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The nullity of harmonic tori in Lie groups

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Abstract. Recently Burstall et al. showed a construction of harmonic tori into Lie groups by using loop algebras. In this paper the auther estimates the nullity of harmonic maps constructed by his method.

§ 1. Introduction

Let $\varphi:(M, g)\rightarrow(N, h)$ be a smooth map of Riemannian manifolds. The energy $E(\varphi)$ of φ is defined by

$$E(\varphi)=\frac{1}{2}\int_{M}|d\varphi|^{2}d_{M},$$

where the differential $d\varphi$ can be viewed as a section of the vector bundle $T^{*}M\otimes\varphi^{-1}TN$ on M and we denote by $|d\varphi|(x)$ its norm at a point x of M, induced by the Riemannian metrics. We say that a map $\varphi:(M, g)\rightarrow(N, h)$ is harmonic if it extremizes the energy on every compact subdomain of M. If M is a circle S^{1}, a map $\varphi: S^{1}\rightarrow N$ is harmonic if and only if it is a closed geodesic parametrized proportionally to arc length. The Euler-Lagrange equation of the energy functional E is

$$\mathrm{Trace}_{g}\nabla d\varphi=0,$$

where we denote by ∇ the connection on $T^{*}M\otimes\varphi^{-1}TN$. The Hessian of a harmonic map φ is given by

$$H_{\varphi}(v, w)=\int_{M}\langle\Delta^{\varphi}v-\mathrm{Trace}_{g}R^{N}(d\varphi, v)d\varphi, w\rangle d_{M},$$

where v and w are sections of the vector bundle $\varphi^{-1}TN$, Δ^{φ} is the Laplacian on sections in the vector bundle $\varphi^{-1}TN$, and R^{N} is the curvatnre of the Levi-Civita connection on N. The operator on sections in $\varphi^{-1}TN$

$$J_{\varphi}=\Delta^{\varphi}-\mathrm{Trace}_{g}R^{N}(d\varphi, \cdot)d\varphi$$

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is called the Jacobi operator and elements of its kernel are called the Jacobi fields. The dimension of its kernel is called the nullity of φ. In the case M is the circle S^1, Jacobi fields is well-known in the theory of geodesics. For the fundamental facts about harmonic maps see Eells-Lemaire [3].

In recent years harmonic maps from a Riemann surface into a Lie group have been investigated extensively. Harmonic spheres into a Lie group were studied by Uhlenbeck [5] and harmonic tori into $SU(2)$ were studied by Hitchin [4]. Burstall [1] showed the construction of harmonic tori into a compact Lie group. On two-dimensional domains, the energy functional is conformally invariant, so we can identify harmonic maps of a 2-torus into a Lie group with harmonic maps of $\boldsymbol{R}^2$ into a Lie group which are doubly periodic with respect to some lattice. In this paper we estimate the nullity of harmonic tori into Lie groups which are constructed by Burstall et all. [2] using loop algebras.

In §2 we review Burstall's construction in [1], [2]. His starting point is the very important observation of Zakharov-Shabat-Uhlenbeck, which states the relation between harmonic maps of $\boldsymbol{R}^2$ into a Lie group G and one parameter families of $\mathfrak{g}^C$-valued 1-form on $\boldsymbol{R}^2$. We identify $\boldsymbol{R}^2$ with the complex line $\boldsymbol{C}$ and let

$$\alpha=\alpha'+\alpha''$$

be the type decomposition of α, where α' is a $\mathfrak{g}^C$-valued (1, 0)-form and α'' is a $\mathfrak{g}^C$-valued (0, 1)-form. Thus $\alpha''=\bar{\alpha}'$, where the conjugation in $\mathfrak{g}^C$ is with respect to the real form $\mathfrak{g}$. For each $\lambda\in\boldsymbol{C}^*$, we define a $\mathfrak{g}^C$-valued 1-form A_λ on $\boldsymbol{R}^2$ by

$$A_\lambda=\frac{1-\lambda}{2}\alpha'+\frac{1-\lambda^{-1}}{2}\alpha''.$$

Then

FACT 1. (Zakharov-Shabat, Uhlenbeck [5]) *The following statements* (i) *and* (ii) *are equivalent.*

(i) *For each $\lambda\in\boldsymbol{C}^*$, A_λ satisfies the Maurer-Cartan equation.*

(ii) *There exists a harmonic map φ of $\boldsymbol{R}^2$ into a Lie group G and the pullback of the Maurer-Cartan form of G bg φ is equal to A_{-1}.*

Thus we can obtain harmonic maps of $\boldsymbol{R}^2$ into a Lie group G if we construct $\mathfrak{g}^C$-valued 1-forms A_λ satisfying the condition of Fact 1 (i). Burstall [1] showed the method to construct such 1-forms by using the flow of some vector fields on the finite subspace of the loop algebra

$$\Omega_{\mathfrak{g}}=\{\xi : S^1\to\mathfrak{g}\,;\ \xi(1)=0\}.$$

An element ξ of $\Omega_{\mathfrak{g}}$ is expanded as follows

$$\xi=\sum_{n\neq 0}(1-\lambda^n)\xi_n \qquad (\xi_n\in\mathfrak{g}^C,\ \bar{\xi}_n=\xi_{-n}).$$

Now for each $d\in\boldsymbol{N}$, we define finite dimensional subspaces of $\Omega_{\mathfrak{g}}$ by

$$\Omega_d=\{\xi\in\Omega_{\mathfrak{g}}\,;\ \xi_n=0\ \ \forall|n|>d\}$$

and define the vector fields X_1, X_2 on Ω_d by

$$\frac{1}{2}(X_1-iX_2)(\xi)=[\xi, 2i(1-\lambda)\xi_d] \qquad \xi\in\Omega_d.$$

Then Burstall proved the following.

FACT 2. (Burstall [1]) *The vector fields X_1, X_2 are complete and commute each other. Using the flow $\xi : \boldsymbol{R}^2\to\Omega_d$ generated by X_1 and X_2, define the $\mathfrak{g}^C$-valued 1-form A_λ on $\boldsymbol{R}^2$ by*

$$A_\lambda=2i(1-\lambda)\xi_d dz-2i(1-\lambda^{-1})\xi_{-d}d\bar{z}$$

then A_λ satisfies the Maurer-Cartan equation for each $\lambda\in\boldsymbol{C}^*$.

Thus we can obtain a harmonic map φ of $\boldsymbol{R}^2$ into a Lie group G when we give a natural number d and an initial condition $\xi(0)\in\Omega_{\mathfrak{g}}$ of the flow. A harmonic map φ is unique up to left multiplication by a constant element of G. If such a harmonic map is doubly periodic then we obtain a harmonic map of a torus $\boldsymbol{R}^2/\Lambda$ into a Lie group G. Burstall showed the sufficient condition for a harmonic map of a torus $\boldsymbol{R}^2/\Lambda$ into a Lie group G obtained by the above construction. Let φ be a harmonic map of a torus $\boldsymbol{R}^2/\Lambda$ into a Lie group G then $\varphi^{-1}d\varphi(\partial/\partial z)$ is viewed as a map from $\boldsymbol{R}^2$ into $\mathfrak{g}^C$. Burstall proved that $\varphi^{-1}d\varphi(\partial/\partial z)$ takes values in a single Ad_{G^C}-orbit in $\mathfrak{g}^C$. Moreover he proved the following fact.

FACT 4. (Burstall et al. [2]) *Let φ be a harmonic map of a torus $\boldsymbol{R}^2/\Lambda$ into a compact semisimple Lie group or a unitary group G. Suppose $\varphi^{-1}d\varphi(\partial/\partial z)$ takes values in a single Ad_{G^C}-orbit of semisimple elements in $\mathfrak{g}^C$, then φ is obtained by the above construction for some $d\in\boldsymbol{N}$.*

In § 3, we estimate the nullity of φ using above d. First we prove that the natural number d in Fact 4 is not unique.

LEMMA 1. *Let d' be any natural number grater than d. Then the* *given harmonic map is also obtained by a flow on* $\Omega_{d'}$.

Now we define natural number $d(\varphi)$ as smallest number having such property. We prove the following theorem.

