))} ; \|\partial_t\phi\|_{L^2(0,T;H^1(\Omega))} ) \leq r \},$$

where  $q \in (3, 6]$ .

Under the assumptions of Theorem 2, (3.17), the first two relations in (3.21) and (3.23) continue to hold. Bearing in mind that (3.18) is a strongly elliptic system, and (3.23), we have, with  $r$  as in Theorem 1,

$$\begin{aligned} \int_0^{\bar{T}} \|u\|_{2,q}^2 dt &\leq c \int_0^{\bar{T}} (|\rho|_\infty^2 |\bar{u}|_\infty^2 |\nabla \bar{u}|_q^2 + |\rho|_\infty^2 |f|_q^2 + |\partial p / \partial \rho|_\infty^2 |\nabla \rho|_q^2 + |\rho|_\infty^2 |\partial_t u|_q^2) dt \leq \\ &\leq c e^{cr\sqrt{\bar{T}}\gamma^4 \bar{T}} + c e^{cr\sqrt{\bar{T}}} \int_0^{\bar{T}} |f|_q^2 dt + \\ (3.32) \quad &+ c \int_0^{\bar{T}} |\partial p / \partial \rho|_\infty^2 (e^{2cr\sqrt{\bar{T}}} |\nabla \rho_0|_q^2 + e^{2cr\sqrt{\bar{T}}} r^2 \bar{T}) dt + \\ &+ c e^{cr\sqrt{\bar{T}} - \|u_0\|_1} r^2 / c \leq \\ &\leq r^2 \end{aligned}$$

for suitable  $c$  and  $\bar{T}$ . We deduce from (3.18), (3.23) and (3.32) that  $GB \subseteq B$ .

The continuity of  $G$  is proved as in Theorem 1. Moreover, a uniqueness theorem holds. Theorem 2 is proved.

#### 4. Global estimates

In this section we establish global estimates which enable us to extend the local solution claimed by Theorem 2 for all  $t \in (0, \infty)$ . In what follows  $k$  and  $c$  are generic constants depending only on  $\Omega$  and  $m_0$ . The proof of Theorem 3 requires many tedious calculations. For the convenience of the reader, we assume  $q=4$ . In this case, in fact, the proof is suitably simplified. We note that, for arbitrary  $q \in (3, 6]$ , mutatis mutandis, the treatment is quite analogous. Now, it may be easily seen that

$$(4.1) \quad |\rho - m_0|_\infty \leq k |\nabla \rho|_4.$$

and

$$(4.2) \quad m_0 - k|\nabla \rho|_4 \leq \rho \leq m_0 + k|\nabla \rho|_4.$$

For the purpose to be cleared up later, define

$$(4.3) \quad \sigma = \nabla(\ln \rho) = \rho^{-1} \nabla \rho.$$

Since

$$|\nabla \rho|_4 \leq |\rho|_\infty |\sigma|_4 \leq (m_0 + k|\nabla \rho|_4) |\sigma|_4,$$

we have

$$(4.4) \quad |\nabla \rho|_4 \leq m_0 |\sigma|_4 (1 - k|\sigma|_4)^{-1}$$

as soon as  $|\sigma|_4$  is sufficiently small ( $|\sigma|_4 < k^{-1}$ ); this is true for the local solution from Theorem 2 if  $r_0$  and  $\bar{T}$  are sufficiently small. Thus (4.4) and (4.2) yield

$$(4.5) \quad \begin{aligned} \alpha(|\sigma|_4) &\equiv m_0[1 - k|\sigma|_4(1 - k|\sigma|_4)^{-1}] \leq \rho \leq \\ &\leq m_0[1 + k|\sigma|_4(1 - k|\sigma|_4)^{-1}] \equiv \beta(|\sigma|_4). \end{aligned}$$

Also, if  $q \in C^2((0, \infty))$ , then

$$(4.6) \quad \begin{aligned} q(\rho) &= q(m_0) + q'(m_0)(\rho - m_0) + \frac{1}{2} q''(m_0 + \vartheta(\rho - m_0))(\rho - m_0)^2, \\ \vartheta &\in (0, 1). \end{aligned}$$

From (4.5) and (4.6) we find

$$(4.7) \quad \begin{aligned} |q(\rho)|_\infty &\leq q(m_0) + km_0 |q'(m_0)| |\sigma|_4 (1 - k|\sigma|_4)^{-1} + \\ &+ \frac{1}{2} [\sup_r \{|q''(r)| : \alpha(|\sigma|_4) \leq r \leq \beta(|\sigma|_4)\}] \cdot \\ &\cdot [m_0 k (1 - k|\sigma|_4)^{-1}]^2 |\sigma|_4^2 \equiv \\ &\equiv c + \tilde{q}(|\sigma|_4) |\sigma|_4^2. \end{aligned}$$

Now, clearly, multiplying the first equation in (2.1) by  $u$ , using (2.2) and integrating over  $\Omega$ , we get the standard energy estimate

$$(4.8) \quad \frac{1}{2} d_t \|\sqrt{\rho} u\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|\nabla \cdot u\|_2^2 \leq c \|\rho\|_\infty^2 \|f\|_2^2 + \left| \int_\Omega \nabla p(\rho) \cdot u dx \right|.$$

(Notice that  $\frac{1}{2} d_t \int_\Omega \rho |u|^2 dx = \int_\Omega (\rho \partial_t u + \rho u \cdot \nabla u) \cdot u dx$ .) We can easily see as above (cf. (4.7)) that

$$(4.9) \quad |\nabla p(\rho)|_2 \leq |\rho p'(\rho)|_\infty |\sigma|_2 \leq c m_0 p'(m_0) |\sigma|_4 + q_0(|\sigma|_4) |\sigma|_4^2,$$

where

$$\begin{aligned} q_0(|\sigma|_4) &= c[m_0 p'(m_0)(1 - k|\sigma|_4)^{-1} + \\ &\quad + m_0^2 \sup_s \{p''(s) ; \alpha(|\sigma|_4) \leq s \leq \beta(|\sigma|_4)\} (1 - k|\sigma|_4)^{-2}]. \end{aligned}$$

Hence

$$\begin{aligned} (4.10) \quad \left| \int_\Omega u \cdot \nabla p(\rho) dx \right| &\leq \varepsilon_1 |u|_2^2 + \varepsilon_1 (m_0 p'(m_0))^4 + c_{\varepsilon_1} |\sigma|_4^4 + \varepsilon_1 + c_{\varepsilon_1} (q_0(|\sigma|_4))^4 |u|_2^4 \leq \\ &\leq \varepsilon_1 c |\nabla u|_2^2 + c \varepsilon_1 + c_{\varepsilon_1} |\sigma|_4^4 + c_{\varepsilon_1} (q_0(|\sigma|_4))^4 |u|_2^4. \end{aligned}$$

Inserting (4.10) into (4.0) with an obvious estimate of  $|\rho|_\infty^2 |f|_2^2$ , we find ( $\varepsilon_1 < 1/(4c)$ )

$$\begin{aligned} (4.11) \quad \frac{1}{2} d_t |\sqrt{\rho} u|_2^2 + \frac{1}{4} |\nabla u|_2^2 + |\nabla \cdot u|_2^2 &\leq \\ &\leq c |f|_2^2 + c |f|_2^3 + \frac{1}{2} c \varepsilon_1 + c_{\varepsilon_1} |\sigma|_4^4 + q_1(|\sigma|_4) (|u|_2^4 + |\sigma|_4^6), \end{aligned}$$

where

$$q_1(|\sigma|_4) = \max \{c(1 + k|\sigma|_4)^{-6}, c_{\varepsilon_1} (q_0(|\sigma|_4))^4\}.$$

Now we proceed to multiply the first equation in (2.1) by  $\partial_t u$  and to integrate over  $\Omega$ . We find

$$\begin{aligned} (4.12) \quad |\sqrt{\rho} \partial_t u|_2^2 + \frac{1}{2} d_t |\nabla u|_2^2 + \frac{1}{2} d_t |\nabla \cdot u|_2^2 &\leq \\ &\leq \left| \int_\Omega \rho u \cdot \nabla u \cdot \partial_t u dx \right| + \left| \int_\Omega \nabla p(\rho) \cdot \partial_t u dx \right| + \left| \int_\Omega \rho f \cdot \partial_t u dx \right|. \end{aligned}$$

Let us estimate the integrals on the right-hand side of (4.12). First, we have

$$\begin{aligned} \left| \int_\Omega \rho u \cdot \nabla u \cdot \partial_t u dx \right| &\leq |\rho|_\infty^{1/2} |\sqrt{\rho} \partial_t u|_2 |\nabla u|_2^{3/2} |\Delta u|_2^{1/2} \leq \\ &\leq \varepsilon_2 |\sqrt{\rho} \partial_t u|_2^2 + c_{\varepsilon_2} [1 + k|\sigma|_4 (1 - k|\sigma|_4)^{-1}] (|\nabla u|_2^{15/4} + |\Delta u|_2^5). \end{aligned}$$

