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Subelliptic estimates for the $\bar{\partial}$-Neumann problem on certain weakly pseudo-convex domains

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§ 0. Introduction

The purpose of this paper is to study on the subelliptic estimates for the $\bar{\partial}$-Neumann problem on certain weakly pseudo-convex domains whose dimensions are greater than or equal to 3 and whose Levi forms have eigenvalues which are not varying smoothly.

Let n be a positive integer and let Ω be a smoothly bounded domain of C^n with a smoothly varying Hermitian metric. For an open set U of C^n and $\varepsilon>0$, we shall consider the following inequality:

$$\|\varphi\|_{\varepsilon}\leq C(\|\bar{\partial}\varphi\|+\|\bar{\partial}^{*}\varphi\|+\|\varphi\|) \quad \text{for all } \varphi\in\mathcal{D}^{p,q}(U). \tag{0.1}$$

Here $1\leq q\leq n-1$, C is a positive constant independent of φ, $\|\cdot\|_{\varepsilon}$ denotes the Sobolev norm of order ε on $\bar{\Omega}$, and $\mathcal{D}^{p,q}(U)$ denotes the space of smooth (p, q)-forms on $\bar{\Omega}$ which are supported in $U\cap\bar{\Omega}$ and belong to the intersection of the domains of $\bar{\partial}$ and $\bar{\partial}^{*}$. We call (0.1) a subelliptic estimate for the $\bar{\partial}$-Neumann problem. It implies the following fact: the (p, q)-form solution u of $\bar{\partial}u=f$ is in the Sobolev space of order $s+\varepsilon$ on U if f is in that of order s on U and u is orthogonal to the null space of $\bar{\partial}$. Thus this estimate is very useful in the investigation of the local regularities of the solutions to the $\bar{\partial}$-problem.

Let $\mathring{z}$ be a point of the boundary $b\Omega$ of Ω and choose a smooth function r so that $dr\neq 0$ and Ω is given by

$$\Omega=\{z; r(z)<0\}. \tag{0.2}$$

Assume that Ω is pseudo-convex near z. In the case when $\varepsilon=1/2$, Kohn [13] proved that there exists a neighborhood U of z such that (0.1) hold for all q when Ω is strongly pseudo-convex at $\mathring{z}$, and Folland, Kohn [5] and Hörmander [8] showed that for given q, (0.1) holds when the Levi form of Ω at $\mathring{z}$ has at most $n-q$ positive eigenvalues. In the case when $0<\varepsilon<1/2$, we cannot know whether (0.1) is valid from the number of positive eigenvalues. Kohn [15] introduced ideals of smooth functions and gave a sufficient condition for (0.1) with some $\varepsilon>0$ to hold in terms of such ideals. Furthermore applying the result of Diederich and Fornæss [4], he showed that when $b\Omega$ is real-analytic in a neighborhood of z, this condition is satisfied if and only if there are no q-dimensional varieties at z which lie in $b\Omega$. On the other hand, to mesure the maximal order of contact of q-dimensional varieties with $b\Omega$ at $\mathring{z}$, D'Angelo [3] and Catlin [2] defined the so-called type $D_q(\mathring{z})$ of $b\Omega$. Using it, Catlin proved that $\varepsilon\leq(D_q(\mathring{z}))^{-1}$ if (0.1) is valid for some neighborhood U of $\mathring{z}$ by applying the result of [1] and that $D_q(\mathring{z})<\infty$ if and only if (0.1) with some $\varepsilon>0$ holds. In some cases, we know some methods to determine the best value of ε for (0.1). When $q=n-1$, Greiner [7] and Kohn [15] proved that if the maximum of all orders of contact of $(n-1)$-dimensional complex manifolds with $b\Omega$ at z is integer m, then (0.1) is valid if $\varepsilon=1/m$, and it is invalid for any $\varepsilon>1/m$. Kohn [14] claimed that if the Levi form is diagonalizable by a suitable base of smooth vector fields, then we can determine the best value of ε by applying the method in proving (0.1) for $n=2$ and $q=1$ to each vector field. When Ω is convex and $q=1$, Fornæss and Sibony in [6] proved (0.1) with $\varepsilon=(D_1(\mathring{z}))^{-1}$. However we know no methods to determine the best value of ε in the other cases.

We shall give pseudo-convex domains which will be considered in the followings. Let $\tilde{r}(w_0, \cdots, w_n)$ be a smooth function on C^{n+1} which is defined on some neighborhood of the origin and satisfies

$$\tilde{r}(0)=0 \quad \text{and} \quad \frac{\partial\tilde{r}}{\partial \operatorname{Im} w_0}(0)\neq 0. \tag{0.3}$$

Setting the domain $\tilde{\Omega}=\{w\in C^{n+1}; \tilde{r}(w)<0\}$, we assume that the Levi form of $\tilde{\Omega}$ at the origin is positive definite. Let l be an integer satisfying $n/2\leq l\leq n$, and let $\{m_i\}_{i=1}^{n}$ be a sequence of positive integers satisfying

$$m_i\leq m_{i+l} \quad \text{for} \quad i=1, \cdots, n-l. \tag{0.4}$$

We introduce functions $f_j(z_j, z_{j-l})$ for $j=l+1, \cdots, n$ as

$$f_j(z_j, z_{j-l})=z_j^{m_j}+\sum_{k=0}^{m_j-1} g_{jk}(z_{j-l})z_j^k, \tag{0.5}$$

where g_{jk} is a holomorphic function defined about the origin and satisfies

$$(0.6) \qquad \frac{\partial^p g_{jk}}{\partial z_{j-l}^n}(0)=0 \quad \text{if} \quad p\leq m_{j-l}(m_j-k)/m_j.$$

Rearrange $\{m_i\}_{i=1}^n$ to $\{\tilde{m}\}_{i=1}^n$ so that $\tilde{m}_i\leq\tilde{m}_{i+1}$. The main result of our paper is the following theorem.

THEOREM 0.1. *Let Ω be a smooth boounded domain in C^{n+1} whose boundary contains the origin. Assume that there exists a neighborhood U of the origin such that*

$$(0.7) \quad \Omega\cap U=\{z\in U; \tilde{r}(z_0, z_1^{m_1}, \cdots, z_l^{m_l}, f_{l+1}(z_{l+1}, z_1), \cdots, f_n(z_n, z_{n-l}))<0\}.$$

The we have the subelliptic estimate (0.1) *with $\varepsilon=1/2\tilde{m}_{n+1-q}$ for some neighborhood of the origin, and* (0.1) *is invalid for any neighborhood of the origin if $\varepsilon>1/2\tilde{m}_{n+1-q}$.*

It follow from (0.3) and (0.5)-(0.7) that $b\Omega$ is smooth near the origin, and the strongly pseudo-convexity of $\tilde{\Omega}$ implies that Ω is pseudo-convex. We give an example:

$$\Omega=\{z\in C^3; \operatorname{Im} z_0>a|z_1|^{2m_1}+2b\operatorname{Re} z_1^{m_1}\bar{z}_2^{m_2}+c|z_2|^{2m_2}\},$$

where $a>0$, $c>0$, $b\neq 0$, and $ac-b^2>0$. Since the eigenvalues of its Levi form are not smooth about the origin, the Levi form is not diagonalizable by smooth vector fields. In view of this example, we can see that our theorem cannot be proven by using the method in [14] even if $f_j=z_j^{m_j}$ in (0.5). Furthermore we remark that when $|z_{j-l}|^{m_{j-l}}/|z_j|^{m_j}$ is large enough, we cannot consider $\sum g_{jk}(z_{j-l})z_j^k$ as a perturbation of $z_j^{m_j}$. Applying Theorem 1 in [1], we can verify easily that (0.1) does not hold if $\varepsilon>1/2\tilde{m}_{n+1-q}$ (see the proof of Theorem 0.1 for the detail). Hence the most part of the proof of our theorem is to prove (0.1) with $\varepsilon=1/2\tilde{m}_{n+1-q}$. Here we shall use the method in the study of subelliptic operators due to Hörmander [10].

The plan of this paper is as follows: In section 1, we review the theory of the tangential Cauchy-Riemann complexes and its relationship with the $\bar{\partial}$-Neumann problem. After this, we shall consider the subelliptic estimates for the tangential Cauchy-Riemann complexes. In section 2, after introducing parameters ρ and τ, we give a neighborhood of each point on the boundary whose size depends on ρ and τ. By investigating local properties of f_j and using the commutators of vector fields as in [9] and [10], we prove some estimates for smooth functions supported on these neighborhoods. In section 3, we construct a locally finite covering of $b\Omega\times R_{\tau}$ with fixed ρ and decompose (p, q)-forms microlocally by pseudo-differential operators whose symbols are supported on open sets of this covering and apply the estimates obtained in section 2. Finally (0.1) is proved by patching local estimates.

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§ 1. Tangential Cauchy-Riemann complexes

In this section, we shall recall the definition of the Cauchy-Riemann complexes on the boundary and reduce the subellipticity of the $\bar{\partial}$-Neumann problems to that of these complexes by applying the results of Sweeney.

For a domain Ω of C^{n+1} with a smooth boundary $b\Omega$, we denote by i the inclusion $b\Omega\rightarrow\bar{\Omega}$. Then the vector bundle $\mathcal{B}^{p,q}(b\Omega)$ on $b\Omega$ can be defined by $\mathcal{B}^{p,q}(b\Omega)=i^{*}\mathcal{D}^{p,q}(\bar{\Omega})$. Moreover the differential operator $\bar{\partial}_{b}$: $\mathcal{B}^{p,q}(b\Omega)\rightarrow\mathcal{B}^{p,q+1}(b\Omega)$ is given so that $\bar{\partial}_{b}{}^{*}(\varphi)=i^{*}(\bar{\partial}\varphi)$ for $\varphi\in\mathcal{D}^{p,q}(\bar{\Omega})$. We call $\{\mathcal{B}^{p,q}(b\Omega)\}_{q}$ the tangential Cauchy-Riemann complex and $\bar{\partial}_{b}$ the tangential Cauchy-Riemann operator. Since the hermitian metric on $\bar{\Omega}$ is induced on $b\Omega$ by i, we can define the adjoint operator $\bar{\partial}_{b}^{*}:\mathcal{B}^{p,q}(b\Omega)\rightarrow\mathcal{B}^{p,q-1}(b\Omega)$ of $\bar{\partial}_{b}$ with respect to this metric.

Let z be a point of $b\Omega$. Then there exists a neighborhood U of z such that we can choose an orthogonal basis $L_{0}, \cdots, L_{n}$ of $T^{1,0}(\bar{\Omega})$ satisfying

$$\partial r(L_{j})=0 \text{ for } j=1, \cdots, n \text{ and } \partial r(L_{0})\neq 0 \text{ on } U, \tag{1.1}$$

for r in (0.2). Restricting $L_{1}, \cdots, L_{n}$ to $b\Omega$, we can see that they form an orthogonal base of $T^{1,0}(b\Omega)$. Let $\tilde{\omega}_{0}, \cdots, \tilde{\omega}_{n}$ be $(1,0)$-forms on $\bar{\Omega}$ satisfying $\tilde{\omega}_{j}(L_{k})=\delta_{jk}$ for $j, k=0, \cdots, n$. Then $i^{*}(\tilde{\omega}_{1}), \cdots, i^{*}(\tilde{\omega}_{n})$ form an orthogonal base of $\mathcal{B}^{1,0}(b\Omega)$. We write $\omega_{j}=i^{*}\tilde{\omega}_{j}$ for $j=1, \cdots, n$. If $\varphi\in\mathcal{B}^{p,q}(b\Omega)$, then on $U\cap b\Omega$, φ can be expressed as

$$\varphi=\sum_{I,J}{}'\varphi_{I,J}\omega_{I}\wedge\bar{\omega}_{J}, \tag{1.2}$$

where $I=(i_{1}, \cdots, i_{p})$ and $J=(j_{1}, \cdots, j_{q})$ with $1\leq i_{k}, j_{k}\leq n$. The symbol $\sum'$ signifies that the summation is restricted to strictly increasing p-tuples I and q-tuples J. The forms ω_{I} and $\bar{\omega}_{J}$ are given by $\omega_{I}=\omega_{i_{1}}\wedge$

$\cdots\wedge\omega_{i_p}$ and $\bar{\omega}_J=\bar{\omega}_{j_1}\wedge\cdots\bar{\omega}_{j_q}$.

In view of the orthogonality of $L_1,\cdots,L_n$, we can verify that $\bar{\partial}_b\varphi$ and $\bar{\partial}_b^*\varphi$ have the following forms:

$$\bar{\partial}_b\varphi=(-1)^p\sum_{I,J}{}'\sum_{j=1}^{n}\bar{L}_j\varphi_{I,J}\omega_I\wedge\bar{\omega}_j\wedge\bar{\omega}_J+\sum f_{H,L}^{I,J}\varphi_{I,J}\omega_H\wedge\bar{\omega}_L, \tag{1.3}$$

$$\bar{\partial}_b^*\varphi=(-1)^{p+1}\sum_{I,K}{}'\sum_{j=1}^{n}L_j\varphi_{I,jK}\omega_I\wedge\bar{\omega}_K+\sum g_{H,L}^{I,J}\varphi_{I,J}\omega_H\wedge\bar{\omega}_K, \tag{1.4}$$

where I and H run through p-tuples, J through q-tuples, L through $(q+1)$-tuples, and K through $(q-1)$-tuples. The coefficients $f_{H,K}^{I,J}$ and $g_{H,K}^{I,J}$ are smooth functions. We put $\varphi_{I,jK}=\varepsilon(jK)\varphi_{I,\langle jK\rangle}$, where $\langle jK\rangle$ denotes the increasingly tuple of jK and $\varepsilon(jK)$ is the sign of the permutation taking jK to $\langle jK\rangle$ (see [5] for the details). The formulas (1.3) and (1.4) are very useful to study the $\bar{\partial}$-Neumann problems from the viewpoint of partial differential equations.

We shall introduce a real coordinate system on $b\Omega$. If $\partial r/\partial\operatorname{Im}z_0\neq 0$ at $\mathring{z}$, then applying the Malgrange preparation theorem (see [12]), we can find real valued smooth functions $f(z)$ and $\phi(\operatorname{Re}z_0,z_1,\cdots,z_n)$ defined on a neighborhood of $\mathring{z}$ such that $\phi(\operatorname{Re}\mathring{z}_0,\mathring{z}_1,\cdots,\mathring{z}_n)=0$, $f(z)>0$ for each z, and r can be represented as

$$r(z)=f(z)(\phi(\operatorname{Re}z_0,\cdots,z_n)-\operatorname{Im}z_0).$$

Therefore putting $x_j=\operatorname{Re}z_j$, $y_j=\operatorname{Im}z_j$ for $j=1,\cdots,n$ and $t=\operatorname{Re}z_0$, $(x_1,\cdots,x_n,y,\cdots,y_n,t)$ can be regarded as a real coordinate system about $\mathring{z}$ on $b\Omega$. We denote by $(\xi_1,\cdots,\xi_n,\eta_1,\cdots,\eta_n,\tau)$ the coordinates on the cotangent space of $(x_1,\cdots,x_n,y_1,\cdots,x_n,t)$.

Here we shall recall the result for the relationship between the subellipticity of the $\bar{\partial}$-Neumann problem and that of the tangential Cauchy-Riemann complex.