MAIN THEOREM. *Under the same hypothesis as in Fact* 4, *the harmonic map* φ *satisfies*

$$d(\varphi) \leqq (\text{The nullity of } \varphi)$$

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§2. A review for Burstall's construction

First we consider harmonic map of $\boldsymbol{R}^2$ into a compact Lie group G. We fix a bi-invariant Riemannian metric on G. Let $\mathfrak{g}$ be the Lie algebra of G and θ be the Maurer-Cartan form of G. Then $\varphi: \boldsymbol{R}^2 \to G$ is harmonic if and only if

$$d^*(\varphi^*\theta) = 0 . \tag{2.1}$$

And $\varphi^*\theta$ satisfies the Maurer-Cartan equation

$$d(\varphi^*\theta) + \frac{1}{2}[\varphi^*\theta \wedge \varphi^*\theta] = 0 \tag{2.2}$$

since θ satisfies it. The equation (2.2) means that the connection $d+\varphi^*\theta$ on the trivial bundle $\boldsymbol{R}^2 \times G$ is flat.

Conversely, given a $\mathfrak{g}$-valued 1-form α on $\boldsymbol{R}^2$ which satisfies (2.2) there exist a gauge transformation $\varphi: \boldsymbol{R}^2 \to G$, unique up to left multiplication by a constant elements of G, such that

$$\varphi^*\theta = \alpha$$

moreover if α satisfies (2.1), φ is a harmonic map. Thus it suffices to consider $\mathfrak{g}$-valued 1-forms α on $\boldsymbol{R}^2$ satisfying (2.1) and (2.2) to obtain harmonic maps of $\boldsymbol{R}^2$ into a Lie group G.

Now we introduce a complex coordinate on $\boldsymbol{R}^2$ and let

$$\alpha = \alpha' + \alpha''$$

be the type decomposition of a $\mathfrak{g}$-valued 1-form α on $\boldsymbol{R}^2$ where α' is a $\mathfrak{g}^C$-valued $(1,0)$-form and α'' is a $\mathfrak{g}^C$-valued $(0,1)$-form, thus $\alpha'' = \bar{\alpha}'$. For any $\lambda\in \boldsymbol{C}^{*}$, We define a $\mathfrak{g}^{C}$-valued 1-form A_{λ} by

$$A_{\lambda}=\frac{1-\lambda}{2}\alpha'+\frac{1-\lambda^{-1}}{2}\alpha''$$

which is $\mathfrak{g}$-valued for $\lambda\in S^{1}$. Then the following fact is well-known.

FACT 1. (Zakharov-Shabat, Uhlenbeck [5]) *The following statements* (i) *and* (ii) *are equivalent.*

(i) *For each* $\lambda\in \boldsymbol{C}^{*}$, A_{λ} *satisfies the Maurer-Cartan equation.*

(ii) *There exists a harmonic map* φ *of* $\boldsymbol{R}^{2}$ *into a Lie group* G *and the pullback of the Maurer-Cartan form of* G *by* φ *is equal to* A_{-1}.

Thus it suffices to consider $\mathfrak{g}^{C}$-valued 1-forms A_{λ} which satisfy the condition of Fact 1 (i) to obtain harmonic maps of $\boldsymbol{R}^{2}$ into a Lie group G.

On loop algebra

$$\Omega_{\mathfrak{g}}=\{\xi:\ S^{1}\rightarrow \mathfrak{g}\ ;\ \xi(1)=0\},$$

we define an inner product by

$$(\xi_{1},\xi_{2})=\int_{S^{1}}(\xi_{1}',\xi_{2}')dvol_{S^{1}}\qquad \text{for}\quad \xi_{1},\xi_{2}\in\Omega_{\mathfrak{g}}.$$

Then any element ξ of $\Omega_{\mathfrak{g}}$ is expanded as follows

$$\xi=\sum_{n\neq 0}(1-\lambda^{n})\xi_{n},$$

where $\xi_{n}\in \mathfrak{g}^{C}$ and $\bar{\xi}_{n}=\xi_{-n}$. For each $d\in\boldsymbol{N}$, we set

$$\Omega_{d}=\{\xi\in\Omega_{\mathfrak{g}}:\ \xi_{n}=0\ \ \forall|n|>d\},$$

which is a finite dimensional linear subspace of $\Omega_{\mathfrak{g}}$. We identify each tangent space of Ω_{d} with itself and we define the vector fields X_{1}, X_{2} on Ω_{d} by

$$\frac{1}{2}(X_{1}-iX_{2})(\xi)=[\xi,2i(1-\lambda)\xi_{d}]\qquad \text{for}\quad \xi\in\Omega_{d}.$$

Then Burstall proved the following.

FACT 2. (Burstall [1]) *The vector fields* X_{1}, X_{2} *are complete and commute each other. Using the flow* $\xi:\boldsymbol{R}^{2}\rightarrow\Omega_{d}$ *generated by* X_{1} *and* X_{2}, *define the* $\mathfrak{g}^{C}$*-valued* 1*-form* A_{λ} *on* $\boldsymbol{R}^{2}$ *by*

$$A_{\lambda}=2i(1-\lambda)\xi_{d}dz-2i(1-\lambda^{-1})\xi_{-d}d\bar{z}$$

then A_λ satisfies the Maurer-Cartan equation for each $\lambda \in \boldsymbol{C}^*$.

Thus we obtain a harmonic map of $\boldsymbol{R}^2$ into a Lie group G satisfying $\varphi^*\theta = A_{-1}$, unique up to left multiplication by a constant element of G after choosing a natural number d and an initial condition $\xi(0) \in \Omega_d$ of the flow.

Let M be a Kähler manifold and φ be a smooth map of M into G. We identify the pullback of the tangent bundle of G with the trivial vector bundle $M \times \mathfrak{g}$ on M by the pullback of the Maurer-Cartan form of G. Then the pullback of the Levi-Civita connection of G is given by

$$\nabla = d + \frac{1}{2} ad\alpha ,$$

where we denote the pullback of the Maurer-Cartan form of G by α. Under the above identification the Jacobi operator J_φ of the harmonic map φ is given by

$$J_\varphi v = \sum_k \left\{ -(\partial_\Delta \bar{\partial}_\Delta - \bar{\partial}_\Delta \partial_\Delta) v(Z_k, \bar{Z}_k) + \frac{1}{4} [\alpha(\bar{Z}_k), [\alpha(Z_k), v]] + \frac{1}{4} [\alpha(Z_k), [\alpha(\bar{Z}_k), v]] \right\},$$

where $\{Z_k\}$ is a local unitary frame for $T^{(1,0)}M$ and v is a map of M into $\mathfrak{g}^C$.

We defined vector fields X_1, X_2 on Ω_d. Let $\xi : \boldsymbol{R}^2 \to \Omega_d$ be a flow generated by X_1 and X_2. Namely ξ is a solution of the following differential equations

$$\begin{cases} \dfrac{\partial \xi}{\partial x} = X_1(\xi) \\[2mm] \dfrac{\partial \xi}{\partial y} = X_2(\xi) . \end{cases}$$

Using the complex coordinate $z = x + iy$, these equations are expressed by

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1-\lambda)\xi_d] .$$

Taking the conjugation we obtain

$$\frac{\partial \xi}{\partial \bar{z}} = [\xi, -2i(1-\lambda^{-1})\xi_{-d}] .$$

Thus the flow ξ satisfies

$$d\xi=[\xi, A_\lambda],$$

where $A_\lambda=2i(1-\lambda)\xi_d dz-2i(1-\lambda^{-1})\xi_{-d}d\bar{z}$.

Let φ be a harmonic map of M into G. Then $\varphi^*\theta$ has a decomposition

$$\varphi^*\theta=(\varphi^*\theta)^{1,0}+(\varphi^*\theta)^{0,1}$$

and for each $\lambda\in\boldsymbol{C}^*$ we set

$$B_\lambda=\frac{1-\lambda}{2}(\varphi^*\theta)^{1,0}+\frac{1-\lambda^{-1}}{2}(\varphi^*\theta)^{0,1}.$$

Then we remark the following fact.

FACT 3. (Burstall, Ferus, Pedit and Pinkall [2]) *Suppose* $Y=\sum_n\lambda^{-n}Y_n$, *where* $Y_n: M\to\mathfrak{g}^C$ *satisfies*

$$dY=[Y, B_\lambda]$$

then each $Y_n: M\to\mathfrak{g}^C$ *is a Jacobi field for the harmonic map* φ.

Next we consider harmonic maps of a torus $\boldsymbol{R}^2/\Lambda$ into a Lie group G. Burstall proved the following.

