

***Global existence of the universal locally trivial family
of analytic subvarieties with locally stable
parametrizations of a compact complex manifold***

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Abstract. We define a notion of analytic subvarieties with locally stable parametrizations of complex manifolds (Definition 1.3), which is a unification and a generalization of closed complex analytic subsets of normal crossing and analytic subvarieties with ordinary singularities ([23]), and prove that; for a given compact complex manifold Y there exists a universal family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ for locally trivial families, i. e., families which are locally products, of such analytic subvarieties of Y , parametrized by (possibly nonreduced) complex spaces; moreover it turns out that the underlying C^∞ structure of the family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ is such that it is C^∞ trivial at a non-singular point of $E(Y)_{\text{red}}$ (the reduction of $E(Y)$) and the C^∞ type of the fiber is constant over each connected component of $E(Y)$ (Theorem 4.8).

Introduction

Let Y be a compact complex manifold. We denote by $E(Y)$ the set of all analytic subvarieties with locally stable parametrizations of Y (cf. Definition 1.3 below). We denote by Z_i an analytic subvariety with a locally stable parametrization of Y corresponding to a “point” $t \in E(Y)$. We define a subset $\tilde{\mathcal{Z}}(Y)$ of the product space $Y \times E(Y)$ by

$$\tilde{\mathcal{Z}}(Y) := \{(y, t) | t \in E(Y), y \in Z_i\}.$$

We denote by $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ the restriction of the projection map $Pr_{E(Y)}: Y \times E(Y) \rightarrow E(Y)$ to $\tilde{\mathcal{Z}}(Y)$. In this paper we shall prove that the sets $E(Y)$ and $\tilde{\mathcal{Z}}(Y)$ have the structure of Hausdorff complex spaces such that $\tilde{\mathcal{Z}}(Y)$ is a closed complex subspace of the product complex space $Y \times E(Y)$ and $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ is a *universal locally trivial family of analytic*

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subvarieties with locally stable parametrizations of Y , parametrized by $E(Y)$. Here a *locally trivial family* means: for each point $p=(y, t) \in Y \times E(Y)$, there exist an open neighborhood $\mathcal{U} \subset Y \times M$ of p and a biholomorphic map $\varphi: \mathcal{U} \rightarrow U \times V$, where we define $U := \mathcal{U} \cap (Y \times t)$ and $V := Pr_M(\mathcal{U})$, such that;

(a) the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\varphi} & U \times V \\ & \searrow Pr_M & \swarrow Pr_V \\ & & V \end{array}$$

commutes,

(b) $\varphi(\mathcal{U} \cap \tilde{\mathcal{Z}}(Y)) = (U \cap \tilde{\mathcal{Z}}(Y)) \times V$, and

(c) $\varphi_{|U \times t} = id_{U \times t}$, the identity map on $U \times t$.

Universal means: for any locally trivial family $\pi: \mathcal{Z} \rightarrow M$ of analytic subvarieties with locally stable parametrizations of Y , parametrized by a complex space M , there exists a unique holomorphic map $f: M \rightarrow E(Y)$ such that $f^*\tilde{\mathcal{Z}}(Y) = \mathcal{Z}$. Here $f^*\tilde{\mathcal{Z}}(Y)$ denotes the pull-back of the family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ by the map $f: M \rightarrow E(Y)$. From the local triviality it follows that the family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ is C^∞ trivial at a point $t \in E(Y)$ whose corresponding point of $E(Y)_{red}$ (the reduction of $E(Y)$) is non-singular. From this fact it follows that the C^∞ type of the fiber of the family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ is constant over each connected component of $E(Y)$. The relation between our family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ and the universal family $\tilde{\pi}_0: \tilde{\mathcal{U}}(Y) \rightarrow D(Y)$ of closed complex subspaces of Y , parametrized by the Douady space $D(Y)$ (cf. [1]) is that the natural inclusion map $\iota: E(Y) \subset D(Y)$ as sets is a holomorphic immersion and $\iota^*\tilde{\mathcal{U}}(Y) = \tilde{\mathcal{Z}}(Y)$. These results are summed up in Theorem 4.8, which might be considered as a generalization of Namba's results in [18] or [19] to higher dimensional singular cases.

Our essential contributions in this paper are to give the definition and examples of analytic subvarieties with locally stable parametrizations and to prove *Relative Normalization Theorem* (Theorem 3.6), which is a generalization of Theorem (10.1) in [23] to the case where a parameter space is possibly a nonreduced complex space. By *Relative Normalization Theorem* it turns out that locally trivial displacements of an analytic subvariety with a locally stable parametrization are equivalent to deformations of a locally stable holomorphic map. Consequently, Flenner's theorem on the existence of the semi-universal family for deformations of a holomorphic map between compact complex spaces, parametrized by a complex space (cf. [3]), implies the existence of the semi-universal family for locally

trivial displacements of an analytic subvariety Z with a locally stable parametrization in a compact complex manifold Y . Furthermore, comparing this semi-universal family with the locally trivial family of displacements of Z in Y , which is obtained by applying Flenner-Kosarew's theorem in [4] to the universal family $\tilde{\pi}_0: \tilde{U}(Y) \rightarrow D(Y)$ of closed complex subspaces of Y , parametrized by the Douady space $D(Y)$, we conclude that the semi-universal family for locally trivial displacements of an analytic subvariety Z with a locally stable parametrization in a compact complex manifold Y is in fact the universal family. Patching up these local universal families, we give a complex structure to the family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$. Following Namba's proof in [19], we can prove the *Hausdorffness* of the space $E(Y)$. The C^∞ triviality of the family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ follows from that of deformations of a locally stable holomorphic map, which has been proved in [24].

Throughout this paper we use the terms of an *analytic variety* and an *analytic subvariety* in the sense of a reduced complex space and a reduced, closed complex subspace, respectively. The author expresses his hearty thanks to his colleague Prof. K. Miyajima of Kagoshima University for useful discussions with him during the preparation of this paper.

§ 1. Definition and examples of analytic subvarieties with locally stable parametrizations of a complex manifold

Let X and Y be complex manifolds, S a finite subset of X , and q a point of Y . A multi-germ $f: (X, S) \rightarrow (Y, q)$ of a holomorphic map at S is an equivalence class of holomorphic maps $g: U \rightarrow Y$ with $g(S) = q$, where U are open neighborhoods of S in X . Throughout this paper we shall interchangeably use a multi-germ of f and a representative g of f . A germ of a parametrized family of multi-germs of holomorphic maps is a multi-germ $F: (X \times \mathbf{C}^r, S \times o) \rightarrow (Y \times \mathbf{C}^r, q \times o)$ of a holomorphic map such that $F(X \times t) \subset Y \times t$ for any t in some open neighborhood of the origin o in \mathbf{C}^r . An unfolding of a multi-germ $f: (X, S) \rightarrow (Y, q)$ of a holomorphic map is a germ of a parametrized family of multi-germs of holomorphic maps $F: (X \times \mathbf{C}^r, S \times o) \rightarrow (Y \times \mathbf{C}^r, q \times o)$ such that $F(x, o) = (f(x), o)$ for $x \in X$. We say that an unfolding $F: (X \times \mathbf{C}^r, S \times o) \rightarrow (Y \times \mathbf{C}^r, q \times o)$ of a multi-germ $f: (X, S) \rightarrow (Y, q)$ of a holomorphic map is *trivial* if there exist germs of t -levels ($t \in \mathbf{C}^r$) preserving analytic automorphisms $G: (X \times \mathbf{C}^r, S \times o) \rightarrow (X \times \mathbf{C}^r, S \times o)$ and $H: (Y \times \mathbf{C}^r, q \times o) \rightarrow (Y \times \mathbf{C}^r, q \times o)$ such that $G_{1, X \times o} = id_X, H_{1, Y \times o} = id_Y$, and $H \circ F \circ G^{-1} = f \times id_{\mathbf{C}^r}$.

1.1 DEFINITION. A multi-germ $f: (X, S) \rightarrow (Y, q)$ of a holomorphic map is said to be *simultaneously stable* if any unfolding of f is trivial. In

particular, if S is one point, say p , a “simultaneously” stable germ $f: (X, p) \rightarrow (Y, q)$ is said to be *stable*.

1.2 DEFINITION. A holomorphic map $f: X \rightarrow Y$ between complex manifolds is said to be *locally stable* if, for any point $q \in Y$ and any finite subset $S \subset f^{-1}(q)$, a multi-germ $f: (X, S) \rightarrow (Y, q)$ is simultaneously stable.

1.3. DEFINITION. An analytic subvariety Z (possibly not of pure dimension) of a complex manifold Y is said to be *with a locally stable parametrization* if

- (i) its normal model X is non-singular, and
- (ii) the composite map $f := \iota \circ n: X \rightarrow Y$ is locally stable, where $n: X \rightarrow Z$ is the normalization map and $\iota: Z \hookrightarrow Y$ is the inclusion map.

1.4 REMARK. The above definition is based on the following facts which have been proved in [23]: If $f: X \rightarrow Y$ is a locally stable holomorphic map between complex manifolds, then it is a *Thom-Boardman map satisfying a certain normal crossing condition* (cf. [23, Theorem 4.2]). For definition see [6, Chapter VI, § 5]). From this fact it follows that if we assume furthermore that $f: X \rightarrow Y$ is a proper map with $\dim X < \dim Y$, and if $X \xrightarrow{f'} Z \hookrightarrow Y$ is the factorization of f , where we define $Z := f(X)$, then $f': X \rightarrow Z$ is the normalization map of Z (cf. [23, Proposition (4.2), Corollary (4.2)]).

In order to give examples of analytic subvarieties with locally stable parametrizations, we quote a proposition from [23] which give a criterion for a multi-germ of a holomorphic map to be simultaneously stable. The proposition is originated from J. N. Mather’s characterization of simultaneously stable map germs by using *contact classes* in [14]. In order to state the proposition and for use later we introduce several symbols. For a holomorphic map $f: X \rightarrow Y$ between complex manifolds and a point $p \in X$, we define the following local \mathbf{C} -algebras:

$$R(f)_p := \mathcal{O}_{X,p} / f^* m_q \cdot \mathcal{O}_{X,p} \quad (q := f(p)),$$

where m_q denotes the maximal ideal of $\mathcal{O}_{Y,q}$;

$$R_k(f)_p := R(f)_p / \mathcal{M}_p^{k+1},$$

where k is a non-negative integer and \mathcal{M} the maximal ideal of $R(f)_p$:

$$\hat{R}(f)_p := \varprojlim_k R_k(f)_p.$$

Suppose we are given a local \mathbf{C} -algebra A , then, for a holomorphic map $f: X \rightarrow Y$ between complex manifolds, we define

$$C(f, A) := \{p \in X \mid R(f)_p \cong A\},$$

$$C_k(f, A) := \{p \in X \mid R_k(f)_p \cong A\},$$

and

$$\hat{C}(f, A) := \{p \in X \mid \hat{R}(f)_p \cong A\}.$$

In particular, if $f: X \rightarrow Y$ is a finite map, then we have $\dim_{\mathbf{C}} R(f)_p < \infty$ for a point $p \in X$ (cf. [2, Theorem (1.11)]). Hence, in this case there is a natural number k_o such that $\mathcal{M}_p^{k_o+1} \subset f^* m_q \cdot \mathcal{O}_{X,p}$. Therefore we have

$$R(f)_p = R_{k_o}(f)_p = R_{k_o+1}(f)_p = \dots = \hat{R}(f)_p \quad (p \in X),$$

and

$$C(f, A) = C_{k_o}(f, A) = C_{k_o+1}(f, A) = \dots = \hat{C}(f, A).$$

1.5 PROPOSITION. *Let $f: X \rightarrow Y$ be a holomorphic map between complex manifolds (possibly X is not pure dimensional), $q \in f(X)$ a point of the image of f , and $S = \{p_1, \dots, p_s\} \subset f^{-1}(q)$ a finite subset of the inverse image of q by f . We define*

$$C_i := \{p \in X \mid R_m(f)_p \cong R_m(f)_{p_i}\} \quad (1 \leq i \leq s),$$

where m denotes the dimension of Y at q . Then the multi-germ $f: (X, S) \rightarrow (Y, q)$ of a holomorphic map is simultaneously stable if, and only if, the following two conditions are satisfied:

(i) for every i ($1 \leq i \leq s$) the germ $f: (X, p_i) \rightarrow (Y, q)$ of the holomorphic map is stable,

(ii) $(df)_{p_1}(T_{C_1, p_1}), \dots, (df)_{p_s}(T_{C_s, p_s})$ are in a general position in the tangent space $T_{Y,q}$ of Y at q , i.e., the codimension of $\{\bigcap_{i=1}^s (df)_{p_i}(T_{C_i, p_i})\}$ in $T_{Y,q}$ is equal to $\sum_{i=1}^s \{\text{the codimension of } (df)_{p_i}(T_{C_i, p_i}) \text{ in } T_{Y,q}\}$, where $(df)_{p_i}: T_{X, p_i} \rightarrow T_{Y,q}$ denote the Jacobian map of f at p_i for $1 \leq i \leq s$ (We should note that; if the condition (i) is satisfied, then by the condition (d) in Theorem (4.1) in [14], at every p_i ($1 \leq i \leq s$) the m -th jet extension $j^m f: X \rightarrow J^m(X, Y)$ of f intersects transversely with the contact classes contact classes $C_{\sigma_i} \subset J^m(X, Y)$ containing the m -th jet $\sigma_i = j^m f(p_i)$, where $J^m(X, Y)$ denotes the m -th jet bundle over $X \times Y$; hence $C_i = (j^m f)^{-1}(C_{\sigma_i})$ is a submanifold in a sufficiently small open neighborhood of p_i in X).

For the definition of contact classes, see [6, Chapter VII, Definition 3.11]. For the proof of the proposition we refer to Proposition (7.1) in [23].

1.6 Example. An analytic subvariety Z of normal crossing of a

complex manifold Y is an analytic subvariety with a locally stable parametrization. Here we say that an analytic subvariety Z of a complex manifold Y is of normal crossing if the following condition is satisfied: For any point $p \in Z$, if we let Z_1, Z_2, \dots, Z_k be all locally irreducible components of Z which are through p , then there exists a local coordinate system $(z_1, \dots, z_{r_1}, z_{r_1+1}, \dots, z_{r_2}, \dots, z_{r_{k-1}+1}, \dots, z_{r_k}, z_{r_k+1}, \dots, z_n)$ of Y with center p such that each Z_α ($1 \leq \alpha \leq k$) is defined by $z_{r_{\alpha-1}+1} = \dots = z_{r_\alpha} = 0$, where we understand $r_0 = 0$. Indeed, by an infinitesimal criterion due to J. N. Mather for a multi-germ of a holomorphic map to be simultaneously stable (cf. [24, Theorem 1.4]), embedding maps are locally stable: hence, by Proposition 1.5, analytic subvarieties of normal crossing are with locally stable parametrizations.

In order to give another example we quote one more proposition from [23].

1.7 PROPOSITION. *Let $f: (X, S) \rightarrow (Y, q)$ be a simultaneously stable multi-germ of a holomorphic map with $\dim X < \dim Y$. Then there exist open neighborhoods U of S in X and V of q in Y with $f(U) \subset V$, enjoying the following properties:*

- (i) $f|_U: U \rightarrow V$ is a finite map, i.e., a proper map and $f|_U^{-1}(q')$ is a finite subset of U for any point q' in $f(U)$;
- (ii) $f|_U: U \rightarrow V$ is a locally stable holomorphic map.

For the proof we refer to Proposition (6.2) in [23]. By (i) above, we can see that; for a *simultaneously stable* multi-germ $f: (X, S) \rightarrow (Y, q)$ of a holomorphic map with $\dim X < \dim Y$, there is an open neighborhood V of q in Y such that $f(X) \cap V$ is an analytic subvariety of V .

1.8 DEFINITION. Let Y be a complex manifold. A *germ* (Y, Z, q) of an analytic subvariety of Y is an equivalence class of analytic subvarieties Z in open neighborhoods D of q in Y with $q \in Z$. Here we say that $Z \subset D \subset Y$ and $Z' \subset D' \subset Y$ are equivalent if there is an open neighborhood D'' of q in $D \cap D'$ such that $Z \cap D'' = Z' \cap D''$. If an analytic subvariety Z' of an open neighborhood of q in Y is a member of an equivalence class (Y, Z, q) , we call Z' a *representative* of (Y, Z, q) and (Y, Z, q) the *germ* of Z' at q .

If there is no fear of confusion, we shall interchangeably use a germ (Y, Z, q) of an analytic subvariety Z of Y at q and a representative Z of a germ (Y, Z, q) .

1.9 DEFINITION. Let $f: (X, S) \rightarrow (Y, q)$ be a simultaneously stable

multi-germ of a holomorphic map with $n := \dim X < m := \dim_q Y$ (the dimension of Y at q). We define $Z := f(X)$. We call a germ (Y, Z, q) of an analytic subvariety of Y an n -dimensional ordinary singularity in an m -dimensional ambient manifold if X is of the same dimension at every points of S .