Second, estimating  $q(\rho) = g(\rho) = \sqrt{\rho} p'(\rho)$  as  $\rho p'(\rho)$  in (4.9), we find

$$\begin{aligned} \left| \int_\Omega \nabla p(\rho) \cdot \partial_t u dx \right| &\leq |g(\rho)|_\infty |\sigma|_2 |\sqrt{\rho} \partial_t u|_2 \leq \\ &\leq c |\sigma|_4 |\sqrt{\rho} \partial_t u|_2 [1 + |\sigma|_4 + \bar{q}_1(|\sigma|_4) |\sigma|_4^2] \leq \\ &\leq \delta + (\varepsilon_2 + \delta) |\sqrt{\rho} \partial_t u|_2^2 + c_\delta |\sigma|_4^4 + c_{\varepsilon_2} (\bar{q}_1(|\sigma|_4))^2 |\sigma|_4^6, \end{aligned}$$



where

$$\begin{aligned}\tilde{q}_1(|\sigma|_4) &= (1 - k|\sigma|_4)^{-1} \\ &+ \sup_s \{(p''(s))^2 : \alpha(|\sigma|_4) \leq s \leq \beta(|\sigma|_4)\} (1 - k|\sigma|_4)^{-2}.\end{aligned}$$

Third, using (4.5) and Young's inequality, we have

$$\left| \int_{\Omega} \rho f \cdot \partial_t u dx \right| \leq |f|_2^{4/3} + 2m_0^2 [1 + k^2 |\sigma|_4^2 (1 - k|\sigma|_4)^{-2}] |\sqrt{\rho} \partial_t u|_2^4.$$

With these estimates we deduce from (4.12)

$$\begin{aligned}(4.13) \quad d_t \left( \frac{1}{2} |\nabla u|_2^2 + \frac{1}{2} |\nabla \cdot u|_2^2 \right) &+ \frac{1}{2} |\sqrt{\rho} \partial_t u|_2^2 \leq \\ &\leq \delta + c_\delta |\sigma|_4^4 + |f|_2^{4/3} + q_2(|\sigma|_4) (|\sigma|_4^6 + |\nabla u|_2^{15/4} + |\Delta u|_2^5 + |\sqrt{\rho} \partial_t u|_2^4)\end{aligned}$$

with a function  $q_2(|\sigma|_4)$  the form of which can be traced in the above estimates.

Now, let us estimate  $|\Delta u|_2$ . From the theory of elliptic systems (see, e.g., [14] or [19]), it is clear that

$$|\Delta u|_2 \leq c |\Delta u + \nabla(\nabla \cdot u)|_2.$$

Directly from the first equation in (2.1), we get

$$\begin{aligned}|\Delta u + \nabla(\nabla \cdot u)|_2^2 &\leq (|\rho \partial_t u|_2 + |\rho u \cdot \nabla u|_2 + |\nabla p|_2 + |\rho f|_2)^2 \leq \\ &\leq (|\sqrt{\rho}|_\infty |\sqrt{\rho} \partial_t u|_2 + c |\rho|_\infty |\nabla u|_2^{3/2} |\Delta u|_2^{1/2} + |\nabla p|_2 + |\rho|_\infty |f|_2)^2 \leq \\ &\leq c(1 + |\sqrt{\rho}|_\infty + |\rho|_\infty + |\rho|_\infty^2) (|\sqrt{\rho} \partial_t u|_2^2 + |\nabla u|_2^6 + |\nabla p|_2^2 + |f|_2^2) + \varepsilon |\Delta u|_2^2.\end{aligned}$$

Employing here the estimates (4.5) and (4.9), ( $\varepsilon > 0$  suitable), we obtain

$$\begin{aligned}(4.14) \quad |\Delta u|_2^2 &\leq c[1 + |\sigma|_4^2 (1 - k|\sigma|_4)^{-2}]^2 [|\sqrt{\rho} \partial_t u|_2^2 + |\nabla u|_2^6 + \\ &+ |\sigma|_4^2 + (q_0(|\sigma|_4))^2 |\sigma|_4^4 + |f|_2^2].\end{aligned}$$

Further, differentiating (2.1) by  $t$ , multiplying the resulting equation by  $\partial_t u$  and integrating over  $\Omega$ , we get

$$\begin{aligned}(4.15) \quad \frac{1}{2} d_t |\sqrt{\rho} \partial_t u|_2^2 &+ |\nabla \partial_t u|_2^2 + |\nabla \cdot \partial_t u|_2^2 \leq 2 \left| \int_{\Omega} \rho (u \cdot \nabla) \partial_t u \cdot \partial_t u dx \right| + \\ &+ \left| \int_{\Omega} \partial_t \rho (u \cdot \nabla u) \cdot \partial_t u dx \right| + \left| \int_{\Omega} \rho \partial_t u \cdot \nabla u \cdot \partial_t u dx \right| + \left| \int_{\Omega} \nabla \partial_t p(\rho) \cdot \partial_t u dx \right| + \\ &+ \left| \int_{\Omega} \partial_t \rho f \cdot \partial_t u dx \right| + \left| \int_{\Omega} \rho \partial_t f \cdot \partial_t u dx \right|.\end{aligned}$$

Estimate the integrals on the right-hand side of (4.15). We find

$$\begin{aligned}
& 2 \left| \int_{\Omega} \rho(u \cdot \nabla) \partial_t u \cdot \partial_t u dx \right| \leq \\
& \leq 2 |\rho|_{\infty}^{1/2} |\sqrt{\rho} \partial_t u|_2 |u|_{\infty} |\nabla \partial_t u|_2 \leq \\
& \leq \varepsilon_3 |\nabla \partial_t u|_2^2 + \varepsilon_3 |\Delta u|_4^2 + c_{\varepsilon_3} [1 + k|\sigma|_4 (1 - k(\sigma)_4)^{-1}]^2 (|\nabla u|_2^4 + |\sqrt{\rho} \partial_t u|_2^8), \\
& \left| \int_{\Omega} \partial_t \rho(u \cdot \nabla u) \partial_t u dx \right| = \left| \int_{\Omega} \rho u \cdot \nabla((u \cdot \Delta u) \cdot \partial_t u) dx \right| \leq \\
& \leq c(|\rho|_{\infty} |u|_6 |\nabla u|_3^2 |\nabla \partial_t u|_2 + |\rho|_{\infty} |u|_6^2 |\Delta u|_2 |\partial_t u|_6 + \\
& \quad + |\rho|_{\infty} |u|_{\infty} |u|_6 |\nabla u|_3 |\nabla \partial_t u|_2) \leq \\
& \leq 3\varepsilon_3 |\nabla \partial_t u|_2^2 + c_{\varepsilon_3} [1 + k|\sigma|_4 (1 - k[\sigma]_4)^{-1}]^2 (|\nabla u|_2^4 + |\Delta u|_2^2), \\
& \left| \int_{\Omega} \rho(\partial_t u \cdot \nabla u) \cdot \partial_t u dx \right| \leq c |\sqrt{\rho}|_{\infty} |\sqrt{\rho} \partial_t u|_2 |\nabla u|_2^{1/2} |\Delta u|_2^{1/2} |\nabla \partial_t u|_2 \leq \\
& \leq \varepsilon_3 |\nabla \partial_t u|_2^2 + \varepsilon_3 |\Delta u|_2^2 + c_{\varepsilon_3} [1 + k|\sigma|_4 (1 - k|\sigma|_4)^{-1}]^2 (|\nabla u|_2^4 + |\sqrt{\rho} \partial_t u|_2^8), \\
& \left| \int_{\Omega} \partial_t \rho f \cdot \partial_t u dx \right| \leq \int_{\Omega} |\rho(u \cdot \nabla) f \cdot \partial_t u| dx + \int_{\Omega} |\rho(u \cdot \nabla) \partial_t u \cdot f| dx \leq \\
& \leq |\rho|_{\infty} |u|_6 |\partial_t u|_6 |\nabla f|_{3/2} + |\rho|_{\infty} |u|_6 |f|_3 |\nabla \partial_t u|_2 \leq \\
& \leq 2\varepsilon_3 |\nabla \partial_t u|_2^2 + c_{\varepsilon_3} (|f|_2^3 + |\nabla f|_{3/2}^2) [|\nabla u|_2^2 + |\nabla u|_2^3 + |\sigma|_4^6 (1 - k|\sigma|_4)^{-6}], \\
& \left| \int_{\Omega} \rho \partial_t f \cdot \partial_t u dx \right| \leq |\rho|_{\infty} |\partial_t f|_{6/5} |\partial_t u|_6 \leq \\
& \leq \varepsilon_3 |\nabla \partial_t u|_2^2 + c_{\varepsilon_3} [|\partial_t f|_{6/5}^2 + |\partial_t f|_{6/5}^3 + |\sigma|_4^6 (1 - k|\sigma|_4)^{-6}],
\end{aligned}$$

and finally

$$\begin{aligned}
& \left| \int_{\Omega} \nabla \partial_t p(\rho) \cdot \partial_t u dx \right| = \left| \int_{\Omega} p'(\rho) (u \cdot \nabla \rho + \rho \nabla \cdot u) \nabla \cdot \partial_t u dx \right| \leq \\
& \leq \varepsilon_3 |\nabla \cdot \partial_t u|_2^2 + c_{\varepsilon_3} |\sigma|_4^4 + c_{\varepsilon_3} \bar{q}_2(|\sigma|_4) |\nabla u|_2^4 + c_{\varepsilon_3} |\nabla \cdot u|_2^2.
\end{aligned}$$

