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Nonpositively curved manifolds with small volume

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Abstruct. In this paper we study nonpositively curved manifold with small volume and prove that such a manifold has locally an Euclidean factor as direct summand.

§ 0. Introduction

Roughly speaking, the set of Riemannian manifolds of nonpositive curvature can be devided into two categories, one of rank one and one of higher rank. By the work of Gromov, Ballmann, Burns-Spatzier, etc. the nonpositively curved manifold of higher rank is fairly well understood, namely such a manifold is isometric to a locally symmetric space. In this paper we concern with rank one manifolds of nonpositive curvature. The most important example of rank one nonpositively curved manifolds is a manifold of (strict) negative curvature. But there are families of rank one nonpositively curved manifolds which do not admit metrics of negative curvature. (See Examples 0.4-0.6 below.) In this paper we will study how much is the difference between these two classes. The examples mentioned above are obtained by taking a product of nonpositively curved manifolds and patch them along their boundaries or corners. So one possible question is whether all nonpositively curved manifolds of rank one are obtained in that way. The main result of this paper is a partial answer to this question in case when the supremum of the injectivity radius of the manifold is sufficiently small. K_M stands for the sectional curvature of M and i_M stands for the injectivity radius of M.

THEOREM 0.1. *There exists a positive number* ε_n *depending only on the dimension* n *such that if* M *is a Riemannian manifold with* $-1\leqq K_M\leqq 0$, $\sup i_M<\varepsilon_n$ *then there exists* $V_i\subset M$ *with the following properties.*

(0.1.1) $\cup V_i+M$.

(0.1.2) *The universal covering space of* V_i *is isometric to a direct product* $W_i\times \boldsymbol{R}^k$, *where* k *is positive.*

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(0.1.3) *Each connected component of the inverse image of* V_i *in the universal covering space of* M *is convex.*

REMARK 0.2. Theorem 0.1 is due to V. Schröeder [8] in case the dimension is 3, and to V. Buyalo [2, 3] in case the dimension is 4.

COROLLARY 0.3. *If* M *satisfies the condition of Theorem* 0.1 *then it admits a polalized* F*-structure* (Cheeger-Gromov [4]).

REMARK 0.4. It follows from Corollary 0.3 that the manifold M there has a sequence of Riemannian metrics g_i such that $-1 \leqq K_{M_{g_i}} \leqq 1$ and that the volume of (M, g_i) converges to 0. In case when the dimension is three it is known (by Schröeder) that M admits a metrics g_i which is of nonpositive curvature in addition. In case the dimension is greater than 3 this property is conjectured (by Buyalo) but is not proved.

Next we recall some examples to illustrate the phenomenon dealed with in this paper.

Example 0.5 (Heintz [7]): Let M_1 and M_2 be 3-dimensional manifolds of constant negative curvature. We assume that M_i are complete noncompact with one end. We may assume that their ends are diffeomorphic to T^2. We patch M_1 and M_2 along their ends and obtain M. One can prove that M admits a metric of nonpositive curvature. Since $\pi_1(M)$ contains $\boldsymbol{Z}^2$, it follows that M does not admit a metric of strictly negative curvature.

Example 0.6 (Gromov [6]): Let Σ be a surface of nonpositive curvature. We assume that Σ is noncompact and its end is isometric to the direct product $S^1 \times [0, \infty)$. We take two copies of the direct product $\Sigma \times S^1$, and patch them by the diffeomorphism: $S^1 \times S^1 \to S^1 \times S^1$, $(x, y) \mapsto (y, x)$, to obtain M. It is easy to see that M admits a metric of nonpositive curvature of arbitrary small volume but does not admit a metric of strictly negative curvature. This manifold is a first example of our theorem.

Example 0.7 (Fukaya-Januszkiewicz, see [5]): Let Σ be as in Example 0.6. We take 6 copies $N_1, \cdots, N_6$ of the direct product $\Sigma \times \Sigma \times S^1$. The end of N_1 is diffeomorphic to $M \times S^1 \times [0, \infty)$, where M is as in Example 0.6. Let $N_{i,1}, N_{i,2} \subset N_i$ be a subset diffeomorphic to $\Sigma \times S^1 \times S^1$. We patch, for example, N_1 and N_2 along $N_{1,2}$ and $N_{2,1}$ by the diffeomorphism: $\Sigma \times \Sigma \times S^1 \to \Sigma \times \Sigma \times S^1$, $(x, s, t) \mapsto (x, t, s)$. The manifold N' we obtain has a singularity whose neighborhood if isometric to the product of $S^1 \times S^1$ and the sum of 6 copies of $\{(x, y)|x \geqq 0, y \geqq 0, x, y < \varepsilon\}$. By modifying the metric in a neighborhood of singularities we get a smooth manifold of nonpositive curvature N. We remark that we can make the volume of N arbitrary small. In this case the covering V_i in our main theorem has 13 members.