FACT 4. (Burstall et al. [2]) *Let* φ *be a harmonic map of a torus* $\boldsymbol{R}^2/\Lambda$ *into a compact semisimple Lie group or a unitary group. Suppose* $\varphi^{-1}d\varphi(\partial/\partial z)$ *takes values in a single* Ad_{G^C} *orbit of semisimple elements in* $\mathfrak{g}^C$, *then* φ *is obtained by the method of Fact 2 for some* $d\in\boldsymbol{N}$. *Namely there exists* $\xi: \boldsymbol{R}^2/\Lambda\to\Omega_d$ *satisfying*

$$\begin{cases} d\xi=[\xi, A_\lambda] \\ A_{-1}=\varphi^*\theta \qquad (i.e.\ 4i\xi_d dz=(\varphi^*\theta)^{1,0}), \end{cases}$$

where A_λ *is as above.*

Thus when we express the flow $\xi: \boldsymbol{R}^2/\Lambda\to\Omega_d$ in Fact 4 in the form

$$\xi=\sum_{|n|\le d}(1-\lambda^n)\xi_n,$$

each ξ_n is a Jacobi field for the given harmonic map φ by Fact 3. In the next section we estimate the nullity of φ by using these Jacobi fields.

§ 3. The estimate of the nullity

Let φ be a harmonic map of a torus $\boldsymbol{R}^2/\Lambda$ into a compact semisimple

Lie group or a unitary group. Suppose that φ satisfies the condition in Fact 4. Then the harmonic map φ is obtained by a flow $\xi: \boldsymbol{R}^2/\Lambda \rightarrow \Omega_d$ generated by X_1 and X_2 for some $d \in \boldsymbol{N}$. The natural number d is not unique for the given harmonic map. In fact we can prove the following.

LEMMA 1. *Let d' be any natural number grater than d. Then the given harmonic map is also obtained by a flow on $\Omega_{d'}$.*

PROOF. Let φ be a harmonic map which is obtained by a flow ξ: $\boldsymbol{R}^2/\Lambda \rightarrow \Omega_d$. Namely $\xi = \sum_{|n| \leq d} (1-\lambda^n)\xi_n$ satisfies

$$\begin{cases} d\xi = [\xi, A_\lambda] \\ (\varphi^*\theta)^{1,0} = 4i\xi_d dz, \end{cases}$$

where $A_\lambda = 2i(1-\lambda)\xi_d dz - 2i(1-\lambda^{-1})\xi_{-d} d\bar{z}$ and θ is the Maurer-Cartan form of G. By using $(\varphi^*\theta)^{1,0} = 4i\xi_d dz$, A_λ is written in the form

$$A_\lambda = \frac{1-\lambda}{2}(\varphi^*\theta)^{1,0} + \frac{1-\lambda^{-1}}{2}(\varphi^*\theta)^{0,1}.$$

Now for any natural number N, we define

$$\tilde{\xi}: \boldsymbol{R}^2/\Lambda \longrightarrow \Omega_{d+N}$$

by

$$\tilde{\xi} = \lambda^N \xi + \lambda^{-N} \bar{\xi}.$$

Then it is easy to verify that $\tilde{\xi}$ satisfies

$$\begin{cases} d\tilde{\xi} = [\tilde{\xi}, A_\lambda] \\ (\varphi^*\theta)^{1,0} = 4i\tilde{\xi}_{d+N} dz. \end{cases}$$

This means that the given harmonic map φ is obtained by the flow $\tilde{\xi}$ on the vector space Ω_{d+N}. ■

Now we define the natural number $d(\varphi)$ as follows for a harmonic map φ which satisfies the condition in Fact 4.

DEFINITION. Let φ be a harmonic map of a torus $\boldsymbol{R}^2/\Lambda$ into a Lie group G which satisfies the condition in Fact 4. The harmonic map φ is obtained by a flow on the vector space Ω_d for some $d \in \boldsymbol{N}$. we denote by $d(\varphi)$ the minimum natural number which has such property.

Then we can estimate the nullity of the harmonic map φ by using above $d(\varphi)$.

THEOREM. *Under the same hypothesis as in Fact 4, the harmonic map φ satisfies*

$$d(\varphi) \leqq (\text{The nullity of } \varphi)$$

In the rest of this paper we prove this theorem. The outline of the proof is the following: By assumption there exists a flow

$$\xi = \sum_{|n| \leq d(\varphi)} (1-\lambda^n)\xi_n,$$

by which the harmonic map φ is constructed. From Fact 3, we can conclude that the coefficients of ξ

$$\xi_1, \xi_2, \cdots, \quad \xi_{d(\varphi)}$$

are Jacobi fields for the harmonic map φ. We prove that these Jacobi fields are linearly independent.

PROOF OF THE THEOREM. Let $\xi: \boldsymbol{R}^2/\Lambda \to \Omega_{d(\varphi)}$ be a flow by which the harmonic map φ is constructed. Then ξ is expanded as follows

$$\xi = \sum_{|n| \leq d(\varphi)} (1-\lambda^n)\xi_n.$$

Each coefficient of ξ

$$\xi_1, \xi_2, \cdots \quad, \xi_{d(\varphi)}$$

is Jacobi field for the harmonic map φ. It suffices to prove that these Jacobi fields are linearly independent. We prove this by contradiction. Assume that there exist some $a_i \in \boldsymbol{C}$ $(1 \leqq i \leqq d(\varphi))$ such that

$$a_1\xi_1 + a_2\xi_2 + \cdots + a_{d(\varphi)}\xi_{d(\varphi)} = 0.$$

We denote $d(\varphi)$ by d for simplicity.

The case for $d=1$ is obvious. In the case for $d=2$, first we prove that a_1 equals to 0 by contradiction. Assume that $a_1 \neq 0$. We define

$$\tilde{\xi}: \boldsymbol{R}^2/\Lambda \longrightarrow \Omega_1$$

by

$$\tilde{\xi} = (1-\lambda)\xi_2 + (1-\lambda^{-1})\xi_{-2}.$$

Then we prove that $\tilde{\xi}$ satisfies the following equations

$$\begin{cases} \dfrac{\partial \tilde{\xi}}{\partial z} = [\tilde{\xi}, 2i(1-\lambda)\tilde{\xi}_1] \\ (\varphi^*\theta)^{1,0} = 4i\tilde{\xi}_1 dz. \end{cases}$$

The latter equation $(\varphi^*\theta)^{1,0}=4i\tilde{\xi}_1dz$ is clearly satisfied, since $\tilde{\xi}_1$ equals to ξ_2. Since

$$\begin{aligned}[\tilde{\xi}, 2i(1-\lambda)\tilde{\xi}_1]&=[(1-\lambda)\xi_2+(1-\lambda^{-1})\xi_{-2}, 2i(1-\lambda)\xi_2]\\ &=(1-\lambda^{-1})(1-\lambda)[\xi_{-2}, 2i\xi_2]\\ &=\{(1-\lambda)+(1-\lambda^{-1})\}[\xi_{-2}, 2i\xi_2],\end{aligned}$$

the equation

$$\frac{\partial\tilde{\xi}}{\partial z}=[\tilde{\xi}, 2i(1-\lambda)\tilde{\xi}_1]$$

is equivalent to the following

$$\frac{\partial\xi_2}{\partial z}=[\xi_{-2}, 2i\xi_2] \tag{3.1}$$

$$\frac{\partial\xi_{-2}}{\partial z}=[\xi_{-2}, 2i\xi_2]. \tag{3.2}$$

Similarly the equation

$$\frac{\partial\xi}{\partial z}=[\xi, 2i(1-\lambda)\xi_2]$$

is equivalent to

$$\frac{\partial\xi_2}{\partial z}=-[\xi_1, 2i\xi_2] \tag{3.3}$$

$$\frac{\partial\xi_1}{\partial z}=2[\xi_1, 2i\xi_2]+[\xi_{-1}, 2i\xi_2]+[\xi_{-2}, 2i\xi_2] \tag{3.4}$$

$$\frac{\partial\xi_{-1}}{\partial z}=[\xi_{-1}, 2i\xi_2]-[\xi_{-2}, 2i\xi_2] \tag{3.5}$$

$$\frac{\partial\xi_{-2}}{\partial z}=[\xi_{-2}, 2i\xi_2]. \tag{3.6}$$

By the assumption the following is satisfied

$$a_1\xi_1+a_2\xi_2=0.$$

Taking bracket with $2i\xi_2$ we obtain

$$a_1[\xi_1, 2i\xi_2]=0.$$

Using the assumption

$$a_1 \neq 0$$

we obtain

$$[\xi_1, 2i\xi_2] = 0. \tag{3.7}$$

Now, if we substitute

$$\xi_1 = -\frac{a_2}{a_1}\xi_2$$

for (3.5), we obtain

$$-\overline{\left(\frac{a_2}{a_1}\right)}\frac{\partial \xi_{-2}}{\partial z} = \left[-\overline{\left(\frac{a^2}{a_1}\right)}\xi_{-2}, 2i\xi_2\right] - [\xi_{-2}, 2i\xi_2].$$

Using (3.6) we can conclude

$$[\xi_{-2}, 2i\xi_2] = 0. \tag{3.8}$$

From (3.7) and (3.8), we see that the equation (3.1) is equivalent to (3.3). The equivalence between (3.2) and (3.6) is obvious. This contradicts to the definition of the number $d = d(\varphi)$. Therefore

$$a_1 = 0.$$

Moreover

$$\xi_2 \neq 0$$

since d equals to 2. So

$$a_2 = 0.$$

Hence Jacobi fields ξ_1 and ξ_2 are linearly independent.