Inserting the above estimates with  $\varepsilon_3 \leq 1/16$  to (4.15), we obtain

$$\begin{aligned}
(4.16) \quad & d_t |\sqrt{\rho} \partial_t u|_2^2 + |\nabla \partial_t u|_2^2 + |\nabla \cdot \partial_t u|_2^2 \leq \\
& \leq c_{\varepsilon_3} |\Delta u|_4^2 + c_{\varepsilon_3} q_3(|\sigma|_4) [|\nabla u|_2^4 + |\nabla u|_2^6 + |\nabla u|_2^{10} + |\sqrt{\rho} \partial_t u|_2^6 + \\
& \quad + |\sqrt{\rho} \partial_t u|_2^8 + |\sigma|_4^6 + |\sigma|_4^{12} + |f|_2^6 + |\sigma|_4^6 (|f|_3^2 + |\nabla f|_{3/2}^2)] +
\end{aligned}$$

$$+ c_{\varepsilon_3} [|\sigma|_4^4 + |\nabla \cdot u|_2^2 + (|f|_3^2 + |\nabla f|_{3/2}^2)(|\nabla u|_2^2 + |\nabla u|_2^3) + \\ + |\partial_t f|_{6/5}^2 + |\partial_t f|_{6/5}^3]$$

where

$$q_3(|\sigma|_4) = (1 - k|\sigma|_4)^{-2} + (q_0(|\sigma|_4))^2(1 - k|\sigma|_4)^{-2} \\ + |\sigma|_4^4(1 - k|\sigma|_4)^{-6} + \bar{q}_2(|\sigma|_4) + (1 - k|\sigma|_4)^{-6}(1 + |\sigma|_4^4).$$

Next, we shall estimate  $|\sigma|_4$ . The equation (2.1)<sub>2</sub> yields

$$-\nabla(\nabla \cdot u) = \partial_t \sigma + \nabla(u \cdot \sigma).$$

Inserting this formula into (2.1)<sub>1</sub>, multiplying the resulting equation by  $\chi^4 |\sigma|^2 \sigma$ , with  $\chi = \lambda^{1/4}$  ( $\lambda > 1$  is a suitable constant), and integrating over  $\Omega$ , we obtain

$$(4.17) \quad \frac{1}{4} d_t |\chi \sigma|_4^4 + \int_{\Omega} \rho p'(\rho) |\chi \sigma|^4 dx = \chi \int_{\Omega} \rho f \cdot \chi \sigma |\chi \sigma|^2 dx + \\ + \chi \int_{\Omega} \Delta u \cdot \chi \sigma |\chi \sigma|^2 dx - \chi \int_{\Omega} \rho \partial_t u \cdot \chi \sigma |\chi \sigma|^2 dx - \\ - \chi \int_{\Omega} \rho u \cdot \nabla u \cdot \chi \sigma |\chi \sigma|^2 dx - \int_{\Omega} \nabla(u \cdot \chi \sigma) \cdot \chi \sigma |\chi \sigma|^2 dx.$$

First, let us see that

$$-\int_{\Omega} \nabla(u \cdot \chi \sigma) \cdot \chi \sigma |\chi \sigma|^2 dx \leq c |\nabla u|_{\infty} |\chi \sigma|_4^4 \leq \varepsilon_4 |\Delta u|_4^2 + c_{\varepsilon_4} |\chi \sigma|_4^8.$$

Further, we have

$$\chi \left| \int_{\Omega} \rho \partial_t u \cdot \chi \sigma |\chi \sigma|^2 dx \right| \leq \chi |\rho|_{\infty} |\partial_t u|_4 |\chi \sigma|_4^3 \leq \\ \leq \varepsilon_4 |\nabla \partial_t u|_2^2 + c_{\varepsilon_4} \chi^2 |\chi \sigma|_4^6 + c_{\varepsilon_4} |\chi \sigma|_4^8 (1 - k|\sigma|_4)^{-2}; \\ \chi \left| \int_{\Omega} \rho u \cdot \nabla u \cdot \chi \sigma |\chi \sigma|^2 dx \right| \leq c \chi |\rho|_{\infty} |u|_{\infty} |\nabla u|_4 |\chi \sigma|_4^3 \leq \\ \leq c \chi |\rho|_{\infty} |\nabla u|_2^{3/4} |\Delta u|_2^{5/4} |\chi \sigma|_4^3 \leq (\text{see 4.14}) \\ \leq c \chi (|\nabla u|_2^4 + |\sqrt{\rho} \partial_t u|_2^4 + |\nabla u|_2^9 + |\chi \sigma|_4^6) + 2\varepsilon_4 |\chi \sigma|_4^4 + \\ + c_{\varepsilon_4} \{(|\chi \sigma|_4^{10} + |\nabla u|_2^6) + (q_0(|\sigma|_4))^5 (|\nabla u|_2^6 + |\chi \sigma|_4^{20})\} + \\ + c \chi |f|_2^4 + \bar{q}(|\sigma|_4) [|\nabla u|_2^4 + |\sqrt{\rho} \partial_t u|_2^4 + |\nabla u|_2^9 + |\chi \sigma|_4^{14} + |\nabla u|_2^6 + \\ + |\chi \sigma|_4^6 + |\chi \sigma|_4^{12} + (q_0(|\sigma|_4))^{5/4} (|\chi \sigma|_4^{10} + |\chi \sigma|_4^{12} + |\nabla u|_2^6) + |f|_2^4]$$

$$(\bar{q}_0(|\sigma|_4) = c(1 - k|\sigma|_4)^{-4}) ;$$

$$\begin{aligned} \chi \left| \int_{\Omega} \Delta u \cdot \chi \sigma |\chi \sigma|^2 dx \right| &\leq \varepsilon_4 |\Delta u|_4^2 + c_{\varepsilon_4} \chi^2 |\chi \sigma|_4^6 ; \\ \chi \left| \int_{\Omega} \rho f \cdot \chi \sigma |\chi \sigma|^2 dx \right| &\leq \chi |\rho|_{\infty} |f|_4 |\chi \sigma|_4^3 \leq \\ &\leq \varepsilon_4 |\chi \sigma|_4^4 + c_{\varepsilon_4} \chi^4 |f|_4^4 + c(|f|_4^2 + |\chi \sigma|_4^8 (1 - k|\sigma|_4)^{-2}) . \end{aligned}$$

If we write  $\rho p'(\rho)$  in the form

$$\rho p'(\rho) = m_0 p'(m_0) + p_1(m_0 + \mathcal{V}(\rho - m_0))(\rho - m_0) ,$$

where

$$p_1(r) = p'(r) + r p''(r), \quad r > 0 ; \quad \mathcal{V} \in (0, 1) ,$$

we can transfer the term with  $p_1$  in the right-hand side of (4.17) and estimate it as follows :

$$\begin{aligned} \left| \int_{\Omega} p_1(m_0 + \mathcal{V}(\rho - m_0))(\rho - m_0) |\chi \sigma|^4 dx \right| &\leq c \bar{p}_1(|\sigma|_4) (1 - k|\sigma|_4)^{-1} |\chi \sigma|_4^5 \\ (\bar{p}_1(|\sigma|_4) &= \sup_r \{p_1(r) ; \alpha(|\sigma|_4) \leq r \leq \beta(|\sigma|_4)\}) . \end{aligned}$$

Thus, we can deduce from (4.17) the estimate

$$\begin{aligned} (4.18) \quad &\frac{1}{4} d_t |\chi \sigma|_4^4 + \frac{1}{2} m_0 p'(m_0) |\chi \sigma|_4^4 \leq 2\varepsilon_4 |\Delta u|_4^2 + \varepsilon_4 |\nabla \partial_t u|_2^2 + \\ &+ c_{\varepsilon_4} (\chi^4 |f|_4^4 + \chi^2 |\chi \sigma|_4^6) + c(|f|_4^2 + |\chi \sigma|_4^8 (1 - k|\sigma|_4)^{-2}) + \\ &+ c_{\varepsilon_4} (|\chi \sigma|_4^8 (1 - k|\sigma|_4)^{-2} + |\chi \sigma|_4^8) + c\chi (|\nabla u|_2^4 + |\sqrt{\rho} \partial_t u|_2^4 + |\nabla u|_2^9 + \\ &+ |\chi \sigma|_4^6) + c_{\varepsilon_4} \{(|\chi \sigma|_4^{10} + |\nabla u|_2^6) + (q_0(|\sigma|_4))^5 (|\nabla u|_2^6 + |\chi \sigma|_4^{20})\} \\ &+ c\chi |f|_2^4 + \bar{q}(|\sigma|_4) [|\nabla u|_2^4 + |\sqrt{\rho} \partial_t u|_2^4 + |\nabla u|_2^9 + |\chi \sigma|_4^{14} + \\ &+ |\nabla u|_2^6 + |\chi \sigma|_4^{12} + (q_0(|\sigma|_4))^{5/4} (|\chi \sigma|_4^{10} + |\chi \sigma|_4^{12} + |\nabla u|_2^6) + \\ &+ |f|_2^4 + |\chi \sigma|_4^6] + c(\bar{p}_1(|\sigma|_4) (1 - k|\sigma|_4)^{-1} |\chi \sigma|_4^5) \leq \\ &\leq 2\varepsilon_4 |\Delta u|_4^2 + \varepsilon_4 |\nabla \partial_t u|_2^2 + c\chi (|\sqrt{\rho} \partial_t u|_2^4 + |\nabla u|_2^4 + \\ &+ |\nabla u|_2^9 + |\chi \sigma|_4^6) + c_{\varepsilon_4} [|\nabla u|_2^6 + \chi^2 |\chi \sigma|_4^6 + |\chi \sigma|_4^8 + |\chi \sigma|_4^{10} + \\ &+ (q_0(|\sigma|_4))^5 (|\nabla u|_2^6 + |\chi \sigma|_4^{20})] + \bar{q}(|\sigma|_4) [|\sqrt{\rho} \partial_t u|_2^4 + |\nabla u|_2^4 + \\ &+ |\nabla u|_2^6 + |\nabla u|_2^9 + |\chi \sigma|_4^5 + |\chi \sigma|_4^6 + c_{\varepsilon_4} |\chi \sigma|_4^8 + |\chi \sigma|_4^{12} + \end{aligned}$$

$$\begin{aligned}
& + |\chi\sigma|_4^{11} + (q_0(|\sigma|_4))^{5/4} (|\chi\sigma|_4^{10} + |\chi\sigma|_4^{12} + |\nabla u|_2^6) + |f|_2^4 \\
& + c\chi|f|_2^4 + c|f|_4^2 + c_{\varepsilon_4}\chi^1|f|_4^4.
\end{aligned}$$