Examples 0.5–0.7 and our main our main theorem support the following.

CONJECTURE 0.8. Let M be a manifold of nonpositive curvature. Then M admits a decomposition $M = \cup M_i$ such that M_i is diffeomorphic to a manifold with corners and M_i is a product of one of the following manifolds: manifold of strict negative curvature (possibly with (totally geodesic) boundaries and/or cornes): locally symmetric space of higher rank (closed).

The organization of this paper is as follows. In section 1 we recall a construction due to (essentially) Buyalo. In section 2 we prove Theorem 0.1 assuming a lemma, which will be proved in section 3.

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§ 1. Buyalo complex

Let M be a manifold with nonpositive curvature, X be its universal covering space, $\Gamma = \pi_1(M)$ and ε be a sufficiently small positive number which we fix later. For $x \in X$, we put

$$\Gamma_\varepsilon(x) = \text{the group generated by} \quad \{\gamma \in \Gamma | d(x, \gamma x) < \varepsilon\}.$$

By Margulis' Lemma $\Gamma_\varepsilon(x)$ has an Abelian subgroup of finite index. Let $\widetilde{AB(\Gamma)}$ be the set of all commensurability classes of almost Abelian subgroups of Γ. By $AB(\Gamma)$ we denote the set of all conjugacy classes of $\widetilde{AB(\Gamma)}$. We put

$$AB(M) = \{[\Gamma_\varepsilon(x)] | x \in X\} \subset \mathcal{AB}(\Gamma),$$

and let $\widetilde{AB(M)}$ be its inverse image in $\widetilde{AB(\Gamma)}$. Now we define a simplicial complex $\mathcal{B}(M)$ as follows. The set of vertexes is $AB(M)$. The set $[\Gamma_0]$, $\cdots, [\Gamma_k]$, spans a k simplexes in $\mathcal{B}(M)$ if, after changing an order and representatives, we have $\Gamma_0 \subset \cdots \subset \Gamma_k$. This complex is essentially the same as one defined in Buyalo [1, 2]. So let us call it Buyalo complex of M.

Example 1.1. Let M be as in Example 0.5. Then $\mathscr{B}(M)$ is

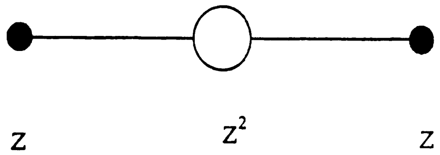

Figure 1.

Example 1.2. Let M be as in Example 0.6, then $\mathscr{B}(M)$ is

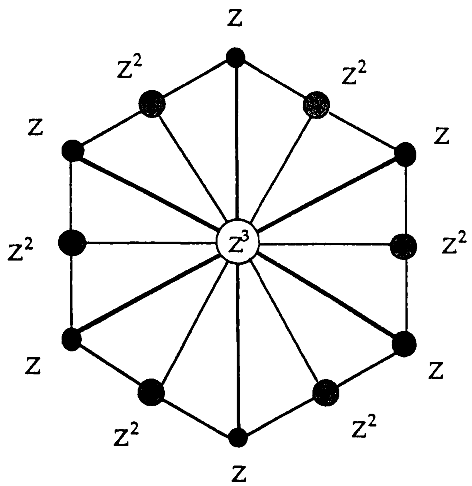

Figure 2.

By these examples it is natural to expect that there is a map from M to $\mathscr{A}\mathscr{B}(M)$. For $\gamma\in\Gamma$, we put

$$c_\gamma=\inf\{d(x,\gamma(x))\mid x\in X\}.$$

For $\Lambda\in\widetilde{AB(M)}$, we put

$$\tilde{C}_\Lambda=\{x\in X\mid\forall\gamma\in\Lambda\, d(\gamma(x),x)=c_\gamma\}$$

By rank Λ we denote the rank of maximal free abelian subgroup of Λ.

For $\Lambda\in\widetilde{AB(M)}, [\Lambda]\in AB(M)$, we put

$$C_{[\Lambda]}=\Gamma\cdot\tilde{C}_\Lambda/\Gamma,$$

$$\widetilde{U_\varepsilon(\Lambda)}=\{x\in X\mid\Gamma_\varepsilon(x)\supset\Lambda\},$$

$$U_\varepsilon([\Lambda])=\Gamma\cdot\widetilde{U_\varepsilon(\Lambda)}/\Gamma$$

Here we remark that Margulis' Lemma implies that $\tilde{C}_\Lambda=\tilde{C}_{\Lambda'}$ if Λ is commensurable to Λ'. Moreover $\gamma\cdot\tilde{C}_\Lambda=C_{\widetilde{\gamma\Lambda\gamma^{-1}}}$. Hence $C_{[\Lambda]}$ is well defined. We put

$$C_k=\bigcup_{\mathrm{rank}\,\Lambda=k}C_{[\Lambda]}\,.$$

$\tilde{C}_k$, $\widetilde{U_\varepsilon(k)}$, $U_\varepsilon(k)$, are defined in a similar way.

