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Solvability of Mizohota and Lewy operators

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Abstract. The Mizohata operator and the Lewy operator have solutions in $C^{\infty}(\boldsymbol{R}_t; \mathcal{G}'(\boldsymbol{R}))$ and $C^{\infty}(\boldsymbol{R}^2; \mathcal{G}'(\boldsymbol{R}_t))$ respectively, where $\mathcal{G}'$ denotes the strong dual space of $\mathcal{G}$ which is the space of real analytic and exponentially decreasing functions. In fact, this space is much smaller than the Aronszajn space.

Introduction.

It is well known that the Mizohata equation

$$\frac{\partial u}{\partial t}+it\frac{\partial u}{\partial x}=f, \qquad f\in C_0^{\infty}(\boldsymbol{R}^2)$$

and the Lewy equation

$$\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}+2i(x+iy)\frac{\partial u}{\partial t}=f, \qquad f\in C_0^{\infty}(\boldsymbol{R}^3)$$

have no solution in the space $\mathcal{D}'$ of distributions, or in the space B of hyperfunctions (see [10] for the historical backgrounds). But, Baouendi [1] showed that the Mizohata equation has a solution in the Aronszajn space of traces of heat kernels. In this paper, we deal with the space $\mathcal{G}(\boldsymbol{R}^n)$ of real analytic and exponentially decreasing functions, which turns out to be invariant under the Fourier transformation in Section 1. Also we show that the differential operators of infinite order act on $\mathcal{G}$ continuously. In Section 2 applying the above result in Section 1 on the Fourier transformation only we show that the Mizohata operator and the Lewy operator have solutions in $C^{\infty}(\boldsymbol{R}_t; \mathcal{G}'(\boldsymbol{R}))$ and $C^{\infty}(\boldsymbol{R}^2; \mathcal{G}'(\boldsymbol{R}_t))$ respectively, where $C^{\infty}(X; Y)$ denotes the space of Y-valued C^{∞} functions on X and $\mathcal{G}'$ denotes the strong dual space of $\mathcal{G}$.

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§ 1. The space $\mathcal{G}(\boldsymbol{R}^n)$ and its dual.

We introduce here the space of real analytic and exponentially decreasing functions and its strong dual. Moreover, we show that the Fourier transformation is an isomorphism of these spaces.

DEFINITION 1.1. We denote by $\mathcal{G}$ or $\mathcal{G}(\boldsymbol{R}^n)$ the set of all $\phi \in C^\infty(\boldsymbol{R}^n)$ such that for any k, $h>0$

$$|\phi|_{k,h} = \sup_{\substack{x\in \boldsymbol{R}^n \\ \alpha \in \boldsymbol{N}_0^n}} \frac{|\partial^\alpha \phi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty \tag{1.1}$$

where $\boldsymbol{N}_0$ is the set of all nonnegative integers. The topology in $\mathcal{G}$ defined by the semi-norms in (1.1) makes $\mathcal{G}$ a Frechet space. In fact, it is the projective limit topology over all $h>0$ and $k>0$.

Furthermore, the space $\mathcal{G}$ is a nuclear Fréchet space and therefore reflexive. We note that every function ϕ in $\mathcal{G}$ is real analytic by the Pringsheim's theorem.

Because of the growth condition (1.1) the space $\mathcal{G}$ is a subspace of the Schwartz space $\mathcal{S}$. Thus we can define the Fourier transformation of $\phi \in \mathcal{G}$ by

$$\hat{\phi}(\xi) = \int_{\boldsymbol{R}^n} e^{-i\langle x,\xi\rangle} \phi(x)dx \tag{1.2}$$

and the Fourier inversion formula

$$\phi(x) = \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{i\langle x,\xi\rangle}\hat{\phi}(\xi)d\xi \tag{1.3}$$

holds.