Finally, we shall estimate  $|\Delta u|_4$  similarly as we have done with  $|\Delta u|_2$  in (4.14). Thus, with the help of (4.14), we find

$$\begin{aligned}
(4.19) \quad |\Delta u|_4^2 & \leq c[1 + |\sigma|_4^2(1 - k|\sigma|_4)^{-2}] \cdot \{|\rho\partial_t u|_4^2 + |\sqrt{\rho}\partial_t u|_2^4 + |\nabla u|_2^3 + \\
& + |\nabla u|_2^4 + |\nabla u|_2^9 + |f|_2^4 + |\sigma|_4^5 + c[(q_0(|\sigma|_4))^{3/2} + (1 - k|\sigma|_4)^{-5} + \\
& + (q_0(|\sigma|_4))^{3/2}(1 - k|\sigma|_4)^{-6}] (|\sqrt{\rho}\partial_t u|_2^{10} + |\nabla u|_2^4 + |\nabla u|_2^6 + \\
& + |\nabla u|_2^{18} + |\sigma|_4^8 + |\sigma|_4^{10} + |\sigma|_4^{16} + |f|_2^{10}) + |f|_4^2\} + c|\nabla p(\rho)|_4^2.
\end{aligned}$$

Estimating  $q(\rho) = \bar{g}(\rho) \equiv \rho p'(\rho)$  as in (4.9), we get

$$\begin{aligned}
|\nabla p(\rho)|_4^2 & \leq |\bar{g}(\rho)|_\infty^2 |\sigma|_4^2 \leq \\
& \leq c|\sigma|_4^2 + q_5(|\sigma|_4)|\sigma|_4^6 \leq \\
& \leq \delta_1 + c_{\delta_1}|\sigma|_4^4 + q_5(|\sigma|_4)|\sigma|_4^6 \\
(q_5(|\sigma|_4) & = \sup_s \{[(p''(s))^2 + (p''(s))^4 + 1] ; \alpha(|\sigma|_4) \leq s \leq \beta(|\sigma|_4)\} \\
& \cdot (1 - k|\sigma|_4)^{-4}).
\end{aligned}$$

Besides,

$$\begin{aligned}
|\rho\partial_t u|_4^2 & = \left( \int_\Omega \rho^{7/2} |\sqrt{\rho}\partial_t u| |\partial_t u|^3 dx \right)^{1/2} \leq \\
& \leq c|\rho|_\infty^{7/4} |\sqrt{\rho}\partial_t u|_2^{1/2} |\partial_t u|_6^{3/2} \leq \\
& \leq \varepsilon_5 |\sqrt{\rho}\partial_t u|_2^2 + c_{\varepsilon_5} |\nabla\partial_t u|_2^2 + (1 - k|\sigma|_4)^{-7} (|\sigma|_4^{11} + |\sqrt{\rho}\partial_t u|_2^4).
\end{aligned}$$

So, from (4.19) with the help of (4.5) we find

$$\begin{aligned}
(4.20) \quad |\Delta u|_4^2 & \leq \varepsilon_5 |\sqrt{\rho}\partial_t u|_2^2 + c_{\varepsilon_5} |\nabla\partial_t u|_2^2 + \delta_1 + c_{\delta_1} |\sigma|_4^4 + \\
& + q_6(|\sigma|_4) (|\nabla u|_2^3 + |\nabla u|_2^4 + |\nabla u|_2^6 + |\nabla u|_2^9 + |\nabla u|_2^{18} + |\sigma|_4^5 + \\
& + |\sigma|_4^6 + |\sigma|_4^8 + |\sigma|_4^{10} + |\sigma|_4^{14} + |\sigma|_4^{16} + |\sqrt{\rho}\partial_t u|_2^4 + \\
& + |\sqrt{\rho}\partial_t u|_2^{10} + |f|_2^4 + |f|_2^{10} + |f|_4^2).
\end{aligned}$$

(The form of  $q_6(|\sigma|_4)$  can be traced in the above estimates.) Let us sum up the above obtained estimates according to the rule  $\lambda_1$  (4.11) + (4.13) +  $\lambda_2$  (4.16) + (4.18) + (4.20), and bearing in mind  $\chi^1 = \lambda$ , and without loss of gen-

erality we assume  $\lambda \geq \sup(1, \lambda_1, \lambda_2)$ , then we get

$$\begin{aligned}
(4.21) \quad & d_t \left[ \frac{1}{2} \lambda_1 |\sqrt{\rho} u|_2^2 + \frac{1}{2} |\nabla u|_2^2 + \frac{1}{2} |\nabla \cdot u|_2^2 + \frac{1}{2} \lambda_2 |\sqrt{\rho} \partial_t u|_2^2 + \frac{1}{4} |\chi \sigma|_4^4 \right] + \\
& + \frac{1}{4} \lambda_1 |\nabla u|_2^2 + \frac{1}{2} \lambda_1 |\nabla \cdot u|_2^2 + \frac{1}{2} |\sqrt{\rho} \partial_t u|_2^2 + \frac{1}{2} \lambda_2 |\nabla \partial_t u|_2^2 + \\
& + \frac{1}{2} \lambda_2 |\nabla \cdot \partial_t u|_2^2 + \frac{1}{2} m_0 p'(m_0) |\chi \sigma|_4^4 + |\Delta u|_4^2 \leq \\
& \leq [(c \varepsilon_3 \lambda_2 + 2 \varepsilon_4) |\Delta u|_4^2 + (\varepsilon_4 + c \varepsilon_5) |\nabla \partial_t u|_2^2 + \varepsilon_5 |\sqrt{\rho} \partial_t u|_2^2 + \\
& + \lambda_2 c_{\varepsilon_3} |\nabla \cdot u|_2^2 + \lambda_2 c_{\varepsilon_3} (|\nabla f|_{3/2}^2 + |f|_3^2) |\nabla u|_2^2 + (\lambda_1 c_{\varepsilon_1} + \lambda_2 c_{\varepsilon_3} + \\
& + c_{\delta} + c_{\delta_1}) |\sigma|_4^4 + \lambda_1 c_{\varepsilon_1} q_1(|\sigma|_4) (|\nabla u|_2^4 + |\sigma|_4^6) + q_2(|\sigma|_4) (|\sigma|_4^6 + \\
& + |\nabla u|_2^{5/4} + |\sqrt{\rho} \partial_t u|_2^4 + |\Delta u|_2^5)] + \lambda_2 c_{\varepsilon_3} |\nabla u|_2^3 (|f|_3^2 + |\nabla f|_{3/2}^2) + \\
& + \lambda_2 c_{\varepsilon_3} q_3(|\sigma|_4) [|\nabla u|_2^4 + |\nabla u|_2^6 + |\nabla u|_2^{10} + |\sqrt{\rho} \partial_t u|_2^6 + \\
& + |\sqrt{\rho} \partial_t u|_2^8 + |\sigma|_4^6 + |\sigma|_2^{12} + |f|_2^6 + |\sigma|_4^6 (|f|_3^2 + |\nabla f|_{3/2}^2)] + \\
& + \lambda_2 c_{\varepsilon_3} |\nabla u|_2^6 + c \chi (|\nabla u|_2^4 + |\nabla u|_2^6 + |\sqrt{\rho} \partial_t u|_2^4 + |\chi \sigma|_4^6) + \\
& + c_{\varepsilon_4} [|\nabla u|_2^6 + \chi^2 |\chi \sigma|_4^6 + |\chi \sigma|_4^8 + |\chi \sigma|_4^{10} + (q_0(|\sigma|_4))^5 (|\nabla u|_2^6 + \\
& + |\chi \sigma|_4^{20})] + \bar{q}(|\sigma|_4) [|\sqrt{\rho} \partial_t u|_2^4 + |\nabla u|_2^4 + |\nabla u|_2^6 + |\nabla u|_2^8 + |\chi \sigma|_4^6 + \\
& + c_{\varepsilon_4} |\chi \sigma|_4^8 + |\chi \sigma|_4^{12} + |\chi \sigma|_4^{14} + (q_0(|\sigma|_4))^{5/2} (|\nabla u|_2^6 + \\
& + |\chi \sigma|_4^{10} + |\chi \sigma|_4^{12}) + |f|_2^4] + q_6(|\sigma|_4) (|\nabla u|_2^3 + |\nabla u|_2^4 + |\nabla u|_2^6 + \\
& + |\nabla u|_2^{18} + |\sigma|_4^5 + |\sigma|_4^6 + |\sigma|_4^8 + |\sigma|_4^{10} + |\sigma|_4^{14} + |\sigma|_4^{16} + |\sqrt{\rho} \partial_t u|_2^4 + \\
& + |\sqrt{\rho} \partial_t u|_2^{10} + |f|_2^2 + c \chi |f|_2^4 + c \lambda_1 (|f|_2^2 + |f|_2^3) + \\
& + c_{\varepsilon_3} \lambda_2 [|\partial_t f|_{6/5}^2 + |\partial_t f|_{6/5}^3 + (|f|_3^2 + |\nabla f|_{3/2}^2)^2] + \\
& + c |f|_4^2 + c_{\varepsilon_4} \chi^4 |f|_4^4 + \delta + \delta_1 + c \varepsilon_1 \lambda_1.
\end{aligned}$$