PROPOSITION 1.1. (Sweeney [17]) *There exists a neighborhood U of $\mathring{z}=(\mathring{x},\mathring{y},\mathring{t})$ such that* (0.1) *holds for (p,q)-forms if and only if there exists a conic neighborhood V of $(\mathring{x},\mathring{y},\mathring{t};\mathring{\xi},\mathring{\eta},\mathring{\tau})\in T^*(b\Omega)$ with $\sigma(L_j(\mathring{x},\mathring{y},\mathring{t};\mathring{\xi},\mathring{\eta},\mathring{\tau})=0$ for $j=1,\cdots,n$ and $\mathring{\tau}>0$ such that for any pseudo-differential operator ψ of order* 0 *with* $\operatorname{supp}(\sigma(\psi))\subset V$, *we have*

$$\|\psi\varphi\|_\varepsilon\leq C(\|\bar{\partial}_b\varphi\|+\|\bar{\partial}_b^*\varphi\|+\|\varphi\|)\quad \text{for}\quad \varphi\in\mathcal{B}^{p,q}(b\Omega),$$

with some $C>0$.

Here $\sigma(P)$ for a pseudo-differential operator P signifies the symbol of P.

Let $\zeta(\tau)$ be a smooth function on $\boldsymbol{R}_\tau$ satisfying $\zeta(\tau)=1$ if $\tau\geq 2$ and $\zeta(\tau)=0$ if $\tau\leq 1$. For $\varepsilon>0$, we introduce the pseudo-differential operator Δ_ε whose symbol is $\tau^\varepsilon\zeta(\tau)$. For an open set U of $b\Omega$, we denote by $\mathscr{B}^{p,q}(U)$ the space of all $\varphi\in\mathscr{B}^{p,q}(b\Omega)$ whose supports are contained in U. The fact that the differential operator $\bar\partial_b\oplus\bar\partial_b^*$ is elliptic outside of V implies that the validity of (1.6) is equivalent to that the following inequality:

$$\|\Delta_\varepsilon\varphi\|\leq C(\|\bar\partial_b\varphi\|+\|\bar\partial_b^*\varphi\|+\|\varphi\|)\quad\text{for}\quad\varphi\in\mathscr{B}^{p,q}(b\Omega\cap U),\tag{1.7}$$

with some $C>0$ and some neighborhood U of $\mathring{z}$.

§ 2. Local subelliptic estimates

In this section, we shall choose a suitable neighborhood for each point of $b\Omega$ for the domain Ω satisfying (0.7), and give some estimates which are satisfied by smooth functions supported on it.

There exists a neighborhood U of the origin in C^{n+1} such that the defining function

$$r(z_0,\cdots,z_n)=\tilde r(z_0,z_1^{m_1},\cdots,z_l^{m_l},f_{l+1}(z_{l+1},z_1),\cdots,f_n(z_n,z_{n-l}))\tag{2.1}$$

of Ω has the form (1.5) on U in view of (0.3). Thus $b\Omega\cap U$ can be regarded as a neighborhood of the origin in $C^n_{(z_1,\dots,z_n)}\times\boldsymbol{R}_t$. We extend the domain of ϕ to $\boldsymbol{C}^n\times\boldsymbol{R}$ so that the values of ϕ and its derivatives are uniformly bounded. Since we shall not use the coordinate z_0 in the followings, we let z signify $(z_1,\cdots,z_n)$ hereafter. For $a>0$, we write $B_a=\{(z,t)\in\boldsymbol{C}^n\times\boldsymbol{R};\ |z|<a,\ |t|<a\}$. Fix a small enough so that B_{2a} is contained in $b\Omega\cap U$. In view of (1.5), we can see that the vector fields

$$L_j=\frac{\partial}{\partial z_j}-\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}\frac{\partial}{\partial t}$$

with $1\leq j\leq n$ are in $T^{1,0}(b\Omega)$ on B_{2a} and satisfy (1.1). Sweeney in [17] proved that the validity of (0.1) is independent of the choice of the hermitian metric on $\bar\Omega$. Thus we choose an hermitian metric so that $L_1,\cdots,L_n$ form an orthogonal base of $T^{1,0}(b\Omega)$, and we shall consider (0.1) only the case that $\bar\Omega$ is equipped with this metric.

Let τ and ρ be positive numbers. For an integer j with $1\leq j\leq l$ $\mathring{z}_j\in\boldsymbol{C}$, we define

$$(2.2)\qquad M_{\mathring{z}_j}^{(j)}(\rho,\tau)=\begin{cases} 2|\mathring{z}_j|^{m_j-1}\left(\dfrac{\tau}{\rho}\right)^{1/2} & \text{if } |\mathring{z}_j|^{m_j}\geq 2\left(\dfrac{\rho}{\tau}\right)^{1/2} \\ \left(\dfrac{\tau}{\rho}\right)^{1/2m_j} & \text{if } |\mathring{z}_j|^{m_j}<2\left(\dfrac{\rho}{\tau}\right)^{1/2}, \end{cases}$$

Next for an integer with $l+1\leq j\leq n$ and $(\mathring{z}_j, \mathring{z}_{j-l})\in \boldsymbol{C}^2$ with $|\mathring{z}_{j-l}|\leq a$, $M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)$ is given by

$$(2.3)\qquad M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)=\begin{cases} 2|\mathring{z}_j|^{m_j-1}\left(\dfrac{\tau}{\rho}\right)^{1/2} & \text{if } |\mathring{z}_j|^{m_j}\geq \max\left(\left(\dfrac{\rho}{\tau}\right)^{1/2}, |\mathring{z}_{j-l}|^{m_{j-l}}\right) \\ \max\limits_{1\leq k\leq m_j}\left|\dfrac{\partial^k f_j}{\partial z_j^k}(\mathring{z}_j, \mathring{z}_{j-l})\left(\dfrac{\tau}{\rho}\right)^{1/2}\right|^{1/k} & \text{otherwise.} \end{cases}$$

For $\mathring{z}=(\mathring{z}_1,\cdots,\mathring{z}_n)\in \boldsymbol{C}^n$ with $|\mathring{z}|\leq a$, we denote $\Lambda_{\mathring{z}}(\rho,\tau)$ the set of integer j satisfying $1\leq j\leq l$ and $|\mathring{z}_j|^{m_j}\geq 2(\rho/\tau)^{1/2}$, and by $\Gamma_{\mathring{z}}(\rho,\tau)$ that of integer j satisfying $l+1\leq j\leq n$ and

$$(2.4)\qquad |\mathring{z}_j|^{m_j}\geq \max\left(2\left(\frac{\rho}{\tau}\right)^{1/2}, |\mathring{z}_{j-l}|^{m_{j-l}}\right).$$

For convenience, we shall abbreviate $M_{\mathring{z}_j}^{(j)}(\rho,\tau)$ and $M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)$ as $M_{\mathring{z}_j}^{(j)}$ and $M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}$ respectively, if there is no confusion. Using $M_{\mathring{z}_j}^{(j)}(\rho,\tau)$ and $M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)$, we set

$$(2.5)\qquad \tilde{\Omega}_{\mathring{z}_j}^{(j)}(\rho,\tau)=\left\{z_j\in \boldsymbol{C};\ |z_j-\mathring{z}_j|<\frac{1}{M_{\mathring{z}_j}^{(j)}(\rho,\tau)}\right\}\quad \text{for } j=1,\cdots,l,$$

$$(2.6)\qquad \hat{\Omega}_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)=\Big\{(z_j, z_{j-l})\in \boldsymbol{C}\times\tilde{\Omega}_{\mathring{z}_{j-l}}^{(j-l)};$$
$$|f_j(z_j, z_{j-l})-f_j(\mathring{z}_j, \mathring{z}_{j-l})|<\frac{|\mathring{z}_j|^{m_j-1}}{AM_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)}\Big\},$$

if (2.4) is satisfied, and otherwise

$$(2.7)\qquad \hat{\Omega}_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)=\left\{(z_j, z_{j-l})\in \boldsymbol{C}\times\tilde{\Omega}_{\mathring{z}_{j-l}}^{(j-l)};\ |z_j-\mathring{z}_j|<\frac{1}{AM_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)}\right\}.$$

Here A is a constant with $A\geq 1$, which will be chosen later on. Furthermore $\tilde{\Omega}'^{(j)}_{\mathring{z}_j}(\rho,\tau)$ for $j=1,\cdots,l$ and $\hat{\Omega}'^{(i)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$ for $j=l+1,\cdots,n$ are given by the forms obtained by replacing $M_{\mathring{z}_j}^{(j)}(\rho,\tau)$ and $M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)$ by $2M_{\mathring{z}_j}^{(j)}(\rho,\tau)$ and $2M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho,\tau)$ respectively in (2.5)-(2.7). Then we have

LEMMA 2.1 *There exists* $C>0$ *such that*

$$M_{\mathring{z}_j}^{(j)}(\rho, \tau)\geq\frac{1}{C}\left(\frac{\tau}{\rho}\right)^{1/2m_j} \quad for\ j=1, \cdots, l, \tag{2.8}$$

and

$$M_{(\mathring{z}_j, \mathring{z}_{j-l})}^{(j)}(\rho, \tau)>\frac{1}{C}\left(\frac{\tau}{\rho}\right)^{1/2m_j} \quad for\ j=l, \cdots, n. \tag{2.9}$$

PROOF. The inequality (2.8) follows from (2.2). If (2.4) is satisfied, then we obtain (2.9) immediately from (2.3). Otherwise, we have

$$M_{(\mathring{z}_j, \mathring{z}_{j-l})}^{(j)}\geq\left|\frac{\partial^{m_j}f_j}{\partial z_j^{m_j}}(\mathring{z}_j, \mathring{z}_{j-l})\left(\frac{\tau}{\rho}\right)^{1/2}\right|^{1/m_j},$$

which gives (2.9). □

LEMMA 2.2. *If* $1\leq j\leq l$ *and* $|\mathring{z}_j|^{m_j}\geq 2(\rho/\tau)^{1/2}$, *then there exists a constant* $C>0$ *such that*

$$\frac{1}{C}|z_j|\leq|\mathring{z}_j|\leq C|z_j| \quad if\ z_j\in\tilde{\Omega}_{\mathring{z}_j}^{(j)}(\rho, \tau)\ or\ \mathring{z}_j\in\tilde{\Omega}_{z_j}^{(j)}(\rho, \tau). \tag{2.10}$$

PROOF. In the case when $z_j\in\tilde{\Omega}_{\mathring{z}_j}^{(j)}(\rho, \tau)$, it follows from (2.2) that

$$|z_j-\mathring{z}_j|\leq\frac{1}{2}|\mathring{z}_j|^{1-m_j}\left(\frac{\rho}{\tau}\right)^{1/2}\leq 2^{(1-2m_j)/m_j}\left(\frac{\rho}{\tau}\right)^{1/2m_j}\leq\frac{1}{4}|\mathring{z}_j|,$$

which gives (2.10). In the other case, we obtain it by a similar argument if $|z_j|^{m_j}\geq 2(\rho/\tau)^{1/2}$. Otherwise, we have

$$2\left(\frac{\rho}{\tau}\right)^{1/2m_j}\geq|z_j|\geq|\mathring{z}_j|-|z_j-\mathring{z}_j|\geq\frac{1}{2}\left(\frac{\rho}{\tau}\right)^{1/2m_j},$$

and

$$2\left(\frac{\rho}{\tau}\right)^{1/2m_j}\leq|\mathring{z}_j|\leq|z_j|+|z_j-\mathring{z}_j|\leq 3\left(\frac{\rho}{\tau}\right)^{1/2m_j},$$

which gives (2.10). □

In order to obtain local estimates, we need the following lemmas.

LEMMA 2.3. *Assume that* $|\mathring{z}_j|^{m_j}\leq\max(|\mathring{z}_{j-l}|^{m_{j-l}}, 2(\rho/\tau)^{1/2})$. *If we choose a small enough, then there exist positive constants* C, C_0, *and an integer* k_0 *with* $1\leq k_0\leq m_j$ *such that*

$$\left|\frac{\partial f_j}{\partial z_{j-l}}(z_j, z_{j-l})\right|\leq Ca^{1/m_j^2}M^{(j-l)}_{\mathring{z}_{j-l}}(\rho, \tau)\left(\frac{\rho}{\tau}\right)^{1/2}, \tag{2.11}$$

$$\left|\frac{\partial^p f_j}{\partial z_j^p}(z_j, z_{j-l})\right|\leq C(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho, \tau))^p\left(\frac{\rho}{\tau}\right)^{1/2} \text{ if } 1\leq p\leq m_j, \tag{2.12}$$

$$\left|\frac{\partial^{k_0} f_j}{\partial z_j^{k_0}}(z_j, z_{j-l})\right|\geq\frac{1}{C}(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho, \tau))^{k_0}\left(\frac{\rho}{\tau}\right)^{1/2}, \tag{2.13}$$

for all $(z_j, z_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho, \tau)$ *and* τ, ρ *with* $\tau\geq C_0\rho$.

PROOF. Firstly, we shall consider the case when $|\mathring{z}_{j-l}|^{m_{j-l}}\leq 2(\rho/\tau)^{1/2}$. It follows from (0.5) and (0.6) that

$$\frac{\partial^{p+q} f_j}{\partial z_j^p\partial z_{j-l}^q}(z_j, z_{j-l})\leq C\sum_{p\leq h\leq m_j(m_{j-l}-q)/m_{j-l}}|z_{j-l}|^{(m_j-h)(1+m_{j-l}m_j)/m_j^2-q}|z_j|^{h-p}. \tag{2.14}$$

By Lemma 2.2 we can verify that if $z_j\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho, \tau)$, then

$$|z_{j-l}|^{m_{j-l}}\leq C\left(\frac{\rho}{\tau}\right)^{1/2} \text{ and } |z_j|^{m_j}\leq C\left(\frac{\rho}{\tau}\right)^{1/2}. \tag{2.15}$$

Consequently, (2.14) with $q=0$ and (2.15) imply that

$$\left|\frac{\partial^p f_j}{\partial z_j^p}(z_j, z_{j-l})\left(\frac{\tau}{\rho}\right)^{1/2}\right|\leq C\left(\frac{\tau}{\rho}\right)^{p/m_j}. \tag{2.16}$$

Combining this with (2.9), we show (2.12). By a similar argument, we can derive (2.11) from (2.8), (2.14) with $p=0$ and $q=1$, and (2.15). Moreover (2.16) gives $M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}\leq C(\tau/\rho)^{1/2m_j}$. Hence we have (2.13) if we put $k_0=m_j$.