Also,

$$\int \hat{\phi}\psi dx = \int \phi\hat{\psi} dx$$

$$\widehat{\varphi * \psi} = \hat{\varphi}\cdot\hat{\psi}, \qquad \text{for all} \quad \varphi, \psi \in \mathcal{G}.$$

LEMMA 1.2. *The spaces $\mathcal{G}$ is dense $\mathcal{S}$.*

PROOF. Since $C_0^\infty(\boldsymbol{R}^n)$ is dense in $\mathcal{S}$ it suffices to show that every function in $C_0^\infty(\boldsymbol{R}^n)$ can be approximated by elements in $\mathcal{G}$ in the topology of $\mathcal{S}$. Let us consider the Cauchy-Weierstrass kernel

$$w_\varepsilon(x)=(2\pi)^{-n/2}\varepsilon^{-n}\exp\left(-\frac{|x|^2}{2\varepsilon^2}\right).$$

Then $w_\varepsilon(x)$ belongs to $\mathcal{G}$ and it is clear that for any $\varphi\in C_0^\infty(\boldsymbol{R}^n)$

$$\lim_{\varepsilon\to 0} w_\varepsilon*\varphi=\varphi \qquad \text{in} \quad \mathcal{S}.$$

Thus it remains to show that $w_\varepsilon*\phi$ belongs to $\mathcal{G}$. For any $k, h>0$

$$\begin{aligned}\sup_{\substack{x\in\boldsymbol{R}^n\\ \alpha\in\boldsymbol{N}_0^n}} &\frac{|\partial^\alpha(w_\varepsilon*\phi)(x)|\exp k|x|}{h^{|\alpha|}\alpha!}\\ &\leqq C(\phi)\sup_{\substack{x\in\boldsymbol{R}^n\\ y\in\operatorname{supp}\phi\\ \alpha\in\boldsymbol{N}_0^n}}\frac{|\partial^\alpha w_\varepsilon(x-y)|\exp k|x|}{h^{|\alpha|}\alpha!}\\ &\leqq C'(\phi)|w_\varepsilon|_{k,h}\end{aligned}$$

which completes the proof.

If ϕ belongs to $\mathcal{G}$ then it can be easily shown that

$$\sup_{\substack{x\in\boldsymbol{R}^n\\ \alpha\in\boldsymbol{N}_0^n}}\frac{|\partial^\alpha\phi(x)|\exp(x, ks)}{h^{|\alpha|}\alpha!}<\infty \tag{1.4}$$

for any k, $h>0$ and $s=(s_1,\cdots,s_n)$, $s_j=\pm 1$, $j=1,\cdots,n$. Furthermore, it defines a family of semi-norms on $\mathcal{G}$ which is equivalent to the semi-norms given by (1.1).

On the other hand, the Pringsheim's theorem shows that every $\phi\in\mathcal{G}$ can be holomorphically extended to an entire function $\tilde{\phi}(z)$ in $\boldsymbol{C}^n$ which is given by $\tilde{\phi}(z)=\phi(x+iy)$, $z=x+iy$. Now we are in a position to state and prove the main theorem in this section.

THEOREM 1.3. *The Fourier transformation* $F: \phi\to\hat{\phi}$ *is a topological isomorphism of* $\mathcal{G}(\boldsymbol{R}^n)$ *with inverse given by the Fourier inversion formula* (1.3).

PROOF. First, we note that for each ζ, $y\in\boldsymbol{R}^n$,

$$\int_{\boldsymbol{R}^n} e^{-i\langle x,\zeta\rangle}\phi(x)dx=\int_{\operatorname{Im} z=y} e^{-i\langle z,\zeta\rangle}\phi(z)dz, \tag{1.5}$$

Because of the growth condition (1.1) of $\phi\in\mathcal{G}$, this can be easily proved by the Cauchy's theorem. It follows from (1.5) that for $k>0$ and $s=$

(s_j), $s_j=\pm 1, j=1, \cdots, n$,

$$\int_{R^n} e^{-i<x,\zeta>}\phi(x-iks)dx=\int_{\mathrm{Im}\, z=-ks} e^{-i<z+iks,\xi>}\phi(z)dz = e^{<\xi,ks>}\hat{\phi}(\xi) \tag{1.6}$$

