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Minimizing Tangent maps from 3-ball to complex projective spaces

By Atsushi FUJIOKA

Abstract. The author studies a sufficient condition that the tangent map of a holomorphic map from S^2 to $\boldsymbol{C}P^n$ is not a minimizing tangent map and gives an example of minimizing tangent map from B^3 to $\boldsymbol{C}P^n$.

§1. Introduction

Let M, N be compact Riemannian manifolds. We assume that M has non-empty boundary and N is isometrically imbedded in $\boldsymbol{R}^k$.

For $u\in C^\infty(M, N)$, we define the energy functional E by

$$E(u)=\int_M |du|^2 d\,Vol_M\,,$$

where

$$|du|^2(x)=\sum_{i=1}^{k}\sum_{\alpha,\beta=1}^{\dim M} g^{\alpha\beta}(x)\frac{\partial u^i}{\partial x^\alpha}(x)\frac{\partial u^i}{\partial x^\beta}(x)$$

$((g^{\alpha\beta}(x)=(g^{\alpha\alpha}(x))^{-1}$ is the metric tensor of $M)$ and $d\,Vol_M$ is the volume form of M.

A map which is a critical point of E is dalled a harmonic map.

If we define a subspace $L^{1,2}(M, N)$ of the Sobolev space $L^{1,2}(M, \boldsymbol{R}^k)$ by

$$L^{1,2}(M, N)=\{u\in L^{1,2}(M, \boldsymbol{R}^k)\,;\, u(x)\in N, a.e. x\in M\}\,,$$

E is extended on $L^{1,2}(M, N)$.

Now, let $\varphi\in L^{1,2}(M, N)$ be fixed and we consider the Dirichlet problem whether we can find a map f such that

$$E(f)=\inf_{\substack{u\in L^{1,2}(M,N)\\ u-\varphi\in L_0^{1,2}(M,\boldsymbol{R}^k)}} E(u)$$

$$f\in L^{1,2}(M, N)$$

$$f-\varphi\in L_0^{1,2}(M, \boldsymbol{R}^k)\,,$$

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where $L_0^{1,2}(M, \boldsymbol{R}^k)$ is the completion of $C_0^\infty(M, \boldsymbol{R}^k)$ in $L^{1,2}(M, \boldsymbol{R}^k)$. We call such f a minimizing harmonic map.

In general, a minimizing harmonic map is not C^∞. So it is a natural question whether a minimizing harmonic map is C^∞ or not. And when a minimizing harmonic map is not C^∞, we consider the question what happens around a singular point where f is not C^∞.

Schoen-Uhlenbeck studied this problem in [13] and they showed that aronud a singular point of a minimizing harmonic map rescaled maps converge to a minimizing tangent map (in abbreviation MTM) $u: B^m \to N$, where $m = \dim M$ and B^m is the unit m-ball.

We shall explain their result more precisely. Let x_0 be a singular point of a minimizing harmonic map f and B^m the geodesic ball with radius 1 centered at x_0. Then there exists a sequence $\{\sigma_i\}_{i=1}^\infty$ which converges to 0 such that $u_i: B^m \to N$ defined by

$$u_i(x) = f(\exp_{x_0}(\sigma_i x))$$

converges to a MTM in $L^{1,2}(B^m, N)$ as $i \to \infty$, where we call $u: B^m \to N$ a MTM provided that u is defined by $u(x) = g(x/|x|)(x \in B^m)$ for a harmonic map g from the standard unit $m-1$ sphere into N (such a map is automatically harmonic) and it is also a minimizing harmonic map.

So it is important to investigate MTM.

Several examples of MTM are known previously. First, Jäger-Kaul [10] showed that $u: B^m \to S^m$ defined by $u(x) = (x/|x|, 0)$ is a MTM if $m \geqq 7$. This is also shown by Schoen-Uhlenbeck [14]. Brezis-Coron-Lieb [1] proved that a non-constant MTM from B^3 to S^2 is defined by $u(x) = R \circ x/|x|$, where R is in $SO(3)$. Further, Lin showed that $u: B^m \to S^{m-1}$ defined by $u(x) = x/|x|$ is a MTM for any m in [12]. Schoen-Uhlenbeck [14] showed that a MTM from B^m to S^n must be a constant map if $m \leqq d(n)$ with $d(3) = 3$, $d(n) = 1 + \min\{n/2, 5\} (n > 3)$. This result leads to the fact that a minimizing harmonic map from any m-dimensional manifold to S^n is always C^∞ if $m \leqq d(n)$.