In the case when $|\mathring{z}_{j-l}|^{m_{j-l}}\geq 2(\rho/\tau)^{1/2}$, we shall show that

$$\left|\left(\frac{1}{M^{(j-l)}_{\mathring{z}_{j-l}}}\right)^q\frac{\partial^{p+q} f_j}{\partial z_j^p\partial z_{j-l}^q}(\mathring{z}_j, \mathring{z}_{j-l})\left(\frac{\tau}{\rho}\right)^{1/2}\right|\leq Ca^{q/m_j^2}(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^p, \tag{2.17}$$

for p, q with $0\leq p\leq m_j, 0\leq q\leq m_{j-l}$, and $p+q\geq 1$. Noting that (2.2) implies that

$$M^{(j-l)}_{\mathring{z}_{j-l}}|\mathring{z}_{j-l}|=2|\mathring{z}_{j-l}|^{m_{j-l}}\left(\frac{\tau}{\rho}\right)^{1/2}\geq 4,$$

we can see that

$$(2.18)\qquad \left|\left(\frac{1}{M^{(j-l)}_{\mathring{z}_{j-l}}}\right)^{q}\frac{\partial^{p+q}f_j}{\partial z_j^p\partial z_{j-l}^q}(\mathring{z}_j,\mathring{z}_{j-j})\left(\frac{\tau}{\rho}\right)^{1/2}\right|$$
$$\leq\frac{C(\tau/\rho)^{1/2}}{M^{(j-l)}_{\mathring{z}_{j-l}}|\mathring{z}_{j-l}|}\sum_{p\leq h\leq m_j(m_{j-l}-q)/m_{j-l}}|z_{j-l}|^{(m_j-h)(1+m_{j-l}m_j)/m_j^2}|z_j|^{h-p}$$
$$\leq Ca^{q/m_j^2}\sum_{p\leq h\leq m_j(m_{j-l}-q)/m_{j-l}}\frac{|\mathring{z}_j|^{h-p}}{|\mathring{z}_{j-l}|^{m_{j-l}h/m_j}},$$

by (2.14) with $(z_j, z_{j-l})=(\mathring{z}_j, \mathring{z}_{j-l})$. Here it follows from Lemma 2.1 that

$$\frac{|\mathring{z}_j|^{h-p}}{|\mathring{z}_{j-l}|^{m_{j-l}h/m_j}}\leq C\left(\frac{|\mathring{z}_j|}{|\mathring{z}_{j-l}|^{m_{j-l}/m_j}}\right)^{h-p}\left(\frac{\tau}{\rho}\right)^{p/2m_j}\leq C(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{p}.$$

Combination (2.18) and this inequality yields (2.16).

Let us take $\varepsilon>0$ satisfying

$$1-\frac{p}{m_j}-\frac{q}{m_{j-l}}\leq-2\varepsilon(p+q)\ \text{ if } p, q\in\boldsymbol{Z} \text{ and } \frac{p}{m_j}+\frac{q}{m_{j-l}}>1.$$

Then it follows from Lemma 2.1 that if we take C_0 large enough, we have

$$(2.19)\qquad (M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{-p}(M^{(j-l)}_{\mathring{z}_{j-l}})^{-q}\left(\frac{\tau}{\rho}\right)^{1/2}$$
$$\leq\left(\left(\frac{\tau}{\rho}\right)^{1/2m_j}\Big/\left(\frac{\tau}{\rho}\right)^{\varepsilon}M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}\right)^{p}\left(\left(\frac{\tau}{\rho}\right)^{1/2m_{j-l}}\Big/\left(\frac{\tau}{\rho}\right)^{\varepsilon}M^{(j-l)}_{\mathring{z}_{j-l}}\right)^{q}$$
$$\leq\frac{a}{2^{p+q}}\ \text{ if } \tau\geq C_0 \text{ and } \frac{p}{m_j}+\frac{q}{m_{j-l}}>1.$$

Hence expanding $\partial f_j/\partial z_j$ into the Taylor series at $(\mathring{z}_j, \mathring{z}_{j-l})$, we observe from (2.17) and (2.19) that

$$\left|\frac{\partial f_j}{\partial z_j}(\mathring{z}_j,\mathring{z}_{j-l})\right|\leq C\left(\sum_{\alpha=1}^{m_j}\sum_{\beta=0}^{m_{j-l}}\left|\frac{(z_j-\mathring{z}_j)M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}}{2}\right|^{\alpha-1}\left|\frac{(z_{j-l}-\mathring{z}_{j-l})M^{(j-l)}_{\mathring{z}_{j-l}}}{2}\right|^{\beta}\right.$$
$$+\sum_{m_j<\alpha\text{ or }m_{j-l}<\beta}\left|\frac{(z_j-\mathring{z}_j)M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}}{2}\right|^{\alpha-1}\left|\frac{(z_{j-l}-\mathring{z}_{j-l})M^{(j-l)}_{\mathring{z}_{j-l}}}{2}\right|^{\beta}$$

$$\times(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{-\alpha}(M^{(j-l)}_{\mathring{z}_{j-l}})^{-\beta}\Big(\frac{\tau}{\rho}\Big)^{1/2}2^{\alpha+\beta}\Big)M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}\Big(\frac{\rho}{\tau}\Big)^{1/2}$$
$$\leq CM^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}\Big(\frac{\rho}{\tau}\Big)^{1/2},$$

if $(z_j, z_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$ and $\tau\geq C_0\rho$ which yields (2.12) with $p=1$. By using the same method, (2.12) with $p=2,\cdots,m_j$ and (2.11) can also be proved from (2.17) and (2.19).

As k_0 in (2.13), we adopt the minimum of the set of integers which consist of k such that $k=m_j$ or it satisfies $1\leq k\leq m_j-1$ and

$$\Big|\frac{\partial^{k'}f_j}{\partial z_j^{k'}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|\leq\frac{1}{m_j}(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{k'-k}\Big|\frac{\partial^{k}f_j}{\partial z_j^{k}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|, \tag{2.20}$$

for $k'=k+1,\cdots,m_j$. We shall prove that

$$\Big|\frac{\partial^{k_0}f_j}{\partial z_j^{k_0}}(\mathring{z}_j,\mathring{z}_{j-l})\Big(\frac{\tau}{\rho}\Big)^{1/2}\Big|\geq\frac{1}{C}(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{k_0}, \tag{2.21}$$

for some $C>0$. Let us take $\bar{k}$ with $1\leq\bar{k}\leq m_j$ so that

$$\Big|\frac{\partial^{\bar{k}}f_j}{\partial z_j^{\bar{k}}}(\mathring{z}_j,\mathring{z}_{j-l})\Big(\frac{\tau}{\rho}\Big)^{1/2}\Big|=(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{\bar{k}}. \tag{2.22}$$

The existence of such $\bar{k}$ follows from the definition (2.3) of $M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}$. If we assume that $k_0<\bar{k}$, then (2.17) with $p=k_0$ and $q=0$, (2.20) with $k=k_0$ and $k'=\bar{k}$, and (2.22) imply that

$$0<\Big|\frac{\partial^{k}f_j}{\partial z_j^{k}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|\leq\frac{1}{m_j}\Big|\frac{\partial^{k}f_j}{\partial z_j^{k}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|$$

which is inconsistent. Hence it follows that $k_0=\bar{k}$ or $k_0>\bar{k}$. In the first case, (2.22) gives (2.21). In the second case, if we assume that

$$\Big|\frac{\partial^{k_0}f_j}{\partial z_j^{k_0}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|\leq\frac{1}{m_j^{m_j}}(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{k_0-\bar{k}}\Big|\frac{\partial^{\bar{k}}f_j}{\partial z_j^{\bar{k}}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|,$$

then this implies that there exists $\tilde{k}$ such that $\tilde{k}<k_0$ and (2.20) with $k=\tilde{k}$ hold. However it contradicts with the minimality of k_0. Hence putting $C=m_j^{m_j}$, we obtain (2.21). Then (2.20) and (2.21) imply that

$$\Big|\sum_{\alpha=0}^{m_j-k_0}\frac{(z_j-\mathring{z}_j)^\alpha}{\alpha!}\frac{\partial^{k_0+\alpha}f_j}{\partial z_j^{k_0+\alpha}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|\geq\Big|\frac{\partial^{k_0}f_j}{\partial z_j^{k_0}}(\mathring{z}_j,\mathring{z}_{j-l})\Big|$$

$$-\sum_{\alpha=1}^{m_j-k_0}\frac{1}{m_j}\left|\frac{(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(z_j-\mathring{z}_j))^\alpha}{\alpha!}\frac{\partial^{k_0}f_j}{\partial z_j^{k_0}}(\mathring{z}_j,\mathring{z}_{j-l})\right|$$
$$\geq\frac{1}{C}\left|\frac{\partial^{k_0}f_j}{\partial z_j^{k_0}}(\mathring{z}_j,\mathring{z}_{j-l})\right|\geq\frac{1}{C}(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{k_0}\left(\frac{\rho}{\tau}\right)^{1/2},$$

for $(z_j, z_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$. Combining this with (2.17) and (2.19)-(2.21), we obtain that

$$\left|\frac{\partial^{k_0}f_j}{\partial z_j^{k_0}}(\mathring{z}_j,\mathring{z}_{j-l})\right|\geq\left|\sum_{\alpha=0}^{m_j-k_0}\frac{(z_j-\mathring{z}_j)^\alpha}{\alpha!}\frac{\partial^{k_0+\alpha}f_j}{\partial z_j^{k_0+\alpha}}(\mathring{z}_j,\mathring{z}_{j-l})\right|$$
$$-\sum_{\alpha=0}^{m_j-k_0}\sum_{\beta=1}^{m_{j-l}}\frac{1}{\alpha!\beta!}\left|(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(z_j-\mathring{z}_j))^\alpha(M^{(j-l)}_{\mathring{z}_{j-l}}(z_{j-l}-\mathring{z}_{j-l}))^\beta\right.$$
$$\left.\times(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{-\alpha-k_0}(M^{(j-l)}_{\mathring{z}_{j-l}})^{-\beta}\frac{\partial^{k_0+\alpha+\beta}f_j}{\partial z_j^{k_0+\alpha}\partial z_{j-l}^\beta}(\mathring{z}_j,\mathring{z}_{j-l})\right|(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{k_0}$$
$$-C\sum_{m_j-k_0<\alpha\text{ or }m_{j-l}<\beta}|(z_j-\mathring{z}_j)^\alpha(z_{j-l}-\mathring{z}_{j-l})^\beta|\geq\frac{1}{C}(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^k\left(\frac{\rho}{\tau}\right)^{1/2},$$

for $(z_j, z_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$ and $\tau\geq C_0\rho$ if a is small enough. This completes the proof. □

We will use the following corollary in the next section.

COROLLARY 2.4. *Let us take a small enough and A large enough Assume that*

$$|\mathring{z}_{j-l}|^{m_{j-l}}\geq 2\left(\frac{\rho}{\tau}\right)^{1/2}\quad and\quad |\mathring{z}_j|^{m_j}\leq|\mathring{z}_{j-l}|^{m_{j-l}},$$

and there exists $(z_j, z_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$ *such that*

$$|z_j|^{m_j}\geq\max\left(|z_{j-l}|^{m_{j-l}},\ 2\left(\frac{\rho}{\tau}\right)^{1/2}\right), \tag{2.23}$$

we have

$$|\mathring{z}_{j-l}|^{m_{j-l}}\leq C|\mathring{z}_j|^{m_j}, \tag{2.24}$$

and

$$\frac{1}{C}|\mathring{z}_j|^{m_j-1}\left(\frac{\tau}{\rho}\right)^{1/2}\leq M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)\leq C|\mathring{z}_j|^{m_j-1}\left(\frac{\tau}{\rho}\right)^{1/2}, \tag{2.25}$$

for some $C>0$.

PROOF. It follows from (0.5), (0.6), and (2.23) that

$$\left|\frac{\partial f_j}{\partial z_j}(z_j, z_{j-l}) - m_j z_j^{m_j-1}\right| \leq \sum_{k=1}^{m_j-1} k|g_{jk}(z_{j-l})z_j^{k-1}| \tag{2.26}$$

$$\leq Ca^{1/m_j} \sum_{k=1}^{m_j-1} |z_j|^{k-1}|z_{j-l}|^{m_{j-l}(m_j-k)/m_j} \leq Ca^{1/m_j}|z_j|^{m_j-1}.$$

Hence in view of (2.12) in Lemma 2.3, we have

$$M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})} \geq \frac{1}{C}\left|\frac{\partial f_j}{\partial z_j}(z_j, z_{j-l})\left(\frac{\tau}{\rho}\right)^{1/2}\right| \geq \frac{1}{C}|z_j|^{m_j-1}\left(\frac{\tau}{\rho}\right)^{1/2}, \tag{2.27}$$

if we take a small enough. It implies that

$$|z_j - \mathring{z}_j| \leq \frac{C}{A}|z_j|^{1-m_j}\left(\frac{\rho}{\tau}\right)^{1/2} \leq \frac{C}{A}|z_j|,$$

so taking A large enough, we have

$$\frac{1}{C}|z_j| \leq |\mathring{z}_j| \leq C|z_j|. \tag{2.28}$$

Combining it with (2.10) with replacing j by $j-l$ in Lemma 2.2 and (2.23), we obtain (2.24). Moreover it follows from (2.14), (2.23), (2.24), and (2.28) that

$$\left|\frac{\partial^p f_j}{\partial^p z_j}(\mathring{z}_j, \mathring{z}_{j-l})\left(\frac{\tau}{\rho}\right)^{1/2}\right| \leq C|z_j|^{m_j-p}\left(\frac{\tau}{\rho}\right)^{1/2}$$

$$\leq \left(|z_j|^{m_j-1}\left(\frac{\tau}{\rho}\right)^{1/2}\right)^p \left(|z_j|^{m_j}\left(\frac{\tau}{\rho}\right)^{1/2}\right)^{1-p} \leq C\left(|z_j|^{m_j-1}\left(\frac{\tau}{\rho}\right)^{1/2}\right)^p.$$

Hence we have $M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})} \leq C|z_j|^{m_j-1}(\tau/\rho)^{1/2}$. Combining it with (2.27) and (2.28) we obtain (2.25). The proof is complete. □

LEMMA 2.5. *If a is small enough, A is large enough, and* (2.4) *is satisfied, then there exists a constant $C>0$ such that for $\rho>0$ and $\tau>0$, we have* (2.28) *for* $(z_j, z_{j-l}) \in \hat{\Omega}^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}(\rho, \tau)$.