§2. Proof of Main theorem

It is easy to see that $\tilde{C}_\Lambda$ is isometric to a direct product $\boldsymbol{R}^k\times\tilde{C}'_\Lambda$, where $k=\mathrm{rank}\,\Lambda$. Thus, the most essential part of the proof of Theorem 0.1 is to show that

$$\bigcup_{\Lambda\in\widetilde{AB(M)}}\tilde{C}_\Lambda=X$$

under our assumption.

Let ε_0 be the Margulis constant.

LEMMA 2.1. *Let $\varepsilon<\varepsilon_0$. If $x\in U_\varepsilon(k)-U_\varepsilon(k+1)$, then there exist a family of geodesics l_x joining x and points in C_k, which depends smoothly on x. The geodesics l_x are contained in $U_\varepsilon(k)-U_\varepsilon(k+1)$.*

PROOF. Choose a lift of x, say $\tilde{x}\in X$. Put $\Lambda=\Gamma_\varepsilon(\tilde{x})$. We have rank $(\Lambda)=k$. Since $\tilde{C}_\Lambda$ is a convex submanifold of X and since X is simply connected and of nonpositive curvature, it follows that there exists a unique minimal geodesic $l:[0.1]\to X$, joining $\tilde{x}$ and C_Λ. Now let $\tilde{x}'$ be another lift of x. We obtain l'. Take $\gamma\in\Gamma$ such $\gamma(\tilde{x})=\tilde{x}'$. It is easy to show that $\gamma(l)=l'$. Hence we can put $l_x=\pi l$ where $\pi:X\to M$ is the projection. To prove that l_x depends continuously on x, we remark that if $x_i\in U_\varepsilon(k)-U_\varepsilon(k+1)$ and if x_i converges to $x\in U_\varepsilon(k)-U_\varepsilon(k+1)$ then we can choose $\tilde{x}_i$ converging to $\tilde{x}$. It follows that $\Gamma_\varepsilon(\tilde{x}_i)=\Gamma_\varepsilon(\tilde{x})$ for sufficiently large i. (Let us remark here that we use Margulis' Lemma essentially in this step.) The lemma follows immediately.

We may assume $l_x(0)=x, l(1)\in C_k$. We define $\phi'_{k,t}:U_\varepsilon(k)-U_\varepsilon(k+1)\to M$ by

$$\phi'_{k,t}(x)=l_x(t)\,.$$

Roughly speaking, the next lemma asserts that we can perturb $\phi'_{k,t}$ so that they are patched together to give a deformation retract from M to $\cup C_k$.

MAINLEMMA 2.2. *There exists* ε_k *independent of* M *such that there exists a continuous family of smooth maps* $\phi_t : M \to M$ *with the following properties.*

$$\phi_0(x) = x . \tag{2.2.1}$$

$$\phi_1(x) \in C_k \quad \text{if} \quad x \in U_{\varepsilon_k}(k) . \tag{2.2.2}$$

The proof of Lemma 2.2 is differed to the next section.

Now we are in the position to prove $\cup C_k = M$. We consider only the case when M is compact. In case it is noncompact we can just use homology of compact support in place of usual homology. We may assume that M is connected. Then $H_n(M; \boldsymbol{Z}_2) = \boldsymbol{Z}_2$ and $H_n(Y; \boldsymbol{Z}_2) = 0$ for an arbitrary proper subset of M. On the other hand, Lemma 2.2 implies that $\cup C_k$ is a deformation retract of M. It follows that $\cup C_k = M$ as required.

Now we are going to prove Theorem 0.1. As V_i we take connected components of $C_k - C_{k+1}$, $k = 1, 2, \cdots, n$. We have already proved (0.1.1). (0.1.2) and (0.1.3) follow immediately from the definition of V_i. The proof of Theorem 0.1 is completed assuming Lemma 2.2.

§3. The main lemma

We prove Lemma 2.2 by an induction on k. We take $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n$, where ε_k depends on $\varepsilon_i, i > k$ and ε_n is the Margulis constant. We put

$$\hat{U}_k = \{\pi(x) \in M \,|\, \operatorname{rand} \Gamma_{\varepsilon_k}(x) \geqq k\} ,$$

and will construct $\phi_t^{(k)} : \hat{U}_k \to \hat{U}_k$ by an induction on k. Note that our assumption implies $\cup \hat{U}_k = M$. Let m is the maximal number such that $\hat{U}_m$ is nonempty. For simplicity, we construct $\phi_t^{(m)}$ and $\phi_t^{(m-1)}$ only. The rest of the construction is quite similar, and is omitted. We put

$$U_k = U_{\varepsilon_k}(k) .$$

First we remark that $\hat{U}_m = U_m$. So we may take $\phi_t^{(m)} = \phi'_{m,t}$.