In this paper we investigate MTM from B^3 to complex projective n-space $\boldsymbol{CP}^n$ with its Fubini-Study metric and obtain the following results.

THEOREM 1. *Let $g: S^2 \to \boldsymbol{CP}^n$ be a holomorphic map of degree>2. Then $f: B^3 \to \boldsymbol{CP}^n$ defined by $f(x) = g(x/|x|)$ is not a MTM.*

THEOREM 2. *Let $\iota: S^2 \to \boldsymbol{CP}^n$ be a totally geodesic embedding. Then $f: B^3 \to \boldsymbol{CP}^n$ defined by $f(x) = \iota(x/|x|)$ is a MTM.*

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§ 2. Preliminaries

In this section we give some basic facts to investigate MTM from B^3 to $\boldsymbol{C}P^n$.

The problem we want to consider is the following.

Problem ●. *Classify all MTM from* B^3 *to* $\boldsymbol{C}P^n$.

In the case of $n=1$, it is studied by Brezis-Coron-Lieb [1] (see § 1).

Let $f: B^3 \to \boldsymbol{C}P^n$ be a MTM, then f satisfies the following condition.

$$f|_{\partial B^3=S^2}: S^2 \longrightarrow \boldsymbol{C}P^n$$

is a harmonic map.

$$E(f)=\inf_{\substack{u\in L^{1,2}(B^3,\, \boldsymbol{C}P^n)\\ u-f\in L_0^{1,2}(B^3,\, \boldsymbol{R}^k)}} E(u)\,.$$

So we may consider the problem ● by the following procedure.

(1) Choose a harmonic map $g: S^2 \to \boldsymbol{C}P^n$.

(2) Define $f: B^3 \to \boldsymbol{C}P^n$ by $f(x)=g(x/|x|)$. (We call f a tangent map.)

(3) Examine whether

$$E(f)=\inf_{\substack{u\in L^{1,2}(B^3,\, \boldsymbol{C}P^n)\\ u-f\in L_0^{1,2}(B^3,\, \boldsymbol{R}^k)}} E(u)$$

holds or not.

So we need to know harmonic maps from S^2 to $\boldsymbol{C}P^n$.

Let's state a definition.

DEFINITION. *A full map from* S^2 *to* $\boldsymbol{C}P^n$ *is one whose image lies in no proper projective subspace.*

The following theorem is well known.

THEOREM (Burns [2], Din-Zakrewski [4], Glaser-Stora [6]).

There is a bijective correspondence between full harmonic maps from S^2 *to* $\boldsymbol{C}P^n$ *and pairs* (φ, r), *where* $\varphi: S^2 \to \boldsymbol{C}P^n$ *is a full holomorphic map and* r *is an integer* $(0 \leqq r \leqq n)$.

We add some more explanation to this theorem. If $r=0$, the corresponding full harmonic map is just a full holomorphic map and if $r=n$, the corresponding map is a full anti-holomorphic map and otherwise the corresponding map is neither a holomorphic nor an anti-holomorphic map i.e. not a $\pm$ holomorphic map. Further, a non-full harmonic map is considerd to be a full harmonic map into $\boldsymbol{C}P^{n'}\subset\boldsymbol{C}P^{n}$ for some $n'<n$.

So a harmonic map from S^2 to $\boldsymbol{C}P^n$ $(n\geqq 2)$ is either

(1) $\pm$ holomorphic

or

(2) non $\pm$ holomorphic.

We divide harmonic maps from S^2 to $\boldsymbol{C}P^n$ in two type for the following reasons.