PROOF. The definition (2.3) of $M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}$ gives

$$M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})} \geq 2^{2m_j - 2/m_j}\left(\frac{\tau}{\rho}\right)^{1/2m_j}. \tag{2.29}$$

If $|z_{j-l}|^{m_{j-l}} \geq 2(\rho/\tau)^{1/2}$, it follows from Lemma 2.2 that

$$|z_{j-l}|^{m_{j-l}} \leq C|\mathring{z}_{j-l}|^{m_{j-l}} \leq C^2|\mathring{z}_j|^{m_j}.$$

Otherwise, recalling (2.2), we can see that

$$\max(|z_{j-l}^{m_{j-l}}|, |\mathring{z}_{j-l}^{m_{j-l}}|) \leq C\left(\frac{\rho}{\tau}\right)^{1/2} \leq C^2|\mathring{z}_j|^{m_j}.$$

Hence we have

$$|g_{jk}(z_{j-l})z_j^k| + |g_{jk}(\mathring{z}_{j-l})\mathring{z}_j^k| \leq Ca^{1/m_j}|\mathring{z}_j|^{m_j}. \tag{2.30}$$

Assuming that $(1-2^{(2-3m_j)/2m_j})|z_j|^{m_j} \geq |\mathring{z}_j|^{m_j}$, we claim from (2.6), (2.29), and (2.30) that

$$\begin{aligned}
&2^{2-2m_j/m_j}|\mathring{z}_j|^{m_j} \geq \left(\frac{\tau}{\rho}\right)^{1/2m_j} \frac{|\mathring{z}_j|^{m_j}}{M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}} \\
&\geq |f_j(z_j, z_{j-l}) - f_j(\mathring{z}_j, \mathring{z}_{j-l})|\,|\mathring{z}_j|\left(\frac{\tau}{\rho}\right)^{1/2m_j} \\
&\geq 2^{1/m_j}\left(|z_j|^{m_j} - |\mathring{z}_j|^{m_j} - \sum_{k=0}^{m_j-1}(|g_{jk}(z_{j-l})z_j^k| + |g_{jk}(\mathring{z}_{j-l})\mathring{z}_j^k|)\right) \\
&\geq 2^{1/m_j}(2^{2-3m_j/2m_j}|z_j|^{m_j} - a^{1/m_j}|z_j|^{m_j}) \geq 2^{8-7m_j/4m_j}|z_j|^{m_j},
\end{aligned}$$

if a is small enough. However it is contradictory, so $|z_j|^{m_j} \leq C|\mathring{z}_j|^{m_j}$ with some $C>0$. Combining this with (2.4) and (2.30), we show that

$$\begin{aligned}
&|z_j^{m_j} - \mathring{z}_j^{m_j}| \\
&\leq \left(|f_j(z_j, z_{j-l}) - f_j(\mathring{z}_j, \mathring{z}_{j-l})| + \sum_{k=0}^{m_j-1}(|g_{jk}(z_{j-l})z_j^k| + |g_{jk}(\mathring{z}_{j-l})\mathring{z}_j^k|)\right) \\
&\leq C\left(\frac{1}{A}\left(\frac{\rho}{\tau}\right)^{1/2} + a|\mathring{z}_j|^{m_j}\right) \leq \frac{1}{2}|\mathring{z}_j|^{m_j},
\end{aligned}$$

by (2.3) and (2.5) if a and $1/A$ are small enough. Hence we obtain (2.28). The proof is complete. □

Applying Lemma 2.2 and 2.5, we can prove

COROLLARY 2.6. *If* (z_j, z_{j-l}) *satisfies* $|z_j|^{m_j} \geq \max(|z_{j-l}|^{m_{j-l}}, 2(\rho/\tau)^{1/2})$ *and there exists* $(\mathring{z}_j, \mathring{z}_{j-l}) \in \hat{\Omega}^{(j)}_{(z_j, z_{j-l})}(\rho, \tau)$ *such that* $|\mathring{z}_{j-l}|^{m_{j-l}} \geq 2(\rho/\tau)^{1/2}$, *then we have* (2.24).

To define a holomorphic vector field on $\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$, we need the following corollary.

COROLLARY 2.7. *If a and $1/A$ are small enough, then there exists a constant $C>0$ such that for ρ and τ with $\tau\geq\rho$, we have*

$$C|\mathring{z}_j|^{m_j-1}\geq\left|\frac{\partial f_j}{\partial z_j}(z_j, z_{j-l})\right|\geq\frac{1}{C}|\mathring{z}_j|^{m_j-1}, \tag{2.31}$$

for (z_j, z_{j-l}) and $(\mathring{z}_j, \mathring{z}_{j-l})$ satisfying (2.4) *and* $(z_j, z_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$.

PROOF. Remark that (2.28) is valid because of Lemma 2.5. If $|\mathring{z}_{j-l}|^{m_{j-l}}\geq 2(\rho/\tau)^{1/2}$, then it follows from (2.10) and (2.28) that

$$|z_{j-l}|^{m_{j-l}}\leq C|z_j|^{m_j}. \tag{2.32}$$

Otherwise, (2.32) follows from (2.2), (2.4), and (2.28). Using and (2.32), we obtain (2.26). Hence combining it with (2.28), (2.31) is proved. This completes the proof. □

Thus if (2.4) is satisfied, then we can define the holomorphic function $d_j(z_j, z_{j-l})$ on $\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$ by the following form:

$$d_j(z_j, z_{j-l})=\frac{\partial f_j}{\partial z_{j-l}}(z_j, z_{j-l})\Big/\frac{\partial f_j}{\partial z_j}(z_j, z_{j-l}).$$

COROLLARY 2.8. *There exists $C>0$ independent of ρ and τ, and $(\mathring{z}_j, \mathring{z}_{j-l})$ such that*

$$|d_j(z_j, z_{j-l})|\leq Ca^{1/m_j}M^{(j-l)}_{\mathring{z}_{j-l}}(\rho,\tau)\left(\frac{\rho}{\tau}\right)^{1/2m_j}, \tag{2.33}$$

and

$$\left|\frac{\partial d_j}{\partial z_j}(z_j, z_{j-l})\right|\leq Ca^{1/m_j}M^{(j-l)}_{\mathring{z}_{j-l}}(\rho,\tau), \tag{2.34}$$

for $(z_j, z_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}(\rho,\tau)$.

PROOF. It follows from (0.6), Lemma 2.5, and Corollary 2.7 that

$$|d_j(z_j, z_{j-l})|\leq Ca^{1/m_j}\sum_{k=1}^{m_{j-l}}|z_{j-l}|^{k-l}|\mathring{z}_j|^{1-(km_j/m_{j-l})}, \tag{2.35}$$

and

$$\left|\frac{\partial d_j}{\partial z_j}(z_j, z_{j-l})\right|\leq Ca^{1/m_j}\sum_{k=1}^{2m_{j-l}-1}|z_{j-l}|^{k-1}|\mathring{z}_j|^{-km_j/m_{j-l}}, \tag{2.36}$$

When $|z_{j-l}|^{m_{j-l}}\leq 2(\rho/\tau)^{1/2}$, (0.4) and (2.8) imply that

(2.37) $$|z_{j-l}|^{k-l}|\mathring{z}_j|^{1-(km_j/m_{j-l})}$$
$$\leq C\left(\frac{\tau}{\rho}\right)^{(m_j-m_{j-l})/2m_{j-l}m_j}\left(|\mathring{z}_j|\left(\frac{\tau}{\rho}\right)^{1/2m_j}\right)^{1-(km_j/m_{j-l})}\leq C\left(\frac{\rho}{\tau}\right)^{1/2m_j}M^{(j-l)}_{\mathring{z}_{j-l}},$$

and

(2.38) $$|z_{j-l}|^{k-1}|\mathring{z}_j|^{-km_j/m_{j-l}}$$
$$\leq C\left(\frac{\tau}{\rho}\right)^{1/2m_{j-l}}\left(|\mathring{z}_j|^{m_j}\left(\frac{\tau}{\rho}\right)^{1/2}\right)^{-k/m_{j-l}}\leq CM^{(j-l)}_{\mathring{z}_{j-l}}.$$

When $|z_{j-l}|^{m_{j-l}}\geq 2(\rho/\tau)^{1/2}$, applying Lemma 2.2, we get

(2.39) $$|z_{j-l}|^{k-1}|\mathring{z}_j|^{1-(km_j/m_{j-l})}\left(\frac{\tau}{\rho}\right)^{1/2m_j}$$
$$\leq C|\mathring{z}_{j-l}|^{m_{j-l}-1}\left(\frac{\tau}{\rho}\right)^{1/2}\left(\frac{|\mathring{z}_j|}{|\mathring{z}_{j-l}|^{m_{j-l}/m_j}}\right)^{1-(km_{j-l}/m_j)}\left(|z_{j-l}|^{m_{j-l}}\left(\frac{\tau}{\rho}\right)^{1/2}\right)^{(1-m_j)/m_j}$$
$$\leq CM^{(j-l)}_{\mathring{z}_{j-l}}.$$

and

(2.40) $$|z_{j-l}|^{k-1}|\mathring{z}_j|^{-km_j/m_{j-l}}$$
$$\leq|\mathring{z}_{j-l}|^{m_{j-l}}\left(\frac{\tau}{\rho}\right)^{1/2}\left(\frac{|\mathring{z}_{j-l}|}{|\mathring{z}_j|^{m_j/m_{j-l}}}\right)^{k}\left(|z_{j-l}|^{-m_{j-l}}\left(\frac{\rho}{\tau}\right)^{1/2}\right)\leq CM^{(j-l)}_{\mathring{z}_{j-l}}.$$

As a consequence, (2.35), (2.37), and (2.39) imply (2.33) and (2.36), (2.38), and (2.40) imply (2.34). □

For $\mathring{z}\in C^n$ with $|\mathring{z}|\leq a$, we give the neighborhood $\tilde{\Omega}_{\mathring{z}}(\rho,\tau)$ and $\tilde{\Omega}'_{\mathring{z}}(\rho,\tau)$ of $\mathring{z}$ by

$$\tilde{\Omega}_{\mathring{z}}(\rho,\tau)=\tilde{\Omega}^{(n-l+1)}_{\mathring{z}_{n-l+1}}\times\cdots\times\tilde{\Omega}^{(l)}_{\mathring{z}_l}\times\hat{\Omega}^{(l+1)}_{(\mathring{z}_{l+1},\mathring{z}_1)}\times\cdots\times\hat{\Omega}^{(n)}_{(\mathring{z}_n,\mathring{z}_{n-l})},$$

and

$$\tilde{\Omega}'_{\mathring{z}}(\rho,\tau)=\tilde{\Omega}'^{(n-l+1)}_{\mathring{z}_{n-l+1}}\times\cdots\times\tilde{\Omega}'^{(l)}_{\mathring{z}_l}\times\hat{\Omega}'^{(l+1)}_{(\mathring{z}_{l+1},\mathring{z}_1)}\times\cdots\times\hat{\Omega}'^{(n)}_{(\mathring{z}_n,\mathring{z}_{n-l})}.$$

Let us define the vector fields $\tilde{L}_j$ on $\tilde{\Omega}_{\mathring{z}}(\rho,\tau)$ as

$$\tilde{L}_j=\begin{cases}L_j-d_{j+l}L_{j+l} & \text{if } j+l\in\Gamma_{\mathring{z}}(\rho,\tau)\\ L_j & \text{if } j+l\notin\Gamma_{\mathring{z}}(\rho,\tau).\end{cases}$$

Applying Lemma 2.3 and Corollary 2.7, we can obtain local estimates stated in the following proposition.

PROPOSITION 2.9. *Let us take a small enough and A lorge enough. Then there exists $C>0$ such that if ρ is large enough and τ satisfies $\tau\geq C_0\rho$, then for $\mathring{z}\in C^n$ with $|\mathring{z}|\leq a$ and smooth functions φ and φ' supported on $\tilde{\Omega}_{\mathring{z}}(\rho, \tau)$, we have*

$$\rho^{1/2}\|M_{\mathring{z}_j}^{(j)}(\rho, \tau)\varphi\|\leq C\|z_j^{m_j-1}\tau^{1/2}\varphi\| \quad \text{for } j\in\Lambda_{\mathring{z}}(\rho, \tau), \tag{2.41}$$

$$\rho^{2-m_j}\|M_{\mathring{z}_j}^{(j)}(\rho, \tau)\varphi\|\leq C(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\|z_j^{m_j-1}\tau^{1/2}\varphi\|) \quad \text{for } j\in\{1, \cdots, l\}\backslash\Lambda_{\mathring{z}}(\rho, \tau), \tag{2.42}$$

$$\rho^{2-m_j}\|M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho, \tau)\varphi\|\leq C\Big(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\Big\|\frac{\partial f_j}{\partial z_j}\tau^{1/2}\varphi\Big\|\Big) \quad \text{for } j\in\{l+1, \cdots, n\}\backslash\Gamma_{\mathring{z}}(\rho, \tau), \tag{2.43}$$

$$\rho^{1/2}\|M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}(\rho, \tau)(\varphi'+d_j\varphi)\|\leq C\Big\|\tau^{1/2}\Big(\frac{\partial f_j}{\partial z_{j-l}}\varphi+\frac{\partial f_j}{\partial z_j}\varphi'\Big)\Big\| \quad \text{for } j\in\Gamma_{\mathring{z}}(\rho, \tau). \tag{2.44}$$

PROOF. It follows from Lemma 2.2 that

$$C|z_j|^{m_j-1}\tau^{1/2}\geq M_{\mathring{z}_j}^{(j)} \text{ if } j\in\Lambda_{\mathring{z}}(\rho, \tau),$$

which gives (2.41). Furthermore applying Corollary 2.7, we obtain (2.44). When $j\in\{1, \cdots, l\}\backslash\Lambda_{\mathring{z}}(\rho, \tau)$, (2.2) implies that

$$(M_{\mathring{z}_j}^{(j)})^{-k}|z_j^{m_j-k}|\tau^{1/2}\leq C\rho^{1/2} \text{ for } k=1, \cdots, m_j,$$

with some $C>0$. Hence we obtain that

$$\begin{aligned}
&\|(M_{\mathring{z}_j}^{(j)})^{-k+1}z_j^{m_j-k}\tau^{1/2}\varphi\|^2\\
&=C((M_{\mathring{z}_j}^{(j)})^{-k+1}[\tilde{L}_j, z^{m_j-k+1}]\tau^{1/2}\varphi, (M_{\mathring{z}_j}^{(j)})^{-k+1}z_j^{m_j-k}\tau^{1/2}\varphi)\\
&=C\{((M_{\mathring{z}_j}^{(j)})^{-k+2}z_j^{m_j-k+1}\tau^{1/2}\varphi, (M_{\mathring{z}_j}^{(j)})^{-k}z_j^{m_j-k}\tilde{L}_j^*\tau^{1/2}\varphi)\\
&-((M_{\mathring{z}_j}^{(j)})^{-k+2}z_j^{m_j-k+1}\tau^{1/2}\tilde{L}_j\varphi, (M_{\mathring{z}_j}^{(j)})^{-k}z_j^{m_j-k}\tau^{1/2}\varphi)\}\\
&\leq C\rho^{1/2}\|(M_{\mathring{z}_j}^{(j)})^{-(k-1)+1}z_j^{m_j-(k-1)}\tau^{1/2}\varphi\|(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\|\varphi\|),
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
&\rho\|M_{\mathring{z}_j}^{(j)}\varphi\|^2=\|(M_{\mathring{z}_j}^{(j)})^{-m_j+1}\tau^{1/2}\varphi\|^2\\
&\leq C\rho^{1/2}\|(M_{\mathring{z}_j}^{(j)})^{-m_j+2}z_j\tau^{1/2}\varphi\|(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\|\varphi\|)\\
&\leq\cdots\\
&\leq C\rho^{1-\varepsilon_k}\|(M_{\mathring{z}_j}^{(j)})^{1-k}z_j^{m_j-k}\tau^{1/2}\varphi\|^{2\varepsilon_k}(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\|\varphi\|)^{2-2\varepsilon_k}
\end{aligned}$$

$$\leq C\rho^{1-\varepsilon_{k-1}}\|(M^{(j)}_{\mathring{z}_j})^{2-k}z_j^{m_j-k+1}\tau^{1/2}\varphi\|^{2\varepsilon_{k-1}}(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\|\varphi\|)^{2-2\varepsilon_{k-1}}$$
$$\leq\cdots$$
$$\leq C\rho^{1-\varepsilon_1}\|z_j^{m_j-1}\tau^{1/2}\varphi\|^{2\varepsilon_1}(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\|\varphi\|)^{2-2\varepsilon_1},$$

by the induction with respect to k, where we put $\varepsilon_j=2^{k-m_j}$. Taking ρ large enough, we obtain (2.42). Furthermore, by using this argument with replacing $z_j^{m_j-1}$ and $M^{(j)}_{\mathring{z}_j}$ by $\partial f_j/\partial z_j$ and $M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}$ respectively, and applying (2.12) in Lemma 2.3, we show that

$$\rho^{2-m_j}\left\|(M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})})^{1-k_0}\frac{\partial^{k_0}f_j}{\partial z_j^{k_0}}\tau^{1/2}\varphi\right\|\leq C\left(\|\tilde{L}_j\varphi\|+\|\tilde{L}_j^*\varphi\|+\left\|\frac{\partial f_j}{\partial z_j}\tau^{1/2}\varphi\right\|\right),$$

when $j\in\{l+1,\cdot,n\}\backslash\Gamma_{\mathring{z}}(\rho,\tau)$. Combination of it and (2.13) in Lemma 2.3 yields (2.43). The proof is complete. □

§ 3. Microlocal decomposition

In this section, we shall decompose (p,q)-forms microlocally by using pseudo-differential operators, and give lower bounds for their L^2 norms derived from Proposition 2.9. In the last part, we shall complete the proof of Theorem 0.1.