But, since $x^\alpha \phi \in \mathcal{G}$ for each $\alpha \in N_0^n$ we obtain from (1.6) that

$$e^{<\xi,ks>}\partial^\alpha \hat{\phi}(\xi)=e^{<\xi,ks>}\widehat{(x^\alpha \phi(x))}(\xi) = \int_{R^n} e^{-i<x,\xi>}(x-iks)^\alpha \phi(x-iks)dx \tag{1.7}$$

On the other hand, for the holomorphic extension of $\phi(x)$ we obtain that for $z=x+iy, |y|\leqq k$,

$$\begin{aligned} |\phi(z)|\exp k|x| &= \exp k|x| \cdot \left| \sum_\alpha \frac{\partial^\alpha \phi(x)}{\alpha!}(iy)^\alpha \right| \\ &= \sum_\alpha \frac{|\partial^\alpha \phi(x)|\exp k|x|}{\alpha! h^{|\alpha|}} |(yh)^\alpha| \\ &\leqq |\phi|_{k,h} 2^n \sum_{j=0}^{\infty} (2|y|h)^j \\ &\leqq 2^n |\phi|_{k,h} \end{aligned} \tag{1.8}$$

if we choose $h>0$ so that $2kh<1$. It follows from (1.7) and (1.8) that

$$\begin{aligned} &\sup_{\substack{\xi \in R^n \\ \alpha \in N_0^n}} \frac{|\partial^\alpha \hat{\phi}(\xi)|\exp(\xi, ks)}{\alpha! h^{|\alpha|}} \\ &\leqq C \sup_{\substack{x \in R^n \\ \alpha \in N_0^n}} \frac{e^{|x|}|x-iks|^{|\alpha|}|\phi(x-iks)|}{\alpha! h^{|\alpha|}} \\ &\leqq C(k,h) \sup_{z \in R^n + i(|y| \leqq k')} |\phi(z)|\exp k'|x| \\ &\leqq C'(k,h)|\phi|_{k',h'} \end{aligned}$$

for some h' and $k'=\max(|ks, 1+2/h)$. This means that for any $k>0$ and $h>0$ there exist $k'>0, h'>0$ and $C=C(k,h)$ such that

$$|\hat{\phi}|_{k,h} \leqq C|\phi|_{k',h'}, \qquad \phi \in \mathcal{G}. \tag{1.9}$$

Then the proof is completed by (1.9) and the Fourier inversion formula.

DEFINITION 1.4. We denote by $\mathcal{G}'$ the strong dual of $\mathcal{G}$. In other words, $u\in\mathcal{G}'$ if and only if there exist $k, h>0$ and $C=C(k, h)>0$ such that

$$|u(\phi)|\leqq C|\phi|_{k,h}, \qquad \phi\in\mathcal{G}. \tag{1.10}$$

It is clear that the space $\mathcal{S}'$ of tempered distributions is a subclass of $\mathcal{G}'$ by Lemma 1.3. We give here other examples:

It is obvious that every analytic functional u belongs to $\mathcal{G}'$, i.e., $A'(\boldsymbol{R}^n)\subset\mathcal{G}'$, since there is a continuous injection:

$$\mathcal{G}\longrightarrow A(\boldsymbol{R}^n).$$

Other examples of elements in $\mathcal{G}'$ are measures $d\mu$ such that for some $\alpha>0$

$$\int e^{-\alpha|x|}d\mu<\infty.$$

For $u\in\mathcal{G}'$ we define the Fourier transformation $\hat{u}$ by

$$\hat{u}(\phi)=u(\hat{\phi}), \qquad \phi\in\mathcal{G}.$$

It follows from Theorem 1.3 that we have the following theorem.

THEOREM 1.5. *The Fourier transformation is an isomorphism of* $\mathcal{G}'(\boldsymbol{R}^n)$.

It is clear that the space $\mathcal{G}$ is stable under the differentiation. Now we give a stronger version.