First if an orientation of S^2 is reversed, a holomorphic map (resp. an anti-holomorphic map) becomes an anti-holomorphic map (resp. a holomorphic map), so we state only about holomorphic maps. Second a holomorphic map has the minimum energy in its homotopy class (Lichnerowicz [11]) and the holomorphicity is invariant under a composition of an automorphism of $\boldsymbol{C}P^n$. This fact is useful to prove theorem 1. And finally by Siu-Yau [15] any non $\pm$ holomorphic map from S^2 to $\boldsymbol{C}P^n$ has not the minimum energy in its homotopy class. This result leads us to conjecture that the tangent map of a non $\pm$ holomorphic map is not a MTM. This is yet obscure and if the first eigenvalue of the Jacobi operator (see [5] for its definition) of a non $\pm$ holomorphic map is less than $-1/4$ (it is obvious that it is less than 0), the tangent map is not a MTM (Dr. H. Nakajima told this fact to the auther).

Theorem 1 tells us a sufficient condition that the tangent map of a holomorphic map is not a MTM. Theorem 2 is an example of non-constant MTM from B^3 to $\boldsymbol{C}P^n$. The auther believes that an example of theorem 2 is the only non-constant MTM from B^3 to $\boldsymbol{C}P^n$.

§3. Estimates of the degree for holomorphic maps

Let $g: S^2\to\boldsymbol{C}P^n$ be a holomorphic map of degree $d(>0)$.

We may assume the holomorphic sectional curvature of $\boldsymbol{C}P^n$ with its Fubini-Study metric is 1.

By the holomorphicity of g, the energy of g depends only on its degree (see §2). So we have $E(g)=8\pi d$.

We identify S^2 with $\boldsymbol{C}\cup\}\infty\}$ by the stereographic projection.

Using the homogeneous coordinates of $\boldsymbol{C}P^n$, we write g as

$$g(z)=[g_0(z):g_1(z):\cdots:g_n(z)]$$

for $z\in\boldsymbol{C}$.

Now suppose that the tangent map f of g is a MTM.

We fix $0<\varepsilon<1$ and choose a function η on $[0,1]$ such that

$$\eta(r)=0 \qquad (0\leqq r\leqq \varepsilon)$$

$$\eta(r)>0 \qquad (\varepsilon<r\leqq 1)$$

$$\eta(1)=1\,.$$

Separating B^3 into the radial direction and the S^2 direction (i.e. corresponding $x\in B^3\backslash\{0\}$ with $|x|\in(0,1]$ and $x/|x|\in S^2$), we define $\varphi: B^3\rightarrow \boldsymbol{C}P^n$ by

$$\varphi(r,z)=[g_0(z):\eta(r)g_1(z):\cdots:\eta(r)g_n(z)]$$

Since f is a MTM,

$$E(f)\leqq E(\varphi)\,.$$

Here if we define $\varphi_r: S^2\rightarrow \boldsymbol{C}P^n (r\in[0,1])$ by

$$\varphi_r(z)=\varphi(r,z) \qquad (z\in \boldsymbol{C})\,,$$

$$\begin{aligned}
E(\varphi)&=\int_{B^3}|d\varphi|^2 d\,Vol_{B^3}\\
&=\int_0^1\int_{S^2}|d\varphi_r|^2 d\,Vol_{S^2}dr+\int_0^1\int_{S^2}\left|d\varphi\left(\frac{\partial}{\partial r}\right)\right|^2 r^2 d\,Vol_{S^2}dr\\
&=\int_0^{\varepsilon}\int_{S^2}|d\varphi_r|^2 d\,Vol_{S^2}dr+\int_{\varepsilon}^1\int_{S^2}|d\varphi_r|^2 d\,Vol_{S^2}dr\\
&\quad+\int_0^1\int_{S^2}\left|d\varphi\left(\frac{\partial}{\partial r}\right)\right|^2 r^2 d\,Vol_{S^2}dr\\
&=0+(1-\varepsilon)\times 8\pi d\\
&\quad+16\int_{\varepsilon}^1\int_C\frac{|\eta'|^2|g_0|^2(|g_1|^2+\cdots+|g_n|^2)r^2dzdr}{\{(|g_1|^2+\cdots+|g_n|^2)\eta^2+|g_0|^2\}^2(1+|z|^2)^2}\,.
\end{aligned}$$