1.10 *Example.* An analytic subvariety Z which has at worst ordinary singularities only, of a complex manifold Y , is an analytic subvariety with a locally stable parametrization. (We call such Z an *analytic subvariety with ordinary singularities*.) Indeed, by definition, for any point p of such an analytic subvariety Z of a complex manifold Y , there exist an open neighborhood V of p in Y and a simultaneously stable multi-germ $f: (U, S) \rightarrow (V, q)$ such that $f(U) = V \cap Z$. Furthermore, by (ii) in Proposition 1.7, shrinking V sufficiently small, we may assume that $f: U \rightarrow V$ is a locally stable holomorphic map. Then the factorization $U \xrightarrow{f'} f(U) = V \cap Z \hookrightarrow V$ of f is the normalization of $f(U)$ (cf. [23, Proposition (4.2), Corollary (4.2)]). Thus $V \cap Z$ is an analytic subvariety with a locally stable parametrization of V ; hence so is Z .

1.11 **REMARK.** By definition a pure dimensional analytic subvariety with a locally stable parametrization is an analytic subvariety with ordinary singularities.

Example 1.10 is rather abstract. J. N. Mather, however, gives in [16] the complete classification of the local \mathbf{C} -algebras A for which there exist stable map germs $f: (X, p) \rightarrow (Y, q)$ of holomorphic maps with $A \cong R(f)_p$, in the case where the pair of positive integers $(\dim X, \dim Y)$ belongs to the so-called “*nice range*” and satisfies the inequality $\dim X < \dim Y$. (We say a pair of positive integers (n, m) belongs to the “*nice range*” if, and only if, it satisfies one of the following conditions: (i) $n < (6/7)m + (8/7)$ and $m - n \geq 4$; (ii) $n < (6/7)m + (9/7)$ and $3 \geq m - n \geq 0$; (iii) $m < 8$ and $m - n = -1$; (iv) $m < 6$ and $m - n = -2$; (v) $m < 7$ and $m - n \leq -3$ (cf. [15])). By this classification together with “*Normal form theorem for stable germs*” ([13, Theorem (5.1)]) and Proposition 1.5, we can calculate the local equations of ordinary singularities in some cases. For instance, if we restrict our considerations to the case where the pair of positive integers $(n, m) := (\dim X, \dim Y)$ satisfies all of the following conditions:

- $$(1.1) \quad \begin{array}{ll} \text{(i)} & (n, m) \text{ belongs to the “nice range”,} \\ \text{(ii)} & n < m, \\ \text{(iii)} & n \leq (2/3)m + 1, \end{array}$$

then the situation is very simple. In this case, according to Mather's classification table of stable map germs in [16], local \mathbf{C} -algebras A for which there exist stable germs $f: (\mathbf{C}^n, o) \rightarrow (\mathbf{C}^m, o)$ with $R(f)_o \cong A$ are only the following three types:

$$A_0 := \mathbf{C}[x]/(x), \quad A_1 := \mathbf{C}[x]/(x^2), \quad A_2 := \mathbf{C}[x]/(x^3),$$

where $\mathbf{C}[x]$ denotes the polynomial ring over \mathbf{C} in one variable x . By "Normal form theorem for stable germ" in [13], we can give the concrete expressions in local coordinates of each stable germs $f: (\mathbf{C}^n, o) \rightarrow (\mathbf{C}^m, o)$ whose local \mathbf{C} -algebras are A_i ($0 \leq i \leq 2$) and give the defining equations of the loci $C_m(f, A)$ ($= C(f, A) = \hat{C}(f, A)$) of such stable germs $f: (\mathbf{C}^n, o) \rightarrow (\mathbf{C}^m, o)$ as follows:

(a) The case $R(f)_o \cong A_0$:

$$\begin{cases} y_i \circ f = x_i & (1 \leq i \leq n) \\ y_i \circ f = 0 & (n+1 \leq i \leq m), \end{cases}$$

$$C_m(f, A_0) = \mathbf{C}^n$$

(b) The case $R(f)_o \cong A_1$:

$$\begin{cases} y_i \circ f = x_i & (1 \leq i \leq n-1) \\ y_n \circ f = x_n^2 \\ y_{n+i} \circ f = x_i x_n & (1 \leq i \leq m-n \leq n-1), \end{cases}$$

$$C_m(f, A_1) = \{x = (x_1, \dots, x_n) \in \mathbf{C}^n \mid x_1 = x_2 = \dots = x_{m-n} = x_n = 0\},$$

where we should assume that $m-n \leq n-1$, i. e., $m \leq 2n-1$.

(c) The case $R(f)_o \cong A_2$:

$$(1.2) \quad \begin{cases} y_i \circ f = x_i & (1 \leq i \leq n-1) \\ y_n \circ f = x_n^3 + x_1 x_n \\ y_{n+i} \circ f = x_{2i} x_n + x_{2i+1} x_n^2 & (1 \leq i \leq m-n), \end{cases}$$

$$C_m(f, A_2) = \{x = (x_1, \dots, x_n) \in \mathbf{C}^n \mid x_1 = x_2 = \dots = x_{2(m-n)+1} = x_n = 0\},$$

where we should assume that $2(m-n)+1 \leq n-1$, i. e., $m \leq (3/2)\{n - (2/3)\}$. In the above expression (x_1, \dots, x_n) and (y_1, \dots, y_m) denote linear coordinate systems on \mathbf{C}^n and \mathbf{C}^m , respectively.

Low-dimensional hypersurfaces with ordinary singularities:

Besides the conditions (i), (ii), (iii) in (1.1) we impose the condition of $m=n+1$. Then the pair of positive integers (n, m) should be one of (1, 2), (2, 3), (3, 4), (4, 5) and (5, 6). In this case, by the concrete expressions in local coordinates of possible stable map germs and by Proposition 1.5, we can calculate the defining equations of ordinary singularities as follows:

1.12 *Example.* A pure 1-dimensional analytic subvariety of a connected 2-dimensional complex manifold is with ordinary singularities if, and only if, it has at worst ordinary double points only as singularities.

1.13 *Example.* A pure 2-dimensional analytic subvariety Z of a connected 3-dimensional complex manifold Y is with ordinary singularities if, and only if, for each singular point p of Z , there exists on Y a local coordinate (x, y, z) with center p such that; in a neighborhood of p , Z is defined by one of the following equations:

- (i) $yz=0$ (double point),
- (ii) $xyz=0$ (triple point),
- (iii) $xy^2-z^2=0$ (cuspidal point).

1.14 *Example.* A pure 3-dimensional analytic subvariety Z of a connected 4-dimensional complex manifold Y is with ordinary singularities if, and only if, for each singular point p of Z , there exists on Y a local coordinate (x, y, z, w) with center p such that; in a neighborhood of p , Z is defined by one of the following equations:

- (i) $zw=0$ (double point),
- (ii) $yzw=0$ (triple point),
- (iii) $xyzw=0$ (quadruple point),
- (iv) $xy^2-z^2=0$ (cuspidal point),
- (v) $w(xy^2-z^2)=0$.

1.15 *Example.* A pure 4-dimensional analytic subvariety Z of a connected 5-dimensional complex manifold Y is with ordinary singularities if, and only if, for each singular point p of Z , there exists on Y a local coordinate (x, y, z, u, v) with center p such that; in a neighborhood of p , Z is defined by one of the following equations, though we omit the local equations for k -ple point:

- (i) a union of k 4-planes which intersect transversely at the origin o

- in \mathbf{C}^5 ($2 \leq k \leq 5$) (k -ple point)
- (ii) $xy^2 - z^2 = 0$ (cuspidal point),
 - (iii) $u(xy^2 - z^2) = 0$,
 - (iv) $uv(xy^2 - z^2) = 0$,
 - (v) $v^3 + 2xzv^2 + (x^2z^2 - 3yzu + xy^2)v - \{z^3u + y(y^2 + xz^2)\}u = 0$.

Although we can also calculate the local equations of 5-dimensional ordinary singularities in a 6-dimensional complex manifold, we refrain from mentioning them here, since they are so complicated (cf. [23, Proposition (7.7)]).

For instance, the equation in (v) in Example 1.15 is calculated as follows: That singularity comes from a stable map germ $f: (\mathbf{C}^4, o) \rightarrow (\mathbf{C}^5, o)$ with $R(f)_o \cong A_2 = \mathbf{C}[x]/(x^3)$. The normal form of this stable map germ f is given by

$$\begin{cases} y_i \circ f = x_i & (1 \leq i \leq 3) \\ y_4 \circ f = x_4^3 + x_1 x_4 \\ y_5 \circ f = x_2 x_4 + x_3 x_4^2, \end{cases}$$

where (x_1, \dots, x_4) and (y_1, \dots, y_5) denote linear coordinate systems on \mathbf{C}^4 and \mathbf{C}^5 , respectively (cf. (1.2)). Substituting $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$ into the last two equations above, we have

$$(1.3) \quad \begin{cases} x_4^3 + y_1 x_4 - y_4 = 0, \\ y_3 x_4^2 + y_2 x_4 - y_5 = 0. \end{cases}$$

We regard this as a simultaneous equation for x_4 with coefficients in the polynomial ring $\mathbf{C}[y_1, \dots, y_5]$. We eliminate x_4 by calculating the *resultant* of the system of equations in (1.3) (cf. [25, Chapter 11]). Then we get the equation in (v) of Example 1.15 by replacing (y_1, \dots, y_5) by (x, y, z, u, v) . For the details of the way to obtain the defining equations of singularities in *Example 1.12*~*Example 1.15* we refer to [23, § 7].

1.16 REMARK. The pair of positive integers $(n, n+1)$ belongs to the “*nice range*” in the sense of J. N. Mather if and only if $n \leq 14$.

High-codimensional analytic subvarieties with ordinary singularities:

If $m \geq 2n - 1$, then the pairs of positive integers (n, m) satisfy the conditions (i), (ii), (iii) in (1.1). Hence, in these cases, by the concrete expressions in local coordinate of possible stable map germs and by Proposi-

tion 1.5, we also have the following examples.

1.17 *Example.* Assume $m \geq 2n + 1$. Then a pure n -dimensional analytic subvariety of a connected m -dimensional complex manifold is with ordinary singularities if, and only if, it is non-singular.

1.18 *Example.* A pure n -dimensional analytic subvariety of a connected $2n$ -dimensional complex manifold is with ordinary singularities if, and only if, it is locally isomorphic to one of the following :

- (i) non-singular point,
- (ii) a union of two n -planes which intersect transversely at a point (ordinary double point).

1.19 *Example.* Assume $n \geq 2$. Then a pure n -dimensional analytic subvariety of a connected $(2n - 1)$ -dimensional complex manifold is with ordinary singularities if, and only if, it is locally isomorphic to one of the following :

- (i) non-singular point,
- (ii) a union of two n -planes which intersect transversely along a line in \mathbb{C}^{2n-1} (ordinary double point),
- (iii) the germ of an analytic subvariety $f(\mathbb{C}^n)$ at the origin o in \mathbb{C}^{2n-1} , where $f: \mathbb{C}^n \rightarrow \mathbb{C}^{2n-1}$ is a proper holomorphic map defined by

$$\begin{cases} y_i \circ f = x_i & (1 \leq i \leq n-1) \\ y_n \circ f = x_n^2 \\ y_{n+i} = x_i x_n & (1 \leq i \leq n-1) \end{cases}$$

(cuspidal point),

- (iii) a union of three 2-planes which intersect transversely at a point in \mathbb{C}^3 (triple points : this occurs only when $n=2$).

1.20 REMARK. The concept of ordinary singularities is closely related to generic linear projections. Indeed, it is well-known that Example 1.12 ~ Example 1.19 (except for Example 1.15, which, we believe, gives a new example) are obtained by projecting algebraic manifolds embedded in the complex projective spaces of sufficiently large dimensions into lower dimensional linear subspaces by *generic* linear projections (cf. [8, Chapter IV, Corollary 3.6, Theorem 3.10], [26, Chapter I, 5], [22], [21]). This is because of the following fact due to J. N. Mather (cf. [17]) : Let X^n be an n -dimensional algebraic manifold embedded in the complex projective space \mathbb{P}^N . If a pair

of positive integers (n, m) with $n < m$ belongs to so-called "nice range", then the restriction $\pi|_{X^n - C} : X^n - C \rightarrow P^m$ of a generic linear projection π to $X^n - C$ ($C :=$ the center of the linear projection $\pi : P^N \rightarrow P^m$, i.e., the locus where π is not defined) is locally stable.

§ 2. Locally trivial family of displacements of an analytic subvariety in a compact complex manifold and its characteristic map

Throughout this section, let Y and Z be an m -dimensional compact complex manifold and an n -dimensional analytic subvariety of Y , respectively.

2.1 DEFINITION. By a *locally trivial family of displacements of Z in Y , parametrized by a complex space*, we mean a quadruple (\mathcal{Z}, π, M, o) , where M is a complex space, \mathcal{Z} a closed complex subspace of the product complex space $Y \times M$, π the restriction to \mathcal{Z} of the projection $Pr_M : Y \times M \rightarrow M$, and o an assigned point of M , which satisfy the following conditions:

(i) for each point $t \in M$, the intersection $\mathcal{Z} \cap (Y \times t) = \pi^{-1}(t)$ is an m -dimensional analytic subvariety of Y ,

(ii) $\pi^{-1}(o)$ coincides with Z ,

(iii) the projection $Pr_M : Y \times M \rightarrow M$ is locally a projection of a product space also on \mathcal{Z} , that is, for each point $p = (y, t) \in Y \times M$, there exist an open neighborhood $\mathcal{U} \subset Y \times M$ of p and an isomorphism $\varphi : \mathcal{U} \rightarrow U \times V$, where we define $U := \mathcal{U} \cap (Y \times t)$ and $V := Pr_M(\mathcal{U})$, such that:

(a) the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\varphi} & U \times V \\ & \searrow Pr_M & \swarrow Pr_V \\ & & V \end{array}$$

commutes,

(b) $\varphi(\mathcal{U} \cap \mathcal{Z}) = (U \cap \mathcal{Z}) \times V$, and

(c) $\varphi|_{U \times t} = id_{U \times t}$.

2.2 DEFINITION. Let $\mathcal{E} = (\mathcal{Z}, \pi, M, o)$ be a locally trivial family of displacements of Z in Y , parametrized by a complex space, let M' be a complex space, and let $h : M' \rightarrow M$ be a holomorphic map with $h(o') = o$, where o' is an assigned point of M' . We define a family $h^*\mathcal{E} = (\mathcal{Z}', \pi', M', o')$ induced by h as follows:

(i) $\mathcal{Z}' := \mathcal{Z} \times_M M'$ (the fiber product over M),

$$(ii) \quad \pi' := Pr_{M'} : \mathcal{Z}' \rightarrow M'.$$

In particular, if M' is a closed complex subspace of M and $h : M' \rightarrow M$ is the inclusion map, then we call the family $h^*\mathcal{E}$ the *restriction of \mathcal{E} to M'* and denote it by $\mathcal{E}|_{M'}$.

2.3 DEFINITION. Let $\mathcal{E} = (\mathcal{Z}, \pi, M, o)$ and $\mathcal{E}' = (\mathcal{Z}', \pi', M', o')$ be two locally trivial families of displacements of Z in Y , parametrized by complex spaces. By a *morphism* (resp. *isomorphism*) from \mathcal{E} into \mathcal{E}' we mean a holomorphic (resp. biholomorphic) map $h : M \rightarrow M'$ with $h(o) = o'$ such that if we define a holomorphic map $H : Y \times M \rightarrow Y \times M'$ by $H := id_Y \times h$, where id_Y denotes the identity map on Y , then $H(Z_t) = Z'_{h(t)}$ ($Z_t := \pi^{-1}(t), Z'_{h(t)} := \pi'^{-1}(h(t))$) for any $t \in M$.

Note that the condition $H(Z_t) = Z'_{h(t)}$ for any $t \in M$ is equivalent to $h^*\mathcal{E}' = \mathcal{E}$. We say that \mathcal{E} and \mathcal{E}' are *equivalent* if there exists an isomorphism from \mathcal{E} onto \mathcal{E}' .

2.4 DEFINITION. We say a locally trivial family $\mathcal{E} = (\mathcal{Z}, \pi, M, o)$ of displacements of Z in Y is *maximal* or *versal at o* , if, for any locally trivial family $\mathcal{E}' = (\mathcal{Z}', \pi', M', o')$ of Z in Y , there exist an open neighborhood N' of o' in M' and a morphism from the restriction $\mathcal{E}'|_{N'}$ of \mathcal{E}' to N' into \mathcal{E} . Furthermore, if it is maximal at every point $t \in M$, we say the family \mathcal{E} is *maximal* or *versal*.

2.5 DEFINITION. We say a locally trivial family $\mathcal{E} = (\mathcal{Z}, \pi, M, o)$ of displacements of Z in Y is *universal* (resp. *semi-universal*) at o if it satisfies the following:

(i) \mathcal{E} is maximal at o , and

(ii) for any locally trivial family $\mathcal{E}' = (\mathcal{Z}', \pi', M', o')$ of Z in Y and a holomorphic map $h : N' \rightarrow M$ from an open neighborhood N' of o' into M , which give a morphism from $\mathcal{E}'|_{N'}$ into \mathcal{E} , the germ (resp. the Jacobian map $(dh)_{o'} : T_{M', o'} \rightarrow T_{M, o}$) of such a holomorphic map h at o' is uniquely determined.

Furthermore, if it is universal (resp. semi-universal) at every point $t \in M$, we say the family \mathcal{E} is *universal* (resp. *semiuniversal*).

We denote by Θ_Y the sheaf of germs of holomorphic tangent vector fields on Y and by $\Theta_Y(\log Z)$ the sheaf of germs of *logarithmic tangent vector fields along Z on Y* , i. e., the subsheaf of Θ_Y consisting of derivations of \mathcal{O}_Y which send \mathcal{I}_Z , the ideal sheaf of Z in \mathcal{O}_Y , into itself.