For $\rho>0$, we write $D_\rho=\{(z,\tau);|z|\leq a/2,\tau\geq 2C_0\rho\}$ and $\tilde{D}_\rho=D_\rho\times\{t;|t|\leq a/2\}$. For $(\mathring{z},\mathring{\tau})\in D_\rho$ let us define the neighborhoods $\Omega_{(\mathring{z},\mathring{\tau})}(\rho)$ and $\Omega'_{(\mathring{z},\mathring{\tau})}(\rho)$ of $(\mathring{z},\mathring{\tau})$ by

$$\Omega_{(\mathring{z},\mathring{\tau})}(\rho)=\{(z,\tau)\in\boldsymbol{C}^n\times\boldsymbol{R};z\in\tilde{\Omega}_{\mathring{z}}(\rho,\mathring{\tau}),|\tau-\mathring{\tau}|<\mathring{\tau}^{1-\delta}\},$$

and

$$\Omega'_{(\mathring{z},\mathring{\tau})}(\rho)=\left\{(z,\tau)\in\boldsymbol{C}^n\times\boldsymbol{R};z\in\tilde{\Omega}'_{\mathring{z}}(\rho,\mathring{\tau}),|\tau-\mathring{\tau}|<\frac{1}{2}\mathring{\tau}^{1-\delta}\right\}.$$

where we choose δ so that

$$0<\delta<\frac{1}{2\tilde{m}_n}. \tag{3.1}$$

Then we have

LEMMA 3.1. *Let $(\mathring{z}_j,\mathring{z}_{j-l})$ and $(\mathring{z}'_j,\mathring{z}'_{j-l})$ be points of $\boldsymbol{C}^2$ satisfying*

$$|\mathring{z}_j|^{m_j}\leq\max\left(|\mathring{z}_{j-l}|^{m_{j-l}},2\left(\frac{\rho}{\tau}\right)^{1/2}\right),$$

$$|\mathring{z}_j'|^{m_j} \geq \max\left(|\mathring{z}_{j-l}'|^{m_{j-l}}|, 2\left(\frac{\rho}{\tau}\right)^{1/2}\right),$$

and

$$(\mathring{z}_j, \mathring{z}_{j-l}) \in \hat{\Omega}^{(j)}_{(\mathring{z}'_j, \mathring{z}'_{j-l})} \text{ or } (\mathring{z}_j', \mathring{z}_{j-l}') \in \hat{\Omega}^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}.$$

Then there exists $C>0$ *such that*

$$\frac{1}{C}(|z_{j-l}-z_{j-l}'|M^{(j-l)}_{\mathring{z}_{j-l}}(\rho, \tau)+|z_j-z_{j-l}'|M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}(\rho, \tau)) \tag{3.2}$$

$$\leq |z_{j-l}-z_{j-l}'|M^{(j-l)}_{\mathring{z}'_{j-l}}(\rho, \tau)+\frac{|f_j(z_j, z_{j-l})-f_j(z_j', z_{j-l}')|}{|\mathring{z}_j'|^{m_j-1}}M^{(j)}_{(\mathring{z}'_j, \mathring{z}'_{j-l})}(\rho, \tau)$$

$$\leq C(|z_{j-l}-z_{j-l}'|M^{(j-l)}_{\mathring{z}_{j-l}}(\rho, \tau)+|z_j-z_j'|M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}(\rho, \tau)),$$

if $(\mathring{z}_j, \mathring{z}_{j-l}) \in \hat{\Omega}^{(j)}_{(z_j, z_{j-l})}(\rho, \tau)$, $(\mathring{z}_j', \mathring{z}_{j-l}') \in \hat{\Omega}^{(j)}_{(z'_j, z'_{j-l})}(\rho, \tau)$, *and* a *and* $1/A$ *are small enough, where* C *is independent of* ρ, τ, a, *and* A.

PROOF. In the case when $(\mathring{z}_j', \mathring{z}_{j-l}') \in \hat{\Omega}^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}(\rho, \tau)$ and $|\mathring{z}_{j-l}|^{m_{j-l}} \geq 2(\rho/\tau)^{1/2}$, or $(\mathring{z}_j, \mathring{z}_{j-l}) \in \hat{\Omega}^{(j)}_{(\mathring{z}'_j, \mathring{z}'_{j-l})}(\rho, \tau)$ and $|\mathring{z}_{j-l}'|^{m_{j-l}} \geq 2(\rho/\tau)^{1/2}$, it follows from Lemma 2.2, Corollary 2.4, and Lemma 2.5 that

$$\begin{aligned} &\max(|z_j|^{m_j}, |z_j'|^{m_j}, |\mathring{z}_j'|^{m_j}|z_{j-l}|^{m_{j-l}}, |z_{j-l}'|^{m_{j-l}}, |\mathring{z}_{j-l}|^{m_{j-l}}, |\mathring{z}_{j-l}'|^{m_{j-l}}) \\ &\leq C\min(|z_j|^{m_j}, |\mathring{z}_j|^{m_j}, |\mathring{z}_j'|^{m_j}, |\mathring{z}_{j-l}'|^{m_{j-l}}), \end{aligned} \tag{3.3}$$

$$\frac{1}{C}M^{(j-l)}_{\mathring{z}_{j-l}} \leq M^{(j-l)}_{\mathring{z}'_{j-l}} \leq CM^{(j-l)}_{\mathring{z}_{j-l}}, \tag{3.4}$$

and

$$\frac{1}{C} \leq M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})} \leq M^{(j)}_{(\mathring{z}'_j, \mathring{z}'_{j-l})} \leq CM^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}. \tag{3.5}$$

The inequality (3.3) implies that

$$|g_{jk}(z_{j-l})z_j^k-g_{jk}(z_{j-l}')z_j'^k||\mathring{z}_j'|^{1-m_j}M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}$$

$$\leq (|g_{jk}(z_{j-l})-g_{jk}(z_{j-l}')||z_j|^k+|g_{jk}(z_{j-l}')||z_j^k-z_j'^k|)\left(\frac{\tau}{\rho}\right)^{1/2}$$

$$\leq Ca^{1/m_j}(|z_j-z_j'||\mathring{z}_j|^{m_j-1}+|z_{j-l}-\mathring{z}_{j-l}'||\mathring{z}_{j-l}'|^{m_{j-l}-1})\left(\frac{\tau}{\rho}\right)^{1/2}$$

$$\leq Ca^{1/m_j}(|z_j-z_j'|M^{(j)}_{(\mathring{z}_j, \mathring{z}_{j-l})}+|z_{j-l}-z_{j-l}'|M^{(j-l)}_{\mathring{z}_{j-l}}).$$

Hence we get

$$(3.6)\qquad \frac{|z_j^{m_j}-z_j'^{m_j}-(f_j(z_j, z_{j-l})-f_j(z_j', z_{j-l}'))|}{|\mathring{z}_j'|^{m_j-1}} M^{(j)}_{(\mathring{z}'_j,\mathring{z}'_{j-l})}$$
$$\leq Ca^{1/m_j}(|z_j-z_j'|M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}+|z_{j-l}-z_{j-l}'|M^{(j-l)}_{\mathring{z}'_{j-l}}).$$

Furthermore, taking a small enough and A large enough, and nothing that $(\rho/\tau)^{1/2}\leq C|z_j|^{m_j}$ in view of (3.3), we observe that

$$(3.7)\qquad |\mathring{z}_j^{m_j}-\mathring{z}_j'^{m_j}|$$
$$\leq |f_j(\mathring{z}_j, \mathring{z}_{j-l})-f_j(\mathring{z}_j', \mathring{z}_{j-l}')|+\sum_{k=0}^{m_j-1}(|g_{jk}(\mathring{z}_{j-k})\mathring{z}_j^k|+|g_{jk}(\mathring{z}_{j-l}')\mathring{z}_j'^k|)$$
$$\leq C\left(\frac{|\mathring{z}_j'|^{m_j-1}}{AM^{(j)}_{(\mathring{z}'_j,\mathring{z}'_{j-l})}}+a^{1/m_j}|z_j|^{m_j}\right)\leq\frac{1}{6}|z_j|^{m_j},$$

when $(\mathring{z}_j', \mathring{z}_{j-l}')\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}$, and that

$$(3.8)\qquad |\mathring{z}_j^{m_j}-\mathring{z}_j'^{m_j}|\leq C|\mathring{z}_j-\mathring{z}_j'||\mathring{z}_j|^{m_j-1}\leq\frac{C}{A}\left(\frac{\rho}{\tau}\right)^{1/2}\leq\frac{1}{6}|z_j|^{m_j}.$$

when $(\mathring{z}_j', \mathring{z}_{j-l}')\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}$. By arguments similar to those used in proving (3.7) and (3.8), we have

$$(3.9)\qquad |z_j'^{m_j}-\mathring{z}_j'^{m_j}|\leq\frac{1}{6}|z_j|^{m_j} \text{ and } |z_j^{m_j}-\mathring{z}_j^{m_j}|\leq\frac{1}{6}|z_j|^{m_j}.$$

Hence combination of (3.7)-(3.9) gives

$$|z_j^{m_j}-z_j'^{m_j}|\leq|z_j^{m_j}-\mathring{z}_j^{m_j}|+|\mathring{z}_j^{m_j}-\mathring{z}_j'^{m_j}|+|\mathring{z}_j'^{m_j}-z_j'^{m_j}|\leq\frac{1}{2}|z_j|^{m_j},$$

which implies that

$$(3.10)\qquad |z_j-z_j'|=|z_j|\left|1-\left(1-\frac{z_j^{m_j}-z_j'^{m_j}}{z_j^{m_j}}\right)^{1/m_j}\right|$$
$$\leq C|\mathring{z}_j'|^{1-m_j}|z_j^{m_j}-z_j'^{m_j}|.$$

Using (3.3) again, we get

$$(3.11)\qquad |z_j^{m_j}-z_j'^{m_j}|\leq C|z_j-z_j'||\mathring{z}_j'|^{m_j-1}.$$

Finally (3.2) follows from (3.4)-(3.6), (3.10), and (3.11).

In the case when $(\mathring{z}_j', \mathring{z}_{j-l}')\in\hat{\Omega}^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}$ and $|\mathring{z}_{j-l}|^{m_{j-l}}\leq 2(\rho/\tau)^{1/2}$, or $(\mathring{z}_j, \mathring{z}_{j-l})\in\hat{\Omega}^{(j)}_{(\mathring{z}'_j,\mathring{z}'_{j-l})}$ and $|\mathring{z}_{j-l}'|^{m_{j-l}}\leq 2(\rho/\tau)^{1/2}$, by Lemma 2.5, we can verify (3.4), (3.5), and the following inequalities:

$$\text{(3.12)}\qquad \frac{1}{C}\max(|z_j|^{m_j}, |z'_j|^{m_j}, |\mathring{z}'_j|^{m_j}, |z_{j-l}|^{m_j-l}, |z'_{j-l}|^{m_j-l},$$

$$|\mathring{z}_{j-l}|^{m_j-l}|\mathring{z}'_{j-l}|^{m_j-l})\leq\left(\frac{\rho}{\tau}\right)^{1/2},$$

$$\text{(3.13)}\qquad \left(\frac{\rho}{\tau}\right)^{1/2}\leq C\min(|z_j|^{m_j}, |\mathring{z}'_j|^{m_j}).$$

It follows from (3.12) and Lemma 2.1 that

$$\frac{|g_{jk}(z_{j-l})z_j^k-g_{jk}(z'_{j-k})z_j'^k|}{|\mathring{z}'_j|^{m_j-1}}M^{(j)}_{(\mathring{z}'_j,\mathring{z}'_{j-l})}$$

$$\leq C\left(|z_j-z'_j|\left(\frac{\tau}{\rho}\right)^{1/2m_j}+|z_{j-l}-z'_{j-l}|\left(\frac{\tau}{\rho}\right)^{1/2m_j-l}\right)$$

$$\leq C(|z_j-z'_j|M^{(j)}_{(\mathring{z}_j,\mathring{z}_{j-l})}+|z_{j-l}-z'_{j-l}|M^{(j-l)}_{\mathring{z}'_{j-l}}).$$

which yields (3.6). In view of (3.12) and (3.13), we obtain (3.10) and (3.11) by using the same argument as that in (3.7)-(3.9). Hence we conclude (3.2) again. The proof is complete. □

COROLLARY 3.2. *If a and $1/A$ are small enough, then for $\rho>0$, there exists a sequence $\{(z^{(\alpha)}, \tau^{(\alpha)})\}_{\alpha=1}^{\infty}$ of points of D_ρ such that*

$$\bigcup_{\alpha=1}^{\infty}\Omega'_{(z^{(\alpha)},\tau^{(\alpha)})}(\rho)\supset\tilde{D}_\rho,$$

and there exists an upper bound independent of $\rho, a,$ and A for the number of $\Omega_{(z^{(\alpha)},\tau^{(\alpha)})}(\rho)$ which have a point in common.