Since $E(f)\leqq E(\varphi)$,

$$\frac{\pi}{2}d\varepsilon\leqq\int_{\varepsilon}^1\int_C\frac{|\eta'|^2|g_0|^2(|g_1|^2+\cdots+|g_n|^2)r^2dzdr}{\{(|g_1|^2+\cdots+|g_n|^2)\eta^2+|g_0|^2\}^2(1+|z|^2)^2}\,.$$

Changing variable and setting $t=\varepsilon/r$, $\alpha(t)=\eta(\varepsilon/r)$ $(t\in[\varepsilon,1])$, we have

$$\frac{\pi}{2}d\leqq\int_{\varepsilon}^1\int_C\frac{|\alpha'|^2|g_0|^2(|g_1|^2+\cdots+|g_n|^2)dzdt}{\{(|g_1|^2+\cdots+|g_n|^2)\alpha^2+|g_0|^2\}^2(1+|z|^2)^2}\,.$$

Letting $\varepsilon\to 0$, we have

$$\frac{\pi}{2}d\leqq\int_0^1\int_C\frac{|\alpha'|^2|g_0|^2(|g_1|^2+\cdots+|g_n|^2)dzdt}{\{(|g_1|^2+\cdots+|g_n|^2)\alpha^2+|g_0|^2\}^2(1+|z|^2)^2}$$

for any function $\alpha:[0,1]\to[0,\infty)$ such that $\alpha(0)=1$, $\alpha(1)=0$.

Set

$$F(s)=\int_0^s\left\{\int_C\frac{|g_0|^2(|g_1|^2+\cdots+|g_n|^2)dz}{\{|g_1|^2+\cdots+|g_n|^2)a^2+|g_0|^2\}^2(1+|z|^2)^2}\right\}^{1/2}da$$

and choose

$$\alpha(t)=F^{-1}(F(1)(1-t))\,.$$

Then we have

$$\frac{\pi}{2}d\leqq F(1)^2\,.$$

So we obtain

$$\left(\frac{\pi}{2}d\right)^{1/2}\leqq\int_0^1\left\{\int_C\frac{|g_0|^2(|g_1|^2+\cdots+|g_n|^2)dz}{\{(|g_1|^2+\cdots+|g_n|^2)s^2+|g_0|^2(1+|z|^2)^2}\right\}^{1/2}ds\,.$$

Let A be an isometry of $\boldsymbol{C}P^n$ then we can identify A with an element of $SU(n+1)$. Since the energy is invariant under a composition of an isometry and f is a MTM, it follows that $A\circ f$ is also a MTM. So if we write $A\circ g:S^2\to\boldsymbol{C}P^n$ as

$$A\circ g(z)=[Ag_0(z):\cdots:Ag_n(z)]$$

and set

$$|g(z)|^2=\sum_{i=0}^n|g_i(z)|^2\,,$$

we have

$$\left(\frac{\pi}{2}d\right)^{1/2}\leqq\int_0^1\left\{\int_C\frac{|Ag_0|^2(|g|^2-|Ag_0|^2)dz}{\{(|Ag_0|^2(1-s^2)+|g|^2s^2\}^2(1+|z|^2)^2}\right\}^{1/2}ds\,. \tag{3.1}$$

Let dm be the biinvariant measure on $SU(n+1)$ with its total volume 1. Integrating the both sides of (3.1) over $SU(n+1)$ with respect to dm, we have by Schwarz inequality

$$\left(\frac{\pi}{2}d\right)^{1/2}\leqq\int_0^1\left\{\int_C\int_{SU(n+1)}\frac{|Ag_0|^2(|g|^2-|Ag_0|^2)dmdz}{\{(|Ag_0|^2(1-s^2)+|g|^2s^2\}^2(1+|z|^2)^2}\right\}^{1/2}ds\,. \tag{3.2}$$

We set $w=g(z)$ and calculate

$$I(w)=\int_{SU(n+1)}\frac{|Aw_0|^2(|w|^2-|Aw_0|^2)}{\{(|Aw_0|^2(1-s^2)+|w|^2s^2\}^2}dm\,.$$

Note that $I(cw)=I(w)$ for $c\in \boldsymbol{C}\backslash\{0\}$ and dm is the biinvariant measure on $SU(n+1)$. So for $\tilde{w}=[1:0:\cdots:0]$ we have

$$I(w)=I(\tilde{w}).$$

And if we set

$$S(U(1)\times U(n))=\left\{\begin{pmatrix} c & 0 \\ 0 & B\end{pmatrix}\in SL(n+1\,;\boldsymbol{C})\,;c\in U(1),B\in U(n)\right\},$$

the integrand of $I(\tilde{w})$ is $S(U(1)\times U(n))$ invariant.