2.6 DEFINITION. We denote by $\mathcal{N}_{Z/N}$ the quotient sheaf $\Theta_Y/\Theta_Y(\log Z)$

and call this the *sheaf of infinitesimal locally trivial displacements of Z in Y* .

We should note that the sheaf $\mathcal{N}_{Z/Y}$ can be considered as a sheaf over Z , because $\mathcal{I}_Z \cdot \theta_Y \subset \theta_Y(\log Z)$.

2.7 PROPOSITION. *For a locally trivial family (\mathcal{Z}, π, M, o) of displacements of Z in Y , we can define a so-called characteristic map*

$$\sigma_o : T_{M,o} \longrightarrow H^0(Y, \mathcal{N}_{Z/Y}),$$

which has functorial property. Here $T_{M,o}$ denotes the Zariski tangent space of M at o .

PROOF. Restricting M to an open neighborhood of o in M if necessary, we may assume the following :

(i) M is a closed complex subspace of a domain D of the complex number space \mathbf{C}^r with a system of local coordinate (t_1, \dots, t_r) and the ideal sheaf \mathcal{I}_M of M in the structure sheaf \mathcal{O}_D of D is generated by a finite number of holomorphic functions $g_1(t), \dots, g_s(t)$ defined in D ;

(ii) the assigned point o of M is the origin of \mathbf{C}^r ;

(iii) Y is covered by a finite number of Stein coordinate neighborhoods Y_i , $i \in I$;

(iv) for each Y_i there exist an open subset $\tilde{\mathcal{U}}_i$ of $Y_i \times D$ and a bi-holomorphic map $\tilde{\varphi}_i : \tilde{\mathcal{U}}_i \rightarrow U_i \times D$, where $U_i = \tilde{\mathcal{U}}_i \cap (Y \times o)$, such that :

(a) the diagram

$$\begin{array}{ccc} \tilde{\mathcal{U}}_i & \xrightarrow{\tilde{\varphi}_i} & U_i \times D \\ \text{Pr}_{D, \tilde{\mathcal{U}}_i} \searrow & & \swarrow \text{Pr}_D \\ & D & \end{array}$$

commutes,

(b) $\tilde{\varphi}_i(\tilde{\mathcal{U}}_i \cap \mathcal{Z}) = (U_i \cap \mathcal{Z}) \times M$,

(c) $\tilde{\varphi}_i|_{U_i \times o} = \text{id}_{U_i \times o}$, the identity map on $U_i \times o$,

(d) $U_i \cap \mathcal{Z}$ is the zero locus of a finite number of holomorphic functions defined in Y_i ;

(v) if we define $\mathcal{U}_i := \tilde{\mathcal{U}}_i \cap (Y \times M)$, then $\{\mathcal{U}_i\}_{i \in I}$ is an open covering of $Y \times M$.

Let (y_i^1, \dots, y_i^m) be a system of local coordinates on Y_i . Since $U_i \subset Y_i$, we may regard (y_i^1, \dots, y_i^m) as a system of local coordinates on U_i . We denote by (y_i, t) both a set of $m+r$ complex numbers $y_i^1, \dots, y_i^m, t_1, \dots, t_r$ and a point in $Y_i \times \mathbf{C}^r$ with the coordinates $(y_i^1, \dots, y_i^m, t_1, \dots, t_r)$. For each

$i \in I$ we express the biholomorphic map $\tilde{\varphi}_i : \tilde{U}_i \rightarrow U_i \times D$ as

$$\tilde{\varphi}_i(y_i, t) = (Y_i^1(y_i, t), \dots, Y_i^m(y_i, t), t)$$

by using local coordinates (y_i, t) , where $Y_i^1(y_i, t), \dots, Y_i^m(y_i, t)$ are holomorphic functions defined in \tilde{U}_i . We have $Y_i^\alpha(y_i, 0) = y_i^\alpha$ ($1 \leq \alpha \leq m$) because of $\tilde{\varphi}_i|_{U_i \times o} = id_{U_i \times o}$. We may consider $(Y_i^1(y_i, t), \dots, Y_i^m(y_i, t), t)$ as another system of local coordinates on \tilde{U}_i . On the intersection $\tilde{U}_i \cap \tilde{U}_j$ the coordinate functions Y_i^1, \dots, Y_i^m are holomorphic functions of Y_j^1, \dots, Y_j^m, t , so we write them as

$$(2.1) \quad Y_i^\alpha = F_{ij}^\alpha(Y_j^1, \dots, Y_j^m, t) \quad (1 \leq \alpha \leq m).$$

We identify $T_{M,o}$, the Zariski tangent space of M at o , with the subspace $\{v \in T_{Cr,o} | v(\mathcal{I}_{M,o}) = 0\}$ of $T_{Cr,o}$, where $\mathcal{I}_{M,o}$ denotes the stalk at o of the ideal sheaf sheaf of M in \mathcal{O}_D . For any $\partial/\partial t \in T_{M,o}$ we define

$$(2.2) \quad \theta_i := \sum_{\alpha=1}^m (\partial Y_i^\alpha / \partial t)|_{t=0} (\partial / \partial y_i^\alpha)$$

$$(\text{where } \partial Y_i^\alpha / \partial t = \sum_{\lambda=1}^m v_\lambda (\partial Y_i^\alpha / \partial t_\lambda) \quad \text{for } \partial / \partial t = \sum_{\lambda=1}^m v_\lambda (\partial / \partial t_\lambda)),$$

which we regard as a holomorphic vector field on $U_i = \tilde{U}_i \cap (Y \times o)$. Then we have

$$(2.3) \quad \begin{aligned} & \theta_i - \theta_j \\ &= \sum_{\alpha=1}^m \{ (\partial Y_i^\alpha / \partial t)|_{t=0} - \sum_{\beta=1}^m (\partial Y_j^\beta / \partial t)|_{t=0} \cdot (\partial F_{ij}^\alpha / \partial Y_j^\beta)|_{t=0} \} (\partial / \partial y_i^\alpha) \end{aligned}$$

on $U_i \cap U_j$. By (2.1) we have

$$(2.4) \quad (\partial Y_i^\alpha / \partial t)|_{t=0} = \sum_{\beta=1}^m (\partial Y_j^\beta / \partial t)|_{t=0} \cdot (\partial F_{ij}^\alpha / \partial Y_j^\beta)|_{t=0} + (\partial F_{ij}^\alpha / \partial t)|_{t=0}.$$

Substituting (2.4) into (2.3), we have

$$(2.5) \quad \theta_i - \theta_j = \sum_{\alpha=1}^m (\partial F_{ij}^\alpha / \partial t)|_{t=0} \cdot (\partial / \partial y_i^\alpha)$$

on $U_i \cap U_j$. Now we shall prove that $\theta_i - \theta_j \in \Gamma(U_i \cap U_j, \Theta_Y(\log Z))$. Since $\tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1}$ give rise to a biholomorphic map from $\tilde{\varphi}_j(\tilde{U}_i \cap \tilde{U}_j \cap \mathcal{Z})$ to $\tilde{\varphi}_i(\tilde{U}_i \cap \tilde{U}_j \cap \mathcal{Z})$, if we pull back the elements of the ideal sheaf $\mathcal{I}_{\tilde{\varphi}_i(\tilde{U}_i \cap \mathcal{Z})} = \mathcal{I}_{(U_i \cap \mathcal{Z}) \times M}$ of the closed complex subspace $\tilde{\varphi}_i(\tilde{U}_i \cap \mathcal{Z}) = (U_i \cap \mathcal{Z}) \times M$ in $\mathcal{O}_{U_i \times D}$ by the map $\tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1} : U_j \times D \rightarrow U_i \times D$, they belong to the ideal sheaf $\mathcal{I}_{\tilde{\varphi}_j(\tilde{U}_j \cap \mathcal{Z})} = \mathcal{I}_{(U_j \cap \mathcal{Z}) \times M}$. For each i let $h_i^1(y_i) = \dots = h_i^{k_i}(y_i) = 0$ be the defining equations of the analytic subvariety $U_i \cap \mathcal{Z}$ in U_i . Then the ideal sheaf $\mathcal{I}_{\tilde{\varphi}_i(\tilde{U}_i \cap \mathcal{Z})}$ is generated by $h_i^1(y_i), \dots, h_i^{k_i}(y_i), g_1(t), \dots, g_s(t)$. The pull-back of $h_i^k(y_i)$ ($1 \leq k \leq k_i$) by the map $\tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1} : U_j \times D \rightarrow U_i \times D$ is calculated as follows :

$$\begin{aligned}
& (\tilde{\varphi}_i \circ \tilde{\varphi}_j^{-1})^*(h_i^k(y_i)) = (\tilde{\varphi}_j^{-1})^*((\tilde{\varphi}_i)^*h_i^k(y_i)) \\
& = (\tilde{\varphi}_j^{-1})^*(h_i^k(Y_i^1(y_i, t), \dots, Y_i^m(y_i, t))) \\
& = (\tilde{\varphi}_j^{-1})^*(h_i^k(F_{ij}^1(Y_j^1(y_j, t), \dots, Y_j^m(y_j, t), t), \dots, F_{ij}^m(Y_j^1(y_j, t), \dots, Y_j^m(y_j, t), t))) \\
& = h_i^k(F_{ij}^1(y_j, t), \dots, F_{ij}^m(y_j, t)).
\end{aligned}$$

Since $h_i^k(F_{ij}^1(y_j, t), \dots, F_{ij}^m(y_j, t))$ belongs to the ideal sheaf $\mathcal{S}_{\tilde{\varphi}_j(\tilde{U}_j \cap Z)} = \mathcal{S}_{(U_j \cap Z) \times M}$, we can express it as

$$\begin{aligned}
(2.6) \quad & h_i^k(F_{ij}^1(y_j, t), \dots, F_{ij}^m(y_j, t)) \\
& = \sum_{\alpha=1}^{k_j} a_j^\alpha(y_j, t) h_j^\alpha(y_j) + \sum_{\beta=1}^s b^\beta(y_j, t) g_\beta(t),
\end{aligned}$$

where $a_j^\alpha(y_j, t), b^\beta(y_j, t)$ are holomorphic functions on $U_j \times D$. We operate $\partial/\partial t = \sum_{\lambda=1}^r v_\lambda(\partial/\partial t_\lambda) \in T_{M,o}$ to the both sides of (2.6) and substitute o for t . Here we should recall that we regard $\partial/\partial t \in T_{M,o}$ as an element of $T_{Cr,o}$ with $(\partial/\partial t)(\mathcal{S}_{M,o}) = 0$. Then, using the identities $Y_j(y_j, o) = y_j$ and $y_i = Y_i(y_i, o) = F_{ij}(Y_j(y_j, o), o) = F_{ij}(y_j, o)$, we have

$$\sum_{r=1}^m (\partial F_{ij}^r/\partial t)(y_j, o) \cdot (\partial h_i^k/\partial y_i^r)(y_i) = \sum_{\alpha=1}^{k_j} (\partial a_j^\alpha/\partial t)(y_j, o) \cdot h_j^\alpha(y_j)$$

for any k with $1 \leq k \leq k_i$. In view of (2.5) this shows that $\theta_i - \theta_j$ sends the ideal sheaf \mathcal{S}_Z of Z in \mathcal{O}_Y into itself on $U_i \cap U_j$. That is, $\theta_i - \theta_j \in \Gamma(U_i \cap U_j, \Theta_Y(\log Z))$. Therefore the collection $\{Q(\theta_i)\}_{i \in I}$ defines an element of $H^0(Y, \mathcal{N}_{Z/Y})$, so an element of $H^0(Z, \mathcal{N}_{Z/Y})$, where Q denotes the map $\Gamma(U_i, \Theta_Y) \rightarrow \Gamma(U_i, \mathcal{N}_{Z/Y})$ induced by the natural homomorphism of \mathcal{O}_Y -modules: $\Theta_Y \rightarrow \mathcal{N}_{Z/Y}$. Obviously, the cohomology class of $H^0(Z, \mathcal{N}_{Z/Y})$ defined by the collection $\{Q(\theta_i)\}_{i \in I}$ is invariant under a refinement of coverings and changing a system of local trivializations $\{\tilde{\varphi}_i: \tilde{U}_i \rightarrow U_i \times D\}_{i \in I}$. Hence it is independent of the choice of them. We define a so-called characteristic map $\sigma_o: T_{M,o} \rightarrow H^0(Z, \mathcal{N}_{Z/Y})$ of the family (\mathcal{Z}, π, M, o) by $\sigma_o(\partial/\partial t) = \{Q(\theta_i)\}_{i \in I}$ for any $\partial/\partial t \in T_{M,o}$. It is almost trivial that thus defined characteristic map has functorial property, because a morphism from a locally trivial family (\mathcal{Z}, π, M, o) of displacements of Z in Y to another one $(\mathcal{Z}', \pi', M', o')$ is a holomorphic map $h: M \rightarrow M'$ such that $h(o) = o'$ and $(id_Y \times h)(Z_i) = Z'_{h(i)}$ for any $t \in M$ by definition. Q. E. D.

2.8 DEFINITION. We say a locally trivial family $\mathcal{E} = (\mathcal{Z}, \pi, M, o)$ of displacements of Z in Y is *effective at o* if its characteristic map $\sigma_o: T_{M,o} \rightarrow H^0(Z, \mathcal{N}_{Z/Y})$ is injective. Furthermore, if it is effective at every point $t \in M$, we say the family \mathcal{E} is *effective*.

2.9 DEFINITION. We call a locally trivial family (\mathcal{Z}, π, M, o) of displacements of Z in Y a *Kuranishi family for locally trivial displacements of Z in Y* if it satisfies the following two conditions:

- (i) it is maximal,
- (ii) it is effective at o .

2.10 PROPOSITION. *The Kuranishi family $\mathcal{E}=(\mathcal{Z}, \pi, M, o)$ for locally trivial displacements of Z in Y is semi-universal at o .*

PROOF. Let $\mathcal{E}'=(\mathcal{Z}', \pi', M', o')$ be another locally trivial family of displacements of Z in Y . Then, since the Kuranishi family $\mathcal{E}=(\mathcal{Z}, \pi, M, o)$ is maximal at o , there exist an open neighborhood N' of o' in M' and a holomorphic map $h: N' \rightarrow M$ with $h(o')=o$ such that $h^*\mathcal{E}=\mathcal{E}'_{|N'}$. By the functorial property of characteristic maps, we have a commutative diagram

$$\begin{array}{ccc} T_{M', o'} & \xrightarrow{(dh)_{o'}} & T_{M, o} \\ \sigma_{o'} \searrow & & \swarrow \sigma_o \\ & H^0(\mathcal{X}, \mathcal{N}_{Z/Y}) & \end{array}$$

where σ_o and $\sigma_{o'}$ denote the characteristic maps of the families \mathcal{E} and \mathcal{E}' , respectively. Since \mathcal{E} is the Kuranishi family, σ_o is injective. Therefore, by the commutativity of the diagram in (2.7) we conclude that the Jacobian map $(dh)_{o'}: T_{M', o'} \rightarrow T_{M, o}$ is uniquely determined. Q. E. D.

§ 3. Relative normalization of a locally trivial family of analytic varieties

3.1 DEFINITION. We call a surjective holomorphic map $\pi: \mathcal{Z} \rightarrow M$ between complex spaces a *locally trivial family of analytic varieties parametrized by a complex space M* , if it satisfies the following conditions:

- (i) the fiber $Z_t := \pi^{-1}(t)$ is an analytic variety for any $t \in M$,
- (ii) for each point $p \in \mathcal{Z}$, there exist open neighborhoods \mathcal{U} of p in \mathcal{Z} , V of $\pi(p)$ in M with $\pi(\mathcal{U})=V$, and a biholomorphic map $\varphi: \mathcal{U} \rightarrow U \times V$, where $U := \mathcal{U} \cap Z_{\pi(p)}$, such that the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\varphi} & U \times V \\ \pi_{|\mathcal{U}} \searrow & & \swarrow Pr_V \\ & V & \end{array}$$

commutes.

In the following, let Z be an analytic variety. We denote by $S(Z)$ the singular locus of Z . We define $\hat{Z} := Z \setminus S(Z)$.

3.2 DEFINITION. A *weakly holomorphic function on Z* is a holomorphic function h on \hat{Z} , which is locally bounded on Z ; that is, for any point $p \in S(Z)$ there exist an open neighborhood U of p in Z and a real number K such that $|h(z)| \leq K$ for any $z \in \hat{U} := U \setminus S(Z)$.

For a locally trivial family $\pi: \mathcal{Z} \rightarrow M$ of analytic varieties parametrized by a complex space M , we define

$$S(\mathcal{Z}/M) := \{p \in \mathcal{Z} \mid \text{the fiber } Z_{\pi(p)} \text{ is singular at } p\}.$$

Since the family $\pi: \mathcal{Z} \rightarrow M$ is locally trivial, the set $S(\mathcal{Z}/M)$ naturally has the structure of a closed complex subspace of \mathcal{Z} . We call $S(\mathcal{Z}/M)$ the *relative singular locus of the family $\pi: \mathcal{Z} \rightarrow M$* . We define $\mathring{\mathcal{Z}} := \mathcal{Z} \setminus S(\mathcal{Z}/M)$ and denote by $\iota_* \mathcal{O}_{\mathring{\mathcal{Z}}}$ the direct image of the structure sheaf $\mathcal{O}_{\mathring{\mathcal{Z}}}$ of $\mathring{\mathcal{Z}}$ by the inclusion map $\iota: \mathring{\mathcal{Z}} \hookrightarrow \mathcal{Z}$. We want to define the sheaf $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ of germs of so-called *weakly holomorphic functions along fibers of the family $\pi: \mathcal{Z} \rightarrow M$* , which is a subsheaf of $\iota_* \mathcal{O}_{\mathring{\mathcal{Z}}}$.