THEOREM 1.6. *Let* $P(x, \partial)=\sum_{|\alpha|=0}^{\infty}a_\alpha(x)\partial^\alpha$ *be a differential operator of infinite order satisfying that for any* $M>0$ *there exist* $L>0$ *and* $B>0$ *such that*

$$\sup_{x\in\boldsymbol{R}^n}|\partial^\beta a_\alpha(x)|\leqq BM^{|\beta|}\beta!L^{|\alpha|}/\alpha!$$

for all α *and* β. *Then the operator* $P(x, \partial): \mathcal{G}\to\mathcal{G}$ *is continuous.*

PROOF. Let ϕ belong to $\mathcal{G}$ and $h, k>0$. Then it follows that

$$\sup_{x\in\boldsymbol{R}^n}|\partial^\beta(a_\alpha(x)\cdot\partial^\alpha\phi(x))|\exp k|x|$$

$$\leqq \sup\left|\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\partial^{\beta-\gamma}a_\alpha(x)\cdot\partial^{\alpha+\gamma}\phi(x)\right|\exp k|x|$$

$$\leqq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}BM^{|\beta-\gamma|}(\beta-\gamma)!\frac{L^{|\alpha|}}{\alpha!}\cdot|\phi|_{k,h}h^{|\alpha+\gamma|}(\alpha+\gamma)!$$

$$\leqq B|\phi|_{k,h}(Lh)^{|\alpha|}\beta!(M+h)^{|\beta|}2^{|\alpha+\beta|}$$

Then it follows that for any $k>0, h>0$,

$$\sup_{x\in R^n} |\partial^\beta(P(x,\partial)\phi(x))| \exp k|x|$$

$$\leq B|\phi|_{k,h}\beta!\,(2M+2h)^{|\beta|}\sum_{|\alpha|=0}^{\infty}(2Lh)^{|\alpha|}$$

Thus, for any $H>0$ if we choose $M>0$ and $h>0$ so small that $2Lh<1/2$ and $2M+2h<H$, then we obtain

$$|P(x,\partial)\phi(x)|_{k,H}\leq B|\phi|_{k,h}$$

which completes the proof.

REMARK. Park and Morimoto [5] introduced the space $Q(\boldsymbol{C}^n)$ of entire functions $f(z)$ satisfying that for each $k>0$ and $z=x+iy$

$$\sup_{z\in R^n+i\{|y|<k\}} |f(z)| \exp k|x|$$

is finite, which is a complex version of $\mathcal{G}(\boldsymbol{R}^n)$. However, it is quite difficult to deal with partial differential equations in the complex space.

§2. Solvability of the Mizohata operator and Lewy operator.

The partial differential operator with constant coefficients is solvable in $A'(\boldsymbol{R}^n)$ (see [8], [10]). Thus it can also be solvable in $\mathcal{G}'$. On the other hand, the following Mizohata equation

$$\frac{\partial u}{\partial t}+it^k\frac{\partial u}{\partial x}=f, \qquad f\in C_0^\infty(\boldsymbol{R}^2) \tag{2.1}$$

has no solution even in $A'(\boldsymbol{R}^2)$ for some $f\in C_0^\infty(\boldsymbol{R}^2)$ if k is an odd number (see [10]). But the equation (2.1) has a solution in the space of Aronszajn's traces. We will show here that (2.1) has a solution in our much smaller space $C^\infty(\boldsymbol{R}_t;\ \mathcal{G}'(\boldsymbol{R}))$.

THEOREM 2.1. *The Mizohata operator has a solution in* $C^\infty(\boldsymbol{R}_t;\ \mathcal{G}'(\boldsymbol{R}))$ *for* $f\in C_0^\infty$.