To transfer the first bracket on the right-hand side of (4.21) to the left with positive constants, we need to choose  $\varepsilon_i, \lambda_i$  so that

$$\lambda_2 c_{\varepsilon_3} (|\nabla f|_{3/2}^2 + |f|_2^3 + 1) < \frac{1}{4} \lambda_1 \quad c \varepsilon_3 \lambda_2 + 2 \varepsilon_4 < 1, \quad \varepsilon_4 + c \varepsilon_5 < \frac{1}{2} \lambda_2,$$

$$\varepsilon_5 < \frac{1}{2}, \quad \lambda_1 c_{\varepsilon_1} + c_{\delta} + \lambda_2 c_{\varepsilon_3} + c_{\delta_1} < m_0 p'(m_0) \lambda / 2.$$

In the choice of the constants  $\lambda_i$  and  $\varepsilon_i$  we proceed as follows :

1. Choose  $\varepsilon_5 = 1/4$  ;
2. Choose  $\lambda_2$  so that  $\lambda_2/2 > c_{\varepsilon_5}$  ;
3. Choose  $\varepsilon_3$  ( $\leq 1/16$ ) so that  $c\varepsilon_3\lambda_2 < 1/2$  ;
4. Choose  $\lambda_1$  so that  $c\lambda_1/4 > \lambda_2 c_{\varepsilon_3} (1 + |\nabla f|_{3/2}^2 + |f|_3^2)$  ;
5. Choose  $\lambda$  so that  $m_0 p'(m_0)\lambda/2 > \lambda_1 c_{\varepsilon_1} + c_\delta + \lambda_2 c_{\varepsilon_3} + c_{\delta_1}$  ;
6. Choose  $\varepsilon_4$  so that  $\varepsilon_4 < \lambda_2/2 - c_{\varepsilon_5}$ ,  $2\varepsilon_4 < 1/2$ .

Note that  $\lambda$  depends on the choice of  $\varepsilon_1$ ,  $\delta$  and  $\delta_1$ . Let us now define a function  $\varphi(t)$  by

$$(4.22) \quad \begin{aligned} \varphi(t) = & \frac{1}{2} \lambda_1 |\sqrt{\rho(t)} u(t)|_2^2 + \frac{1}{2} |\nabla u(t)|_2^2 + \frac{1}{2} |\nabla \cdot u(t)|_2^2 \\ & + \frac{1}{2} \lambda_2 |\sqrt{\rho(t)} \partial_t u(t)|_2^2 + \frac{1}{4} \lambda |\sigma(t)|_4^4. \end{aligned}$$

Then from (4.21) it can be easily seen that there exist a constant  $c_0 > 0$ , a continuous increasing function  $c_1(\varphi)$  on  $[0, k^{-1})$  with  $\lim_{\varphi \rightarrow 1/k-0} c_1(\varphi) = +\infty$ , where  $k$  is the constant from (4.5), an integer  $m$  and numbers  $a_i > 1$ ,  $i=1, \dots, m$ , such that

$$(4.23) \quad \varphi'(t) + c_0 \varphi(t) - \sqrt{\lambda} c_1(\varphi(t)) \sum_{i=1}^m \varphi(t)^{a_i} \leq c_2,$$

$c_2$  being given by

$$(4.24) \quad \begin{aligned} c_2 = & c\lambda_1 (|f|_2^2 + |f|_3^3 + \varepsilon_1) + \delta + |f|_2^{4/3} + c_{\varepsilon_3} \lambda_2 [|\partial_t f|_{5/5}^2 + \\ & + |\partial_t f|_{5/5}^3 + (|f|_3^2 + |\nabla f|_{3/2}^2)^2] + c_{\varepsilon_4} \lambda |f|_4^4 + \\ & + c(|f|_2^4 + |f|_2^{10} + |f|_4^2) + \delta_1. \end{aligned}$$

Now we can apply the argument as, e.g., in [31], Lemma 4.10. It is clear that there exist a  $\gamma \in (0, k^{-1})$ ,  $\varepsilon_1$ ,  $\delta$ ,  $\delta_1$  and  $\sup_{0 \leq t < \infty} \{\|f(t)\|_1, |\partial_t f(t)|_2\}$  sufficiently small such that

$$(4.25) \quad \begin{aligned} c_0 \gamma - \sqrt{\lambda} c_1(\gamma) \sum_{i=1}^m \gamma^{a_i} & > 0, \\ c_2 - c_0 \gamma + \sqrt{\lambda} c_1(\gamma) \sum_{i=1}^m \gamma^{a_i} & < 0. \end{aligned}$$

Take the initial data so small that  $\varphi(0) = \varphi_0 \leq \gamma$  and then prove that

$$(4.26) \quad \varphi(t) \leq \gamma \quad \text{for all } t \text{ for which } \varphi(t) \text{ exists.}$$

Suppose that the contrary is true. Then there exists a  $t^*$  such that  $\varphi(t^*) > \gamma$ . Define

$$\bar{t} = \inf \{t > 0; \varphi(t) > \gamma\}.$$

Clearly  $\varphi(\bar{t}) = \gamma$  and from (4.23), (4.25) we get  $\varphi'(\bar{t}) < 0$ . But this implies  $\varphi(t_1) < \gamma$  for some  $t_1 > \bar{t}$ , which is a contradiction to the definition of  $\bar{t}$ .

So for the solution from Theorem 2 with sufficiently small data we have proved (see (4.26), (4.22))

$$(4.27) \quad \sup_{0 \leq t \leq T(r_0, m_0)} \{|\sqrt{\rho(t)}u(t)|_2, |\nabla u(t)|_2, |\nabla \cdot u(t)|_2, \\ |\sqrt{\rho(t)}\partial_t u(t)|_2, |\sigma(t)|_4\} < \infty.$$

## 5. The global existence theorem

In this section we shall prove the theorem 3 on the base of the global estimates which have been derived in Sec. 4. In fact, in Sec. 4 we have almost proved the following lemma.

LEMMA 5.1. *Let  $u(t), \rho(t)$  be a solution from Theorem 2 on  $(0, T)$ ,  $T > 0$  with  $r_0 + \sup_{0 \leq t < T} \{\|f(t)\|_1, |\partial_t f(t)|_2\}$  sufficiently small and  $q \in (3, 6]$ . Then*

$$(5.1) \quad u \in L^\infty(0, T; H^2(\Omega)); \quad \partial_t u \in L^\infty(0, T; L^2(\Omega));$$

$$(5.2) \quad 0 < \alpha \leq \rho(x, t) \leq \beta < \infty; \quad ((x, t) \in Q_T), \nabla \rho \in L^\infty(0, T; L^q(\Omega));$$

$$(5.3) \quad \partial_t \rho \in L^\infty(0, T; L^q(\Omega)).$$

PROOF OF LEMMA. Take the  $\gamma$  from (4.25), and choose  $r_0$  so small that  $\varphi(0) = \varphi_0 \leq \gamma$ . Then (4.26) holds. From (4.26) and (4.5) the relations (5.2) are obtained immediately. Further, by (4.26) and (5.2) we have  $u \in L^\infty(0, T; H_0^1(\Omega))$ ,  $\partial_t u \in L^\infty(0, T; L^2(\Omega))$ . Hence by (4.14),  $u \in L^\infty(0, T; H^2(\Omega))$ . Since by the equation of continuity we have

$$|\partial_t \rho|_q \leq |u|_\infty |\nabla \rho|_q + |\rho|_\infty |\nabla \cdot u|_q,$$

the relation (5.3) is a consequence of (5.1), (5.2), which have already been proved.