In the case the number d is larger than or equal to 3, we set

$$k = \min_{1 \leqq n \leqq d}\{n;\ a_n \neq 0\}.$$

First we prove that

$$a_k = 0$$

by contradiction in the case $1 \leqq k \leqq d-2$. Now assume that $a_k \neq 0$. We set

$$a'_n = -\frac{a_n}{a_k}$$

for $k+1 \leqq n \leqq d$, and define $\alpha_i^{(k)}$ by

$$\begin{cases} \alpha_1^{(k)} = 1 + a'_{k+1} \\ \alpha_i^{(k)} = a'_{i+k} - a'_{i+k-1} \qquad 2 \leqq i \leqq d-k-2 \\ \alpha_{d-k-1}^{(k)} = a'_{d-1} - a'_{d-2} + a'_d. \end{cases} \tag{3.9}$$

Using this $\alpha_i^{(k)}$, we define

$$\xi^{(k)}: \boldsymbol{R}^2/\Lambda \longrightarrow \Omega_{d-k}$$

by

$$\begin{aligned}\xi^{(k)} = & \sum_{n=2}^{d-k}(1-\lambda^n)\left\{\xi_{k+n} - \sum_{i=1}^{d-k-n}\alpha_i^{(k)}\xi_{i+k+n}\right\} \\ & +(1-\lambda)\left\{\xi_{k+1} - \sum_{i=1}^{d-k-1}\alpha_i^{(k)}\xi_{i+k+1} + \xi_k - \sum_{i=k+1}^{d-1}a_i'\xi_i\right\} \\ & +(1-\lambda^{-1})\left\{\xi_{-(h+1)} - \sum_{i=1}^{d-k-1}\overline{\alpha_i^{(k)}}\xi_{-(i+k+1)} + \xi_{-k} - \sum_{i=k+1}^{d-1}\bar{a}_i'\xi_{-i}\right\} \\ & +\sum_{n=2}^{d-k}(1-\lambda^{-n})\left\{\xi_{-(k+n)} - \sum_{i=1}^{d-k-n}\overline{\alpha_i^{(k)}}\xi_{-(i+k+n)}\right\}.\end{aligned}$$

Then we claim the following.

CLAIM. Let $\xi^{(k)}$ be as above. The harmonic map φ is obtained by the flow $\xi^{(k)}$. Namely $\xi^{(k)}$ satisfies

$$\begin{cases} \dfrac{\partial \xi^{(k)}}{\partial z} = [\xi^{(k)}, 2i(1-\lambda)\xi_{d-k}^{(k)}] \\ (\varphi^*\theta)^{1,0} = 4i\xi_{d-k}^{(k)}dz. \end{cases}$$

From this claim we can see

$$a_k = 0$$

because this claim contradicts to the definition of $d(\varphi)$. We prove this claim first in the case $k=1$ using the following lemma.

LEMMA 2. *Assume that*

$$a_1 \neq 0$$

then

$$\sum_{|n|\leq d}[\xi_n, \xi_d] + \sum_{i=1}^{d-1}a_{i+1}'[\xi_i, \xi_d] = 0 \tag{3.10}$$

$$[\xi_{-2}, \xi_d] = \sum_{i=3}^{d}\overline{a_{i-1}'}[\xi_{-i}, \xi_d] \tag{3.11}$$

$$[\xi_1, \xi_d] = \sum_{i=2}^{d-1}a_i'[\xi_i, \xi_d] \tag{3.12}$$

$$[\xi_{-1}, \xi_d] = \sum_{i=2}^{d} \bar{a}'_i [\xi_{-i}, \xi_d]. \tag{3.13}$$

PROOF OF LEMMA 2. The equation

$$\frac{\partial \xi}{\partial z} = [\xi, 2i(1-\lambda)\xi_d]$$

is equivalent to

$$\begin{cases} \dfrac{\partial \xi_d}{\partial z} = -[\xi_{d-1}, 2i\xi_d] \\ \dfrac{\partial \xi_{d-1}}{\partial z} = [\xi_{d-1}, 2i\xi_d] - [\xi_{d-2}, 2i\xi_d] \\ \dfrac{\partial \xi_{d-2}}{\partial z} = [\xi_{d-2}, 2i\xi_d] - [\xi_{d-3}, 2i\xi_d] \\ \quad\vdots \\ \dfrac{\partial \xi_2}{\partial z} = [\xi_2, 2i\xi_d] - [\xi_1, 2i\xi_d] \\ \dfrac{\partial \xi_1}{\partial z} = [\xi_1, 2i\xi_d] + \sum_{|n| \leq d} [\xi_n, 2i\xi_d] \end{cases} \tag{3.14}$$

$$\begin{cases} \dfrac{\partial \xi_{-1}}{\partial z} = [\xi_{-1}, 2i\xi_d] - [\xi_{-2}, 2i\xi_d] \\ \dfrac{\partial \xi_{-2}}{\partial z} = [\xi_{-2}, 2i\xi_d] - [\xi_{-3}, 2i\xi_d] \\ \quad\vdots \\ \dfrac{\partial \xi_{-(d-1)}}{\partial z} = [\xi_{-(d-1)}, 2i\xi_d] - [\xi_{-d}, 2i\xi_d] \\ \dfrac{\partial \xi_{-d}}{\partial z} = [\xi_{-d}, 2i\xi_d]. \end{cases} \tag{3.15}$$

From the assumption

$$a_1\xi_1 + a_2\xi_2 + \cdots + a_d\xi_d = 0, \tag{3.16}$$

we obtain

$$a_1\frac{\partial \xi_1}{\partial z} + a_2\frac{\partial \xi_2}{\partial z} + \cdots + a_d\frac{\partial \xi_d}{\partial z} = 0. \tag{3.17}$$

From (3.14), (3.16) and (3.17) we obtain

$$a_1\sum_{|n|\leqq d}[\xi_n, 2i\xi_d]-a_2[\xi_1, 2i\xi_d]-a_3[\xi_2, 2i\xi_d]-\cdots-a_d[\xi_{d-1}, 2i\xi_d]=0\,.$$

By the definition

$$a_i'=-\frac{a_i}{a_1}(2\leqq i\leqq d)\,,$$

we can conclude (3.10) immediately. Similarly from

$$\bar{a}_1\xi_{-1}+\bar{a}_2\xi_{-2}+\cdots+\bar{a}_d\xi_{-d}=0 \tag{3.18}$$

we obtain

$$\bar{a}_1\frac{\partial\xi_{-1}}{\partial z}+\bar{a}_2\frac{\partial\xi_{-2}}{\partial z}+\cdots+\bar{a}_d\frac{\partial\xi_{-d}}{\partial z}=0\,. \tag{3.19}$$

From (3.15), (3.18) and (3.19), we obtain

$$\bar{a}_1[\xi_{-2}, 2i\xi_d]+\bar{a}_2[\xi_{-3}, 2i\xi_d]+\cdots+\overline{a_{d-1}}[\xi_{-d}, 2i\xi_d]=0\,.$$

Then we can conclude (3.11) immediately. Also from (3.16) and (3.18) we obtain

$$\begin{cases} \xi_1=\sum_{i=2}^{d} a_i'\xi_i \\ \xi_{-1}=\sum_{i=2}^{d}\overline{a_i'}\xi_{-i}\,. \end{cases}$$

Taking bracket with ξ_d, we can conclude (3.12) and (3.13).

Now we shall prove the claim for $k=1$ by use of Lemma 2.