PROOF. Taking the Fourier transformation on the both sides of the equation (2.1) with respect to x variable we obtain that

$$\frac{\partial \hat{u}}{\partial t}-t^k\xi\hat{u}=\hat{f}(\xi,t) \tag{2.2}$$

and supp $\hat{f}(\xi, t)\subset \boldsymbol{R}_\xi\times[-T, T]$, for some $T>0$. Then the ordinary differential equation (2.2) has a solution

$$\hat{u}(\xi, t)=\int_0^t \hat{f}(\xi, s)\exp\left[\frac{\xi}{k+1}(t^{k+1}-s^{k+1})\right]ds. \tag{2.3}$$

Note that $\hat{u}(\xi, t)$ is a C^∞ function of ξ and t. Since $\hat{f}$ is bounded there exists a constant $C=C(T)$ such that

$$|\hat{u}(\xi, t)|\leqq C\exp[(t^{k+1}+T^{k+1})|\xi|/(k+1)]$$

for all $(\xi, t)\in \boldsymbol{R}^2$. Therefore, $\hat{u}(\xi, t)$ is of exponential growth with respect to ξ variable for each $t>0$. Then $\hat{u}(\xi, t)$ belongs to $\mathcal{G}'(\boldsymbol{R}_\xi)$ for each $t>0$, so that the inverse transform of $\hat{u}$ with respect to ξ variable gives the required solution of the given equation.

Now consider the solvability of Lewy equation

$$\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}+2i(x+iy)\frac{\partial u}{\partial t}=f(x, y, t), \qquad f\in C_0^\infty. \tag{2.4}$$

It is well known that this equation is also not solvable even in the space $\mathcal{D}'$ of distributions.

THEOREM 2.2. *The Lewy operator has a solution in* $C^\infty(\boldsymbol{R}^2;\ \mathcal{G}'(\boldsymbol{R}_t))$ *for* $f\in C_0^\infty$.

PROOF. Putting $z=x+iy$ and performing the Fourier transformation with respect to the t variable we obtain that

$$2\frac{\partial \hat{u}}{\partial \bar{z}}-2z\tau\hat{u}=\hat{f}(z, \bar{z}, \tau) \tag{2.5}$$

and supp $\hat{f}\subset\{(x, y)||z|\leqq T\}\times \boldsymbol{R}_\tau$ for some $T>0$. Furthermore, it follows that

$$\frac{\partial}{\partial \bar{z}}(e^{-\tau|z|^2}\hat{u})=\frac{1}{2}e^{-\tau|z|^2}\hat{f}$$

Then

$$\hat{u}(z, \bar{z}, \tau)=\frac{1}{2\pi}e^{\tau|z|^2}\left[\frac{1}{z}*\hat{f}(z, \bar{z}, \tau)e^{-\tau|z|^2}\right]$$

where $*$ denotes the convolution with respect to x, y variables. Since $\hat{f}=0$ for $|z|>T$, $\hat{u}$ is well defined and C^∞ function. Since $\hat{f}e^{-\tau|z|^2}$ is bounded and of exponential growth with respect to τ, $\hat{u}(z, \bar{z}, \tau)$ is a C^∞

function which is of exponential growth with respect to τ. Thus $\hat{u}$ belongs to $\mathcal{G}'(\boldsymbol{R}_\tau)$ for each z, so that the inverse transform of $\hat{u}$ with respect to the τ variable gives the solution of (2.4).

REMARK. Recently, we obtain that every C^∞-function $U(x,t)$ in $\boldsymbol{R}_+^{n+1}$ defines a Fourier hyperfunction $u\in\mathcal{F}'$ (an element $u\in\mathcal{G}'$ which will be called a Fourier ultrahyperfunction resp.) in the sense that $\lim_{t\to 0} U(x,t)=u$ if and only if $U(x,t)$ satisfies the conditions:

(i) $(\partial_t-\Delta)U(x,t)=0$ in $\boldsymbol{R}_+^{n+1}$

(ii) For every $k>0$ there exists $C>0$ (There exist $k>0$ and $C>0$ resp.) such that

$$|U(x,t)|\leqq C\exp k\left(|x|+t+\frac{1}{t}\right),\quad t>0.$$

From these facts it follows that $\mathcal{G}'$ is slightly bigger than the space of Fourier hyperfunctions. The other problems related to the solvability of more general equations and structure theorems will be considered in a forthcoming paper. (See [2], [4] and [5] for the related results.)

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