PROOF OF THEOREM 3. Let  $u_0, \rho_0, f$  be as assumed. Take  $\gamma \in (0, k^{-1})$  with the constant  $k$  from (4.5) and such that

$$c_0 \gamma - \sqrt{\lambda} c_1(\gamma) \sum_{i=1}^m \gamma^{\alpha_i} > 0.$$



Further, take  $\varepsilon_1, \delta_1, \delta$  and  $\sup_{0 \leq t < \infty} \{\|f(t)\|_1, |\partial_t f(t)|_2\}$  so small that (4.25) holds with the constant  $c_2$  given by (4.25). Then we take the data  $u_0, \rho_0$  such that  $\varphi_0 = \varphi(0) \leq \gamma$ , where  $\varphi(t)$  is given by (4.22). By Theorem 2 for  $r_0 = \|u_0\|_2 + |\nabla \rho_0|_q$  there is a  $T = T(r_0, m_0)$  such that the solution exists on  $(0, T)$ . Moreover, let  $T$  be a maximal number for which there exists a solution of (2.1)–(2.3) on  $(0, T)$  of the class specified in Theorem 2. Suppose that  $T < \infty$ . Then there exist  $T_n > 0$  such that  $T_n \nearrow T$  and, by (4.25),  $\varphi(T_n) \leq \gamma$  for all  $n$ . Hence there exists an  $r_1 > 0$  independent of  $n$  such that  $\|u(T_n)\|_1 + |\nabla \rho(T_n)|_q \leq r_1$  for all  $n$ .

According to the Theorem 2 there exists an  $\eta > 0$  independent of  $n$  such that for any  $n$  the solution exists on  $[T_n, T_n + \eta)$ . But this implies  $T_{n_0} + \eta > T$  for some  $n_0$ , which is a contradiction to the definition of  $T$ .

## 6. Behavior of the global solution as $t \rightarrow \infty$

In this section we shall prove the Theorem 4. We use an accommodated version of the procedure which have been used in [15] to prove a similar result in one dimension (in fact, it is difficult to compare the result proved in [15] and Theorem 4 since in [15] the data need not be small, but the first estimate in (5.2) is assumed a priori).

PROOF OF THEOREM 4. By the Theorem 3, under the conditions of Theorem 4, there exists a global solution  $(u(t), \rho(t))$  of (2.1)–(2.3) and satisfies the global estimate (4.26) with  $\varphi$  given by (4.22) for all  $t \in [0, \infty)$ . First, let us prove

$$(6.1) \quad u(t) \rightarrow 0 \text{ strongly in } L^2(\Omega) \text{ as } t \rightarrow \infty.$$

Proceeding as in deriving the estimate (4.8), we obtain

$$(6.2) \quad \frac{1}{2} d_t \int_{\Omega} \rho |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla \cdot u|^2 dx - \int_{\Omega} p(\rho) \nabla \cdot u dx = \int_{\Omega} \rho f \cdot u dx$$

Denote by  $P(r)$  ( $r > 0$ ) a positive function such that  $d_r^2 P(r) = r^{-1} d_r p(r)$ ,  $r > 0$ . Then from the second equation in (2.1) we have

$$(6.3) \quad \int_{\Omega} \nabla p(\rho) \cdot u dx = \int_{\Omega} P''(\rho) \nabla \rho \cdot \rho u dx = - \int_{\Omega} P'(\rho) \nabla \cdot (\rho u) dx = d_t \int_{\Omega} P(\rho) dx.$$

Besides, we have also

$$(6.4) \quad \int_{\Omega} \rho f \cdot u dx = - \int_{\Omega} \nabla g \cdot \rho u dx = \int_{\Omega} g \nabla \cdot (\rho u) dx = - d_t \int_{\Omega} g \rho dx.$$

From (6.2), (6.3) and (6.4) we find

$$(6.5) \quad d_t \int_{\Omega} \left[ \frac{1}{2} \rho |u|^2 + P(\rho) + g\rho \right] dx + \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla \cdot u|^2 dx = 0.$$

Integrating (6.5) with respect to  $t$  from 0 to  $t$ , we get

$$(6.6) \quad \begin{aligned} & \frac{1}{2} |\sqrt{\rho(t)} u(t)|_2^2 + \int_{\Omega} P(\rho(t)) dx + \int_{\Omega} g\rho(t) dx + \int_{\Omega} |\nabla u(s)|_2^2 ds + \\ & + \int_0^t |\nabla \cdot u(s)|_2^2 ds = \frac{1}{2} |\sqrt{\rho_0} u_0|_2^2 + \int_{\Omega} P(\rho_0) dx + \int_{\Omega} g\rho_0 dx, \quad t > 0. \end{aligned}$$

Since  $g$  is determined up to an additive constant, we can assume  $g(x) \geq 0$ ,  $x \in \Omega$ . Then (6.6) implies

$$(6.7) \quad \begin{aligned} & \sqrt{\rho} u \in L^\infty(0, \infty; L^2(\Omega)); \quad P(\rho) \in L^\infty(0, \infty; L^1(\Omega)); \\ & g\rho \in L^\infty(0, \infty; L^1(\Omega)); \quad u \in L^\infty(0, \infty; L^2(\Omega)); \\ & \nabla u \in L^2(0, \infty; L^2(\Omega)); \quad \nabla \cdot u \in L^2(0, \infty; L^2(\Omega)). \end{aligned}$$

In particular, since  $\int_0^\infty |\nabla u|_2^2 dt < \infty$ , denoting  $\sigma(t) = \left( \int_{t-1}^t |\nabla u(s)|_2^2 ds \right)^{1/2}$ , we have

$$(6.8) \quad \lim_{t \rightarrow \infty} \sigma(t) = 0.$$

Let us now integrate (6.2) with respect to  $t$  from  $s$  to  $t$  ( $0 \leq s \leq t < \infty$ ) and the result integrate with respect to  $s$  from  $t-1$  to  $t$ . We obtain

$$(6.9) \quad \begin{aligned} & \frac{1}{2} |\sqrt{\rho(t)} u(t)|_2^2 + \int_{t-1}^t \int_s^t |\nabla u(\tau)|_2^2 d\tau ds + \int_{t-1}^t \int_s^t |\nabla \cdot u(\tau)|_2^2 d\tau ds \\ & = \frac{1}{2} \int_{t-1}^t |\sqrt{\rho(s)} u(s)|_2^2 ds + \int_{t-1}^t \int_s^t \int_{\Omega} [\rho f \cdot u + p(\rho) \nabla \cdot u] dx d\tau ds. \end{aligned}$$

Estimating the right-hand side of (6.9) with the help of (5.2), we find

$$\begin{aligned} & \left| \int_{t-1}^t \int_s^t \int_{\Omega} \rho f \cdot u dx d\tau ds \right| \leq \\ & \leq |\rho|_\infty |f|_2 \left( \int_{t-1}^t \int_{\Omega} |u|^2 dx d\tau \right)^{1/2} \leq \\ & \leq c |f|_2 \sigma(t), \end{aligned}$$

and, quite analogously,

$$\left| \int_{t-1}^t \int_s^t \int_{\Omega} p(\rho) \nabla \cdot u dx d\tau ds \right| \leq c \sup_{\alpha \leq r \leq \beta} p(r) \sigma(t)$$

with a suitable constant  $c$  of imbedding. Thus, (6.9) and (5.2) yield

$$\|u(t)\|_2^2 \leq c(1 + \|f\|_2) \sigma(t),$$

from which (6.1) follows immediately.

Define a function  $w(x, t)$  on  $Q_{\infty}$  as the solution of the problem

$$\begin{aligned} \Delta w &= \nabla \cdot (\rho f) \quad \text{in } Q_{\infty}, \\ \frac{\partial w}{\partial n} &= \rho f \cdot n \quad \text{on } \partial \Omega \times [0, \infty), \\ \int_{\Omega} w dx &= 0. \end{aligned} \tag{6.10}$$

Now, let us prove

$$\rho(t) \rightarrow \bar{\rho} \quad \text{in } L^2(\Omega) \quad \text{as } t \rightarrow \infty, \tag{6.11}$$

where

$$\begin{aligned} \nabla p(\bar{\rho}) &= \bar{\rho} f \quad \text{in } \Omega, \\ \int_{\Omega} \bar{\rho} dx &= \int_{\Omega} \bar{\rho}_0 dx. \end{aligned} \tag{6.12}$$

As a first step, we shall prove

$$\left| \int_{\Omega} [p(\rho(t)) - w(t)](h - Mh) dx \right| \leq c \sigma(t) \|h\|_2, \quad h \in L^2(\Omega). \tag{6.13}$$

with a constant  $c$  independent of  $h$ , where we have denoted

$$Mh = (\text{meas } \Omega)^{-1} \int_{\Omega} h dx.$$

To this purpose we shall estimate at first the integral

$$I(t) = \int_{t-1}^t \varphi(s-t) \int_{\Omega} [p(\rho(s)) - w(s)](h - Mh) dx ds,$$

where  $\varphi \in C_0^{\infty}(-1, 0)$  is a fixed function such that  $\int_{-1}^0 \varphi(\tau) d\tau = 1$ . Let  $\phi$  be a solution of the problem ( $h \in L^2(\Omega)$ , arbitrary)

$$\Delta \phi = h - Mh \quad \text{in } \Omega,$$

$$(6.14) \quad \frac{\partial \phi}{\partial n} = 0 \quad \text{in } \partial \Omega ,$$

$$\int_{\Omega} \phi dx = 0 .$$

The standard results on boundary-value problems for elliptic equations imply

$$(6.15) \quad \|\phi\|_2 \leq c \|h - Mh\|_2$$

with a constant  $c$  independent of  $h$  (note that  $\|h - Mh\|_2 \leq \|h\|_2$ ). With this substitution, we find