First, we consider the case where both the following two conditions are satisfied:

- (i) \mathcal{Z} is a product family, namely, $\mathcal{Z} := Z \times M$ and $\pi := \text{Pr}_M: \mathcal{Z} = Z \times M \rightarrow M$, the projection to M , where Z is an analytic variety,
- (ii) M is a closed complex subspace of a domain D in the complex number space \mathbf{C}^m .

In this case we define

$$(3.1) \quad \tilde{\mathcal{O}}_{\mathcal{Z}/M} := \mathcal{O}_{\mathcal{Z}} \otimes_{\mathcal{O}_{Z \times D}} [\tilde{\mathcal{O}}_{Z \times D} / (\pi'^* \mathcal{I}_M) \cdot \tilde{\mathcal{O}}_{Z \times D}],$$

where $\tilde{\mathcal{O}}_{Z \times D}$ denotes the sheaf of germs of weakly holomorphic functions on $Z \times D$, $\pi' := \text{Pr}_D: \mathcal{Z}' = Z \times D \rightarrow D$ the projection to D , and \mathcal{I}_M the ideal sheaf of M in \mathcal{O}_D .

3.3 LEMMA. *Let Z and W be analytic varieties. Let M (resp. N) be a closed complex subspace of a domain D (resp. D') of the complex number space \mathbf{C}^n (resp. \mathbf{C}^m). We define $\mathcal{Z} := Z \times M$, $\pi_1 := \text{Pr}_M: \mathcal{Z} = Z \times M \rightarrow M$, $\mathcal{W} := W \times N$, and $\pi_2 := \text{Pr}_N: \mathcal{W} := W \times N \rightarrow N$. Suppose we are given open subsets \mathcal{U} and \mathcal{V} of \mathcal{Z} and \mathcal{W} , respectively, such that there exists a fiber preserving biholomorphic map $f: \mathcal{U} \rightarrow \mathcal{V}$, namely if we define $U := \pi_1(\mathcal{U})$ and $V := \pi_2(\mathcal{V})$, there exists a biholomorphic map $f_0: U \rightarrow V$ such*

that the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & \mathcal{C}\mathcal{V} \\ \pi_1 \downarrow & f_0 & \downarrow \pi_2 \\ U & \longrightarrow & V \end{array}$$

commutes. Then there naturally exists a $\mathcal{O}_{\mathcal{C}\mathcal{V}}$ -isomorphism

$$\alpha : \tilde{\mathcal{O}}_{\mathbb{W}/N|\mathcal{C}\mathcal{V}} \xrightarrow{\cong} f_*(\tilde{\mathcal{O}}_{Z/M|U}).$$

PROOF. Let $p=(z_o, t_o)$ be a point of \mathcal{U} . We define $q := f(p) = (w_o, s_o) \in \mathcal{C}\mathcal{V}$. We shall show that there naturally exists an isomorphism of $\mathcal{O}_{\mathbb{W},q}$ -modules

$$\alpha_q : \tilde{\mathcal{O}}_{\mathbb{W}/N,q} \longrightarrow \tilde{\mathcal{O}}_{Z/M,p}.$$

We write f as $f(x, t) := (f_1(x, t), f_0(t)) \in \mathcal{C}\mathcal{V}$ for $(x, t) \in \mathcal{U}$. We take open neighborhoods \tilde{U} of p in $Z \times D$, W' of w_o in W and \tilde{V} of t_o in D' , and holomorphic maps $F_1 : \tilde{U} \rightarrow W'$ and $F_0 : \pi'_1(\tilde{U}) \rightarrow \tilde{V}$, where $\pi'_1 := P_{\tau_D} : Z \times D \rightarrow D$, satisfying the following conditions:

- (i) $\tilde{U} \cap (Z \times M) \subset \mathcal{U}$,
- (ii) $(W' \times \tilde{V}) \cap (W \times N) \subset \mathcal{C}\mathcal{V}$,
- (iii) $f_1(\tilde{U} \cap (Z \times M)) \subset W'$ and the following diagram commutes;

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{F_1} & W' \\ \uparrow & f_1 & \parallel \\ \tilde{U} \cap (Z \times M) & \longrightarrow & W' \end{array}$$

- (iv) $f_0(\pi'_1(\tilde{U}) \cap U) \subset \tilde{V} \cap V$ and the following diagram commutes;

$$\begin{array}{ccc} \pi'_1(\tilde{U}) & \xrightarrow{F_0} & \tilde{V} \\ \uparrow & f_0 & \uparrow \\ \pi'_1(\tilde{U}) \cap U & \longrightarrow & \tilde{V} \cap V \end{array}$$

We define

$$F(z, t) := (F_1(z, t), F_0(t)) \quad ((z, t) \in \tilde{U}).$$

Then F is a holomorphic map from \tilde{U} into $W' \times \tilde{V}$ and the following diagram commutes:

$$(3.2) \quad \begin{array}{ccc} \tilde{\mathcal{U}} & \xrightarrow{F} & W' \times \tilde{V} \\ \uparrow & f & \uparrow \\ \tilde{\mathcal{U}} \cap (Z \times M) & \longrightarrow & W' \times (\tilde{V} \cap V). \end{array}$$

We say F generate f at p . Let $a \in \tilde{\mathcal{O}}_{\mathcal{W}/N, q}$. We take $A \in \tilde{\mathcal{O}}_{W \times D', q}$ whose image by the map $\tilde{\mathcal{O}}_{W \times D', q} \rightarrow \tilde{\mathcal{O}}_{\mathcal{W}/N, q}$ is a . Obviously, F^*A (the pull-back of A by F) $\in \tilde{\mathcal{O}}_{Z \times D, p}$. We define

$$\alpha_q(a) := \text{the image of } F^*A \text{ by the map } \tilde{\mathcal{O}}_{Z \times D, p} \longrightarrow \tilde{\mathcal{O}}_{Z/M, p}.$$

We can see that $\alpha_q(a)$ does not depend on the choice of a holomorphic map which generates f at p as follows:

Let F' be another holomorphic map which generates f at p . There is a non-zero divisor $d \in \mathcal{O}_{W \times D', q}$, a so-called *universal denominator at q* , with a property $d \cdot \tilde{\mathcal{O}}_{W \times D', q} \subset \mathcal{O}_{W \times D', q}$ (cf. [20, Chapter III, Corollary 2 of Theorem 6]). Hence we can represent A as $A = B/d$, where $B \in \mathcal{O}_{W \times D', q}$. Then we have

$$(3.3) \quad \begin{aligned} F^*A - F'^*A &= F^*B/F^*d - F'^*B/F'^*d \\ &= [(F^*B - F'^*B)F'^*d + F'^*B(F'^*d - F^*d)]/F^*d \cdot F'^*d \end{aligned}$$

Since $F^*B - F'^*B, F'^*d - F^*d \in (\pi_1'^* \mathcal{I}_M)_p (\pi_1' := Pr_D : Z \times D \rightarrow D)$, by (3.3) we conclude that the image of $F^*A - F'^*A$ by the map $\tilde{\mathcal{O}}_{Z \times D, p} \rightarrow \tilde{\mathcal{O}}_{Z/M, p}$ is zero as required.

THE PROOF OF INJECTIVITY OF α_q : By the same argument as used in deriving the diagram in (3.2), we can take open neighborhoods $\tilde{\mathcal{C}}\tilde{\mathcal{V}}$ of $q = (w_o, s_o)$ in $W' \times \tilde{V}$, Z' of z_o in Z , \tilde{U} of t_o in D with $Z' \times \tilde{U} \subset \tilde{\mathcal{U}}$, and a holomorphic map $G : \tilde{\mathcal{C}}\tilde{\mathcal{V}} \rightarrow Z' \times \tilde{U}$ such that the following diagram commutes:

$$(3.4) \quad \begin{array}{ccc} Z' \times \tilde{U} & \xleftarrow{G} & \tilde{\mathcal{C}}\tilde{\mathcal{V}} \\ \uparrow & f^{-1} & \uparrow \\ Z' \times (\tilde{U} \cap U) & \xleftarrow{\quad} & \tilde{\mathcal{C}}\tilde{\mathcal{V}} \cap (W \times N). \end{array}$$

Suppose $a \in \tilde{\mathcal{O}}_{\mathcal{W}/N, q}$ satisfies $\alpha_q(a) = 0$. As before we take $A \in \tilde{\mathcal{O}}_{W \times D', q}$ whose image by the map $\tilde{\mathcal{O}}_{W \times D', q} \rightarrow \tilde{\mathcal{O}}_{\mathcal{W}/N, q}$ is a and represent it as $A = B/d$, where $d, B \in \mathcal{O}_{W \times D', q}$. Then $\alpha_q(a) = 0$ implies $F^*d \cdot F^*A = F^*B \in (\pi_1'^* \mathcal{I}_M)_p$. Then we have $G^*F^*d \cdot G^*F^*A = G^*F^*B \in (\pi_2'^* \mathcal{I}_N)_q (\pi_2' := Pr_D : W \times D' \rightarrow D')$. Hence

$$(3.5) \quad G^*F^*A = G^*F^*B/G^*F^*d \in (\pi_2'^* \mathcal{I}_N)_q \cdot \tilde{\mathcal{O}}_{W \times D', q}.$$

On the other hand, we have

$$(3.6) \quad \begin{aligned} G^*F^*A - A &= G^*F^*B/G^*F^*d - B/d \\ &= [(G^*F^*B - B)d - B(G^*F^*d - d)]/(G^*F^*d) \cdot d \end{aligned}$$

Since $G^*F^*B - B$, $G^*F^*d - d \in (\pi_2^* \mathcal{J}_N)_q$, by (3.6) we have

$$(3.7) \quad G^*F^*A - A \in (\pi_2^* \mathcal{J}_N)_q \cdot \tilde{\mathcal{O}}_{W \times D', q}.$$

From (3.5) and (3.7) it follows $A \in (\pi_2^* \mathcal{J}_N)_q \cdot \tilde{\mathcal{O}}_{W \times D', q}$, that is, $a = 0$. Therefore we conclude that α_q is injective.

THE PROOF OF SURJECTIVITY OF α_q : Let b be any element of $\tilde{\mathcal{O}}_{Z/M, p}$. We take $B \in \tilde{\mathcal{O}}_{Z \times D, p}$ whose image by the map $\tilde{\mathcal{O}}_{Z \times D, p} \rightarrow \tilde{\mathcal{O}}_{Z/M, p}$ is b . We define

$$a := \text{the image of } G^*B \text{ by the map } \tilde{\mathcal{O}}_{W \times D', q} \longrightarrow \tilde{\mathcal{O}}_{qW/N, q}.$$

Then

$$\alpha_q(a) = \text{the image of } F^*G^*B \text{ by the map } \tilde{\mathcal{O}}_{Z \times D, p} \longrightarrow \tilde{\mathcal{O}}_{Z/M, p}.$$

By the same argument used in deriving (3.7), we can derive

$$(3.8) \quad F^*G^*B - B \in (\pi_1^* \mathcal{J}_M)_p \cdot \tilde{\mathcal{O}}_{Z \times D, p}$$

from the commutativity of the diagrams in (3.2) and (3.4). By (3.8) we conclude that $\alpha_q(a) = b$, that is, α_q is surjective. Q. E. D.

Next, we shall show that, for a locally trivial family $\pi: \mathcal{Z} \rightarrow M$ of analytic varieties parametrized by a complex space, we can define the sheaf $\tilde{\mathcal{O}}_{Z/M}$ of germs of so-called *weakly holomorphic functions along fibers* by patching up the sheaves defined locally by (3.1) on sufficiently small open neighborhoods of each point of \mathcal{Z} . We take an open covering $\{\mathcal{U}_i\}_{i \in I}$ of \mathcal{Z} satisfying the following conditions for each $i \in I$:

(i) there exist an open subset V_i of M with $\pi(\mathcal{U}_i) = V_i$, a point $t_i \in V_i$, and a biholomorphic map $\varphi_i: \mathcal{U}_i \rightarrow U_i \times V_i$, where $U_i := \mathcal{U}_i \cap \pi^{-1}(t_i)$, such that the diagram

$$\begin{array}{ccc} \mathcal{U}_i & \xrightarrow{\varphi_i} & U_i \times V_i \\ \pi|_{\mathcal{U}_i} \searrow & & \swarrow Pr_{V_i} \\ & & V_i \end{array}$$

commutes,

(ii) there is an embedding $e_i : V_i \rightarrow D_i$, where D_i is a domain of the complex number space \mathbf{C}^{n_i} .

For such an open covering $\{\mathcal{U}_i\}_{i \in I}$ of \mathcal{Z} , we define $V'_i := e_i(V_i)$; $\mathcal{U}'_i := U_i \times V'_i$; $\pi'_i := Pr_{V'_i} : \mathcal{U}'_i \rightarrow V'_i$ (the projection to the second factor); $\varphi'_i := (id_{U_i} \times e'_i) \circ \varphi_i : \mathcal{U}_i \rightarrow \mathcal{U}'_i$, where $id_{U_i} : U_i \rightarrow U_i$ denotes the identity map on U_i and $V_i \xrightarrow{e'_i} V'_i \subset D_i$ a factorization of e_i ; $\mathring{\mathcal{U}}_i := \mathcal{U}_i \setminus S(\mathcal{Z}/M)$; $\mathring{U} := U \setminus S(U)$; $\mathring{\mathcal{U}}'_i := \mathring{U} \times V'_i$; and $\mathring{\varphi}'_i := \mathring{\varphi}'_i \circ \mathring{\varphi}_i : \mathring{\mathcal{U}}_i \rightarrow \mathring{\mathcal{U}}'_i$. We denote by $\kappa_i : \mathring{\mathcal{U}}'_i \subset \mathcal{U}'_i$ the inclusion map. The relation among these symbols is illustrated as follows :

$$(3.9) \quad \begin{array}{ccc} \mathcal{U}_i & \xrightarrow{\varphi'_i} & \mathcal{U}'_i \\ \swarrow \mathcal{U}_i & \searrow \varphi'_i & \swarrow \mathcal{U}'_i \\ \mathcal{U}'_i & \xrightarrow{\varphi'_i} & \mathcal{U}'_i \\ \downarrow \pi_{|\mathcal{U}_i} & & \downarrow \pi'_{|\mathcal{U}'_i} \\ V_i & \xrightarrow{e'_i} & V'_i \end{array}$$

We denote by $\overline{\varphi'_i}^{-1} : \mathcal{O}_{\mathcal{Z}|\mathcal{U}_i} = \mathcal{O}_{U_i} \rightarrow (\varphi'_i)^{-1} \ast \mathcal{O}_{U'_i}$ the isomorphism of \mathcal{O}_{U_i} -modules associated to a biholomorphic map $\varphi'_i : \mathcal{U}'_i \rightarrow \mathcal{U}_i$. This isomorphism induces an isomorphism $\iota \ast \mathcal{O}_{\mathcal{Z}|\mathcal{U}_i} \rightarrow (\varphi'_i)^{-1} \ast \kappa_i \ast \mathcal{O}_{\mathring{\mathcal{U}}'_i}$ of \mathcal{O}_{U_i} -modules, because φ'_i induces a biholomorphic map between $\mathring{\mathcal{U}}'_i$ and $\mathring{\mathcal{U}}_i$. We denote this isomorphism by the same letter, namely,

$$\overline{\varphi'_i}^{-1} : \iota \ast \mathcal{O}_{\mathcal{Z}|\mathcal{U}_i} \longrightarrow (\varphi'_i)^{-1} \ast \kappa_i \ast \mathcal{O}_{\mathring{\mathcal{U}}'_i}.$$

Over \mathcal{U}_i we define the sheaf $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ of germs of *weakly holomorphic functions along fibers of the family* $\pi : \mathcal{Z} \rightarrow M$ by

$$(3.10) \quad \tilde{\mathcal{O}}_{\mathcal{Z}/M|\mathcal{U}_i} := (\overline{\varphi'_i}^{-1})^{-1} [(\varphi'_i)^{-1} \ast \tilde{\mathcal{O}}_{\mathcal{U}'_i/V'_i}].$$

For this definition to make a sense we need to show that if $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, then

$$(3.11) \quad (\overline{\varphi'_i}^{-1})^{-1} [(\varphi'_i)^{-1} \ast \tilde{\mathcal{O}}_{\mathcal{U}'_i/V'_i}] = (\overline{\varphi'_j}^{-1})^{-1} [(\varphi'_j)^{-1} \ast \tilde{\mathcal{O}}_{\mathcal{U}'_j/V'_j}] \text{ over } \mathcal{U}_i \cap \mathcal{U}_j.$$

Suppose $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$, then by (3.9) we have the following commutative diagram :

$$\begin{array}{ccc} \varphi'_i(\mathcal{U}_i \cap \mathcal{U}_j) & \xrightarrow{\varphi'_j \circ \varphi'^{-1}_i} & \varphi'_j(\mathcal{U}_i \cap \mathcal{U}_j) \\ \downarrow \pi'_i & & \downarrow \pi'_j \\ e_i(\pi(\mathcal{U}_i \cap \mathcal{U}_j)) & \xrightarrow{e_j \circ e^{-1}_i} & e_j(\pi(\mathcal{U}_i \cap \mathcal{U}_j)). \end{array}$$