PROOF OF THE CLAIM IN THE CASE $k=1$:
By the definition, $\xi^{(1)}$ is in the form

$$\xi^{(1)}=\sum_{|n|\leqq d-1}(1-\lambda^n)(\sum_j c_{n,j}^{(1)}\xi_j)$$

for some $c_{n,j}^{(1)}\in \boldsymbol{C}$. Namely $\xi_n^{(1)}$ is written in the form

$$\xi_n^{(1)}=\sum_j c_{n,j}^{(1)}\xi_j\,.$$

The equation

$$(\varphi^*\theta)^{1,0}=4i\xi_{d-1}^{(1)}dz$$

is clearly satisfied since $\xi_{d-1}^{(1)}=\xi_d$. So we shall prove the equation

$$\frac{\partial \xi^{(1)}}{\partial z}=[\xi^{(1)}, 2i(1-\lambda)\xi^{(1)}_{d-1}].$$

This equation is equivalent to

$$(3.20)\quad \begin{cases} \dfrac{\partial \xi^{(1)}_{d-1}}{\partial z}=-[\xi^{(1)}_{d-2}, 2i\xi^{(1)}_{d-1}] \\ \dfrac{\partial \xi^{(1)}_{d-2}}{\partial z}=[\xi^{(1)}_{d-2}, 2i\xi^{(1)}_{d-1}]-[\xi^{(1)}_{d-3}, 2i\xi^{(1)}_{d-1}] \\ \quad\vdots \\ \dfrac{\partial \xi^{(1)}_{3}}{\partial z}=[\xi^{(1)}_{3}, 2i\xi^{(1)}_{d-1}]-[\xi^{(1)}_{2}, 2i\xi^{(1)}_{d-1}] \end{cases}$$

$$(3.21)\quad \frac{\partial \xi^{(1)}_{2}}{\partial z}=[\xi^{(1)}_{2}, 2i\xi^{(1)}_{d-1}]-[\xi^{(1)}_{1}, 2i\xi^{(1)}_{d-1}]$$

$$(3.22)\quad \frac{\partial \xi^{(1)}_{1}}{\partial z}=[\xi^{(1)}_{1}, 2i\xi^{(1)}_{d-1}]+\sum_{|n|\leqq d-1}[\xi^{(1)}_{n}, 2i\xi^{(1)}_{d-1}]$$

$$(3.23)\quad \frac{\partial \xi^{(1)}_{-1}}{\partial z}=[\xi^{(1)}_{-1}, 2i\xi^{(1)}_{d-1}]-[\xi^{(1)}_{-1}, 2i\xi^{(1)}_{d-1}]$$

$$(3.24)\quad \begin{cases} \dfrac{\partial \xi^{(1)}_{-2}}{\partial z}=[\xi^{(1)}_{-2}, 2i\xi^{(1)}_{d-1}]-[\xi^{(1)}_{-3}, 2i\xi^{(1)}_{d-1}] \\ \quad\vdots \\ \dfrac{\partial \xi^{(1)}_{-(d-2)}}{\partial z}=[\xi^{(1)}_{-(d-2)}, 2i\xi^{(1)}_{d-1}]-[\xi^{(1)}_{-(d-1)}, 2i\xi^{(1)}_{d-1}] \\ \dfrac{\partial \xi^{(1)}_{-(d-1)}}{\partial z}=[\xi^{(1)}_{-(d-1)}, 2i\xi^{(1)}_{d-1}]. \end{cases}$$

First we shall consider $\xi^{(1)}_n$ for $3\leqq n\leqq d-1$. After substituting

$$\xi^{(1)}_n=\sum_{j=2}^{d} c^{(1)}_{n,j}\xi_j, \quad \xi^{(1)}_{n-1}=\sum_{j=2}^{d} c^{(1)}_{n-1,j}\xi_j \quad \text{and} \quad \xi^{(1)}_{d-1}=\xi_d$$

for

$$\frac{\partial \xi^{(1)}_{n}}{\partial z}=[\xi^{(1)}_{n}, 2i\xi^{(1)}_{d-1}]-[\xi^{(1)}_{n-1}, 2i\xi^{(1)}_{d-1}],$$

we obtain

$$(3.25)\qquad \sum_{j=2}^{d} c_{n,j}^{(1)} \frac{\partial \xi_j}{\partial z} = \left[\sum_{j=2}^{d} c_{n,j}^{(1)} \xi_j, 2i\xi_d\right] - \left[\sum_{j=2}^{d} c_{n-1,j}^{(1)} \xi_j, 2i\xi_d\right].$$

On the other hand,

$$\xi : \boldsymbol{R}^2/\Lambda \longrightarrow \Omega_d$$

satisfies

$$\frac{\partial \xi_j}{\partial z} = [\xi_j, 2i\xi_d] - [\xi_{j-1}, 2i\xi_d]$$

for $2 \leqq j \leqq d$. Also after substituting these for (3.25), we obtain

$$\sum_{j=2}^{d} c_{n,j}^{(1)}([\xi_j, 2i\xi_d] - [\xi_{j-1}, 2i\xi_d])$$

$$= \left[\sum_{j=2}^{d} c_{n,j}^{(1)} \xi_j, 2i\xi_d\right] - \left[\sum_{j=2}^{d} c_{n-1,j}^{(1)} \xi_j, 2i\xi_d\right].$$

So that

$$(3.26)\qquad \sum_{j=2}^{d} c_{n,j}^{(1)} [\xi_{j-1}, \xi_d] = \left[\sum_{j=2}^{d} c_{n-1,j}^{(1)} \xi_j, \xi_d\right].$$

Namely we must prove

$$(3.27)\qquad \left[\sum_{j=2}^{d} c_{n,j}^{(1)} \xi_{j-1} - \sum_{j=2}^{d} c_{n-1,j}^{(1)} \xi_j, \xi_d\right] = 0.$$

Now from the definition of $c_{n,j}^{(1)}$ we obtain

$$\sum_{j=2}^{d} c_{n,j}^{(1)} \xi_{j-1} - \sum_{j=2}^{d} c_{n-1,j}^{(1)} \xi_j$$

$$= \left(\xi_n - \sum_{i=1}^{d-n-1} \alpha_i^{(1)} \xi_{i+n}\right) - \left(\xi_n - \sum_{i=1}^{d-n} \alpha_i^{(1)} \xi_{i+n}\right)$$

$$= \alpha_{d-n}^{(1)} \xi_d .$$

So (3.27) is clearly satisfied. We have proved that $\xi_n^{(1)}$ satisfies (3.20).

Next we shall consider $\xi_2^{(1)}$. Similarly, we see that (3.21) is equivalent to the following equation

$$(3.28)\qquad \left[\sum_{j=2}^{d} c_{2,j}^{(1)} \xi_{j-1} - \sum_{j=1}^{d} c_{1,j}^{(1)} \xi_j, \xi_d\right] = 0 .$$

Also from the definition of $c_{n,j}^{(1)}$ we obtain

$$\sum_{j=2}^{d} c_{2,j}^{(1)}\xi_{j-1}-\sum_{j=1}^{d} c_{1,j}^{(1)}\xi_j$$

$$=\left(\xi_2-\sum_{i=1}^{d-3}\alpha_i^{(1)}\xi_{i+2}\right)-\left(\xi_2-\sum_{i=1}^{d-2}\alpha_i^{(1)}\xi_{i+2}+\xi_1-\sum_{i=2}^{d-1}a_i'\xi_i\right)$$

$$=\alpha_{d-2}^{(1)}\xi_d-\left(\xi_1-\sum_{i=2}^{d-1}a_i'\xi_i\right)$$

Therefore the equation (3.28) is satisfied from Lemma 2 (3.12). We have proved that $\xi_2^{(1)}$ satisfies (3.21).