Hence if we define a function G on $\boldsymbol{CP}^n=SU(n+1)/S(U(1)\times U(n))$ by

$$G([z_0:\cdots:z_n])=\frac{\dfrac{|z_0|^2}{|z_0|^2+\cdots+|z_n|^2}\left(1-\dfrac{|z_0|^2}{|z_0|^2+\cdots+|z_n|^2}\right)}{\left\{\dfrac{|z_0|^2}{|z_0|^2+\cdots+|z_n|^2}(1-s^2)+s^2\right\}^2},$$

we have by [8 p. 369 theorem 1.7]

$$I(w)=\int_{CP^n} G\,d\,Vol_{CP^n}.$$

(Here we normalize $d\,Vol_{CP^n}$ by $\int_{CP^n} d\,Vol_{CP^n}=1$.)

So

$$\begin{aligned} I(w)&=\int_{CP^n}\frac{\dfrac{|z_1|^2+\cdots+|z_n|^2}{|z^0|^2}}{\left\{1+s^2\dfrac{|s_1|^2+\cdots+|z_n|^2}{|z_0|^2}\right\}^2}d\,Vol_{CP^n} \\ &=\int_{R^{2n}}\frac{|x|^2}{(1+s^2|x|^2)^2}\,\frac{n!}{\pi^n}\,\frac{dx_1\cdots dx_{2n}}{(|x|^2+1)^{n+1}} \\ &=\int_0^\infty \frac{2nr^{2n+1}}{(1+s^2r^2)^2(1+r^2)^{n+1}}dr\,. \end{aligned}$$

Setting

$$G_n(s)=\int_0^\infty \frac{2nr^{2n+1}}{(1+s^2r^2)^2(1+r^2)^{n+1}}dr\,,$$

we have from (3.2)

$$d\leqq 2\left\{\int_0^1 \sqrt{G_n(s)}\,ds\right\}^2.$$

We set $I_n=\int_0^1\sqrt{G_n(s)}\,ds$.

REMARK. The main idea of the above discussion follows Brezis-Coron-Lieb [1]. In [1] they proved $d<2$ for $n=1$ and discussed in the case of $d=1$. If $n\geqq 2$, we can estimate the degree of g by the following lemma.

LEMMA. $I_2, I_3<\sqrt{\frac{3}{2}}$.

The proof will be given in the next section.

By this lemma we have the estimates of I_n in the following.

PROPOSITION. $I_n<\sqrt{\frac{n}{2}}$ *for* $n\geqq 3$.

PROOF. Since

$$G_n(s)=\int_0^\infty \frac{2nr^{2n+1}}{(1+s^2r^2)^2(1+r^2)^{n+1}}\,dr\,,$$

$$G_n(s)<\frac{n}{n-1}G_{n-1}(s)\,.$$

So

$$G_n(s)<\frac{n}{3}G_3(s) \qquad (u\geqq 3)\,. \quad \blacksquare$$

We can now prove the first main theorem.

THEOREM 1. *Let $g: S^2\to CP^n$ be a holomorphic map of degree>2. Then the tangent map of g is not a MTM.*

PROOF. If the tangent map of g is a MTM for $n\geqq 3$, we get degree $g<n$ from the above proposition. Hence g is not a full holomorphic map by [7 p. 173]. So g is a holomorphic map into $CP^{n'}\subset CP^n$ for $n'<n$.

Then we can use the estimate of the above proposition for $CP^{n'}$ and finally since $I_2<\sqrt{3/2}$, g must be a holomorphic map of degree$\leqq 2$ into a totally geodesic submanifold CP^2 in CP^n. $\blacksquare$

§4. The proof of the lemma

In this section we shall prove the lemma in §3.