PROOF. For $\rho>0$, $\tau>0$, and $j=1,\cdots,l$, we can find s sequence $z^{(1)}$, $z^{(2)},\cdots,z^{(\alpha)},\cdots$ in C^n_z so that $|z^{(\alpha)}|\leq a$,

$$\bigcup_{\alpha=1}^{\infty}\tilde{\Omega}'_{z^{(\alpha)}}(\rho,\tau)\supset\{z\in C^n;\ |z|\leq a\},$$

and

$$z^{(\alpha)}\notin\tilde{\Omega}'_{z^{(\beta)}}(\rho,\tau)\ \text{ when }\ \alpha>\beta.$$

The existence of such a sequence follows from the fact that if a sequence satisfies the second condition, then there exists a lower bound for the distance of two points of it. For $\mathring{z}$, let $\alpha(1),\cdots,\alpha(p)$ be integers satisfying $\mathring{z}\in\tilde{\Omega}'_{z^{(\alpha(i))}}(\rho,\tau)$ for each i. We introduce the new coordinate w_j given by

$$w_j=\frac{z_j-\mathring{z}_j}{M_{\mathring{z}_j}^{(j)}} \quad \text{for} \quad j=1, \cdots, l,$$

$$w_j=\frac{z_j-\mathring{z}_j}{AM_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}} \quad \text{for} \quad j\in\{l+1, \cdots, n\}\backslash\Gamma_{\mathring{z}}(\rho, \tau),$$

and

$$w_j=\frac{f_j(z_j, z_{j-l})-f_j(\mathring{z}_j, \mathring{z}_{j-l})}{|\mathring{z}_j|^{m_j-1}AM_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}} \quad \text{for} \quad j\in\Gamma_{\mathring{z}}(\rho, \tau).$$

Then using Lemma 2.2, 2.3, 2.5 and Corollary 2.4, 2.6, 2.7, we can verify that for each $i=1, \cdots, p$,

$$\frac{1}{C}M_{z_j(\alpha(i))}^{(j)}\leq M_{\mathring{z}_j}^{(j)}\leq CM_{z_j(\alpha(i))}^{(j)} \quad \text{if} \quad j=1, \cdots, l,$$

and

$$\frac{1}{C}M_{(z_j^{(\alpha(i))},z_{j-l}^{(\alpha(i))})}^{(j)}\leq M_{(\mathring{z}_j,\mathring{z}_{j-l})}^{(j)}\leq CM_{(z_j^{(\alpha(i))},z_{j-l}^{(\alpha(i))})}^{(j)} \quad \text{if} \quad j=l+1, \cdots, n,$$

where we can choose C so that it is independent of ρ, a, and A. Combining these inequalities with Lemma 3.1, we obtain that $|w_j^{(\alpha(i))}|<C_1$ and $|w_j^{(\alpha(i))}-w_j^{(\alpha(i'))}|\geq 1/C_1$ for some j if $i\neq i'$. Here C_1 is independent of ρ, a, and A. Hence there exists an upper bound for p. Finally, we take a sequence $\{\tau^{(\beta)}\}_{\beta=1}^{\infty}$ so that $\tau^{(1)}=C_0\rho$ and $\tau^{(\beta+1)}=\tau^{(\beta)}+\frac{1}{2}(\tau^{(\beta)})^{1-\delta}$, and rearrange the double sequence $\{(z^{(\alpha)}, \tau^{(\beta)})\}_{\alpha,\beta}$ to $\{(z^{(\alpha)}, \tau^{(\alpha)})\}_{\alpha=1}^{\infty}$. Then we can see that this sequence satisfies the conditions in the statement in this corollary. The proof is complete. □

For $\alpha=1, 2, \cdots$, we write

$$M_{\alpha}^{(j)}=M_{z_j^{(\alpha)}}^{(j)},\ M_{\alpha}^{(j)}=M_{(z_j^{(\alpha)},z_{j-l}^{(\alpha)})}^{(j)},$$
$$\Lambda_{\alpha}(\rho)=\Lambda_{z^{(\alpha)}}(\rho, \tau^{(\alpha)}), \quad \Gamma_{\alpha}(\rho)=\Gamma_{z^{(\alpha)}}(\rho, \tau^{(\alpha)}),$$
$$\Omega_{\alpha}(\rho)=\Omega_{(z^{(\alpha)},\tau^{(\alpha)})}(\rho), \quad \Omega'_{\alpha}(\rho)=\Omega'_{(z^{(\alpha)},\tau^{(\alpha)})}(\rho),$$

and set

(3.14)
$$\begin{aligned}&\phi_{\alpha}(z, t, \tau)\\ &=\Big(\prod_{j=1}^{l}\chi\Big(\frac{|z_j-z_j^{(\alpha)}|}{M_{\alpha}^{(j)}}\Big)\Big)\Big(\prod_{j\in\{l+1,\cdots,n\}\backslash\Gamma_{\alpha}(\rho,\tau)}\chi\Big(\frac{|z_j-z_j^{(\alpha)}|}{M_{\alpha}^{(j)}}\Big)\Big)\\ &\times\Big(\prod_{j\in\Gamma_{\alpha}(\rho,\tau)}\chi\Big(\frac{|f_j(z_j, z_{j-l})-f_j(z_j^{(\alpha)}, z_{j-l}^{(\alpha)})|}{|z_j^{(\alpha)}|^{m_j-1}M_{\alpha}^{(j)}}\Big)\Big)\chi\Big(\frac{t}{2a}\Big)\chi\Big(\frac{|\tau-\tau^{(\alpha)}|}{(\tau^{(\alpha)})^{1-\delta}}\Big),\end{aligned}$$

$$d_{\alpha,j}(z_j, z_{j-l}) = \begin{cases} d_j(z_j, z_{j-l}) & \text{if} \quad j \in \Gamma_\alpha(\rho) \\ 0 & \text{if} \quad j \notin \Gamma_\alpha(\rho), \end{cases}$$

and

(3.15) $$L_{\alpha,j} = L_j - d_{\alpha,j+l} L_{j+l} \quad \text{for} \quad j = 1, \cdots, l,$$

where χ is a smooth function on $\boldsymbol{R}_s$ satisfying

$$\chi(s) = 0 \text{ if } |s| \geq 1 \text{ and } \chi(s) = 1 \text{ if } |s| \leq \frac{1}{2}.$$

Moreover we denote by Ψ_α the pseudo-differential operator $\psi_\alpha(z, t, D_t)$. Using $L_1, \cdots, L_n$, we define smooth functions $c_{jk}(z, t)$ for $j, k = 1, \cdots, n$ as

$$c_{jk}(z, t) = \partial\bar{\partial} r(L_j, \bar{L}_k)|_{(z,t)}.$$

Then we get

(3.16) $$[L_j, \bar{L}_k] = c_{jk} D_t .$$

In what follows, we shall use the notation $S(m, g)$, which was introduced by Hörmander in [11] and [12] as a symbol class of pseudo-differential operators. It follows from (3.14) that $\{\psi_\alpha\}_\alpha$ is a bounded subset in the sense of the seminorms of $S(1, g)$ with

$$g = |dt|^2 + \frac{|d\tau|^2}{(1+|\tau|^2)^{1-\delta}}$$

if we fix z, and there exists an upper bound independent of z for the seminorms of ψ_α. Moreover we give a sequence $\{\gamma_\alpha(z, t, \tau)\}_{\alpha=1}^\infty$ of smooth functions by

$$\gamma_\alpha = \psi_\alpha \prod_{\beta<\alpha} (1 - \psi_\beta^2).$$

Then

(3.17) $$\sum_\alpha \gamma_\alpha \psi_\alpha = 1 \quad \text{on } \tilde{D}_\rho$$

and Corollary 3.2 implies that $\{\gamma_\alpha\}$ is a bounded subset of $S(1, g)$. For α, we introduce the differential operator Θ_α by

$$\Theta_\alpha = (D_t - \tau^{(\alpha)})(\tau^{(\alpha)})^{-1/2}.$$

Using these notations, we obtain

LEMMA 3.3. *There exists $C > 0$ independent of ρ such that*

$$(3.18)\qquad \sum_{\alpha}(\|[\Psi_{\alpha}L_{\alpha,j}](\varphi+d_{\alpha,j}\varphi')\|+\|[\Psi_{\alpha}\bar{L}_{\alpha,j}](\varphi+d_{\alpha,j}\varphi')\|)$$
$$\leq C\Big(\sum_{\alpha}\|M_{\alpha}^{(j)}\Psi_{\alpha}(\varphi+d_{\alpha,j}\varphi')\|+\|\varphi\|+\|\varphi'\|\Big),$$

$$(3.19)\qquad \sum_{\alpha}\Big\|\frac{1}{M_{\alpha}^{(k)}}[[\Psi_{\alpha},\bar{L}_{\alpha,k}],L_{\alpha,j}](\varphi+d_{\alpha,j}\varphi')\Big\|$$
$$\leq C\Big(\sum_{\alpha}\|M_{\alpha}^{(j)}\Psi_{\alpha}(\varphi+d_{\alpha,j}\varphi')\|+\|\varphi\|+\|\varphi'\|\Big),$$

$$(3.20)\qquad \sum_{\alpha}\Big\|\frac{M_{\alpha}^{(j)}}{M_{\alpha}^{(k)}}[\Psi_{\alpha},\bar{L}_{\alpha,k}](\varphi+d_{\alpha,j}\varphi')\Big\|$$
$$\leq C\Big(\sum\|M_{\alpha}^{(j)}\Psi_{\alpha}(\varphi+d_{\alpha,j}\varphi')\|+\|\varphi\|+\|\varphi'\|\Big),$$

$$(3.21)\qquad \sum_{\alpha}\Big\|\frac{1}{M_{\alpha}^{(j)}}[\Psi_{\alpha},L_{\alpha,j}]\varphi\Big\|\leq C\Big(\sum_{\alpha}\|\Psi_{\alpha},\varphi\|+\|\varphi\|\Big),$$

$$(3.22)\qquad \sum_{\alpha}\|\Theta_{\alpha}\Psi_{\alpha}\varphi\|\leq C\Big(\rho^{-\delta}\sum_{\alpha}\|(\tau^{(\alpha)})^{1/2}\Psi_{\alpha}\varphi\|+\|\varphi\|\Big),$$

$$(3.23)\qquad \sum_{\alpha}\|\Theta_{\alpha}(\tau^{(\alpha)})^{-1/2}M_{\alpha}^{(j)}\Psi_{\alpha}\varphi\|\leq C\Big(\rho^{-\delta}\sum_{\alpha}\|M_{\alpha}^{(j)}\Psi_{\alpha}\varphi\|+\|\varphi\|\Big),$$

for any smooth functions φ *and* φ' *supported in* $B_{a/2}$.

PROOF. Before the proof, we shall introduce a notation. For symbols $a(z,t,\tau)$ and $b(z,t,\tau,)$ we denote by $a\sharp b(z,t,\tau)$ the symbol of the composition of pseudo-differential opersators $a(z,t,D_t)$ and $b(z,t,D_t)$.

Firstly, we shall prove (3.18). By symbolic calculation, $\sigma([\Psi_{\alpha},L_j])$ can be express as

$$(3.24)\qquad \sigma([\Psi_{\alpha},L_j])$$
$$=-\frac{\partial\psi_{\alpha}}{\partial z_j}+\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}\frac{\partial\psi_{\alpha}}{\partial t}-i\tau\Big(\psi_{\alpha}\sharp\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}-\frac{\psi_{\alpha}\partial\phi/\partial z_j}{i+\partial\phi/\partial t}\Big).$$

It follows from Theorem 3.6 in [11] and (3.14) that there exists C independent of α and j such that

$$(3.25)\qquad \Big|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}\Big(\psi_{\alpha}\sharp\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}-\frac{\psi_{\alpha}\partial\phi/\partial z_j}{i+\partial\phi/\partial t}\Big)\Big|\leq C(|\tau|+1)^{(\delta-1)(1+\nu)}.$$

Furthermore we have

$$(3.26)\qquad \Big|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}\Big(\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}\frac{\partial\psi_{\alpha}}{\partial t}\Big)\Big|\leq C(|\tau|+1)^{(\delta-1)\nu},$$

from (3.14). Combination of (3.24)-(3.26) yields

$$\left|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}\left(\sigma([\Psi_{\alpha}, L_j])+\frac{\partial\phi_{\alpha}}{\partial z_j}\right)\right|\leq C(|\tau|+1)^{\delta+(\delta-1)\nu}. \tag{3.27}$$

In addition, we observe that

$$\left|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}\left(\frac{\partial}{\partial z_j}-d_{\alpha,j+l}\frac{\partial}{\partial z_{j+l}}\right)\phi_{\alpha}\right|\leq CM_{\alpha}^{(j)}(|\tau|+1)^{(\delta-1)\nu} \tag{3.28}$$

because

$$\left(\frac{\partial}{\partial z_j}-d_{\alpha,j+l}\frac{\partial}{\partial z_{j+l}}\right)\chi(|f_{j+l}(z_{j+l}, z_j)|)=0 \quad \text{when } j+l\in\Gamma_{\alpha}(\rho).$$

If $j+l\notin\Gamma_{\alpha}(\rho)$, in view of Lemma 2.1, (3.1) and (3.27), and (3.28), we can see that

$$\left|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}\sigma([\Psi_{\alpha}, L_{\alpha,j}])\right|\leq CM_{\alpha}^{(j)}(|\tau|+1)^{(\delta-1)\nu}. \tag{3.29}$$

In the case when $j+l\in\Gamma_{\alpha}(\rho)$, $\sigma([\Psi_{\alpha}, L_{\alpha,j}])$ is of the form

$$\begin{aligned}\sigma([\Psi_{\alpha}, L_{\alpha,j}])=&-\left(\frac{\partial\phi_{\alpha}}{\partial z_j}-d_{j+l}\frac{\partial\phi_{\alpha}}{\partial z_{j+l}}\right)\\&+\left(\sigma([\Psi_{\alpha}, L_j])+\frac{\partial\phi_{\alpha}}{\partial z_j}\right)-d_{j+l}\left(\sigma([\Psi_{\alpha}, L_{j+l}])+\frac{\partial\phi_{\alpha}}{\partial z_{j+l}}\right).\end{aligned} \tag{3.30}$$

Since it follows from Lemma 2.1 and (3.27) that the second and third terms in the right-hand side of (3.30) and their derivatives are estimated by the right-hand side of (3.28), we show that (3.29) holds also in this case. On the other hand, by integration by part we observe that

$$\begin{aligned}\phi_{\alpha}\#\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}=&\frac{1}{2\pi}\int e^{-is\eta}\left(1-\left(\frac{\partial}{\partial\eta}\right)^{2}\right)\eta^{-M-2}\phi_{\alpha}(z, t, \eta+\tau)\\&\times(1+s^2)^{-1}\left(-i\frac{\partial}{\partial s}\right)^{M+2}\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}(z, s+t)dsd\eta,\end{aligned}$$

for any integer $M>0$, which implies that if $(z, t, \tau)\notin\Omega_{\alpha}(\rho)$, then

$$\left|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}\phi_{\alpha}\#\frac{\partial\phi/\partial z_j}{i+\partial\phi/\partial t}\right|\leq C(\tau^{(\alpha)})^{-M}(|\tau|+1)^{(\delta-1)\nu}. \tag{3.31}$$

For α, we set

$$r_{j;\alpha}^{0}=\left(1-\sum_{\Omega_{\alpha}(\rho)\cap\Omega_{\beta}(\rho)\neq\phi}r_{\beta}\phi_{\beta}\right)\sigma([\Psi_{\alpha}, L_{\alpha,j}]).$$

Here by Corollary 3.2 we can see that the number of β with $\Omega_{\alpha}(\rho)\cap\Omega_{\beta}(\rho)\neq\phi$ is smaller than a constant independent of α and ρ, and (3.17) implies that

$$1-\sum_{\Omega_{\alpha}(\rho)\cap\Omega_{\beta}(\rho)\neq\phi}\gamma_{\beta}\psi_{\beta}\equiv 0 \quad \text{on } \Omega_{\alpha}(\rho). \tag{3.32}$$

Then it follows from (3.14), (3.24), and (3.30)-(3.32) that for $M>0$, there exists $C>0$ such that

$$\left|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}r_{j;\alpha}^{0}\right|\leq C(\tau^{(\alpha)})^{-M}(|\tau|+1)^{(\delta-1)\nu} \tag{3.33}$$

For integer p and α with $p\geq 1$, we introduce the set $\kappa(\alpha, p, \rho)$ of p-tuples as follows:

$$\kappa(\alpha, p, \rho)$$
$$=\{(\beta_1, \cdots, \beta_p);\ \Omega_{\alpha}(\rho)\cap\Omega_{\beta_1}(\rho)\neq\phi,\ \Omega_{\beta_{i-1}}(\rho)\cap\Omega_{\beta_i}(\rho)\neq\phi \text{ for all } i\}.$$