Third we shall prove that $\xi_1^{(1)}$ satisfies the equation (3.22). After substituting

$$\xi_n^{(1)}=\sum_j c_{n,j}^{(1)}\xi_j \quad (|n|\leqq d-1)$$

for (3.22), we obtain

$$\sum_{j=2}^{d} c_{1,j}^{(1)}\frac{\partial\xi_j}{\partial z}+c_{1,1}^{(1)}\frac{\partial\xi_1}{\partial z}=\left[\sum_{j=1}^{d} c_{1,j}^{(1)}\xi_j,\, 2i\xi_d\right]+\sum_{|n|\leqq d-1}\left[\sum c_{n,j}^{(1)}\xi_j,\, 2i\xi_d\right]. \tag{3.29}$$

On the other hand

$$\xi:\ \boldsymbol{R}^2/\Lambda \longrightarrow \Omega_d$$

satisfies

$$\frac{\partial\xi_j}{\partial z}=[\xi_j, 2i\xi_d]-[\xi_{j-1}, 2i\xi_d] \quad (2\leqq j\leqq d) \tag{3.30}$$

$$\frac{\partial\xi_1}{\partial z}=[\xi_1, 2i\xi_d]+\sum_{|n|\leqq d}[\xi_n, 2i\xi_d]. \tag{3.31}$$

After substituting (3.30) and (3.31) for (3.29), we obtain

$$\sum_{j=2}^{d} c_{1,j}^{(1)}([\xi_j, 2i\xi_d]-[\xi_{j-1}, 2i\xi_d])+c_{1,1}^{(1)}([\xi_1, 2i\xi_d]+\sum_{|n|\leqq d}[\xi_n, 2i\xi_d])$$

$$=\left[\sum_{j=1}^{d} c_{1,j}^{(1)}\xi_j, 2i\xi_d\right]+\sum_{|n|\leqq d-1}[\sum_j c_{n,j}^{(1)}\xi_j, 2i\xi_d].$$

So that

$$-\sum_{j=2}^{d} c_{1,j}^{(1)}[\xi_{j-1}, \xi_d]+c_{1,1}^{(1)}\sum_{|n|\leqq d}[\xi_n, \xi_d]=\sum_{|n|\leqq d-1}[\sum_j c_{n,j}^{(1)}\xi_j, \xi_d].$$

Now from the definition of $c_{n,j}^{(1)}$ this becomes

$$-\left[\xi_1-\sum_{i=1}^{d-2}\alpha_i^{(1)}\xi_{i+1}-\sum_{i=2}^{d-1}a_i'\xi_{i-1}, \xi_d\right]+\sum_{|n|\leqq d}[\xi_n, \xi_d] \tag{3.32}$$

$$=\left[\sum_{n=1}^{d-1}\xi_{1+n}-\sum_{n=1}^{d-1}\sum_{i=1}^{d-1-n}\alpha_i^{(1)}\xi_{i+1+n}+\xi_1-\sum_{i=2}^{d-1}a_i'\xi_i,\ \xi_d\right]$$

$$+\left[\sum_{n=1}^{d-1}\xi_{-(1+n)}-\sum_{n=1}^{d-1}\sum_{i=1}^{d-1-n}\overline{\alpha_i^{(1)}}\xi_{-(i+1+n)}+\xi_{-1}-\sum_{i=2}^{d-1}\bar{a}_i'\xi_{-i},\ \xi_d\right].$$

The term $\sum_{n=1}^{d-1}\sum_{i=1}^{d-1-n}\alpha_i^{(1)}\xi_{i+1+n}$ in right hand side is written

$$\sum_{n=1}^{d-1}\sum_{i=1}^{d-1-n}\alpha_i^{(1)}\xi_{i+1+n}=\sum_{i=1}^{1}\alpha_i^{(1)}\xi_3+\sum_{i=1}^{2}\alpha_i^{(1)}\xi_4+\cdots+\sum_{i=1}^{d-2}\alpha_i^{(1)}\xi_d \tag{3.33}$$

$$=\sum_{l=3}^{d}\sum_{i=1}^{l-2}\alpha_i^{(1)}\xi_l.$$

From the definition (3.9) of $\alpha_i^{(k)}$ we obtain

$$\begin{cases}\sum_{i=1}^{l-2}\alpha_i^{(1)}=1+a_{l-1}' & (3\leqq l\leqq d-1)\\ \sum_{i=1}^{d-2}\alpha_i^{(1)}=1+a_{d-1}'+a_d'.\end{cases} \tag{3.34}$$

After substituting (3.34) for (3.33) we obtain

$$\sum_{n=1}^{d-1}\sum_{i=1}^{d-1-n}\alpha_i^{(1)}\xi_{i+1+n}=\sum_{l=3}^{d-1}(1+a_{l-1}')\xi_l+(1+a_{d-1}'+a_d')\xi_d \tag{3.35}$$

$$=\sum_{i=3}^{d}(1+a_{i-1}')\xi_i+a_d'\xi_d.$$

Similarly we obtain

$$\sum_{n=1}^{d-1}\sum_{i=1}^{d-1-n}\overline{\alpha_i^{(1)}}\xi_{-(i+1+n)}=\sum_{i=3}^{d}\overline{(a_{i-1}'}+1)\xi_{-i}+\bar{a}_d'\xi_{-d}. \tag{3.36}$$

The term $\sum_{i=1}^{d-2}\alpha_i^{(1)}\xi_{i+1}$ in left hand side of (3.32) is written

$$\sum_{i=1}^{d-2}\alpha_i^{(1)}\xi_{i+1}$$

$$=\alpha_1^{(1)}\xi_2+\sum_{i=2}^{d-3}\alpha_i^{(1)}\xi_{i+1}+\alpha_{d-2}^{(1)}\xi_{d-1}$$

$$=(a_2'+1)\xi_2+\sum_{i=2}^{d-3}(a_{i+1}'-a_i')\xi_{i+1}+(a_{d-1}'-a_{d-2}'+a_d')\xi_{d-1} \tag{3.37}$$

$$=\xi_2+a'_d\xi_{d-1}+\sum_{i=1}^{d-2}a'_{i+1}\xi_{i+1}-\sum_{i=2}^{d-2}a'_i\xi_{i+1}$$

$$=\xi_2+a'_d\xi_{d-1}+\sum_{i=2}^{d-1}a'_i\xi_i-\sum_{i=3}^{d-1}a'_{i-1}\xi_i.$$

After substituting (3.35), (3.36) and (3.37) for (3.32) we obtain

$$\left[-\xi_1+\sum_{i=2}^{d-1}a'_i\xi_i+\sum_{i=1}^{d-1}a'_{i+1}\xi_i,\ \xi_d\right]+\sum_{|n|\leq d}[\xi_n,\xi_d] \tag{3.38}$$

$$=\left[\xi_1-\sum_{i=2}^{d-1}a'_i\xi_i,\ \xi_d\right]+\left[\xi_{-2}-\sum_{i=3}^{d}\overline{a'_{i-1}}\xi_{-i}+\xi_{-1}-\sum_{i=2}^{d}\bar{a}'_i\xi_{-i},\ \xi_d\right].$$

This equation is clearly satisfied from (3.10), (3.11), (3.12), and (3.13).

Forth we shall prove that $\xi_{-1}^{(1)}$ satisfies the equation (3.23). After substituting

$$\xi_{-1}^{(1)}=\sum_{j=1}^{d}c_{-1,-j}^{(1)}\xi_{-j},\quad \xi_{-2}^{(1)}=\sum_{j=2}^{d}c_{-2,-j}^{(1)}\xi_{-j},\quad \text{and}\quad \xi_{d-1}^{(1)}=\xi_d$$

for

$$\frac{\partial\xi_{-1}^{(1)}}{\partial z}=[\xi_{-1}^{(1)},2i\xi_{d-1}^{(1)}]-[\xi_{-2}^{(1)},2i\xi_{d-1}^{(1)}],$$

we obtain

$$\sum_{j=1}^{d}c_{-1,-j}^{(1)}\frac{\partial\xi_{-j}}{\partial z}=\left[\sum_{j=1}^{d}c_{-1,-j}^{(1)}\xi_{-j},\ 2i\xi_d\right]-\left[\sum_{j=2}^{d}c_{-2,-j}^{(1)}\xi_{-j},\ 2i\xi_d\right]. \tag{3.39}$$

On the other hand

$$\xi:\ \boldsymbol{R}^2/\Lambda\longrightarrow\Omega_d$$

satisfies

$$\frac{\partial\xi_{-j}}{\partial z}=[\xi_{-j},2i\xi_d]-[\xi_{-(j+1)},2i\xi_d] \tag{3.40}$$

for $1\leqq j\leqq d$. Also after substituting (3.40) for (3.39) we obtain

$$\sum_{j=1}^{d}c_{-1,-j}^{(1)}([\xi_{-j},2i\xi_d]-[\xi_{-(j+1)},2i\xi_d])$$

$$=\left[\sum_{j=1}^{d}c_{-1,-j}^{(1)}\xi_{-j},\ 2i\xi_d\right]-\left[\sum_{j=2}^{d}c_{-2,-j}^{(1)}\xi_{-j},\ 2i\xi_d\right].$$

So that

$$\sum_{j=1}^{d}c_{-1,-j}^{(1)}[\xi_{-(j+1)},\xi_d]=\left[\sum_{j=2}^{d}c_{-2,-j}^{(1)}\xi_{-j},\ \xi_d\right]. \tag{3.41}$$