That is, $\varphi'_j \circ \varphi'^{-1}_i$ gives a fiber preserving biholomorphic map from $\varphi'_i(\mathcal{U}_i \cap \mathcal{U}_j)$ to $\varphi'_j(\mathcal{U}_i \cap \mathcal{U}_j)$. Hence by Lemma 3.3 there is naturally a $\mathcal{O}_{\varphi'_j(\mathcal{U}_i \cap \mathcal{U}_j)}$ -isomorphism

$$(3.12) \quad \alpha_{ij} : \tilde{\mathcal{O}}_{\mathcal{U}'_j/V'_j|\varphi'_j(\mathcal{U}_i \cap \mathcal{U}_j)} \longrightarrow (\varphi'_j \circ \varphi'^{-1}_i)_* \tilde{\mathcal{O}}_{\mathcal{U}'_i/V'_i|\varphi'_i(\mathcal{U}_i \cap \mathcal{U}_j)}.$$

From this we obtain the following commutative diagram :

$$(3.13) \quad \begin{array}{ccc} (\varphi'^{-1}_j)_* \tilde{\mathcal{O}}_{\mathcal{U}'_j/V'_j|\mathcal{U}_i \cap \mathcal{U}_j} & \xrightarrow{(\varphi'^{-1}_j)^{-1}} & \iota_* \mathcal{O}_{\mathbb{Z}^1|\mathcal{U}_i \cap \mathcal{U}_j} \\ \downarrow & & \parallel \\ (\varphi'^{-1}_i)_* \tilde{\mathcal{O}}_{\mathcal{U}'_i/V'_i|\mathcal{U}_i \cap \mathcal{U}_j} & \xrightarrow{(\varphi'^{-1}_i)^{-1}} & \iota_* \mathcal{O}_{\mathbb{Z}^1|\mathcal{U}_i \cap \mathcal{U}_j}, \end{array}$$

where the vertical arrows on the left hand side is an isomorphism of $\mathcal{O}_{\mathcal{U}_i \cap \mathcal{U}_j}$ -modules induced by α_{ij} in (3.12). Therefore, by (3.13) we conclude that the equality in (3.11) certainly holds over $\mathcal{U}_i \cap \mathcal{U}_j$ as required. Obviously, the sheaf $\tilde{\mathcal{O}}_{\mathbb{Z}^1|M}$ defined above is invariant under a refinement of covering and changing a system of local trivializations $\{\varphi_i : \mathcal{U}_i \rightarrow \tilde{U}_i \times V_i\}_{i \in I}$. Hence it is independent of the choice of them.

3.4 PROPOSITION. *The sheaf $\tilde{\mathcal{O}}_{\mathbb{Z}^1|M}$ of germs of weakly holomorphic functions along fibers of a locally trivial family $\pi : \mathcal{Z} \rightarrow M$ of analytic varieties parametrized by a complex space is a coherent $\mathcal{O}_{\mathbb{Z}}$ -modules.*

PROOF. The quetion being local, we may assume that $\mathcal{Z} = Z \times M$ and $\pi = Pr_M : \mathcal{Z} = Z \times M \rightarrow M$ (the projection to M), where Z is an analytic variety and M a closed complex subspace of a domain D in the complex number space \mathbb{C}^n . Then, by (3.1) we conclude the sheaf $\tilde{\mathcal{O}}_{\mathbb{Z}^1|M}$ is a coherent $\mathcal{O}_{\mathbb{Z}}$ -module, because the sheaf $\tilde{\mathcal{O}}_{Z \times D}$ of germs of weakly holomorphic functions on a reduced complex space $Z \times D$ is a coherent $\mathcal{O}_{Z \times D}$ -module (cf. [20, Chapter VI, Theorem 4]). Q. E. D.

By this proposition, for a locally trivial family $\pi : \mathcal{Z} \rightarrow M$ of analytic varieties parametrized by a complex space, we can make an analytic spectrum of the sheaf $\tilde{\mathcal{O}}_{\mathbb{Z}^1|M}$ (cf. [10], [2, Chaper 1, 1.14]). We denote it by $\zeta : \mathcal{Z}^* \rightarrow \mathcal{Z}$ and for brevity we write

$$\mathcal{Z}^* := \text{Specan } \tilde{\mathcal{O}}_{\mathbb{Z}^1|M}$$

for it.

3.5 LEMMA. *Let $\pi: \mathcal{Z} = Z \times M \rightarrow M$ be a product family of an analytic variety Z , parametrized by a complex space M . Then, concerning its analytic spectrum $\zeta: \mathcal{Z}^* \rightarrow \mathcal{Z}$ we have the following:*

For any point $p = (z, t) \in \mathcal{Z} = Z \times M$, there are open neighborhoods U of z in Z , V of t in M , and a biholomorphic map $\phi: \zeta^{-1}(U \times V) \rightarrow U^ \times V$ such that the diagram*

$$\begin{array}{ccc} \zeta^{-1}(U \times V) & \xrightarrow{\phi} & U^* \times V \\ \zeta \searrow & & \swarrow n \times id_V \\ & U \times V & \end{array}$$

commutes, where $n: U^ \rightarrow U$ denotes the normalization of U .*

PROOF. The question being local, we may assume that M is a closed complex subspace of a domain D in the complex number space \mathbf{C}^N . Let $g_0 = 1, g_1, \dots, g_l$ be the generators of $\mathcal{O}_{Z \times D, p}$ -module $\tilde{\mathcal{O}}_{Z \times D, p}$ and $h_0 = 1, h_1, \dots, h_l$ the images of $g_0 = 1, g_1, \dots, g_l$ by the map $\tilde{\mathcal{O}}_{Z \times D, p} \rightarrow \tilde{\mathcal{O}}_{\mathcal{Z}/M, p}$, respectively (cf. (3.1)). Here $\mathcal{O}_{Z \times D, p}, \tilde{\mathcal{O}}_{Z \times D, p}, \tilde{\mathcal{O}}_{\mathcal{Z}/M, p}$ denote the stalks of the sheaves $\mathcal{O}_{Z \times D}, \tilde{\mathcal{O}}_{Z \times D}, \tilde{\mathcal{O}}_{\mathcal{Z}/M}$ at $p \in \mathcal{Z} = Z \times M \subset Z \times D$. Then $h_0 = 1, h_1, \dots, h_l$ generate $\tilde{\mathcal{O}}_{\mathcal{Z}/M, p}$ as a $\mathcal{O}_{\mathcal{Z}, p}$ -module. First, we consider *Specan* $\tilde{\mathcal{O}}_{Z \times D}$ in place of $\mathcal{Z}^* = \text{Specan } \tilde{\mathcal{O}}_{\mathcal{Z}/M}$. In the following we denote by $\mathbf{R}[x_1, \dots, x_l]$ the polynomial ring in n variables over a local \mathbf{C} -algebra \mathbf{R} . We have a canonical epimorphism

$$(3.14) \quad \rho: \mathcal{O}_{Z \times D, p}[x_1, \dots, x_l] \longrightarrow \tilde{\mathcal{O}}_{Z \times D, p} \text{ with } \rho(x_i) = g_i \ (1 \leq i \leq l).$$

For each i ($1 \leq i \leq l$) and any natural number k we define

$$M_k^{(i)} := \mathcal{O}_{Z \times D, p} + \mathcal{O}_{Z \times D, p} \cdot g_i + \dots + \mathcal{O}_{Z \times D, p} \cdot g_i^k \subset \tilde{\mathcal{O}}_{Z \times D, p}.$$

Since $\tilde{\mathcal{O}}_{Z \times D, p}$ is a finitely generated $\mathcal{O}_{Z \times D, p}$ -module, it is Noetherian. Hence we have

$$M_k^{(i)} = M_{k+1}^{(i)}$$

for sufficiently large k . Therefore, for each i ($1 \leq i \leq l$) there are $\lambda_0, \dots, \lambda_k \in \mathcal{O}_{Z \times D, p}$ such that

$$g_i^{k+1} = \lambda_0 + \lambda_1 g_i + \dots + \lambda_k g_i^k.$$

This implies that the kernel of ρ contains monic polynomials

$$f_1 \in \mathcal{O}_{Z \times D, p}[x_1], \dots, f_l \in \mathcal{O}_{Z \times D, p}[x_l].$$

Hence ρ induces an epimorphism

$$\sigma : \mathcal{O}_{Z \times D, p}[x_1, \dots, x_l]/(f_1, \dots, f_l) \longrightarrow \tilde{\mathcal{O}}_{Z \times D, p}$$

of finitely generated $\mathcal{O}_{Z \times D, p}$ -modules. Such an epimorphism σ can be extended to an open neighborhood of $p=(z, t)$ in $Z \times D$. Namely, taking sufficiently small open neighborhoods U of z in Z and D' of t in D , for every i ($1 \leq i \leq l$) we may find representatives

$$G_i \in \Gamma(U \times D', \tilde{\mathcal{O}}_{Z \times D}) \quad \text{and} \quad F_i \in \Gamma(U \times D', \mathcal{O}_{Z \times D})[x_i]$$

of g_i and f_i such that

- (i) $F_i(G_i) \equiv 0$,
- (ii) a morphism

$$\tilde{\sigma} : \mathcal{O}_{U \times D'} \otimes_{\mathcal{C}} \mathcal{C}[x_1, \dots, x_l]/(F_1, \dots, F_l) \longrightarrow \tilde{\mathcal{O}}_{U \times D'}$$

of $\mathcal{O}_{U \times D'}$ -modules defined by $\tilde{\sigma}(\bar{x}_i) = G_i$ ($1 \leq i \leq l$) is surjective, where (F_1, \dots, F_l) denotes the ideal sheaf of the $\mathcal{O}_{U \times D'}$ -module $\mathcal{O}_{U \times D'} \otimes_{\mathcal{C}} \mathcal{C}[x_1, \dots, x_l]$ generated by the sections F_1, \dots, F_l over $U \times D'$ and \bar{x}_i the residue class of x_i in $\mathcal{O}_{U \times D'} \otimes_{\mathcal{C}} \mathcal{C}[x_1, \dots, x_l]/(F_1, \dots, F_l)$,

- (iii) $\tilde{\sigma}$ coincides with σ at p .

Furthermore, shrinking U and D' sufficiently small if necessary, we may find $K_1, \dots, K_m \in \Gamma(U \times D', \mathcal{O}_{Z \times D})[x_1, \dots, x_l]$ such that the induced homomorphism

$$\mathcal{O}_{U \times D'} \otimes_{\mathcal{C}} \mathcal{C}[x_1, \dots, x_l]/(F_1, \dots, F_l, K_1, \dots, K_m) \longrightarrow \tilde{\mathcal{O}}_{U \times D'}$$

is an isomorphism of $\mathcal{O}_{U \times D'}$ -modules. Therefore, by definition *Spesan* $\tilde{\mathcal{O}}_{U \times D'}$ is a closed complex subspace of $U \times D' \times \mathcal{C}^l$ defined by the ideal sheaf $(F_1, \dots, F_l, K_1, \dots, K_m)$ of $\mathcal{O}_{U \times D' \times \mathcal{C}^l}$. We define $V := D' \cap M$ and denote by $\bar{F}_1, \dots, \bar{F}_l, \bar{K}_1, \dots, \bar{K}_m$ the images of $F_1, \dots, F_l, K_1, \dots, K_m$ by the map

$$\Gamma(U \times D', \mathcal{O}_{Z \times D})[x_1, \dots, x_l] \longrightarrow \Gamma(U \times V, \mathcal{O}_{Z \times M})[x_1, \dots, x_l],$$

respectively. We denote by $\pi' : U \times D' \rightarrow D'$ the projection to the second factor. Then we have the following commutative diagram of $\mathcal{O}_{U \times D'}$ -modules :

$$\begin{array}{ccc}
 & 0 & 0 \\
 & \downarrow & \downarrow \\
 \pi'^* \mathcal{I}_V \otimes_C \mathbf{C}[x_1, \dots, x_l] & & \longrightarrow \pi'^* \mathcal{I}_V \cdot \tilde{\mathcal{O}}_{U \times D'} \\
 /(\pi'^* \mathcal{I}_V \otimes_C \mathbf{C}[x_1, \dots, x_l]) \cap (F_1, \dots, F_l, K_1, \dots, K_m) & & \\
 \downarrow & & \downarrow \\
 (3.15) \quad \mathcal{O}_{U \times D'} \otimes_C \mathbf{C}[x_1, \dots, x_l] / (F_1, \dots, F_l, K_1, \dots, K_m) & \longrightarrow & \tilde{\mathcal{O}}_{U \times D'} \\
 \downarrow & & \downarrow \\
 \mathcal{O}_{U \times V} \otimes_C \mathbf{C}[x_1, \dots, x_l] / (\bar{F}_1, \dots, \bar{F}_l, \bar{K}_1, \dots, \bar{K}_m) & \longrightarrow & \tilde{\mathcal{O}}_{U \times V|V} \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

where both vertical lines are exact and the first and the second horizontal arrows are isomorphisms of $\mathcal{O}_{U \times D'}$ -modules. Hence the last horizontal arrow in (3.15) is an isomorphism of $\mathcal{O}_{U \times V}$ -modules. This shows that $(U \times V)^* := \text{Specan } \tilde{\mathcal{O}}_{U \times V|V}$ is a closed complex subspace of $U \times V \times \mathbf{C}^l$ defined by the ideal sheaf $(\bar{F}_1, \dots, \bar{F}_l, \bar{K}_1, \dots, \bar{K}_m)$ of $\mathcal{O}_{U \times V \times \mathbf{C}^l}$. From this fact we infer that $(U \times V)^*$ is a closed complex subspace of $(U \times D')^* := \text{Specan } \tilde{\mathcal{Q}}_{U \times D'}$ ($\subset U \times D' \times \mathbf{C}^l$) defined by the ideal sheaf $\nu^* \pi'^* \mathcal{I}_V$ of $\mathcal{O}_{(U \times D')^*}$, where $\nu: (U \times D')^* \rightarrow U \times D'$ denotes the restriction to $(U \times D')^*$ of the projection $Pr_{U \times D'}: U \times D' \times \mathbf{C}^l \rightarrow U \times D'$. Here it should be noticed that the map $\nu: (U \times D')^* \rightarrow U \times D'$ is nothing but the normalization of $U \times D'$ (cf. [20]). On the other hand, $n \times id_{D'}: U^* \times D' \rightarrow U \times D'$ is also the normalization of $U \times D'$. Therefore, by the uniqueness of the normal model we have a biholomorphic map over $U \times D'$:

$$\begin{array}{ccc}
 (U \times D')^* & \xrightarrow{\psi} & U^* \times D' \\
 \searrow \nu & & \swarrow n \times id_{D'} \\
 & U \times D' &
 \end{array}$$

which induces a biholomorphic map over $U \times V$:

$$\begin{array}{ccc}
 (U \times V)^* & \xrightarrow{\psi} & U^* \times V \\
 \searrow \nu_{(U \times V)^*} & & \swarrow n \times id_V \\
 & U \times V &
 \end{array}$$

This completes the proof, because $\zeta^{-1}(U \times V)$ is biholomorphic to $(U \times V)^*$ over $U \times V$.

3.6 THEOREM (Relative Normalization Theorem). *Let $\pi: \mathcal{Z} \rightarrow M$ be a locally trivial family of analytic varieties parametrized by a complex space. Then there exists a locally trivial family $\pi_1: \mathcal{X} \rightarrow M$ of analytic varieties parametrized by the same complex space M and a surjective holomorphic map $\nu: \mathcal{X} \rightarrow \mathcal{Z}$ over M (i. e., $\pi_1 = \pi \circ \nu$) which enjoy the following properties:*

$$(i) \quad \nu_t: X_t \rightarrow Z_t \quad (X_t := \pi_1^{-1}(t), Z_t := \pi^{-1}(t), \nu_t := \nu|_{X_t}: X_t \rightarrow Z_t)$$

is the normalization of Z_t for any $t \in M$,

(ii) *the holomorphic map $\nu: \mathcal{X} \rightarrow \mathcal{Z}$ is locally trivial in the following sense; for any point $p \in \mathcal{Z}$, there exist open neighborhoods \mathcal{U} of p in \mathcal{Z} , V of $\pi(p)$ in M , a biholomorphic map $\varphi: \mathcal{U} \rightarrow U \times V$ over V , where $U := \mathcal{U} \cap Z_{\pi(p)}$, and a biholomorphic map $\psi: \nu^{-1}(\mathcal{U}) \rightarrow U^* \times V$ over V , where $U^* := \nu^{-1}(\mathcal{U}) \cap X_{\pi(p)}$, such that the diagram*

$$\begin{array}{ccc} \nu^{-1}(\mathcal{U}) & \xrightarrow{\psi} & U^* \times V \\ \nu \downarrow & \varphi & \downarrow \nu_{\pi(p)} \times id_V \\ \mathcal{U} & \xrightarrow{\varphi} & U \times V \end{array}$$

commutes.

Furthermore, the family $\pi_1: \mathcal{X} \rightarrow M$ and the surjective holomorphic map $\nu: \mathcal{X} \rightarrow \mathcal{Z}$ over M are uniquely determined up to biholomorphic maps over M .