$$\begin{aligned} I(t) &= - \int_{t-1}^t \varphi(s-t) \int_{\Omega} [w(s) - p(\rho(s))] \Delta \phi dx ds = \\ &= - \int_{t-1}^t \varphi(s-t) \int_{\Omega} [\rho \partial_s u(s) + \rho(s) u(s) \cdot \nabla u(s) - \Delta u(s) - \nabla(\nabla \cdot u(s))] \cdot \nabla \phi dx ds \\ &= - \int_{t-1}^t \varphi'(s-t) \int_{\Omega} \rho(s) u(s) \cdot \nabla \phi dx ds - \int_{t-1}^t \varphi(s-t) \int_{\Omega} \rho(s) u_i(s) u_j(s) \partial_i \partial_j \phi dx ds + \\ &\quad + \int_{t-1}^t \varphi(s-t) \int_{\Omega} \partial_i u_j(s) \partial_i \partial_j \phi dx ds + \int_{t-1}^t \varphi(s-t) \int_{\Omega} (\nabla \cdot u(s)) (h - Mh) dx ds \leq \\ &\leq \|\rho\|_{L^\infty(Q_\infty)} \left( \int_{-1}^0 \varphi'(\tau)^2 d\tau \right)^{1/2} \|\nabla \phi\|_2 \cdot \left( \int_{t-1}^t |u(s)|_2^2 ds \right)^{1/2} + \\ &\quad + c \|\rho\|_{L^\infty(Q_\infty)} \left( \int_{-1}^0 \varphi(\tau)^2 d\tau \right)^{1/2} \|\phi\|_2 \left( \int_{t-1}^t |u(s)|_4^2 ds \right)^{1/2} + \\ &\quad + \left( \int_{-1}^0 \varphi(\tau)^2 d\tau \right)_{t-1}^{1/2} \|h - Mh\|_2 \left( \int_{t-1}^t |\nabla \cdot u(t)|_2^2 ds \right)^{1/2} + \\ &\quad + \left( \int_{-1}^0 \varphi(\tau)^2 d\tau \right)^{1/2} \|\phi\|_2 \left( \int_{t-1}^t |\nabla u(s)|_2^2 ds \right)^{1/2} \leq \\ &\leq c \|\varphi\|_1 \|h\|_2 \sigma(t) , \end{aligned}$$

where we have used (6.15). Since  $\varphi$  is a fixed function, we have

$$(6.16) \quad |I(t)| \leq c \sigma(t) \|h\|_2, \quad t \geq 0$$

with some constant  $c$  independent of  $h$  and  $t$ . Next, we shall prove (6.13)

with the help of (6.16). Since  $\int_{-1}^0 \varphi ds = 1$ , we have

$$\begin{aligned}
(6.17) \quad & \left| \int_{\Omega} [p(\rho(t)) - w(t)](h - Mh) dx \right| \leq \\
& \leq |I(t)| + \left| \int_{t-1}^t \varphi(s-t) \int_{\Omega} [p(\rho(t)) - p(\rho(s))](h - Mh) dx ds \right| + \\
& + \left| \int_{t-1}^t \varphi(s-t) \int_{\Omega} [\rho(t) - \rho(s)] f \cdot \nabla \phi dx ds \right|.
\end{aligned}$$

The integral  $I(t)$  is estimated in (6.16). Besides, we have

$$\rho(t) - \rho(s) = \int_s^t \partial_{\tau} \rho(\tau) d\tau = - \int_s^t \nabla \cdot (\rho(\tau) u(\tau)) d\tau.$$

Hence

$$\begin{aligned}
(6.18) \quad & \left| \int_{t-1}^t [\varphi(s-t) \int_{\Omega} [\rho(t) - \rho(s)] f \cdot \nabla \phi dx ds \right| = \\
& = \left| \int_{t-1}^t \int_s^t \varphi(s-t) \int_{\Omega} \rho(\tau) u(\tau) \cdot \nabla (f \cdot \nabla \phi) dx d\tau ds \right| \leq \\
& \leq |\rho|_{L^{\infty}(\mathcal{Q}_{\infty})} |\varphi|_2 \|\nabla f\|_{3/2} \|\nabla \phi\|_6 + c \|f\|_3 \|\phi\|_2 \left( \int_{t-1}^t |u(s)|_6^2 ds \right)^{1/2} \leq \\
& \leq c \|f\|_{1,3/2} \|\phi\|_2 \sigma(t).
\end{aligned}$$

The second term in (6.17) is estimated similarly as above. Write it in the form

$$\begin{aligned}
(6.19) \quad & \int_{t-1}^t \varphi(s-t) \int_{\Omega} [p(\rho(t)) - p(\rho(s))](h - Mh) dx ds = \\
& = \int_{t-1}^t \varphi(s-t) \int_{\Omega} d_{\tau} p(\rho(\tau)) (h - Mh) d\tau ds dx.
\end{aligned}$$

Multiplying the equation (2.1)<sub>2</sub> by  $p'(\rho)$ , we get

$$\partial_t p(\rho) = -p'(\rho) u \cdot \nabla \rho - \rho p'(\rho) \nabla \cdot u.$$

This, together with (5.2), yields in (6.19)

$$\begin{aligned}
(6.20) \quad & \left| \int_{t-1}^t \varphi(s-t) \int_s^t \int_{\Omega} d_{\tau} p(\rho(\tau)) (h - Mh) dx d\tau ds \right| \leq \\
& \leq \left| \int_{t-1}^t \varphi(s-t) \int_s^t \int_{\Omega} p'(\rho(\tau)) u(\tau) \cdot \nabla \rho(\tau) (h - Mh) dx d\tau ds \right| + \\
& + \left| \int_{t-1}^t \varphi(s-t) \int_s^t \int_{\Omega} \rho(\tau) p'(\rho(\tau)) \nabla \cdot u (h - Mh) dx d\tau ds \right| \leq
\end{aligned}$$

$$\begin{aligned}
&\leq c|\varphi|_{\infty}\sigma(t)\left(\int_{t-1}^t|\nabla\rho(s)|_3^2ds\right)^{1/2}\sup_{\alpha\leq r\leq\beta}p'(r)|h|_2+ \\
&\quad +c|\varphi|_2\sup_{\alpha\leq r\leq\beta}rp'(r)|h|_2\sigma(t)\leq \\
&\leq c\sigma(t)|h|_2.
\end{aligned}$$

Substituting (6.18), (6.19), (6.20) into (6.17), we get (6.13). From (6.13) we have

$$p(\rho(t)) - w(t) - Mp(\rho(t)) \rightarrow 0 \quad \text{in } L^2(\Omega) \text{ as } t \rightarrow \infty.$$

Since (5.2) holds, there exist  $t_n \rightarrow \infty$  such that  $\rho(t_n) \rightarrow \bar{\rho}$  weakly in  $L^2(\Omega)$ . Further, as the function  $w(t)$  is a solution of (6.10) for any  $t$ , the sequence  $\{w(t_n)\}$  is compact in  $L^2(\Omega)$ . Choosing a subsequence converging to  $w_{\infty} \in L^2(\Omega)$ , we find  $w_{\infty} = A(\bar{\rho}f)$ , where  $A$  stands for the solution operator for (6.10). Thus  $p(\rho(t_n)) - Mp(\rho(t_n)) \rightarrow w_{\infty}$  in  $L^2(\Omega)$  and a.e. in  $\Omega$ . By (5.2),  $t_n$  can also be chosen in such a way that  $Mp(\rho(t_n))$  converges to some  $p_{\infty}$ . This yields  $\rho(t_n) \rightarrow p^{-1}(w_{\infty} + p_{\infty})$  a.e. in  $\Omega$  and in  $L^2(\Omega)$  strongly. Hence  $\bar{\rho} = p^{-1}(w_{\infty} + p_{\infty})$ , or  $p(\bar{\rho}) = A(\bar{\rho}f) + p_{\infty}$ . This is equivalent to

$$\int_{\Omega} [\nabla p(\bar{\rho}) - \bar{\rho}f] \cdot \nabla \theta dx = 0 \quad \text{for any } \theta \in C^{\infty}(\Omega), \quad M\theta = 0,$$

which yields

$$(6.21) \quad \nabla p(\bar{\rho}) = (Id - R)(\bar{\rho}f),$$

where  $R$  is the projection of  $L^2(\Omega)$  onto the closure in  $L^2(\Omega)$  of the space of divergence free vector functions.

Now, prove that  $R(\bar{\rho}f) = 0$ . To this purpose, let us take an arbitrary  $\theta \in C_0^{\infty}(\Omega)$ ,  $\eta = R\theta$ . Then we have

$$\int_{\Omega} R(\rho(s)f) \cdot \theta dx = \int_{\Omega} [\rho(s)\partial_s u(s) + \rho(s)u(s) \cdot \nabla u(s) - \Delta u(s) - \nabla(\nabla \cdot u(s))] \cdot \eta dx.$$

Similarly as above in deriving (6.16), (6.17), we can prove

$$\left| \int_{t-1}^t \dot{\varphi}(s-t) \int_{\Omega} R(\rho(s)f) \cdot \theta dx ds \right| \leq c|\varphi'|_2 \|\eta\|_1 \sigma(t), \quad t \geq 0,$$

from which

$$(6.22) \quad \left| \int_{\Omega} R(\rho(t)f) \cdot \theta dx \right| \leq c|\varphi'|_2 \|\eta\|_1 \sigma(t), \quad t \geq 0.$$

Since  $\|\eta\|_1 \leq c\|\theta\|_1$ , from (6.22) we get  $R(\rho(t)f) \rightarrow 0$  as  $t \rightarrow \infty$  in  $H^{-1}(\Omega)$  strongly. Then we have, as  $t_n \rightarrow \infty$ ,  $R(\rho(t_n)f) \rightarrow 0$  in  $H^{-1}(\Omega)$  and  $R(\rho(t_n)f) \rightarrow R(\bar{\rho}f)$  in

$L^2(\Omega)$ .