First by direct computation we have

$$G_n(s)=\sum_{k=0}^{n-2}\frac{n(k+1)}{(n-k)(n-k-1)(1-s^2)^{k+2}}$$

$$-\frac{2n^2s^2+2n}{(1-s^2)^{n+2}}\log s-\frac{n(n+1)}{(1-s^2)^{n+1}}\,.$$

(Using a parameter β and differentiating $\int_0^\infty \frac{2nr}{(1+s^2r^2)^2(1+\beta r^2)}dr$ by β, we can calculate $G_n(s)$ inductively.)

Expanding this in powers of $(1-s^2)$, we have

$$G_n(s)=\sum_{k=0}^{\infty}\frac{n(k+1)}{(n+k+1)(n+k+2)}(1-s^2)^k$$

for $0<s\leqq 1$.

First we estimate I_2.

PROPOSITION. $I_2<\sqrt{\frac{3}{2}}$.

PROOF. Setting $t=1-s^2$, we have

$$(1-s^2)^2G_2(s)=\sum_{n=2}^{\infty}\frac{2(n-1)}{(n+1)(n+2)}t^n .$$

And

$$\{\log(1-t)\}^2=2\sum_{n=2}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k}\right)t^n .$$

Here we find

$$\frac{n-1}{(n+1)(n+2)}<0.242\times\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k}$$

for $n\geqq 2$ i.e.

$$\frac{n(n+1)}{(n+2)(n+3)}<0.242\times\sum_{k=1}^{n}\frac{1}{k}$$

for $n\geqq 1$.

Indeed, for $1\leqq n\leqq 36$ we can check this directly and for $n\geqq 37$ we have

$$\frac{n(n+1)}{(n+2)(n+3)}<1<0.242\times\sum_{k=1}^{36}\frac{1}{k}<0.242\times\sum_{k=1}^{n}\frac{1}{k}.$$

So the above inequality holds for $n\geqq 37$.

Hence we have

$$(1-s^2)^2G_2(s)<0.242\times\{\log(1-t)\}^2 .$$

So

$$(1-s^2)^2G_2(s)<0.242\times 4(\log s)^2 .$$

And we get

$$G_2(s)<\sqrt{\frac{121}{125}}\frac{\log s}{s^2-1}.$$

Therefore

$$I_2=\int_0^1 \sqrt{G(s)}\,ds$$

$$<\sqrt{\frac{121}{125}}\int_0^1 \frac{\log s}{s^2-1}ds$$

$$=\sqrt{\frac{121}{125}}\frac{\pi^2}{8}$$

$$<\sqrt{\frac{3}{2}}. \quad \blacksquare$$

Next we estimate I_3

PROPOSITION. $I_3<\sqrt{\frac{3}{2}}$.

PROOF. Setting $t=s^2$, $f_0(t)=(1-t)^2G_3(s)/(\log t)^2$, we have

$$f_0(t)=3\times\frac{-\frac{3t+1}{(1-t)^3}\log t+\frac{1}{6}+\frac{1}{1-t}-\frac{1}{(1-t)^2}}{(\log t)^2}.$$

We investigate increase and decrease of $f_0(t)$.
Differentiating f_0 by $\log t$, we have

$$f_0'(t)=-\frac{18t(t+1)(\log t)^2+3(1-t)(t^2+4t-1)\log t+(1-t)^2(t^2-8t-17)}{t(1-t)^4(\log t)^3}.$$

Setting

$$f_1(t)=(\log t)^2+\frac{(1-t)(t^2+4t-1)}{6t(t+1)}\log t+\frac{(1-t)^2(t^2-8t-17)}{18t(t+1)},$$

we have

$$f_1'(t)=\frac{-t+10t^3+16t^2+14t+1}{6t^2(t+1)^2}\log t$$

$$-\frac{-t^5+5t^4+16t^3+10t^2-23t-7}{9t(t+1)^2}.$$

And if we set

$$f_2(t)=\log t-\frac{2}{3}\frac{-t^5+5t^4+16t^2+10t^2-23t-7}{-t^4+10t^3+16t^2+14t+1},$$

we have

$$f_2'(t)=\frac{(1-t)^5(t+1)(2t^3-35t^2-54t+3)}{3t(-t^4+10t^3+16t^2+14t+1)^2}\,.$$