For $(\beta_1, \cdots, \beta_p)\in\kappa(\alpha, p, \rho)$, $g_{j;\alpha;\beta_1,\cdots,\beta_p}^{p}$ is given by

$$g_{j;\alpha;\beta_1}^{1}=\psi_{\beta_1}\gamma_{\beta_1}\sigma([\Psi_{\alpha}, L_{\alpha,j}])-(\gamma_{\beta_1}\sigma([\Psi_{\alpha}, L_{\alpha,j}]))\#\psi_{\beta_1}$$

when $p=1$, and

$$g_{j;\alpha;\beta_1,\cdots,\beta_p}^{p}=\psi_{\beta_p}\gamma_{\beta_p}g_{j;\alpha;\beta_1,\cdots,\beta_{p-1}}^{p-1}-(\gamma_{\beta_p}g_{j;\alpha;\beta_1,\cdots,\beta_{p-1}}^{p-1})\#\psi_{\beta_p}.$$

Furthermore, $r_{j;\alpha;\beta_1,\cdots,\beta_p}^{p}$ is given by

$$r_{j;\alpha;\beta_1,\cdots,\beta_p}^{p}=\Big(1-\sum_{\Omega_{\gamma}(\rho)\cap\Omega_{\beta_p}(\rho)\neq\phi}\gamma_{\gamma}\psi_{\gamma}\Big)g_{j;\alpha;\beta_1,\cdots,\beta_p}^{p}.$$

Applying Theorem 3.6 in [11] again, we can verify from (3.14) and (3.29) that

$$\left|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}g_{j;\alpha;\beta_1,\cdots,\beta_p}^{p}\right|\leq CM_{\alpha}^{(j)}(|\tau|+1)^{(\delta-1)(\nu+p)}, \tag{3.34}$$

and by an argument similar to that in the proof of (3.33), we obtain that

$$\left|\frac{\partial^{\mu+\nu}}{\partial t^{\mu}\partial\tau^{\nu}}g_{j;\alpha;\beta_1,\cdots,\beta_p}^{p}\right|\leq C(\tau^{(\alpha)})^{-M}(|\tau|+1)^{(\delta-1)\nu} \text{ if } (z, t, \tau)\notin\Omega_{\alpha}(\rho), \tag{3.35}$$

and

$$\left| \frac{\partial^{\mu+\nu}}{\partial t^\mu \partial \tau^\nu} r^p_{j;\alpha;\beta_1,\cdots,\beta_p} \right| \le C(\tau^{(\alpha)})^{-M}(|\tau|+1)^{(\delta-1)\nu}. \tag{3.36}$$

As a result, it follows from (3.29) and (3.33)-(3.36) that

$$\begin{aligned}
&\|(\gamma_\beta \sigma([\Psi_\alpha, L_{\alpha,j}]))(z, t, D_t)\Psi_\alpha(\varphi + d_{\alpha,j}\varphi')\| \le C\|M^{(j)}_\alpha \Psi_\alpha(\varphi + d_{\alpha,j}\varphi')\|,\\
&\|(\gamma_{\beta_p} g^{p-1}_{j;\alpha;\beta_1,\cdots,\beta_{p-1}})(z, t, D_t)\Psi_\alpha(\varphi + d_{\alpha,j}\varphi')\|\\
&\le C(\tau^{(\alpha)})^{(\delta-1)(p-1)}\|M^{(j)}_\alpha \Psi_\alpha(\varphi + d_{\alpha,j}\varphi')\|\\
&\|(\psi_{\beta_p}\gamma_{\beta_p} g^{p-1}_{j;\alpha;\beta_1,\cdots,\beta_{p-1}})(z, t, D_t)(\varphi + d_{\alpha,j}\varphi')\|\\
&\le C(\tau^{(\alpha)})^{(\delta-1)(p-1)}\|M^{(j)}_\alpha(\varphi + d_{\alpha,j}\varphi')\| \le C(\tau^{(\alpha)})^{(\delta-1)(p-1)+1}(\|\varphi\| + \|\varphi'\|),
\end{aligned}$$

and

$$\begin{aligned}
&\|r^p_{j;\alpha;\beta_1,\cdots,\beta_p}(z, t, D_t)(\varphi + d_{\alpha,j}\varphi')\|\\
&\le C(\tau^{(\alpha)})^{-M}\|\varphi + d_{\alpha,j}\varphi'\| \le C(\tau^{(\alpha)})^{-M+1}(\|\varphi\| + \|\varphi'\|).
\end{aligned}$$

Combining them with the form

$$\begin{aligned}
\sigma([\Psi_\alpha, L_{\alpha,j}]) &= r^0_{j;\alpha} + \sum_{\beta\in\kappa(\alpha,1,\rho)} ((\gamma_\beta \sigma([\Psi_\alpha, L_{\alpha,j}]))\sharp\psi_\beta + r^1_{j;\alpha;\beta})\\
&+ \sum_{p=2}^{m-1}\Big(\sum_{(\beta_1,\cdots,\beta_p)\in\kappa(\alpha,p,\rho)} (\gamma_{\beta_p} g^{p-1}_{j;\alpha;\beta_1,\cdots,\beta_{p-1}})\sharp\psi_{\beta_p} + r^p_{j;\alpha;\beta_1,\cdots,\beta_p}\Big)\\
&+ \sum_{(\beta_1,\cdots,\beta_n)\in\kappa(\alpha,m,\rho)} \psi_{\beta_m}\gamma_{\beta_m} g^{m-1}_{j;\alpha;\beta_1\cdots,\beta_{m-1}},
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_\alpha \|[\Psi_\alpha, L_{\alpha,j}](\varphi + d_{\alpha,j}\varphi')\| \le C\sum_\alpha (\|M^{(j)}_\alpha \Psi_\alpha(\varphi + d_{\alpha,j}\varphi')\|\\
&+ ((\tau^{(\alpha)})^{-M+1} + (\tau^{(\alpha)})^{(\delta-1)m+1})(\|\varphi\| + \|\varphi'\|)).
\end{aligned}$$

Similarly we can prove the above inequality with replacing $L_{\alpha,j}$ by $\bar{L}_{\alpha,j}$. Hence taking m and M large enough, we obtain (3.18). Here we remark that it follows from Corollary 3.2 that there exists an upper bound independent of α and ρ for the number of elements of $\kappa(\alpha, p, \rho)$. By the argument as that used in the above, we can prove (3.19)-(3.21) by Lemma 2.1, (3.1), and (3.14). Finally using the form

$$\sigma(\Theta_\alpha \Psi_\alpha) = (\tau - \tau^{(\alpha)})(\tau^{(\alpha)})^{-1/2}\psi_\alpha - i\frac{\partial\psi_\alpha}{\partial t},$$

we get

$$\left| \frac{\partial^{\mu+\nu}}{\partial t^\mu \partial \tau^\nu} \sigma(\Theta_\alpha \Psi_\alpha) \right| \le C\rho^{-\alpha}(\tau^{(\alpha)})^{1/2}(|\tau|+1)^{(\delta-1)\nu},$$

and verify that $\sigma(\Theta_\alpha\Psi_\alpha)=0$ outside of $\Omega_\alpha(\rho)$. Hence we obtain also (3.22) and (3.23) by a similar method. The proof is complete. □
Applying this lemma, we have

PROPOSITION 3.4. *For large ρ and A, there exists $C>0$ such that for n-tuples $(\varphi_1, \cdots, \varphi_n)$ of smooth functions supported on $B_{a/2}$, we have*

$$(3.37)\quad \sum_\alpha \Big|\sum_{j,k} ((c_{jk}D_t\Psi_\alpha\varphi_j, \Psi_\alpha\varphi_k)-(\Psi_\alpha L_j\varphi_j, \Psi_\alpha L_k\varphi_k)+(\Psi_\alpha\bar{L}_k\varphi_j, \Psi_\alpha\bar{L}_j\varphi_k))\Big| \leq C\Big(\sum_\alpha(\|\Psi_\alpha\sum_j L_j\varphi_j\|^2+\sum_j\|M_\alpha^{(j)}\Psi_\alpha(\varphi_j+d_{\alpha,j}\varphi_{j-l})\|^2)+\sum_j\|\varphi_j\|^2\Big).$$

PROOF. By using commutators of L_j, $\bar{L}_j$ and Ψ_α we observe that

$$(\Psi_\alpha L_j\varphi_j, \Psi_\alpha L_k\varphi_k)=-(\bar{L}_k L_j\Psi_\alpha\varphi_j, \Psi_\alpha\varphi_k)+([\Psi_\alpha, L_j]\varphi_j, \Psi_\alpha L_k\varphi_k) -([\Psi_\alpha, L_j]\varphi_j, ([\Psi_\alpha, L_k]+\bar{a}_k\Psi_\alpha)\varphi_k)+(\Psi_\alpha L_j\varphi_j, ([\Psi_\alpha, L_k]+\bar{a}_k\Psi_\alpha)\varphi_k),$$

and

$$(\Psi_\alpha\bar{L}_k\varphi_j, \Psi_\alpha\bar{L}_j\varphi_k)=-(L_j\bar{L}_k\Psi_\alpha\varphi_j, \Psi_\alpha\varphi_k) +(\Psi_\alpha\varphi_j, ([[\Psi_\alpha, \bar{L}_j], L_k]+\bar{a}_k[\Psi_\alpha, \bar{L}_j]+[a_j\Psi_\alpha, L_k]+a_j\bar{a}_k\Psi_\alpha)\varphi_k) -(\Psi_\alpha\varphi_j, ([\Psi_\alpha, \bar{L}_j]+a_j\Psi_\alpha)L_k\varphi_k)+([\Psi_\alpha, \bar{L}_k]\varphi_j, ([\Psi_\alpha, \bar{L}_j]+a_k\Psi_\alpha)\varphi_k) -([\Psi_\alpha, \bar{L}_k]L_j\varphi_j, \Psi_\alpha\varphi_k)+([[\Psi_\alpha, \bar{L}_k], L_j]\varphi_j, \Psi_\alpha\varphi_k),$$

where we put

$$(3.38)\quad a_j(z, t)=L_j^*+\bar{L}_j.$$

For convenience, we write

$$(3.39)\quad \varphi_{\alpha,j}=\varphi_j+d_{\alpha,j}\varphi_{j-l} \quad\text{and}\quad a_{\alpha,j}=a_j-\bar{d}_{\alpha,j+l}a_{j+l}.$$

Then noting that

$$\sigma([\Psi_\alpha, L_{\alpha,j}])=\sigma([\Psi_\alpha, L_j])-d_{\alpha,j+l}\sigma([\Psi_\alpha, L_{j+l}]),$$

we can verify that

$$(3.40)\quad \Big|\sum_{j,k}((c_{jk}D_t\Psi_\alpha\varphi_j, \Psi_\alpha\varphi_k)-(\Psi_\alpha L_j\varphi_j, \Psi_\alpha L_k\varphi_k)+(\Psi_\alpha\bar{L}_k\varphi_j, \Psi_\alpha\bar{L}_j\varphi_k))\Big| \leq \|\Psi_\alpha\sum_j L_j\varphi_j\|^2+\sum_j\Big(\|[\Psi_\alpha, \bar{L}_{\alpha,j}]\varphi_{\alpha,j}\|^2+\|\bar{a}_{\alpha,j}\Psi_\alpha\varphi_{\alpha,j}\|^2 +\|M_\alpha^{(j)}\Psi_\alpha\varphi_{\alpha,j}\|^2+\Big\|\frac{1}{M_\alpha^{(j)}}[\Psi_\alpha, \bar{L}_{\alpha,j}]\sum_k L_k\phi_k\Big\|^2+\Big\|\frac{a_{\alpha,j}}{M_\alpha^{(j)}}\Psi_\alpha\sum_k L_k\varphi_k\Big\|^2\Big) +\sum_{j,k}\Big(\Big\|\frac{1}{M_\alpha^{(j)}}[[\Psi_\alpha, \bar{L}_{\alpha,j}], L_{\alpha,k}]\varphi_{\alpha,k}\Big\|^2$$

$$+\left\|\frac{1}{M_\alpha^{(j)}}(a_{\alpha,j}[\Psi_\alpha, L_{\alpha,k}]+\bar{a}_{\alpha,k}[\Psi_\alpha, \bar{L}_{\alpha,j}]+(L_{\alpha,k}a_{\alpha,j})\Psi_\alpha)\varphi_{\alpha,k}\right\|^2$$

$$+\left\|\frac{M_\alpha^{(k)}}{M_\alpha^{(j)}}[\Psi_\alpha, \bar{L}_{\alpha,j}]\varphi_{\alpha,k}\right\|^2+\left\|\frac{M_\alpha^{(k)}}{M_\alpha^{(j)}}a_{\alpha,j}\Psi_\alpha\varphi_{\alpha,k}\right\|^2\Bigg).$$

In view of Lemma 3.3 and the fact that

$$|a_{\alpha,j}|\leq CM_\alpha^{(j)} \quad \text{and} \quad |L_{\alpha,k}a_{\alpha,k}|\leq CM_\alpha^{(j)}M_\alpha^{(k)},$$

because of Corollary 2.8, we can see that the summation of the right-hand side of (3.40) with respect to α is estimated by that of (3.37). The proof is complete. □

By Proposition 2.9 and Lemma 3.3, we obtain

PROPOSITION 3.5. *There exists* $\varepsilon>0$ *and* $C>0$ *independent of* ρ *such that for n-tuples* $(\varphi_1, \cdots, \varphi_n)$ *of smooth functions supported on* $B_{a/2}$, *we have*

$$\rho^\varepsilon\sum_{\alpha,j}\|M_\alpha^{(j)}\Psi_\alpha(\varphi_j+d_{\alpha,j}\varphi_{j-l})\|^2 \tag{3.41}$$

$$\leq C\Big(\sum_{\alpha,j,k}(\mathrm{Re}(c_{jk}D_t\Psi_\alpha\varphi_j, \Psi_\alpha\varphi_k)+\|\Psi_\alpha\bar{L}_j\varphi_k\|^2)+\sum_j\|\varphi_j\|^2\Big),$$

for large ρ, A, *and small* a *satisfying*

$$a<\rho^{-1/2}. \tag{3.42}$$

PROOF. Let

$$\tilde{c}_{jk}(w)=\partial\bar{\partial}\tilde{r}\Big(\frac{\partial}{\partial w_j}-\frac{\partial\tilde{r}/\partial w_j}{\partial\tilde{r}/\partial w_0}\frac{\partial}{\partial w_0}, \frac{\partial}{\partial\bar{w}_k}-\frac{\partial\tilde{r}/\partial\bar{w}_k}{\partial\tilde{r}/\partial\bar{w}_0}\frac{\partial}{\partial\bar{w}_0}\Big)\Big|_w.$$

for $w\in b\tilde{\Omega}$. Then the strongly pseudo-convexity of $\tilde{\Omega}$ implies that $(\tilde{c}_{jk}(w))_{jk}$ is positive definite for each $w\in b\tilde{\Omega}$. In view of (2.1), we observe that

$$c_{jk}=\sum_{p,q}\tilde{c}_{pq}(t+i\phi(t,z),f_1(z), \cdots, f_n(z))\frac{\partial f_p}{\partial z_j}\overline{\frac{\partial f_q}{\partial z_k}}, \tag{3.43}$$

where we write

$$f_j(z)=z_j^{m_j} \quad \text{for} \quad j=1, \cdots, l, \tag{3.44}$$

for the sake of simplicity. This implies that there exists $C>0$ such that

$$\sum_{j=1}^{l}\|(\tau^{(\alpha)})^{1/2}z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2+\sum_{j=l+1}^{n}\left\|(\tau^{(\alpha)})^{1/2}\left(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\right)\right\|^2$$
$$\leq C\sum_{p,q}\left(\tilde{c}_{pq}\tau^{(\alpha)}\sum_j\frac{\partial f_p}{\partial z_j}\Psi_\alpha\varphi_j,\ \sum_k\frac{\partial f_q}{\partial z_k}\Psi_\alpha\varphi_k\right)$$
$$\leq C\Big(\sum_{j,k}\mathrm{Re}(c_{jk}D_t\Psi_\alpha\varphi_j,\Psi_\alpha\varphi_k)+\sum_{j=1}^{l}\|\Theta_\alpha z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2$$
$$+\sum_{j=l+1}^{n}\left\|\Theta_\alpha\left(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\right)\right\|^2\Big)$$
$$+\frac{1}{2}\Big(\sum_{j=1}^{l}\|(\tau^{(\alpha)})^{1/2}z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2+\sum_{j=l+1}^{n}\left\|(\tau^{(\alpha)})^{1/2}\left(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\right)\right\|^2\Big).$$

In fact, $\sum_{j,k}\mathrm{Re}(c_{jk}\tau^{(\alpha)}\varphi_j,\varphi_k)$ is non-negative because of the pseudo-convexity of $\tilde{\Omega}$. Combining this inequality with (3.22) in Lemma 3.3, we obtain that

$$\sum_\alpha\Big(\sum_{j=1}^{l}\|(\tau^{(\alpha)})^{1/2}z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2+\sum_{j=l+1}^{n}\left\|(\tau^{(\alpha)})^{1/2}\left(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\right)\right\|^2\Big) \tag{3.45}$$
$$\leq C\Big(\sum_\alpha\sum_{j,k}\mathrm{Re}(c_{jk}D_t\Psi_\alpha\varphi_j,\Psi_\alpha\varphi_k)+\sum_j\|\varphi_j\|^2\Big),$$

for ρ is large enough.