PROOF. Let $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ be the sheaf of germs of weakly holomorphic functions along fibers of the family $\pi: \mathcal{Z} \rightarrow M$. By Proposition 3.4 $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ is a coherent $\mathcal{O}_{\mathcal{Z}}$ -module. Let $\nu: \mathcal{X} \rightarrow \mathcal{Z}$ be the analytic spectrum of $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$. We define $\pi_1 := \pi \circ \nu: \mathcal{X} \rightarrow M$. We will show that the surjective holomorphic map $\nu: \mathcal{X} \rightarrow \mathcal{Z}$ over M and the family $\pi_1: \mathcal{X} \rightarrow M$ enjoy the properties (i) and (ii) in the theorem.

We take a point $p \in \mathcal{Z}$. By local triviality of the family $\pi: \mathcal{Z} \rightarrow M$, there exist open neighborhoods \mathcal{U} of p in \mathcal{Z} , V of $\pi(p)$ in M with $\pi(\mathcal{U}) = V$, and a biholomorphic map $\varphi: \mathcal{U} \rightarrow U \times V$ over V , where $U := \mathcal{U} \cap Z_{\pi(p)}$. Furthermore, shrinking \mathcal{U} sufficiently small if necessary, we have an embedding $e: V \rightarrow D$, where D is a domain of the complex number space \mathbb{C}^n . We define $V' := e(V)$, $\mathcal{U}' := U \times V'$, $\pi' := Pr_{V'}: \mathcal{U}' \rightarrow V'$, the projection of the second factor, and $\varphi' := (id_U \times e') \circ \varphi: \mathcal{U} \rightarrow \mathcal{U}'$, where $id_U: U \rightarrow U$ denotes

the identity map on U and $V \xrightarrow{e'} V' \hookrightarrow D$ the factorization of e . Then, by the definition of the sheaf $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ there exists an isomorphism

$$\overline{(\varphi'^{-1})}^{-1} : (\varphi'^{-1})_* \tilde{\mathcal{O}}_{\mathcal{U}'/V'} \longrightarrow \tilde{\mathcal{O}}_{\mathcal{Z}/M \wr \mathcal{U}}$$

of $\mathcal{O}_{\mathcal{U}}$ -modules (cf. (3.10)). Hence we have the following commutative diagram :

$$(3.16) \quad \begin{array}{ccc} \nu^{-1}(\mathcal{U}) & \xleftarrow{\cong} & \text{Specan } \tilde{\mathcal{O}}_{\mathcal{U}'/V'} \\ \nu \downarrow & \xleftarrow{\cong \varphi'^{-1}} & \downarrow \nu' \\ \mathcal{U} & \xleftarrow{\cong} & \mathcal{U}' \\ \pi \downarrow & \xleftarrow{\cong e'^{-1}} & \downarrow \pi' \\ V & \xleftarrow{\cong} & V' \end{array}$$

On the other hand, by Lemma 3.5, shrinking \mathcal{U} sufficiently small if necessary, we have the following commutative diagram :

$$(3.17) \quad \begin{array}{ccccc} & & \text{id}_{V'} \times e' & & \\ & & \cong & & \\ \text{Specan } \tilde{\mathcal{O}}_{\mathcal{U}'/V'} & \xrightarrow{\cong} & U^* \times V' & \xleftarrow{\cong} & U^* \times V \\ \nu' \downarrow & & \downarrow n \times \text{id}_{V'} & \text{id}_{U^*} \times e' & \downarrow n \times \text{id}_V \\ \mathcal{U}' & \xrightarrow{\cong} & \mathcal{U}' & \xleftarrow{\cong} & U \times V \\ \pi' \downarrow & & \downarrow \pi' & e' & \downarrow Pr_V \\ V' & \xrightarrow{\cong} & V' & \xleftarrow{\cong} & V \end{array}$$

Therefore, by (3.16) and (3.17) we conclude that the surjective holomorphic map $\nu : \mathcal{X} \rightarrow \mathcal{Z}$ and the family $\pi_1 : \mathcal{X} \rightarrow M$ certainly enjoy the properties (i) and (ii) in the theorem.

Let $\pi'_1 : \mathcal{Y} \rightarrow M$ and $\mu : \mathcal{Y} \rightarrow \mathcal{Z}$ be another locally trivial family of analytic varieties parametrized by the complex space M and a surjective holomorphic map over M which enjoy the properties (i) and (ii) in the theorem. Then, by the property (i), $\mu : \mathcal{Y} \rightarrow \mathcal{Z}$ is a finite map; hence $\mu : \mathcal{Y} \rightarrow \mathcal{Z}$ is biholomorphic to $\text{Specan } \mu_* \mathcal{O}_{\mathcal{Y}}$ over \mathcal{Z} (cf. [10]). By the property (ii) there is naturally an isomorphism $\tilde{\mathcal{O}}_{\mathcal{Z}/M} \cong \mu_* \mathcal{O}_{\mathcal{Y}}$ of $\mathcal{O}_{\mathcal{Z}}$ -modules. From these facts it follows that $\mu : \mathcal{Y} \rightarrow \mathcal{Z}$ is biholomorphic to $\nu : \mathcal{X} \rightarrow \mathcal{Z}$ over \mathcal{Z} , since $\nu : \mathcal{X} \rightarrow \mathcal{Z}$ is $\text{Specan } \tilde{\mathcal{O}}_{\mathcal{Z}/M}$. This completes the proof.

3.7 DEFINITION. We call the locally trivial family $\pi_1 : \mathcal{X} \rightarrow M$ of analytic varieties and the surjective holomorphic map $\nu : \mathcal{X} \rightarrow \mathcal{Z}$ over M

in Theorem 3.6, which are determined up to isomorphisms over M for a locally trivial family $\pi : \mathcal{Z} \rightarrow M$ of analytic varieties parametrized by a complex space M , a *relative normalization* of $\pi : \mathcal{Z} \rightarrow M$.

§ 4 Global existence of the universal locally trivial family

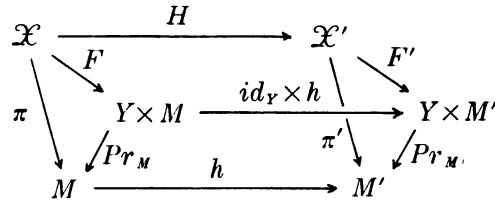
First, we recall the definition of a family of deformations of a holomorphic map between complex manifolds with a target manifold fixed, parametrized by a complex space, and that of a morphism between such two families.

4.1 DEFINITION. Let $f : X \rightarrow Y$ be a holomorphic map between complex manifolds. By a family of deformations of $f : X \rightarrow Y$ with Y fixed, we mean a sextuple $(\mathcal{X}, F, \pi, M, o, \varphi)$ satisfying the following conditions :

- (i) $\pi : \mathcal{X} \rightarrow M$ is a surjective smooth holomorphic map,
- (ii) $F : \mathcal{X} \rightarrow Y \times M$ is a holomorphic map such that $\pi = Pr_M \circ F$, where $Pr_M : Y \times M \rightarrow M$ denotes the projection,
- (iii) o is an assigned point of M ,
- (iv) $\varphi : X \rightarrow \pi^{-1}(o)$ is a biholomorphic map for which $f = id_Y^{-1} \circ F|_{\pi^{-1}(o)} \circ \varphi$ holds, where $F|_{\pi^{-1}(o)} : \pi^{-1}(o) \rightarrow Y \times o$ denotes the restriction of F to the fiber $\pi^{-1}(o)$ and $id_Y : Y \rightarrow Y \times o$ the identity map.

4.2 DEFINITION. Let $\mathcal{F} = (\mathcal{X}, F, \pi, M, o, \varphi)$ and $\mathcal{F}' = (\mathcal{X}', F', \pi', M', o', \varphi')$ be two families of deformations of a holomorphic map $f : X \rightarrow Y$ between complex manifolds with Y fixed, parametrized by complex spaces. By a morphism (resp. an isomorphism) from \mathcal{F} into \mathcal{F}' , we mean a pair of holomorphic (resp. biholomorphic) maps $H : \mathcal{X} \rightarrow \mathcal{X}'$ and $h : M \rightarrow M'$ with $h(o) = o'$ such that :

- (i) the diagram



commute, where id_Y denotes the identity map on Y ,

- (ii) $H(X_t) = X'_{h(t)}$, for any $t \in M$, where $X_t := \pi^{-1}(t)$ and $X'_{h(t)} := \pi'^{-1}(h(t))$.

For families of deformations of holomorphic maps, *versality*, *univer-*

sality, and semi-universality are also defined as in Definition 2.4 and Definition 2.5 (cf. [9], [3]).

4.3 LEMMA. *Let Z be an analytic subvariety with a locally stable parametrization of a compact complex manifold Y , $n: X \rightarrow Z$ the normalization of Z , and $f := \iota \circ n: X \rightarrow Y$ the composite of the normalization map $n: X \rightarrow Z$ and the inclusion map $\iota: Z \hookrightarrow Y$. Let $(\mathcal{X}, F, \pi, M, o, \varphi)$ be a family of deformations of $f: X \rightarrow Y$ with Y fixed, parametrized by a complex space. We define $\mathcal{Z} := F(\mathcal{X})$ and let $\mathcal{X} \xrightarrow{F'} \mathcal{Z} \hookrightarrow Y \times M$ be the factorization of F . Then there is an open neighborhood N of o in M such that:*

(i) $F_t: X_t \rightarrow Y \times t$ ($X_t := \pi^{-1}(t)$, $F_t := F|_{X_t}: X_t \rightarrow Y = Y \times t$) is a locally stable map for any $t \in N$,

(ii) $F'|_{\pi^{-1}(N)}: \pi^{-1}(N) \rightarrow \pi^{-1}(N)$ is the relative normalization (cf. Definition 3.7) of the family $\pi_{1|\pi^{-1}(N)}: \pi^{-1}(N) \rightarrow N$ of analytic varieties, where $\pi_1 := Pr_{M|\mathcal{Z}}: \mathcal{Z} \rightarrow M$, the restriction to \mathcal{Z} of the projection map $Pr_M: Y \times M \rightarrow M$.

PROOF. For any point $q \in Z \times o = F_0(X_0)$, we define $S_q := F_0^{-1}(q)$. Then S_q is a finite subset of X_0 , because $F'_0: X_0 \rightarrow Z \times o$ is the normalization of Z . By the definition of an analytic subvariety with a locally stable parametrization, $f: X \rightarrow Y$ is locally stable; hence, so is $F_0: X_0 \rightarrow Y$. Hence, taking into consideration that $F: \mathcal{X} \rightarrow Y \times M$ is a proper map, we infer that there exists an open neighborhood $\mathcal{C}\mathcal{V}_q$ of q in $Y \times M$ such that: if we define $\mathcal{U}_q := F^{-1}(\mathcal{C}\mathcal{V}_q)$ and $W_q := Pr_M(\mathcal{C}\mathcal{V}_q)$, then there exist biholomorphic maps $\varphi_q: \mathcal{U}_q \rightarrow U_q \times W_q$ and $\psi_q: \mathcal{C}\mathcal{V}_q \rightarrow V_q \times W_q$ over W_q , where we define $U_q := \mathcal{U}_q \cap X_0$ and $V_q := \mathcal{C}\mathcal{V}_q \cap (Y \times o)$, such that the diagram

$$(4.1) \quad \begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\varphi_q} & U_q \times W_q \\ F \downarrow & & \downarrow F_0 \times id_{W_q} \\ \mathcal{C}\mathcal{V}_q & \xrightarrow{\psi_q} & V_q \times W_q \end{array}$$

commutes. Since $\{\mathcal{C}\mathcal{V}_q\}_{q \in Z \times o}$ covers $Z \times o$, and since $Z \times o$ is compact, we may extract a finite subcover $\{\mathcal{C}\mathcal{V}_{q_\mu}\}_{\mu=1, \dots, N}$ of the covering $\{\mathcal{C}\mathcal{V}_q\}_{q \in Z \times o}$. Then $\{\mathcal{U}_{q_\mu}\}_{\mu=1, \dots, N}$ covers X_0 . Since $\pi: X \rightarrow M$ and $Pr_M: Y \times M \rightarrow M$ are proper maps, there exists an open neighborhood N of o in M such that $\pi^{-1}(N) \subset \bigcup_{\mu=1}^N \mathcal{U}_{q_\mu}$ and $Pr_M^{-1}(N) \subset \bigcup_{\mu=1}^N \mathcal{C}\mathcal{V}_{q_\mu}$. Then, for any $t \in N$ and $q \in Z_t$, there exists $\mathcal{C}\mathcal{V}_{q_\mu}$ with $q \in \mathcal{C}\mathcal{V}_{q_\mu}$ ($1 \leq \mu \leq N$). If we let $\phi_{q_\mu}(q) := (q', t) \in V_q \times N$, then by (4.1) we have $q' \in Z \times o$ and $F^{-1}(q) = F_t^{-1}(q) = \phi_{q_\mu}^{-1}(F_0^{-1}(q'), t)$. Hence the multi-germ $F_t: (X_t, F_t^{-1}(q)) \rightarrow (Y \times t, q)$ of a holomorphic map is equivalent to $F_0: (X_0, F_0^{-1}(q')) \rightarrow (Y \times o, q')$. Therefore $F_t: X_t \rightarrow Y \times t$ is simul-

taneously stable at $F_t^{-1}(q)$, because $F'_0: X_0 \rightarrow Y \times o$ is locally stable (cf. Definition 1.1 and Definition 1.2). Since q is any point of $Z_t := \pi_1^{-1}(t)$ for $t \in N$, we conclude that $F_t: X_t \rightarrow Y \times t$ is locally stable for any $t \in N$.

The neighborhood N of o in M also has the property (ii) in the lemma. Indeed, from the fact that the multi-germ $F_t: (X_t, F_t^{-1}(q)) \rightarrow (Y \times t, q)$ of a holomorphic map ($t \in N, q \in Z_t$) is equivalent to $F_0: (X_0, F_0^{-1}(q')) \rightarrow (Y \times o, q')$ for a certain $q' \in Z_0$, it follows that the multi-germ $F'_t: (X_t, F'^{-1}_t(q)) \rightarrow (Z_t, q)$ of a holomorphic map is equivalent to $F'_0: (X_0, F'^{-1}_0(q')) \rightarrow (Z_0, q')$. Since $F'_0: (X_0, F'^{-1}_0(q')) \rightarrow (Z_0, q')$ is the normalization of a germ of analytic variety (Z_0, q') , so is $F'_t: (X_t, F'^{-1}_t(q)) \rightarrow (Z_t, q)$. Therefore, since q is any point of Z_t for $t \in N$, we conclude that $F'_t: X_t \rightarrow Z_t$ is the normalization of Z_t for any $t \in N$. Q. E. D.

4.4 THEOREM. *Let Z be an analytic subvariety with a locally stable parametrization of a compact complex manifold Y . Then there exists the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ for locally trivial displacements of Z in Y (cf. Definition 2.9) such that $\bar{Z}_t := \bar{\pi}^{-1}(t)$ is an analytic subvariety with a locally stable parametrization of Y for any $t \in \bar{M}$. Furthermore, if $H^1(Z, \mathcal{N}_{Z|Y}) = 0$, then the parameter space \bar{M} is non-singular and the characteristic map $\sigma_{\bar{o}}: T_{\bar{o}}(\bar{M}) \rightarrow H^0(Z, \mathcal{N}_{Z|Y})$ is bijective.*

PROOF. Let $n: X \rightarrow Z$ be the normalization of Z and $\iota: Z \subset Y$ the inclusion map. We define $f := \iota \circ n: X \rightarrow Y$. We denote by $(\mathbf{GC})^0$ the dual category of germs of complex spaces. We define two deformation functors D and L from $(\mathbf{GC})^0$ to *Set*, the category of sets, by:

$$D: (M, o) \rightarrow \{\text{isomorphism classes of the families of deformations of the holomorphic map } f: X \rightarrow Y \text{ with } Y \text{ fixed, parametrized by } (M, o)\},$$

$$L: (M, o) \rightarrow \{\text{isomorphism classes of the locally trivial families of displacements of } Z \text{ in } Y, \text{ parametrized by } (M, o)\},$$

where (M, o) denotes a germ of a complex space. Given a family $(\mathcal{X}, F, \pi, M, o, \varphi)$ of deformations of the map $f: X \rightarrow Y$ with Y fixed, parametrized by (M, o) , we define $\mathcal{Z} := F(\mathcal{X})$ and $\pi_1 := Pr_{M|\mathcal{Z}}: \mathcal{Z} \rightarrow M$, the restriction to \mathcal{Z} of the projection map $Pr_M: Y \times M \rightarrow M$. Then, since $f: X \rightarrow Y$ is a locally stable holomorphic map, $(\mathcal{Z}, \pi_1, M, o)$ is a locally trivial family of displacements of Z in Y , parametrized by (M, o) . By the definitions of morphisms in the two categories (cf. Definition 2.3 and Definition 4.2), we conclude that the above correspondence $(\mathcal{X}, F, \pi, M, o, \varphi) \rightarrow (\mathcal{Z}, \pi_1, M, o)$ give a *natural transformation* between the functors D and L . Furthermore,

by Theorem 3.6 and Lemma 4.3 we infer that this correspondence is a *natural equivalence*. By Flenner's theorem [3, Theorem (8.5)] there exists a versal family $(\bar{\mathcal{X}}, \bar{F}, \bar{\pi}, \bar{M}, \bar{\delta}, \bar{\varphi})$ of deformations of $f: X \rightarrow Y$ with Y fixed such that the characteristic map

$$(4.2) \quad \bar{\tau}_o: T_o(\bar{M}) \longrightarrow \mathbf{Ext}_{\bar{\mathcal{O}}_X}^1(L_{X/Y}, \bar{\mathcal{O}}_X)$$

is injective, where $L_{X/Y}$ denotes the complex of $\bar{\mathcal{O}}_X$ -modules of length -1 : $0 \rightarrow f^*\bar{\Omega}_Y \rightarrow \bar{\Omega}_X \rightarrow 0$ and $\mathbf{Ext}_{\bar{\mathcal{O}}_X}^*$ *global hyperext* (cf. [7]). We define $\bar{\mathcal{Z}} := \bar{F}(\bar{\mathcal{X}})$ and let $\bar{\mathcal{X}} \xrightarrow{\bar{F}'} \bar{\mathcal{Z}} \hookrightarrow Y \times \bar{M}$ be the factorization of \bar{F} . By Lemma 4.3 we may assume that the family $(\bar{\mathcal{X}}, \bar{F}, \bar{\pi}, \bar{M}, \bar{\delta}, \bar{\varphi})$ satisfies the following:

(i) $\bar{F}_t: \bar{X}_t \rightarrow Y \times t$ ($\bar{X}_t := \bar{\pi}^{-1}(t)$, $\bar{F}_t := \bar{F}|_{\bar{X}_t}: \bar{X}_t \rightarrow Y = Y \times t$) is a locally stable map for any $t \in \bar{M}$,

(ii) $\bar{F}': \bar{\mathcal{X}} \rightarrow \bar{\mathcal{Z}}$ is the relative normalization of the family $\bar{\pi}_1: \bar{\mathcal{Z}} \rightarrow \bar{M}$, where $\bar{\pi}_1 := Pr_{M|\bar{X}}: \bar{\mathcal{Z}} \rightarrow \bar{M}$, the restriction to $\bar{\mathcal{Z}}$ of the projection map $Pr_M: Y \times \bar{M} \rightarrow \bar{M}$. From this it follows that the family $(\bar{\mathcal{Z}}, \bar{\pi}_1, \bar{M}, \bar{\delta})$ is a locally trivial family of analytic subvarieties with locally stable parametrizations of Y . It is maximal (=versal), because the functors D and L are equivalent (cf. Definition 2.4). We shall show that the family $(\bar{\mathcal{Z}}, \bar{\pi}_1, \bar{M}, \bar{\delta})$ is effective at $\bar{\delta}$ by proving that: there is naturally an isomorphism

$$(4.3) \quad H^p(X, \mathcal{I}_{X/Y}) \xrightarrow{\cong} \mathbf{Ext}_{\bar{\mathcal{O}}_X}^{p+1}(L_{X/Y}, \bar{\mathcal{O}}_X) \quad \text{for } p \geq 0,$$

where $\mathcal{I}_{X/Y}$ is a \mathcal{O}_X -module defined by

$$0 \longrightarrow \Theta_X \longrightarrow f^*\Theta_Y \longrightarrow \mathcal{I}_{X/Y} \longrightarrow 0,$$

where Θ_X (resp. Θ_Y) denotes the sheaf of germs of holomorphic vector fields on X (resp. on Y); hence the characteristic map in (4.2) coincides with the one defined by Horikawa in [9]. We take an injective resolution I^\cdot of $\bar{\mathcal{O}}_X$ and define a complex $\mathcal{H}om_{\bar{\mathcal{O}}_X}(L_{X/Y}, I^\cdot)$ by

$$\mathcal{H}om_{\bar{\mathcal{O}}_X}^n(L_{X/Y}, I^\cdot) := \prod_{p \in \mathbb{Z}} \mathcal{H}om_{\bar{\mathcal{O}}_X}(L_{X/Y}, I^{p+n})$$

and

$$d^n := \prod_{p \in \mathbb{Z}} (d_{L, X/Y}^{p-1} + (-1)^{n+1} d_I^{p+n}).$$

Then the *local hyperext* and the *global hyperext* are given by

$$\mathcal{E}xt_{\bar{\mathcal{O}}_X}^q(L_{X/Y}, \bar{\mathcal{O}}_X) := \mathcal{H}^q(\mathcal{H}om_{\bar{\mathcal{O}}_X}(L_{X/Y}, I^\cdot))$$

and

$$\mathbf{E}xt_{\bar{\mathcal{O}}_X}^q(L_{X/Y}, \bar{\mathcal{O}}_X) := H^q(\Gamma(X, \mathcal{H}om_{\bar{\mathcal{O}}_X}(L_{X/Y}, I^\cdot)),$$

respectively (cf. [7]). By the same arguments as in the case of the usual *ext* functor, we can prove that there exists a spectral sequence for which

$$(4.4) \quad E_2^{p,q} = H^p(X, \mathcal{E}xt^q(L_{X/Y}, \mathcal{O}_X)) \longrightarrow E_\infty^{p+q} = \mathbf{Ext}_{\mathcal{O}_X}^{p+q}(L_{X/Y}, \mathcal{O}_X)$$

(cf. [5, Théorème 7.3.3]). Using isomorphisms

$$\mathcal{H}^q(\mathcal{H}om_{\mathcal{O}_X}(L_{X/Y}, I)) \longrightarrow \mathcal{H}^q(\mathcal{H}om_{\mathcal{O}_X}(L_{X/Y}, \mathcal{O}_X)), \quad (q \geq 0)$$

the local hyperext $\mathcal{E}xt_{\mathcal{O}_X}^q(L_{X/Y}, \mathcal{O}_X)$ is calculated as follows :

$$\mathcal{E}xt_{\mathcal{O}_X}^q(L_{X/Y}, \mathcal{O}_X) = \begin{cases} 0 & \text{for } q \neq 1, \\ \text{Coker}\{\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(f^*\Omega_Y, \mathcal{O}_X)\} \\ \cong \text{Coker}\{\Theta_X \longrightarrow f^*\Theta_Y\} \cong \mathcal{I}_{X/Y} & \text{for } q = 1. \end{cases}$$

Hence by the spectral sequence in (4.4), we obtain the isomorphism in (4.3). Then the effectivity at \bar{o} of the family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ follows from the commutativity of the following diagram, which was proved in [23] ([23, Proposition (9.3)]):

$$(4.5) \quad \begin{array}{ccc} & \xrightarrow{\tau_{\bar{o}}} & H^0(X, \mathcal{I}_{X/Y}) \\ T_{\bar{o}}(\bar{M}) & & \downarrow \dagger \\ & \xrightarrow{\sigma_{\bar{o}}} & H^0(Z, \mathcal{N}_{Z/Y}), \end{array}$$

where $\tau_{\bar{o}}$ denotes the characteristic map defined in [9] and \dagger the isomorphism induced by the one of \mathcal{O}_Y -modules $\mathcal{N}_{Z/Y} \xrightarrow{\sim} \dagger_* \mathcal{I}_{X/Y}$ (cf. [23, Proposition (9.1)]).

Finally, we shall prove the latter part of the theorem. We have

$$(4.6) \quad H^p(X, \mathcal{I}_{X/Y}) \xrightarrow{\sim} H^p(Z, \mathcal{N}_{Z/Y}) \quad \text{for } p \geq 0$$

because of the isomorphism of \mathcal{O}_Y -modules $\mathcal{N}_{Z/Y} \xrightarrow{\sim} \dagger_* \mathcal{I}_{X/Y}$ mentioned above. Hence by (4.3) and (4.6) we have

$$(4.7) \quad \mathbf{Ext}_{\mathcal{O}_X}^p(L_{X/Y}, \mathcal{O}_X) \xrightarrow{\sim} H^{p-1}(Z, \mathcal{N}_{Z/Y}) \quad \text{for } p \geq 1.$$

Therefore if $H^1(Z, \mathcal{N}_{Z/Y}) = 0$, then we have $\mathbf{Ext}_{\mathcal{O}_X}^2(L_{X/Y}, \mathcal{O}_X) = 0$, to which the obstruction classes in Flenner's theory of deformations of holomorphic maps belong; hence the parameter space \bar{M} of the Flenner's versal family $(\bar{\mathcal{X}}, \bar{F}, \bar{\pi}, \bar{M}, \bar{o}, \bar{\varphi})$, or the parameter space of the family $(\bar{\mathcal{Z}}, \bar{\pi}_1, \bar{M}, \bar{o})$, is non-singular. Furthermore, if $H^1(Z, \mathcal{N}_{Z/Y}) = 0$, the characteristic map $\tau_{\bar{o}}: T_{\bar{o}}(\bar{M})$

$\rightarrow \mathbf{Ext}_{\mathcal{O}_X}^1(L_{X/Y}, \mathcal{O}_X)$ is bijective: hence by (4.5) and (4.3) so is the characteristic $\sigma_{\bar{o}}: T_{\bar{o}}(\bar{M}) \rightarrow H^0(Z, \mathcal{N}_{Z/Y})$ of the family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$. Q. E. D.

4.5 REMARK. The above theorem might be considered as a generalization of K. Kodaira's result in [12] and M. Namba's one in [18] or [19] to higher dimensional cases.

4.6 DEFINITION. We say a locally trivial family (\mathcal{Z}, π, M, o) of displacements of an analytic subvariety Z in a compact complex manifold Y is *injective* if $\pi^{-1}(t) \neq \pi^{-1}(t')$ for any $t, t' \in M$ with $t \neq t'$.

4.7 THEOREM. *Let Z be an analytic subvariety with a locally stable parametrization of a compact complex manifold Y . Then the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ for locally trivial displacements of Z in Y is;*

- (i) *injective and*
- (ii) *universal*

in a sufficiently small open neighborhood of \bar{o} in \bar{M} .

PROOF. We denote by $D(Y)$ the Douady space of closed complex subspaces of Y , by $\tilde{\pi}_o: \tilde{\mathcal{U}}(Y) \rightarrow D(Y)$ the universal family of closed complex subspaces of Y , and by \bar{o} the point of $D(Y)$ corresponding to the analytic subvariety Z . By Corollary (0.2) in [4], for any point $z \in Z$, there exists a locally closed complex subspace N_z of $D(Y)$ containing the point \bar{o} , which enjoy the following property:

If $\alpha: (T, o) \rightarrow (D(Y), \bar{o})$ is a holomorphic map between germs of complex spaces, then the induced family $(\alpha^ \tilde{\mathcal{U}}(Y), z) \rightarrow (T, o)$ of deformations of the germ of complex space (Z, z) is isomorphic to the trivial deformation $(Z, z) \times (T, o) \rightarrow (T, o)$ if, and only if, α factorizes over (N_z, \bar{o}) .*

We define

$$\begin{aligned} \tilde{N} &:= \bigcap_{z \in Z} N_z \quad (\text{the intersection as complex subspaces}), \\ \tilde{\mathcal{Z}} &:= \tilde{\mathcal{U}}(Y)|_{\tilde{N}} \quad (\text{the restriction of } \tilde{\mathcal{U}}(Y) \text{ to } \tilde{N}), \\ \tilde{\pi} &:= \tilde{\pi}_o|_{\tilde{\mathcal{Z}}}: \tilde{\mathcal{Z}} \rightarrow \tilde{N} \quad (\text{the restriction of } \tilde{\pi}_o: \tilde{\mathcal{U}}(Y) \rightarrow D(Y) \text{ to } \tilde{\mathcal{Z}}). \end{aligned}$$

Then, by the definition of the family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \tilde{N}, \bar{o})$ it is a locally trivial family of displacements of Z in Y that is injective and maximal at \bar{o} . We shall prove that the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ for locally trivial displacements of Z in Y is isomorphic to the family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \tilde{N}, \bar{o})$ in a sufficiently small open neighborhood of \bar{o} in \bar{M} . The injectivity of the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ follows from this fact. By the maximality at \bar{o} of the family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \tilde{N}, \bar{o})$, there exist an open neighborhood U of \bar{o}

in \bar{M} and a holomorphic map $\varphi: U \rightarrow \tilde{N}$ with $\varphi(\bar{o}) = \bar{o}$ such that $\varphi^* \tilde{\mathcal{Z}} = \bar{\mathcal{Z}}$ over U ; and by the maximality at \bar{o} of the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$, there exist an open neighborhood V of \bar{o} in \tilde{N} and a holomorphic map $\psi: V \rightarrow \bar{M}$ with $\psi(\bar{o}) = \bar{o}$ such that $\psi^* \bar{\mathcal{Z}} = \tilde{\mathcal{Z}}$ over V . Hence we have $(\varphi \circ \psi)^* \tilde{\mathcal{Z}} = \psi^*(\varphi^* \tilde{\mathcal{Z}}) = \psi^*(\bar{\mathcal{Z}}) = \tilde{\mathcal{Z}}$ over $\psi^{-1}(U) \cap V$; hence $(\iota_{\tilde{N}} \circ \varphi \circ \psi)^* \tilde{\mathcal{U}}(Y) = \tilde{\mathcal{Z}} = (\iota_{\tilde{N}} \circ id_{\tilde{N}})^* \tilde{\mathcal{U}}(Y)$ on $\psi^{-1}(U) \cap V$, where $id_{\tilde{N}}$ denote the identity map on \tilde{N} and $\iota_{\tilde{N}}$ the inclusion map $\tilde{N} \hookrightarrow D(Y)$. Then by the universality of the family $\tilde{\pi}_o: \tilde{\mathcal{U}}(Y) \rightarrow D(Y)$, we conclude that $\iota_{\tilde{N}} \circ \varphi \circ \psi = \iota_{\tilde{N}} \circ id_{\tilde{N}}$, and so $\varphi \circ \psi = id_{\tilde{N}}$ on $\psi^{-1}(U) \cap V$. From this fact it follows that $\varphi: U \rightarrow \tilde{N}$ is surjective as a map between topological spaces and that the homomorphism of $\mathcal{O}_{\tilde{N}}$ -modules $\mathcal{O}_{\tilde{N}} \rightarrow \varphi_* \mathcal{O}_{\bar{M}}$ is injective in an open neighborhood of \bar{o} in \bar{M} . On the other hand we have $(\psi \circ \varphi)^* \bar{\mathcal{Z}} = \varphi^*(\psi^* \bar{\mathcal{Z}}) = \varphi^* \tilde{\mathcal{Z}} = \bar{\mathcal{Z}}$ over $\varphi^{-1}(V) \cap U$. Hence by the semi-universality at \bar{o} of the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, o)$ (cf. Proposition 2.10), we have $d(\psi \circ \varphi)_o = d(id_{\bar{M}})_o$ (dh_p denotes the Jacobian map at $p \in \bar{M}$ of a holomorphic map $h: M \rightarrow N$ between complex spaces). Then, since $d(\psi \circ \varphi)_o = (d\psi)_o \cdot (d\varphi)_o = id_{T_{\bar{o}}(\bar{M})}$, we infer that the Jacobian map $(d\varphi)_o: T_{\bar{o}}(\bar{M}) \rightarrow T_{\bar{o}}(\tilde{N})$ is injective. Hence by Proposition (2.4) in [2], $\varphi: U \rightarrow \tilde{N}$ is an immersion at \bar{o} , and so an embedding in a sufficiently small open neighborhood of \bar{o} in \bar{M} . Therefore, since $\varphi: U \rightarrow \tilde{N}$ is surjective as a map between topological spaces, and since the homomorphism of $\mathcal{O}_{\tilde{N}}$ -modules $\mathcal{O}_{\tilde{N}} \rightarrow \varphi_* \mathcal{O}_{\bar{M}}$ is injective in an open neighborhood of \bar{o} in \tilde{N} as proved above, we conclude that φ is a homeomorphism and that the homomorphism of $\mathcal{O}_{\tilde{N}}$ -modules $\mathcal{O}_{\tilde{N}} \rightarrow \varphi^* \mathcal{O}_{\bar{M}}$ is an isomorphism in a sufficiently small open neighborhood of \bar{o} in \tilde{N} . Consequently, φ is a biholomorphic map near \bar{o} , and so the Kuranishi family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \tilde{N}, \bar{o})$ is isomorphic to the family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{N}, \bar{o})$ over an open neighborhood of \bar{o} in \bar{M} ; hence it is injective there.

Next, we shall prove that the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ is universal in a sufficiently small open neighborhood of \bar{o} in \bar{M} . Let $\varphi: U \rightarrow V$ be a biholomorphic map from an open neighborhood U of \bar{o} in \bar{M} into an open neighborhood V of \bar{o} in \tilde{N} such that $\varphi(\bar{o}) = \bar{o}$ and $\varphi^* \tilde{\mathcal{Z}} = \bar{\mathcal{Z}}$ over U . Let t_o be any point of U . Suppose we are given a locally trivial [family (\mathcal{Z}, π, M, o) of displacements of $\bar{Z}_{t_o} := \pi^{-1}(t_o)$ in Y , then, since the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ is maximal at every point of \bar{M} , there is a holomorphic map $h: M \rightarrow \bar{M}$ such that $h(o) = t_o$ and $\mathcal{X} = h^* \bar{\mathcal{Z}} = h^*(\varphi^* \tilde{\mathcal{Z}}) = (\varphi \circ h)^*(\iota_{\tilde{N}}^* \tilde{\mathcal{U}}(Y)) = (\iota_{\tilde{N}} \circ \varphi \circ h)^* \tilde{\mathcal{U}}(Y)$ in an open neighborhood of o in M . Suppose there is another holomorphic map $g: M \rightarrow \bar{M}$ such that $g(o) = t_o$ and $\mathcal{X} = g^* \bar{\mathcal{Z}} = (\iota_{\tilde{N}} \circ \varphi \circ g)^* \tilde{\mathcal{U}}(Y)$ in an open neighborhood of o in M , then by the universality of the family $\tilde{\pi}_o: \tilde{\mathcal{U}}(Y) \rightarrow D(Y)$ we have $\iota_{\tilde{N}} \circ \varphi \circ h = \iota_{\tilde{N}} \circ \varphi \circ g$; hence $\varphi \circ h = \varphi \circ g$, and so $h = g$ in an open neighborhood of o in M . That is, the Kuranishi family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ is universal at any point $t \in U$. Q. E. D.