To define $\phi_t^{(m-1)}$, we choose ε_{m-1} so that if $x \in U_{m-1}$ then the geodesic $\phi'_{m-1,t} | t \in [0, 1]\}$ is contained in $\hat{U}_m \cup U_{m-1}$. (See Figure 3 below.)

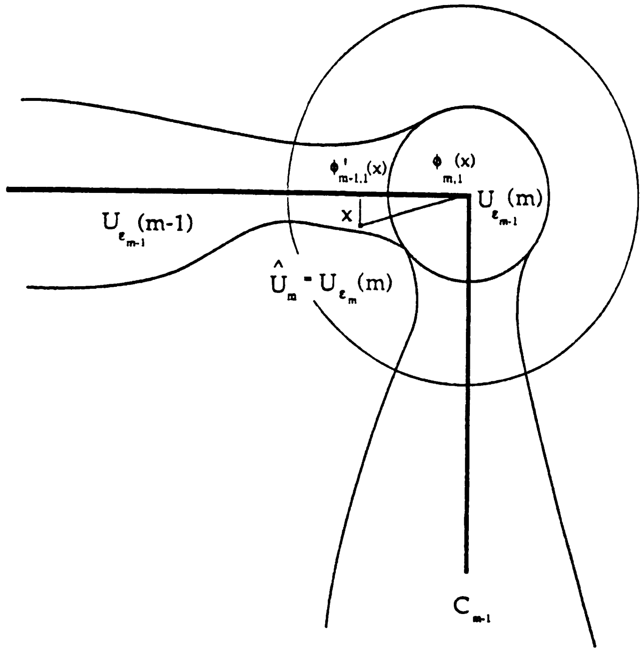

Figure 3.

Let $x \in \hat{U}_m \cap U_{m-1}$. Then by changing ε_{m-1} if necessary we may assume that there exists a geodesic μ_x in C_{m-1} joining $\phi'_{m-1,1}(x)$ and $\phi_{m,1}(x)$. We may assume that μ_x depends smoothly on x, and $\mu_x(0) = \phi'_{m-1,1}(x)$, $\mu_x(1) = \phi'_{m,1}(x)$. We may assume also that there exists a family geodesics $l_{s,x}$ joining x and $\mu_x(s)$ which depends smoothly on s and $x \in \hat{U}_m \cap U_{m-1}$ and which is contained in $\hat{U}_m$. We choose a smooth function χ on $\hat{U}_m \cup U_{m-1}$ such that

$$\chi(x) \begin{cases} = 0 & \text{if } x \in U_{m-1} - \hat{U}_m, \\ = 1 & \text{if } x \notin U_{\varepsilon_{m-1}}(m-1), \\ \in [0,1] & \text{everywhere.} \end{cases}$$

Now we define $\phi_t^{(m-1)}$ by

$$\phi_t^{(m-1)}(x) = \begin{cases} \phi'_{m-1,t}(x) & \text{if } x \in U_{m-1} - \hat{U}_m \\ \phi_t^{(m)}(x) & \text{if } x \notin U_{\varepsilon_{m-1}}(m-1) \\ l_{\chi(x),x}(t) & \text{otherwise}. \end{cases}$$

It is easy to see that this map has required properties. The proof of Lemma 2.2 is now completed.

References

[1] Ballmann, W., Gromov, M. and V. Schöeder, "Manifolds of nonpositive curvature", Birkhäuser, Boston, 1985.

[2] Buyalo, V., Collapsing manifolds of nonpositive curvature I, Leningrad Math. J. 1 (1990), 1135-1155.

[3] Buyalo, V., Collapsing manifolds of nonpositive curvature II, Leningrad Math. J. 1 (1990), 1371-1399.

[4] Cheeger, J. and M. Gromov, Collapsing Riemannian manifold while keeping their curvature bounded I, J. Differential Geom. 23 (1986), 309-346.

[5] Fukaya, K., Hausdorff Convergence of Riemannian manifolds and its applications, Advanced studies in Pure Math. 18-I (1990), 143-238, North Holland.

[6] Gromov, M., Manifolds of negative curvature, J. Differential Geom. 13 (1978), 223-230.

[7] Heintze, E., Mannigfaltigkeiten negativier Krümmung, Habilitationsschrift, Univ. Bonn.

[8] Schröeder, V., Rigidity of nonpositively curved graph manifolds, Math. Ann. 274 (1986), 19-26.

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