Namely we must prove

$$\left[\sum_{j=1}^{d} c^{(1)}_{-1,-j}\xi_{-(j+1)}-\sum_{j=2}^{d} c^{(1)}_{-2,-j}\xi_{-j},\ \xi_d\right]=0. \tag{3.42}$$

Now from the definition of $c^{(1)}_{n,j}$ we obtain

$$\begin{aligned}&\sum_{j=1}^{d} c^{(1)}_{-1,-j}\xi_{-(j+1)}-\sum_{j=2}^{d} c^{(1)}_{-2,-j}\xi_{-j}\\&=\left(\xi_{-3}-\sum_{i=1}^{d-2}\overline{\alpha^{(1)}_i}\xi_{-(i+3)}+\xi_{-2}-\sum_{i=2}^{d-1}\bar{a}'_i\xi_{-(i+1)}\right)\\&\qquad\qquad-\left(\xi_{-3}-\sum_{i=1}^{d-3}\overline{\alpha^{(1)}_i}\xi_{-(i+3)}\right)\\&=\xi_{-2}-\sum_{i=2}^{d-1}\bar{a}'_i\xi_{-(i+1)}.\end{aligned}$$

Therefore the equation (3.42) is satisfied from Lemma 2 (3.11). We have proved that $\xi^{(1)}_{-1}$ satisfies (3.23).

Finally we shall prove that $\xi^{(1)}_{-n}$ satisfies the equation (3.24). Similarly (3.24) is equivalent to

$$\left[\sum_{j=2}^{d} c^{(1)}_{-n,-j}\xi_{-(j+1)}-\sum_{j=2}^{d} c^{(1)}_{-(n+1),-j}\xi_{-j},\ \xi_d\right]=0. \tag{3.43}$$

From the definition of $c^{(1)}_{n,j}$ we obtain

$$\begin{aligned}&\sum_{j=2}^{d} c^{(1)}_{-n,-j}\xi_{-(j+1)}-\sum_{j=2}^{d} c^{(1)}_{-(n+1),-j}\xi_{-j}\\&=\left(\xi_{-(n+2)}-\sum_{i=1}^{d-n-2}\overline{\alpha^{(1)}_i}\xi_{-(i+n+2)}\right)\\&\quad-\left(\xi_{-(n+2)}-\sum_{i=1}^{d-n-2}\overline{\alpha^{(1)}_i}\xi_{-(i+n+2)}\right)\\&=0.\end{aligned}$$

Therefore the equation (3.43) is clearly satisfied. We have proved that $\xi^{(1)}_{-n}$ satisfies (3.24) for $2\leqq n\leqq d-1$. So we have proved the claim in the case $k=1$.

We shall prove the claim in the case $2\leqq k\leqq d-2$, using the following lemma, which can be proved similarly as Lemma 2.

LEMMA 3. *Fix any k for $2\leqq k\leqq d-2$. Assume that*

$$a_1=a_2=\cdots=a_{k-1}=0, \quad a_k\neq 0,$$

then

$$[\xi_{k-1},\xi_d]=\sum_{i=k}^{d-1}a'_{i+1}[\xi_i,\xi_d] \tag{3.44}$$

$$[\xi_{-(k+1)},\xi_d]=\sum_{i=k+2}^{d}\overline{a'_{i-1}}[\xi_{-i},\xi_d] \tag{3.45}$$

$$[\xi_k,\xi_d]=\sum_{i=k+1}^{d-1}a'_i[\xi_i,\xi_d] \tag{3.46}$$

$$[\xi_{-k},\xi_d]=\sum_{i=k+1}^{d}\bar{a}'_i[\xi_{-i},\xi_d]. \tag{3.47}$$

PROOF OF THE CLAIM IN THE CASE $2\leqq k\leqq d-2$: The equation

$$\frac{\partial\xi^{(k)}}{\partial z}=[\xi^{(k)},2i(1-\lambda)\xi^{(k)}_{d-k}]$$

is equivalent to the following

$$\begin{cases}\dfrac{\partial\xi^{(k)}_{d-k}}{\partial z}=-[\xi^{(k)}_{d-k-1},2i\xi^{(k)}_{d-k}]\\[2mm] \dfrac{\partial\xi^{(k)}_{d-k-1}}{\partial z}=[\xi^{(k)}_{d-k-1},2i\xi^{(k)}_{d-k}]-[\xi^{(k)}_{d-k-2},2i\xi^{(k)}_{d-k}]\\[2mm] \dfrac{\partial\xi^{(k)}_{d-k-2}}{\partial z}=[\xi^{(k)}_{d-k-2},2i\xi^{(k)}_{d-k}]-[\xi^{(k)}_{d-k-3},2i\xi^{(k)}_{d-k}]\\ \quad\vdots\\ \dfrac{\partial\xi^{(k)}_{2}}{\partial z}=[\xi^{(k)}_{2},2i\xi^{(k)}_{d-k}]-[\xi^{(k)}_{1},2i\xi^{(k)}_{d-k}]\\[2mm] \dfrac{\partial\xi^{(k)}_{1}}{\partial z}=[\xi^{(k)}_{1},2i\xi^{(k)}_{d-k}]+\displaystyle\sum_{|n|\leqq d-k}[\xi^{(k)}_{n},2i\xi^{(k)}_{d-k}]\\[2mm] \dfrac{\partial\xi^{(k)}_{-1}}{\partial z}=[\xi^{(k)}_{-1},2i\xi^{(k)}_{d-k}]-[\xi^{(k)}_{-2},2i\xi^{(k)}_{d-k}]\\[2mm] \dfrac{\partial\xi^{(k)}_{-2}}{\partial z}=[\xi^{(k)}_{-2},2i\xi^{(k)}_{d-k}]-[\xi^{(k)}_{-3},2i\xi^{(k)}_{d-k}]\\ \quad\vdots\end{cases}$$

$$\left|\begin{array}{l}\dfrac{\partial\xi_{-(d-k-1)}^{(k)}}{\partial z}=[\xi_{-(d-k-1)}^{(k)},2i\xi_{d-k}^{(k)}]-[\xi_{-(d-k)}^{(k)},2i\xi_{d-k}^{(k)}]\\ \dfrac{\partial\xi_{-(d-k)}^{(k)}}{\partial z}=[\xi_{-(d-k)}^{(k)},2i\xi_{d-k}^{(k)}].\end{array}\right.$$

Since we can prove similarly as in the case $k=1$ that $\xi_n^{(k)}(n\neq 1)$ satisfies above equation, we shall only prove that $\xi_1^{(k)}$ satisfies the equation

$$\frac{\partial\xi_1^{(k)}}{\partial z}=[\xi_1^{(k)},2i\xi_{d-k}^{(k)}]+\sum_{|n|\leq d-k}[\xi_n^{(k)},2i\xi_{d-k}^{(k)}]. \tag{3.48}$$

After substituting

$$\xi_n^{(k)}=\sum_j c_{n,j}^{(k)}\xi_j$$

for the equation (3.48), we obtain

$$\sum_{j=2}^d c_{1,j}^{(k)}\frac{\partial\xi_j}{\partial z}=\left[\sum_{j=2}^d c_{1,j}^{(k)}\xi_j,2i\xi_d\right]+\sum_{|n|\leq d-k}[\sum_j c_{n,j}^{(k)}\xi_j,2i\xi_d]. \tag{3.49}$$

On the other hand

$$\xi: \boldsymbol{R}^2/\Lambda\longrightarrow\Omega_d$$

satisfies

$$\frac{\partial\xi_j}{\partial z}=[\xi_j,2i\xi_d]-[\xi_{j-1},2i\xi_d]\quad(2\leqq j\leqq d). \tag{3.50}$$

After substituting (3.50) for the equation (3.49), we obtain

$$\sum_{j=2}^d c_{1,j}^{(k)}([\xi_j,2i\xi_d]-[\xi_{j-1},2i\xi_d])$$

$$=\left[\sum_{j=2}^d c_{1,j}^{(k)}\xi_j,2i\xi_d\right]+\sum_{|n|\leq d-k}[\sum_j c_{n,j}^{(k)}\xi_j,2i\xi_d].$$

So that

$$-\sum_{j=2}^d c_{1,j}^{(k)}[\xi_{j-1},\xi_d]=\sum_{|n|\leq d-k}[\sum_j c_{n,j}^{(k)}\xi_j,\xi_d].$$