So we get increases and decreases of f_2, f_1 and f_0 as in the tables below

t	0		α_2		1
$f_2'(t)$		$+$	0	$-$	
$f_2(t)$		$\nearrow$	maximal	$\nwarrow$	0

t	0		α_1		1
$f_1'(t)$		$-$	0	$+$	
$f_1(t)$		$\nwarrow$	minimal	$\nearrow$	0

t	0		α_0		1
$f_0'(t)$		$+$	0	$-$	
$f_0(t)$		$\nearrow$	maximal	$\nwarrow$	

We cannot get the precise value of α_0, α_2 and α_2, but we can get $0.0129<\alpha_0<0.0130$ by approximating α_0 from f_2'.

If we set $u=1-s^2$, we have

$$(1-s^2)^2G_3(s)=\sum_{n=2}^{\infty}\frac{3(n-1)}{(n+2)(n+3)}u^n\,.$$

Comparing this with

$$\{\log(1-u)\}^2=2\sum_{n=2}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}\right)u^n\,,$$

we get

$$G_3(s)<\frac{159}{125}\left(\frac{\log s}{1-s^2}\right)^2.$$

(We use the same method as in the case of I_2 and obtain

$$\frac{n(n+1)}{(n+3)(n+4)}<0.212\times\sum_{k=1}^{n}\frac{1}{k}.$$

We calculate directly for $n\geqq 63$ and for $n>63$ we have

$$\frac{n(n+1)}{(n+3)(n+4)}<1<0.212\times\sum_{k=1}^{n}\frac{1}{k}.$$

Since we find $f_0(0.0129)$, $f_0(0.0130)<0.282$, we have for $0<a_4<\cdots<a_1<\sqrt{0.0129}<\sqrt{a_0}<\sqrt{0.0130}<b_1<\cdots<b_6<1$

$$\begin{aligned}\int_0^1\sqrt{G_3(s)}ds\leqq&\sqrt{\frac{159}{125}}\times\frac{\log\sqrt{0.0129}}{(\sqrt{0.0129})^2-1}\times(\sqrt{0.0130}-\sqrt{0.0129})\\&+\sqrt{\frac{141}{125}}\int_0^1\frac{\log s}{s^2-1}ds\\&-\left\{\sqrt{\frac{141}{125}}-f_0(a_1)^{1/2}\right\}\int_0^{a_1}\frac{\log s}{s^2-1}ds\\&-\sum_{i=1}^{3}\{f_0(a_i)^{1/2}-f_0(a_{i+1})^{1/2}\}\int_0^{a_{i+1}}\frac{\log s}{s^2-1}ds\\&-\left\{\sqrt{\frac{141}{125}}-f_0(b_1)^{1/2}\right\}\int_{b_1}^1\frac{\log s}{s^2-1}ds\\&-\sum_{i=1}^{4}\{f_0(b_i)^{1/2}-f_0(b_{i+1})^{1/2}\}\int_{b_{i+1}}^1\frac{\log s}{s^2-1}ds.\end{aligned}$$

Here for $0<a<1$,

$$\begin{aligned}\int_0^a\frac{\log s}{s^2-1}ds&=-\frac{1}{2}\log\frac{1+a}{1-a}\log a+\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)^2}\\&>-\frac{1}{2}\log\frac{1+a}{1-a}\log a+a+\frac{a^3}{9}+\frac{a^5}{25}+\frac{a^7}{49},\end{aligned}$$

$$\begin{aligned}\int_a^1\frac{\log s}{s^2-1}ds&=\frac{\pi^2}{8}+\frac{1}{2}\log\frac{1+a}{1-a}\log a-\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)^2}\\&>\frac{\pi^2}{8}+\frac{1}{2}\log\frac{1+a}{1-a}\log a-a-\frac{a^3}{9}-\frac{a^5}{25}-0.09\times a^7.\end{aligned}$$

Using these inequalities and setting $a_1=0.05$, $a_2=0.026$, $a_3=0.012$, $a_4=0.004$, $b_1=0.29$, $b_2=0.43$, $b_3=0.58$, $b_4=0.75$, $b_5=0.9$, we get

$$\int_0^1\sqrt{G_3(s)}\,ds<\sqrt{\frac{3}{2}},\quad\blacksquare$$

REMARK. The auther calculated the above estimate by computer. Computing by Fortran using single and double precision, he checked the error is enough less to get the estimate $I_3<\sqrt{3/2}$.