From now on, applying Proposition 2.9, we shall estimate the left-hand side of (3.45) from below. Using a_j in (3.38) and $a_{\alpha,j}$ in (3.39), we have

$$L_{\alpha,j}L_{\alpha,j}{}^*+L_{\alpha,j}{}^*L_{\alpha,j} \tag{3.46}$$
$$=c_{jj}D_t+2\bar{L}_{\alpha,j}{}^*\bar{L}_{\alpha,j}-2\bar{L}_{\alpha,j}{}^*a_j-2\bar{a}_j\bar{L}_{\alpha,j}+|a_j|^2-(L_ja_j)\quad\text{if } j+l\notin\Gamma_\alpha(\rho),$$

and

$$L_{\alpha,j}L_{\alpha,j}{}^*+L_{\alpha,j}{}^*L_{\alpha,j} \tag{3.47}$$
$$=(m_j)^2|z_j|^{2m_j-2}\tilde{c}_{jj}D_t+2\bar{L}_{\alpha,j}{}^*\bar{L}_{\alpha,j}-2((L_{j+l}d_{j+l})+\bar{a}_{\alpha,j})\bar{L}_{\alpha,j}$$
$$-2\bar{L}_{\alpha,j}{}^*((\overline{L_{j+l}d_{j+l}})+\bar{a}_{\alpha,j})-(L_{\alpha,j}a_{\alpha,j})\quad\text{if } j+l\in\Gamma_\alpha(\rho),$$

in view of (3.15), (3.16), and (3.43). If $1\leq j\leq l$ and $j+l\notin\Gamma_\alpha(\rho)$, then (2.11) in Lemma 2.3 and (3.42) imply that there exists $C>0$ independent of a, A, and ρ such that

$$\left|\frac{\partial f_{j+l}}{\partial z_j}\right|(\tau^{(\alpha)})^{1/2}\leq CM_\alpha^{(j)}\quad\text{on } \Omega_\alpha(\rho)\quad\text{if } j+l\notin\Gamma_\alpha(\rho). \tag{3.48}$$

Then combining (3.46)-(3.48) with (2.33) in Corollary 2.8 and (3.43), we obtain that

$$\begin{aligned}(3.49)\qquad &\|L_{\alpha,j}\Psi_\alpha\varphi_j\|^2+\|L_{\alpha,j}{}^*\Psi_\alpha\varphi_j\|^2\le C(\|\Psi_\alpha\bar{L}_{\alpha,j}\varphi_j\|^2+\|M_\alpha^{(j)}\Psi_\alpha\varphi_j\|^2\\ &+\|(\tau^{(\alpha)})^{1/2}z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2+\|\Theta_\alpha(\tau^{(\alpha)})^{-1/2}M_\alpha^{(j)}\Psi_\alpha\varphi_j\|^2\\ &+\|\Theta_\alpha z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2)\quad \text{if } 1\le j\le l,\end{aligned}$$

and

$$\begin{aligned}(3.50)\qquad &\|L_{\alpha,j}\Psi_\alpha\varphi_j\|^2+\|L_{\alpha,j}{}^*\Psi_\alpha\varphi_j\|^2\le C\Big(\|\Psi_\alpha\bar{L}_{\alpha,j}\varphi_j\|^2+\|M_\alpha^{(j-l)}\Psi_\alpha\varphi_{j-l}\|^2\\ &+\Big\|(\tau^{(\alpha)})^{1/2}\Big(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\Big)\Big\|^2+\|\Theta_\alpha M_\alpha^{(j-l)}\Psi_\alpha\varphi_{j-l}\|^2\\ &+\Big\|\Theta_\alpha\Big(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\Big)\Big\|^2\Big)\quad \text{if } l+1\le j\le n \text{ and } j\notin\Gamma_\alpha(\rho).\end{aligned}$$

As a result, (2.42) and (3.49) imply that

$$\begin{aligned}(3.51)\qquad &\rho^{2^{1-m_j}}\|M_\alpha^{(j)}\Psi_\alpha\varphi_{\alpha,j}\|^2\le C(\|\Psi_\alpha\bar{L}_{\alpha,j}\varphi_j\|^2+\|M_\alpha^{(j)}\Psi_\alpha\varphi_j\|^2\\ &+\|(\tau^{(\alpha)})^{1/2}z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2+\|\Theta_\alpha z_j^{m_j-1}\Psi_\alpha\varphi_j\|^2\\ &+\|\Theta_\alpha(\tau^{(\alpha)})^{-1/2}M_\alpha^{(j)}\Psi_\alpha\varphi_j\|^2)\quad \text{if } 1\le j\le l \text{ and } j\notin\Lambda_\alpha(\rho),\end{aligned}$$

and (2.43) and (3.50) imply that

$$\begin{aligned}(3.52)\qquad &\rho^{2^{1-m_j}}\|M_\alpha^{(j)}\Psi_\alpha\varphi_{\alpha,j}\|^2\le C\Big(\|\Psi_\alpha\bar{L}_j\varphi_j\|^2+\|M_\alpha^{(j-l)}\Psi_\alpha\varphi_{j-l}\|^2\\ &+\Big\|(\tau^{(\alpha)})^{1/2}\Big(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\Big)\Big\|^2+\|\Theta_\alpha M_\alpha^{(j-l)}\Psi_\alpha\varphi_{j-l}\|^2\\ &+\Big\|\Theta_\alpha\Big(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\Big)\Big\|^2\Big)\quad \text{if } l+1\le j\le n \text{ and } j\notin\Gamma_\alpha(\rho).\end{aligned}$$

Furthermore if $j\notin\Lambda_\alpha(\rho)$ and $j+l\in\Gamma_\alpha(\rho)$, then we have $|d_{\alpha,j+l}|\le C$ in view of (2.2) and (2.33), which implies that

$$(3.53)\qquad \|\Psi_\alpha\bar{L}_{\alpha,j}\varphi_j\|\le C(\|\Psi_\alpha\bar{L}_j\varphi_j\|+\|\Psi_\alpha\bar{L}_{j+l}\varphi_j\|).$$

If $j\notin\Lambda_\alpha(\rho)$ and $j+l\notin\Gamma_\alpha(\rho)$, then (3.53) is satisfied because $L_{\alpha,j}=L_j$. Combining (3.51)-(3.53) with (3.22) and (3.23), and applying (2.41) if $j\in\Lambda_\alpha(\rho)$ and (2.44) if $j\in\Gamma_\alpha(\rho)$, we obtain that

$$\begin{aligned}(3.54)\qquad &\rho^{2^{1-\bar{m}_n}}\sum_{\alpha,j}\|M_\alpha^{(j)}\varphi_{\alpha,j}\|^2\le C\Big(\sum_\alpha\Big(\sum_{j,k}\|\Psi_\alpha\bar{L}_j\varphi_k\|^2+\sum_{j=1}^{l}\|(\tau^{(\alpha)})^{1/2}z_j{}^{m_j-1}\Psi_\alpha\varphi_j\|^2\\ &+\sum_{j=l+1}^{n}\Big\|(\tau^{(\alpha)})^{1/2}\Big(\frac{\partial f_j}{\partial z_j}\Psi_\alpha\varphi_j+\frac{\partial f_j}{\partial z_{j-l}}\Psi_\alpha\varphi_{j-l}\Big)\Big\|^2\Big)+\sum_j\|\varphi_j\|^2\Big),\end{aligned}$$

if ρ is large enough.

Finally (3.41) is given by (3.45) and (3.54) with $\varepsilon=2^{1-\bar{m}_n}$. The proof is complete. □

PROOF OF THEOREM 0.1. It follows from (1.3) and (1.4) that

$$\sum_{\alpha}(\|\Psi_\alpha\bar{\partial}_b\varphi\|^2+\|\Psi_\alpha\bar{\partial}_b^*\varphi\|^2)=\sum_J{}'\sum_{\alpha,j}\|\Psi_\alpha\bar{L}_j\varphi_j\|^2 + \sum_K{}'\sum_{\alpha,j,k}(\mathrm{Re}(\Psi_\alpha L_j\varphi_{jK},\Psi_\alpha L_k\varphi_{kK})-\mathrm{Re}(\Psi_\alpha\bar{L}_k\varphi_{jK},\Psi_\alpha\bar{L}_j\varphi_{jK})). \tag{3.55}$$

Taking ρ and A large enough and a small enough so that (3.42) is satisfied, and applying Proposition 3.4 and 3.5 to the n-tuples $(\varphi_{1K},\cdots,\varphi_{nK})$, we show that

$$\rho^\varepsilon\sum_{\alpha,j}\|M_\alpha^{(j)}\Psi_\alpha(\varphi_{jK}+d_{\alpha,j}\varphi_{j-lK})\|^2$$
$$\leq C\Big(\sum_{\alpha,j,k}(\mathrm{Re}(\Psi_\alpha L_j\varphi_{jK},\Psi_\alpha L_k\varphi_{kK})-\mathrm{Re}(\Psi_\alpha\bar{L}_k\varphi_{jK},\Psi_\alpha\bar{L}_j\varphi_{kK})$$
$$+\|\Psi_\alpha\bar{L}_j\varphi_{kK}\|^2)+\sum_\alpha\|\Psi_\alpha\sum_j L_j\varphi_{jK}\|^2+\sum_j\|\varphi_{jK}\|^2\Big) \quad \text{for each } K.$$

Combining it with (3.55), we obtain that

$$\rho^\varepsilon\sum_{\alpha,j}\sum_K{}'\|M_\alpha^{(j)}\Psi_\alpha(\varphi_{jK}+d_{\alpha,j}\varphi_{j-lK})\|^2\leq C\Big(\sum_\alpha(\|\Psi_\alpha\bar{\partial}_b\varphi\|^2+\|\Psi_\alpha\bar{\partial}_b^*\varphi\|^2)+\|\varphi\|^2\Big). \tag{3.56}$$

Since $\{\psi_\alpha\}_\alpha$ is a symbol in $S(1,g)$ with values in $\mathcal{L}(l^2,\boldsymbol{C})$, it follows that

$$\sum_\alpha(\|\Psi_\alpha\bar{\partial}_b\varphi\|^2+\|\Psi_\alpha\bar{\partial}_b^*\varphi\|^2)\leq C(\|\bar{\partial}_b\varphi\|^2+\|\bar{\partial}_b^*\varphi\|^2). \tag{3.57}$$

Using Lemma 2.1 and (2.33) in Corollary 2.8, we show that

$$\sum_j\|(\tau^{(\alpha)})^{1/2m_j}\Psi_\alpha\varphi_{jK}\|^2\leq C\sum_j\|M_\alpha^{(j)}\Psi_\alpha(\varphi_{jK}+d_{\alpha,j}\varphi_{j-lK})\|^2, \tag{3.58}$$

where C may depend on ρ. Using the method in the proof of Lemma 3.3, for α and $M>0$, we can find symbols $h_\alpha(z,\tau)$ and $r_\alpha(z,\tau)$ such that

$$\sum_\alpha\tau^{1/2m_j}\gamma_\alpha\psi_\alpha=\sum_\alpha(h_\alpha\#(\tau^{(\alpha)})^{1/2m_j}\psi_\alpha+r_\alpha),$$

and

$$\|h_\alpha(z,D_t)\varphi\|\leq C\|\varphi\| \quad \text{and} \quad \|r_\alpha(z,D_t)\varphi\|\leq C(\tau^{(\alpha)})^{-M}\|\varphi\|,$$

for $\varphi\in C_0^\infty(\boldsymbol{C}_z^n\times\boldsymbol{R}_t)$. Hence

$$(3.59)\qquad \|\Delta_{1/2m_j}\varphi_{jK}\|^2 \le C(\|\varphi_{jK}\|^2 + \sum_\alpha \|(\tau^{(\alpha)})^{1/2m_j}\Psi_\alpha\varphi_{jK}\|^2).$$

Finally combination of (3.56)-(3.59) yields

$$\sum_j \sum_K{}' \|\Delta_{1/2m_j}\varphi_{jK}\|^2 \le C(\|\bar\partial_b\varphi\|^2 + \|\bar\partial_b^*\varphi\|^2 + \|\varphi\|^2),$$

so we get (1.7) with $\varepsilon = 1/2\tilde{m}_{n+1-q}$.

Next we shall prove that (0.1) does not hold if $\varepsilon > 1/2\tilde{m}_{n+1-q}$. Rearrange $\{1, \cdots, n\}$ to $\{i(1), \cdots, i(n)\}$ so that $m_{i(j)} = \tilde{m}_j$. Using f_j in (3.44) for $j = 1, \cdots, l$, we can verify that $|r(z)| \le C \sum_j |f_j(z)|^2$ on the complex hypersurface

$$(3.60)\qquad z_0 = \frac{1}{\dfrac{\partial\tilde r}{\partial w_0}(0)}\left(\sum_{j=1}^n \frac{\partial\tilde r}{\partial w_j}(0) f_j(z) + \frac{1}{2}\sum_{j,k}\frac{\partial\tilde r}{\partial w_j}(0)\frac{\partial\tilde r}{\partial w_k}(0) f_j(z) f_k(z)\right).$$

Hence on q-dimensional submanifold V^q defined by (3.60) and $z_{i(1)} = \cdots = z_{i(n-q)} = 0$, we have

$$|r(z)| \le C \sum_{j=n+1-q}^n |z_{i(j)}|^{2\tilde m_j} \le C|z|^{2\tilde m_{n+1-q}},$$

from (0.5) and (0.6). Since V^q passes through the origin, we can apply Theorem 1 in Catlin [1]. This completes the proof. □

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