Now from the definition of $c_{n,j}^{(k)}$ this becomes

$$-\left[\xi_k-\sum_{i=1}^{d-k-1}\alpha_i^{(k)}\xi_{i+k}+\xi_{k-1}-\sum_{i=k+1}^{d-1}a_i'\xi_{i-1},\xi_d\right]$$

$$=\left[\sum_{n=1}^{d-k}\xi_{k+n}-\sum_{n=1}^{d-k}\sum_{i=1}^{d-k-n}\alpha_i^{(k)}\xi_{i+k+n}+\xi_k-\sum_{i=k+1}^{d-1}a_i'\xi_i,\xi_d\right] \tag{3.51}$$

$$+\left[\sum_{n=1}^{d-k}\xi_{-(k+n)}-\sum_{n=1}^{d-k}\sum_{i=1}^{d-k-n}\overline{\alpha_i^{(k)}}\xi_{-(i+k+n)}+\xi_{-k}-\sum_{i=k+1}^{d-1}\bar{a}'_i\xi_{-i},\xi_d\right].$$

The term $\sum_{n=1}^{d-k}\sum_{i=1}^{d-k-n}\alpha_i^{(k)}\xi_{i+k+n}$ in right hand side is written

$$\begin{aligned}&\sum_{n=1}^{d-k}\sum_{i=1}^{d-k-n}\alpha_i^{(k)}\xi_{i+k+n}\\ (3.52)\qquad &=\sum_{i=1}^{1}\alpha_i^{(k)}\xi_{k+2}+\sum_{i=1}^{2}\alpha_i^{(k)}\xi_{k+3}+\cdots+\sum_{i=1}^{d-k-1}\alpha_i^{(k)}\xi_d\\ &=\sum_{l=k+2}^{d}\sum_{i=1}^{l-k-1}\alpha_i^{(k)}\xi_l\,.\end{aligned}$$

From the definition (3.9) of $\alpha_i^{(k)}$ we obtain

$$(3.53)\qquad \begin{cases}\displaystyle\sum_{i=1}^{l-k-1}\alpha_i^{(k)}=1+a'_{l-1} & (k+2\leqq l\leqq d-1)\\ \displaystyle\sum_{i=1}^{d-k-1}\alpha_i^{(k)}=1+a'_{d-1}+a'_d\,.\end{cases}$$

After substituting (3.53) for (3.52) we obtain

$$\begin{aligned}(3.54)\qquad \sum_{n=1}^{d-k}\sum_{i=1}^{d-k-n}\alpha_i^{(k)}\xi_{i+k+n}&=\sum_{l=k+2}^{d-1}(1+a'_{l-1})\xi_l+(1+a'_{d-1}+a'_d)\xi_d\\ &=\sum_{i=k+2}^{d}(1+a'_{i-1})\xi_i+a'_d\xi_d\,.\end{aligned}$$

Similarly we obtain

$$(3.55)\qquad \sum_{n=1}^{d-k}\sum_{i=1}^{d-k-n}\overline{\alpha_i^{(k)}}\xi_{-(i+k+n)}=\sum_{i=k+2}^{d}(1+\overline{a'_{i-1}})\xi_{-i}+\bar{a}'_d\xi_{-d}\,.$$

The term $\sum_{i=1}^{d-k-1}\alpha_i^{(k)}a_{i+k}$ in left hand side of (3.51) is written as

$$\begin{aligned}&\sum_{i=1}^{d-k-1}\alpha_i^{(k)}\xi_{i+k}=\alpha_1^{(k)}\xi_{1+k}+\sum_{i=2}^{d-k-2}\alpha_i^{(k)}\xi_{i+k}+\alpha_{d-k-1}^{(k)}\xi_{d-1}\\ (3.56)\qquad &=(1+a'_{k+1})\xi_{k+1}+\sum_{i=2}^{d-k-2}(a'_{i+k}-a'_{i+k-1})\xi_{i+k}+(a'_{d-1}-a'_{d-2}+a'_d)\xi_{d-1}\\ &=\xi_{k+1}+a'_d\xi_{d-1}+\sum_{i=1}^{d-k-1}a'_{i+k}\xi_{i+k}-\sum_{i=2}^{d-k-1}a'_{i+k-1}\xi_{i+k}\end{aligned}$$

$$=\xi_{k+1}+a'_d\xi_{d-1}+\sum_{i=k+1}^{d-1}a'_i\xi_i-\sum_{i=k+2}^{d-1}a'_{i-1}\xi_i\,.$$

After substituting (3.54), (3.55) and (3.56) for (3.51), we obtain

$$\left[-\xi_k+\sum_{i=k+1}^{d-1}a'_i\xi_i-\xi_{k-1}+\sum_{i=k}^{d-1}a'_{i+1}\xi_i,\ \xi_d\right]$$

$$=\left[\xi_k-\sum_{i=k+1}^{d-1}a'_i\xi_i,\ \xi_d\right]+\left[\xi_{-(k+1)}-\sum_{i=k+2}^{d}\overline{a'_{i-1}}\xi_{-i}+\xi_{-k}-\sum_{i=k+1}^{d}\bar{a}'_i\xi_{-i},\ \xi_d\right].$$

This equation is clearly satisfied from (3.44), (3.45), (3.46), and (3.47). So we have proved that $\xi_1^{(k)}$ satifies the equation (3.48). We have completely proved the claim.

We assumed that there exist some $a_i\in\boldsymbol{C}$ $(1\leqq i\leqq n)$ such that

$$a_1\xi_1+a_2\xi_2+\cdots+a_d\xi_d=0\,.$$

We can conclude

$$a_i=0\quad(1\leqq i\leqq d-2)$$

from the claim. So we obtain

$$a_{d-1}\xi_{d-1}+a_d\xi_d=0\,. \tag{3.57}$$

It suffices to prove that

$$a_{d-1}=0$$

to see that the Jacobi fields $\{\xi_i\}_{1\leqq i\leqq d}$ are linearly independent, since ξ_d is not equal to zero.

We shall prove

$$a_{d-1}=0$$

by contradiction. Assume that

$$a_{d-1}\neq 0$$

then we obtain

$$\xi_{d-1}=a'_d\xi_d\quad\left(a'_d=-\frac{a_d}{a_{d-1}}\right).$$

Taking bracket with ξ_d, we can conclude

$$[\xi_{d-1},\xi_d]=0\,.$$

Now we define

$$\xi^{(d-1)}:\ \boldsymbol{R}^2/\Lambda\longrightarrow\Omega_1$$

by

$$\xi^{(d-1)}=(1-\lambda)\xi_d+(1-\lambda^{-1})\xi_{-d}\,.$$

We shall prove that $\xi^{(d-1)}$ satisfies the following equation

$$\frac{\partial\xi^{(d-1)}}{\partial z}=[\xi^{(d-1)},2i(1-\lambda)\xi_1^{(d-1)}]\,. \tag{3.58}$$

This equation is equivalent to

$$\begin{cases}\dfrac{\partial\xi_d}{\partial z}=[\xi_{-d},2i\xi_d]\\[2ex] \dfrac{\partial\xi_{-d}}{\partial z}=[\xi_{-d},2i\xi_d]\,.\end{cases} \tag{3.59}$$

On the other hand

$$\xi:\ \boldsymbol{R}^2/\Lambda\longrightarrow\Omega_d$$

satisfies the equation

$$\begin{cases}\dfrac{\partial\xi_d}{\partial z}=-[\xi_{d-1},2i\xi_d]\\[2ex] \dfrac{\partial\xi_{-d}}{\partial z}=[\xi_{-d},2i\xi_d]\,.\end{cases} \tag{3.60}$$

From (3.15) and (3.57) we obtain

$$\overline{a_{d-1}}[\xi_{-d},\xi_d]=0\,.$$

Using the assumption of a contradiction, we can conclude

$$[\xi_{-d},\xi_d]=0\,.$$

From this we can see that $\xi^{(d-1)}$ satisfies the equation (3.58). This contradicts to the fact that $d=d(\varphi)\geqq 3$. So that we can conclude

$$a_{d-1}=0\,.$$

We have proved that the Jacobi fields $\{\xi_i\}_{1\leq i\leq d}$ for the harmonic map φ are linearly independent and we have completely proved the theorem. ■

The above theorem is meaningful for a harmonic map φ which has large $d(\varphi)$. But in the present the author doesn't know as if there exist those harmonic map of a torus $\boldsymbol{R}^2/\Lambda$ into a Lie group which has large $d(\varphi)$.

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