In the following, let Y be a compact complex manifold. We denote by $E(Y)$ the set of all analytic subvarieties with locally stable parametrizations of Y . We denote by Z_t an analytic subvariety with a locally stable parametrization of Y corresponding to a "point" $t \in E(Y)$. We define a subset $\tilde{\mathcal{Z}}(Y)$ of the product space $Y \times E(Y)$ by

$$\tilde{\mathcal{Z}}(Y) := \{(y, t) \mid t \in E(Y), y \in Z_t\}.$$

We denote by $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ the restriction of the projection map $Pr_{E(Y)}: Y \times E(Y) \rightarrow E(Y)$ to $\tilde{\mathcal{Z}}(Y)$.

4.8 THEOREM. *$E(Y)$ and $\tilde{\mathcal{Z}}(Y)$ have the structure of Hausdorff complex spaces which enjoy the following properties:*

(i) *$\tilde{\mathcal{Z}}(Y)$ is a closed complex subspace of the product complex space $Y \times E(Y)$ and $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ is a locally trivial family of analytic subvarieties with locally stable parametrizations of Y parametrized by $E(Y)$.*

(ii) (*Universality*) *Given a locally trivial family $\pi: \mathcal{Z} \rightarrow M$ of analytic subvarieties with locally stable parametrizations of Y parametrized by a complex space M , there exists a unique holomorphic map $f: M \rightarrow E(Y)$ such that $f^*\tilde{\mathcal{Z}}(Y) = \mathcal{Z}$.*

(iii) *We denote by $D(Y)$ the Douady space of closed complex subspaces of Y and by $\tilde{\pi}_0: \tilde{\mathcal{U}}(Y) \rightarrow D(Y)$ the universal family of closed complex subspaces of Y (cf. [1]). Then the inclusion map $\iota: E(Y) \rightarrow D(Y)$ is a holomorphic immersion and $\iota^*\tilde{\mathcal{U}}(Y) = \tilde{\mathcal{Z}}(Y)$.*

(iv) (*C^∞ triviality*) *If $t_0 \in E(Y)$ is a point whose corresponding point of $E(Y)_{red}$ (the reduction of $E(Y)$) is non-singular, then there exist an open neighborhood N of t_0 in $E(Y)$ and a diffeomorphism $\Psi: Y \times N \rightarrow Y \times N$ over N (i. e., $Pr_N \circ \Psi = Pr_N$) such that $\Psi(Z_{t_0} \times N) = \tilde{\pi}^{-1}(N)$.*

(v) (*C^∞ type constancy*) *If t and t' are two points of any connected component of $E(Y)$, then there exists a diffeomorphism $\varphi: Y \rightarrow Y$ such that $\varphi(Z_t) = Z_{t'}$.*

PROOF. (i) Let t_0 be a point of $E(Y)$ and Z_{t_0} a corresponding analytic subvariety with a locally stable parametrization of Y . Then, by Theorem 4.4 and Theorem 4.7 we have the universal family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \bar{M}, \bar{o})$ for locally trivial displacements of Z_{t_0} in Y . By Theorem 3.6 and Lemma 4.3, shrinking \bar{M} around \bar{o} if necessary, we may assume that the universal family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \bar{M}, \bar{o})$ is a locally trivial family of analytic subvarieties with locally stable parametrizations of Y . Since the family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \bar{M}, \bar{o})$ is injective (cf. Definition 4.6 and Theorem 4.7), there are injective maps $\Phi: \tilde{\mathcal{Z}} \rightarrow \tilde{\mathcal{Z}}(Y)$ and $\varphi: \bar{M} \rightarrow E(Y)$ such that the diagram

$$\begin{array}{ccc}
 & \phi & \\
 \bar{\mathcal{Z}} & \longrightarrow & \tilde{\mathcal{Z}}(Y) \\
 \bar{\pi} \downarrow & \varphi & \downarrow \tilde{\pi} \\
 \bar{M} & \longrightarrow & E(Y)
 \end{array}$$

commutes. Hence the family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ gives a structure of a locally trivial family of analytic subvarieties with locally stable parametrizations of Y to $\bar{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ around Z_{i_0} . By the uniqueness of the universal family $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ up to isomorphisms, these local structures patch up to give a global structure of locally trivial family of analytic subvarieties with locally stable parametrizations of Y to $\bar{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$. Since $\bar{\mathcal{Z}}$ is a closed complex subspace of the product complex space $Y \times \bar{M}$, $\tilde{\mathcal{Z}}(Y)$ is that of the product complex space $Y \times E(Y)$ with respect to the above defined structure of a complex space.

Next, we shall prove that $E(Y)$ is a Hausdorff space. We denote by $E(Y)_c$ the underlying topological space of the complex space $E(Y)$. We define a Hermitian metric on Y and denote by $d(p, q)$ the distance between two points p, q of Y with respect to this Hermitian metric. Using this metric, we define

$$d'(t, t') := \sup_{p \in Z_t} d(p, Z_{t'}) + \sup_{q \in Z_{t'}} d(Z_t, q)$$

for any two points t, t' of $E(Y)$. Then d' defines a metric on $E(Y)$. We denote by $E(Y)_d$ the topological space $E(Y)$ whose topology is defined by this metric d' . Since $E(Y)_d$ is obviously Hausdorff, the following claim implies the Hausdorffness of $E(Y)_c$.

CLAIM. The identity map $Id: E(Y)_c \rightarrow E(Y)_d$ is continuous.

PROOF OF CLAIM: It suffices to prove the following: Let t_o be a point of $E(Y)$ and $(\bar{\mathcal{Z}}, \bar{\pi}, \bar{M}, \bar{o})$ the universal family for locally trivial displacements of Z_{i_0} in Y . Then, for a given positive number ε there exists an open neighborhood V of \bar{o} in \bar{M} such that $d'(\bar{Z}_o, \bar{Z}_t) \leq \varepsilon$ ($\bar{Z}_o = Z_{i_0}$) for any $t \in V$, where $\bar{Z}_t := \bar{\pi}^{-1}(t)$. We denote by $P_Y: \bar{\mathcal{Z}} \rightarrow Y$ the composite of the inclusion map $\iota: \bar{\mathcal{Z}} \hookrightarrow Y \times \bar{M}$ and the projection map $Pr_Y: Y \times \bar{M} \rightarrow Y$. Let $\{\Delta(p_i, \varepsilon)\}_{i=1, \dots, m}$ be a finite open covering of $Z_{i_0} = Pr_Y(\bar{Z}_o)$ in Y , where p_i 's are points of Z_{i_0} and

$$\Delta(p_i, \varepsilon) := \{y \in Y \mid d(y, p_i) < \varepsilon\}$$

for $i=1, \dots, m$. Then $\{P_Y^{-1}(\Delta(p_i, \varepsilon))\}_{i=1, \dots, m}$ gives a finite open covering of $\bar{Z}_o (= Z_{i_0})$ in $\bar{\mathcal{Z}}$. We take a finite open covering $\{\mathcal{U}_j\}_{j=1, \dots, m'}$ of \bar{Z}_o in $\bar{\mathcal{Z}}$

which satisfies the following conditions: for each j ,

- (i) there exists $i(j)$ ($1 \leq i(j) \leq m$) with $\mathcal{U}_j \subset P_Y^{-1}(\Delta(p_{i(j)}, \varepsilon/4))$,
- (ii) there is a biholomorphic map $\varphi_j: \mathcal{U}_j \rightarrow (\mathcal{U}_j \cap \bar{Z}_o) \times V_j$, where $V_j := Pr_{\bar{M}}(\mathcal{U}_j)$ ($Pr_{\bar{M}}: Y \times \bar{M} \rightarrow \bar{M}$, the projection to \bar{M}), such that
 - (a) the diagram

$$\begin{array}{ccc}
 \mathcal{U}_j & \xrightarrow{\varphi_j} & (\mathcal{U}_j \cap \bar{Z}_o) \times V_j \\
 \searrow Pr_{\bar{M}} & & \swarrow Pr_{V_j} \\
 & & V_j
 \end{array}$$

commutes,

- (b) $\varphi_{j|_{\mathcal{U}_j \cap \bar{Z}_o}}: \mathcal{U}_j \cap \bar{Z}_o \rightarrow (\mathcal{U}_j \cap \bar{Z}_o) \times o$ is the idently map.

We define $V := \bigcap_{j=1}^{m'} V_j$. Let p be a point of \bar{Z}_o . Suppose p belongs to \mathcal{U}_j . Then, for any point $t \in V$, $\varphi_j^{-1}(p, t) \in \mathcal{U}_j \cap \bar{Z}_i \subset P_Y^{-1}(\Delta(p_{i(j)}, \varepsilon/4))$; and since both p and $\varphi_j^{-1}(p, t)$ belongs to $P_Y^{-1}(\Delta(p_{i(j)}, \varepsilon/4))$, we have $d(p, \varphi_j^{-1}(p, t)) \leq \varepsilon/2$; hence $d(p, \bar{Z}_i) \leq \varepsilon/2$, because $\varphi_j^{-1}(p, t) \in \bar{Z}_i$. Since p and t are any points of \bar{Z}_o and V , respectively, we have

$$(4.8) \quad \sup_{p \in \bar{Z}_o} d(p, \bar{Z}_i) \leq \varepsilon/2 \quad \text{for any } t \in V.$$

On the other hand, let t be any point of V and q any point of \bar{Z}_i . Suppose q belongs to \mathcal{U}_j . Then, since $q \in \mathcal{U}_j \subset P_Y^{-1}(\Delta(p_{i(j)}, \varepsilon/4))$, we have $d(p_{i(j)}, q) \leq \varepsilon/4$. Hence $d(\bar{Z}_o, q) \leq \varepsilon/4$, because $p_{i(j)} \in \bar{Z}_o$. Since q and t are any points of \bar{Z}_i and V , respectively, we have

$$(4.9) \quad \sup_{q \in \bar{Z}_i} d(\bar{Z}_o, q) \leq \varepsilon/4 \quad \text{for any } t \in V.$$

By (4.8) and (4.9) we have

$$d'(\bar{Z}_o, \bar{Z}_i) = \sup_{p \in \bar{Z}_o} \varphi(p, \bar{Z}_i) + \sup_{q \in \bar{Z}_i} d(\bar{Z}_o, q) \leq (3/4)\varepsilon < \varepsilon.$$

This completes the proof of the claim.

(ii) By the definition of the locally trivial family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ of analytic subvarieties with locally stable parametrizations of Y , for any point p of the parameter space M of a given locally trivial family $\pi: \mathcal{Z} \rightarrow M$ of analytic subvarieties with locally stable parametrizations of Y , there are an open neighborhood N_p of p in M and a unique holomorphic map $\varphi_p: N_p \rightarrow E(Y)$ such that $\varphi_p^* \tilde{\mathcal{Z}}(Y) = \mathcal{Z}_{|N_p}$. By the uniqueness of the holomorphic map φ_p , these φ_p 's pach up to a global holomorphic map $\varphi: M \rightarrow E(Y)$.

- (iii) This follows from the proof in Theorem 4.7 of the injectivity

and the universality of the Kuranishi family $(\tilde{\mathcal{Z}}, \tilde{\pi}, \tilde{M}, \tilde{o})$ for locally trivial displacements of an analytic subvariety with a locally stable parametrization in Y .

(iv) Let $\tilde{\pi}_1: \tilde{\mathcal{X}}(Y) \rightarrow E(Y)$ and $\nu: \tilde{\mathcal{X}}(Y) \rightarrow \tilde{\mathcal{Z}}(Y)$ be the relative normalization of the locally trivial family $\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ (cf. Theorem 3.6). $\nu: \tilde{\mathcal{X}}(Y) \rightarrow \tilde{\mathcal{Z}}(Y)$ is a surjective holomorphic map over $E(Y)$ for which $\nu_t: X_t \rightarrow Z_t$ is the normalization map for any $t \in E(Y)$, where $X_t := \tilde{\pi}_1^{-1}(t)$, $Z_t := \tilde{\pi}^{-1}(t)$ and $\nu_t := \nu|_{X_t}: X_t \rightarrow Z_t$. We define $\tilde{F}: \tilde{\mathcal{X}}(Y) \rightarrow Y \times E(Y)$ to be the composite of the map $\nu: \tilde{\mathcal{X}}(Y) \rightarrow \tilde{\mathcal{Z}}(Y)$ and the inclusion map $\iota: \tilde{\mathcal{Z}}(Y) \hookrightarrow Y \times E(Y)$. Then $(\tilde{\mathcal{X}}(Y), \tilde{F}, Y \times E(Y), \tilde{\pi}_1, E(Y))$ is a family of locally stable holomorphic maps into Y parametrized by $E(Y)$. Hence, by Theorem 3.4 in [24], for any point $t_0 \in E(Y)$ whose corresponding point of $E(Y)_{red}$ is non-singular, there exist an open neighborhood N of t_0 in $E(Y)$ and diffeomorphisms $\phi: X_{t_0} \times N \rightarrow \tilde{\pi}_1^{-1}(N)$, $\Psi: Y \times N \rightarrow Y \times N$ such that the diagram

$$\begin{array}{ccc}
 X_{t_0} \times N & \xrightarrow{\phi} & \tilde{\pi}_1^{-1}(N) \\
 \downarrow \text{Pr}_N & \searrow f_{t_0} \times id_N & \downarrow \tilde{F} \\
 & Y \times N & \xrightarrow{\Psi} Y \times N \\
 \downarrow \text{Pr}_N & \swarrow \text{Pr}_N & \downarrow \text{Pr}_N \\
 N & \xlongequal{\quad} & N
 \end{array}$$

commutes, where $f_{t_0}: X_{t_0} \rightarrow Y$ denotes the composite of the normalization map $\nu_{t_0}: X_{t_0} \rightarrow Z_{t_0}$ and the inclusion map $\iota_{t_0}: Z_{t_0} \hookrightarrow Y$. Hence we have $\Psi(Z_{t_0} \times N) = \tilde{\pi}_1^{-1}(N)$, because $Z_{t_0} = f_{t_0}(X_{t_0})$ and $\tilde{\pi}_1^{-1}(N) = \tilde{F}^{-1}(\tilde{\pi}_1^{-1}(N))$.

(v) Let E_1 be a connected component of $E(Y)$. Since E_1 is arcwise-connected, it suffices to prove that, given a point $t_0 \in E_1$, there is an open neighborhood N_{t_0} of t_0 in E_1 such that for any $t \in N_{t_0}$ there is a diffeomorphism $\varphi: Y \rightarrow Y$ with $\varphi(Z_t) = Z_{t_0}$. Since the problem is local, we may assume that E_1 is a closed complex subspace of a relatively compact open domain D of the complex number space \mathbb{C}^n containing the origin o and that $t_0 = o$. Furthermore, taking D sufficiently small if necessary, we may assume that all of the irreducible components of E_1 as a complex space contain the origin o . Let $b: \hat{D} \rightarrow D$ be an embedded resolution of $(E_1)_{red}$. Namely, b is a surjective holomorphic map from a complex manifold \hat{D} into D such that $F' := b^{-1}((E_1)_{red})$ is an analytic subset of simple normal crossing. Then, for any $t \in (E_1)_{red}$ there are an irreducible component F' of $F' := b^{-1}((E_1)_{red})$ and a point $t' \in F'$ with $b(t') = t$. Note that $F' \cap b^{-1}(o) \neq \emptyset$ because $o \in b(F')$. We take a point, say t'_0 , of $F' \cap b^{-1}(o)$. We denote $\tilde{\pi}': \tilde{\mathcal{Z}}(Y)' \rightarrow F'$ the pull-back of the family $\tilde{\pi}_{red}: \tilde{\mathcal{Z}}(Y)_{red} \rightarrow E(Y)_{red}$ (the reduction of

$\tilde{\pi}: \tilde{\mathcal{Z}}(Y) \rightarrow E(Y)$ by a holomorphic map $b': F' \rightarrow (E_1)_{red}$, where $F' \xrightarrow{b'} (E_1)_{red} \hookrightarrow D$ denote the factorization of $b_{|F'}: F' \rightarrow D$. Then, $\tilde{\pi}': \tilde{\mathcal{Z}}(Y)' \rightarrow F'$ is a locally trivial family of analytic subvarieties with locally stable parametrizations of Y parametrized by F' . Since F' is non-singular and arcwise-connected, by Theorem 3.4 in [24] there is a diffeomorphism $\varphi: Y \rightarrow Y$ with $\varphi(Z'_i) = Z'_{i_0}$, where $Z'_i := \tilde{\pi}'^{-1}(t')$ and $Z'_{i_0} := \tilde{\pi}'^{-1}(t'_0)$. On the other hand, Z'_i and Z'_{i_0} are biholomorphic to Z_i and Z_{i_0} , respectively. Consequently, we have done the proof. Q. E. D.

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