Hence  $R(\bar{\rho}f)=0$ . Therefore, by (6.21) we obtain the first relation in (6.12) with  $\alpha \leq \bar{\rho} \leq \beta$  a. e. in  $\Omega$ . Now, if  $\pi$  is such a function that  $\pi'(r)=r^{-1}p'(r)$ , then

$$\nabla \pi(\bar{\rho}) = \nabla g.$$

Hence we have  $\pi(\bar{\rho})=g+k$  with some constant  $k$ , or  $\bar{\rho}=\pi^{-1}(g+k)$ . As we have

$$\int_{\Omega} \rho(x, t_n) dx = \int_{\Omega} \rho_0(x) dx,$$

we have also

$$\int_{\Omega} \bar{\rho}(x) dx = \int_{\Omega} \rho_0(x) dx.$$

Since the function  $a(k)=\int_{\Omega} \pi^{-1}(g+k) dx$  is increasing, the function  $\bar{\rho}$  is uniquely determined by a unique constant  $k^*$  satisfying

$$\int_{\Omega} \pi^{-1}(g+k^*) dx = \int_{\Omega} \rho_0 dx.$$

(6.11) is thus proved. It is elementary from  $p'(\bar{\rho})\nabla \bar{\rho} = \bar{\rho}f$  that  $\bar{\rho} \in H^{2,3/2}(\Omega)$ . The proof of Theorem 4 is complete.

## 7. Proof of Theorem 5

First, to prove the existence of a solution of (3.8), we consider an approximation  $\rho_0^\varepsilon$  of  $\rho_0$  such that

$$\rho_0^\varepsilon \in H^2(\Omega), \quad 0 < \varepsilon \leq \rho_0^\varepsilon \leq 1, \quad \rho_0^\varepsilon \rightarrow \rho_0, \quad \text{as } \varepsilon \rightarrow 0,$$

strongly in  $H^2(\Omega)$ , and  $\rho_0 \leq \rho_0^\varepsilon$ .

Then there exists a solution of (3.8) for the initial density  $\rho_0^\varepsilon$ . In what follows, we denote simply by  $(u, \rho)$  the solution of (3.8) with the data  $\rho_0^\varepsilon, u_0$ . We notice that (3.21) holds with  $\alpha = \varepsilon$ .

Now we consider (3.12) in the following form

$$(7.1) \quad \begin{aligned} & \frac{1}{2} |\nabla u(t)|_2^2 + \frac{1}{4} |\nabla \cdot u(t)|_2^2 + \frac{2}{3} \int_0^t |\sqrt{\rho} \partial_\tau u|_2^2 d\tau \leq \frac{1}{2} |\nabla u(0)|_2^2 + \\ & + \frac{1}{2} |\nabla \cdot u(0)|_2^2 + c_\delta t \int_0^t |\partial_\tau p|_2^2 d\tau + \delta \left( \sup_{0 \leq \tau \leq t} |\nabla u|_2 \right)^2 + |p(\rho(t))|_2^2 \end{aligned}$$

$$+ |p(\rho(0))|_2^2 + |\nabla \cdot u(0)|_2^2 + c \sup_{0 \leq \tau \leq t} |\rho|_\infty \int_0^t (|\bar{u}|_6^2 |\nabla \bar{u}|_3^2 + |f|_2^2) d\tau,$$

whence ( $\delta < 1/4$ )

$$\sqrt{\rho} \partial_t u \in L^2(Q_T).$$

Next the integration of (3.14) with respect to  $t$  implies

$$\begin{aligned} & \sup_{0 \leq \tau \leq t} |\sqrt{\rho} \partial_\tau u|_2^2 + \int_0^t |\nabla \partial_\tau u|_2^2 d\tau \leq \\ & \leq |\sqrt{\rho_0} \partial_t u|_2^2 + c \sup_{0 \leq \tau \leq t} |\rho|_\infty^2 \int_0^t (|\bar{u}|_\infty^2 |\nabla \bar{u}|_2^2 |\nabla \bar{u}|_3^2 + \\ & + |\bar{u}|_6^2 |\bar{u}|_6^2 |\nabla(\nabla \bar{u})|_2^2) d\tau + c \int_0^t (|\rho|_\infty |\sqrt{\rho} \partial_\tau \bar{u}|_2^2 |\nabla \bar{u}|_3^2 + |\sqrt{\rho}|_\infty |\bar{u}|_\infty \cdot \\ & \cdot |\nabla \partial_\tau \bar{u}|_2 |\sqrt{\rho} \partial_\tau u|_2 + |\sqrt{\rho} \bar{u}|_\infty^2 |\sqrt{\rho} \partial_\tau u|_2^2 + |\sqrt{\rho}|_\infty |\partial_\tau f|_2 |\sqrt{\rho} \partial_\tau u|_2 + \\ (7.2) \quad & + |\partial_\tau p|_2^2 + |\partial_\tau \rho|_2^2 |f|_3^2) d\tau \leq \\ & \leq c(|u_0 \cdot \nabla u_0|_2^2 + |(\Delta u_0 + \nabla(\nabla \cdot u_0) + \nabla p(\rho_0))/\sqrt{\rho_0}|_2^2 + |f(0)|_2^2) + \\ & + c \sup_{0 \leq \tau \leq t} |\rho|_\infty^2 \int_0^t (|\bar{u}|_\infty^2 |\nabla \bar{u}|_2^2 |\nabla \bar{u}|_3^2 + |\bar{u}|_6^4 |\nabla(\nabla \bar{u})|_2^2) d\tau + \\ & + c \sup_{0 \leq \tau \leq t} |\rho|_\infty \int_0^t (|\bar{u}|_\infty |\nabla \partial_\tau \bar{u}|_2 |\sqrt{\rho} \partial_\tau u|_2 + |\bar{u}|_\infty^2 |\sqrt{\rho} \partial_\tau u|_2^2 + \\ & + |\sqrt{\rho} \partial_\tau \bar{u}|_2^2 |\nabla \bar{u}|_3^2) dt + c \sup_{0 \leq \tau \leq t} |\sqrt{\rho}|_\infty \int_0^t |\partial_\tau f|_2 |\sqrt{\rho} \partial_\tau u|_2 d\tau + \\ & + c \int_0^t (|\partial_\tau p|_2^2 + |\partial_\tau \rho|_2^2 |f|_3^2) d\tau. \end{aligned}$$

The inequality (7.2) implies

$$(7.3) \quad \partial_t \nabla u \in L^2(Q_T), \quad \sqrt{\rho} \partial_t u \in L^\infty(0, T; L^2(\Omega)).$$

Furthermore, from (3.18), bearing in mind (7.3), we have

$$u \in L^2(0, T; H^3(\Omega)) \cap L^\infty(0, T; H^2(\Omega)).$$

Now let us prove that  $GB \subseteq B$ , where  $B$  is as in the proof of Theorem 1 with

$$r^2 = \tilde{c} e^{\|u_0\|_1} (|(\Delta u_0 + \nabla(\nabla \cdot u_0) + \nabla p(\rho_0))/\sqrt{\rho_0}|_2^2 + |u_0 \cdot \nabla u_0|_2^2 + |f(0)|_2^2).$$

$\tilde{c} > 1$  is a suitable constant.



From (7.2), as in the Theorem 1, for suitable  $\bar{T}$ , we have

$$(7.5) \quad \sup_{0 \leq t \leq \bar{T}} |\sqrt{\rho(t)}u(t)|_2^2 + \int_0^T |\nabla \partial_t u|_2^2 d\tau \leq \frac{1}{10} r^2 e^{-\|u_0\|_1}.$$

Thanks to (3.21) and (7.3), we have

$$\nabla(\rho \partial_t u) = \nabla \rho \partial_t u + \rho \nabla \partial_t u \in L^2(Q_T).$$

Now repeating the calculations to obtain (3.24), we have, for suitable  $\bar{T}$ , that

$$GB \subseteq B.$$

The continuity of  $G$  in  $L^2(Q_T)$  is obtained as in Theorem 1.

Now, the solutions  $u^\varepsilon$  belong to  $B$ , uniformly with respect to  $\varepsilon$ , and  $\rho^\varepsilon$  satisfy (3.21) with  $\alpha=0$  uniformly with respect to  $\varepsilon$ . Hence, passing to the limit  $\varepsilon \rightarrow 0$ , it is routine matter to obtain a solution  $(u, \rho)$  of (2.1).

The uniqueness of the solution is proved in just the same manner as in Theorem 1.

Theorem 5 is completely proved.

*Acknowledgement.* The authors would like to express their thanks to the referee for valuable comments.

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(Received August 5, 1991)

(Revised August 11, 1992)

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