§ 5. An example of MTM

Since $\pi_2(\boldsymbol{C}P^n)\neq 0$, from Schoen-Uhlenbeck [13] there exists a non-constant MTM from B^3 to $\boldsymbol{C}P^n$. In this section we give an example of non-constant MTM.

Let $\pi:\boldsymbol{C}^{n+1}\to\boldsymbol{C}P^n$ be a natural projection and V be a (complex) 2-dimensional subspace in $\boldsymbol{C}^{n+1}$. We give $\boldsymbol{C}^{n+1}$ a Hermitian metric and let $\tilde{\pi}_V$ be the orthogonal projection from $\boldsymbol{C}^{n+1}$ into V. Let $P^1(V)$ be the projectification of V and $p: V\to P^1(V)$ be the natural projection and $V^\perp$ be the orthogonal complement.

Then it is easy to see that there exists a map $\pi_V:\boldsymbol{C}P^n\backslash\pi(V^\perp)\to P^1(V)$ such that the following commutative diagram holds.

$$\begin{array}{ccc} \boldsymbol{C}^{n+1}\backslash V^\perp & \xrightarrow{\tilde{\pi}_V} & V \\ \pi\downarrow & & \downarrow p \\ \boldsymbol{C}P^n\backslash\pi(V^\perp) & \xrightarrow[\pi_V]{} & P^1(V) \end{array}$$

THEOREM 2. *Let $\iota: S^2\to\boldsymbol{C}P^n$ be a totally geodesic embedding. Then the tangent map of ι is a MTM.*

PROOF. First for $v\in T\boldsymbol{C}P^n$ there exists a constant $c>0$ such that

$$c|v|^2=\int_{V\in G_2(\boldsymbol{C}^{n+1})}|d\pi_V(v)|^2dG\,,$$

where dG is a biinvariant measure on Grassmannian manifold $G_2(\boldsymbol{C}^{n+1})$ with its total volume 1. We find that $c<\infty$ from computation below and Fubini's theorem (see also Coron-Gulliver [3 p. 85].)

Hence if we set $v=du(\partial/\partial x^\alpha)$ $(\alpha=1,2,3)$ for $u: B^3\to\boldsymbol{C}P^n$, it follows that

$$c|du|^2=\int_{V\in G_2(\boldsymbol{C}^{n+1})}|d(\pi_V\circ u)|^2dG\,.$$

So

$$cE(u)=\int_{V\in G_2(\boldsymbol{C}^{n+1})}E(\pi_V\circ u)dG\,.$$

If we choose $u=u_0(x)=u_0(r,z)=[z:1:0:\cdots:0]$, we have $c=1$ since $E(u_0)=E(\pi_V\circ u_0)=8\pi$ for almost all V.

Since the energy is invariant under a composition of an isometry, from [9 p. 334 theorem 11.1] we may assume that the tangent map of ι is u_0.

And since $E(\pi_V \circ u) \geqq E(\pi_V \circ u_0)$ for any $u \in L^{1,2}(B^3, \boldsymbol{CP}^n)$ such that $u - u_0 \in L_0^{1,2}(B^3, \boldsymbol{R}^k)$ ([3 theorem 1.1]), we have

$$\begin{aligned} E(u) &= \int_{G_2(C^{n+1})} E(\pi_V \circ u) dG \\ &\geqq \int_{G_2(C^{n+1})} E(\pi_V \circ u_0) dG \\ &= E(u_0) . \quad \blacksquare \end{aligned}$$

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Department of Mathematical Sciences
University of Tokyo
Hongo, Tokyo
113 Japan