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On diffusion in viscous fluids. Existence and uniqueness of solutions

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Abstract. We consider isothermic flow of a mixtuле of two viscous fluids with densities ρ_1 and ρ_2, and velocities u_1 and u_2, respectively. We assume that the total density $\rho=\rho_1+\rho_2$ is constant. Then the diffusion effect is associated only with changes of pressure and concentrations of the components of the mixture. The motion of the mixture can be described by a closed system of equations involving mean mass velocity vector $u=(\rho_1 u_1+\rho_2 u_2)/\rho$, pressure p, and concentration of one of its components c, $(c=\rho_1/\rho)$.

We assume that the mixture occupies a bounded domain Ω in R^3, and prove existence and uniqueness of solutions (u, p, c) of a boundary-value problem for the equations governing its stationary motion, in Sobolev spaces $W^{2,q}(\Omega)\times W^{1,q}(\Omega)\times W^{1,2}(\Omega), q>3$.

1. Introduction and Results

In this paper we consider a boundary-value problem for the equations describing stationary motion of a mixture of two viscous fluids, with the absence of heat transfer. The mixture has constant density and occupies a bounded domain Ω in R^3, with boundary $\partial\Omega$. The boundary-value problem reads.

$$-\nu\Delta u+(u\cdot\nabla)u+(1/\rho)\nabla p=f+cg \qquad \text{in } \Omega, \quad (1.1)$$

$$\operatorname{div} u=0 \qquad \text{in } \Omega, \quad (1.2)$$

$$-\operatorname{div}(D(c)\nabla c)+u\cdot\nabla c=(1/\rho)\operatorname{div}(K(c)\nabla p) \qquad \text{in } \Omega, \quad (1.3)$$

$$u=0 \qquad \text{on } \partial\Omega, \quad (1.4)$$

$$c=h \qquad \text{on } \partial\Omega, \quad (1.5)$$

The unknown functions $u=(u_1, u_2, u_3)$, p and c denote mean mass velocity

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vector, pressure, and concentration of one of the components (say, first component) of the mixture, respectively; $f=(f_1, f_2, f_3)$ is the external force per unit mass acting on the first component, and $g=(g_1, g_2, g_3)$ is the difference between the forces per unit mass acting on the second and the first component. If the only external force to act is that of gravitation then $g=0$. D and K denote diffusion and barodiffusion coefficients, $\nu=const>0$ is the viscosity coefficient.

By ∇, Δ and div we denote the usual gradient, Laplacian and divergence operators, so that Δu, $(u\cdot\nabla)u$ and ∇p are vectors with components Δu_i, $u_j(\partial/\partial x_j)u_i$ and $(\partial/\partial x_i)p$, respectively ($i=1,2,3$; repeated indices are summed); $\operatorname{div} u=(\partial/\partial x_i)u_i$, $u\cdot\nabla c=u_j(\partial/\partial x_j)c$, etc.

We assume that functions D and K are defined on the real line and satisfy

$$0<m\leq D(t)\leq M \qquad \text{for each} \quad t\in R, \tag{1.6}$$

$$K(t)=0 \quad \text{if} \quad t\leq 0 \quad \text{or} \quad t\geq 1, \tag{1.7}$$

where m, M are some positive constants, $m\leq M$. The first assumption is in agreement with the thermodynamical constraint $D>0$, and the second reflects the fact that in pure fluid the diffusion flux equals zero [8].

From the definition of c as the concentration of the first component of the mixture we conclude that

$$0\leq c\leq 1. \tag{1.8}$$

In fact, a maximum principle for equation (1.3), together with (1.7) and the assumption $0\leq h\leq 1$ on $\partial\Omega$, give (1.8) for each solution c of (1.3), (1.5).

We notice that if $c\equiv 0$ or 1 in Ω then, in view of (1.7), equations (1.1)-(1.3) reduce to the Navier-Stokes equations for one of the components of constant density ρ.

For the thorough discussion of the diffusion phenomenon and derivation of equations (1.1)-(1.3) we refer to [6], [8], [10], [15].

Below, for convenience of the reader, we sketch the derivation of equations (1.1)-(1.3). Let ρ_i and u_i denote the density and the velocity of the i-th component of the mixture, respectively $(i=1,2)$. For stationary motion the law of conservation of mass of the i-th component reads

$$\operatorname{div}(\rho_i u_i)=0, \qquad (i=1,2). \tag{1.9}$$

Denote by ρ and u the total density and the mean mass velocity of the mixture: $\rho=\rho_1+\rho_2$, $u=(\rho_1 u_1+\rho_2 u_2)/\rho$. Summing the equations in (1.9) we obtain $\operatorname{div}(\rho u)=0$. If $\rho=const$, the last equation reduces to (1.2). Now, denote by J the diffusion flux of the first component: $J=\rho_1(u_1-u)$. Proceeding formally we obtain from (1.9)

$$\rho u\cdot\nabla c=-\operatorname{div} J, \tag{1.10}$$

where $c=\rho_1/\rho$. By thermodynamical considerations [8]

$$-J=\alpha\nabla z, \tag{1.11}$$

$$\alpha\nabla z=\rho D(c)\nabla c+K(c)\nabla p, \tag{1.12}$$

where α is a positive constant and z denotes the chemical potential of the mixture. Combining (1.10)-(1.12) we get (1.3). We obtain equation (1.1), roughly speaking, by adding equations of momentum for the two components. The procedure, however, is rather complicated and we shall not reproduce it, refering the reader to the literature quoted above.

A motivation to study the above model of diffusion comes from the fact that it is the basis of many other, and much more involved, models of mixtures which are of considerable importance in the applied sciences (see [10] and the literature quoted there). Fluid suspensions (for example blood) [11], [12] belong to this type of mixtures. Their densities and temperatures are constant and the diffusion effect is due to changes of pressure and concentrations of the components.

Before stating the results we introduce some notations:

—Ω: a bounded open subset of R^3, locally situated on one side of its boundary $\partial\Omega$, a manifold of class C^2;

—L^q=usual $L^q(\Omega)$ space $(1\le q\le\infty)$, with norm

$$|\varphi|_q=\left(\int_\Omega|\varphi|^q\right)^{1/q}\qquad\text{for}\quad q<\infty,$$

and the obvious modification for $q=\infty$;

—H_0^1=closure of the set of smooth functions of compact supports in Ω, in the norm

$$\|\varphi\|_1=|\nabla\varphi|_2=\left(\sum_{i=1}^3\int_\Omega\left|\frac{\partial\varphi}{\partial x_i}\right|^2\right)^{1/2};$$

—$\tilde{V}$=the set of smooth and divergence free vector functions in R^3, compactly supported in Ω;

—V=closure of $\tilde{V}$ in the norm

$$\|u\|_1=|\nabla u|_2=\left(\sum_{i,j=1}^{3}\int_\Omega\left|\frac{\partial u_j}{\partial x_j}\right|^2\right)^{1/2};$$

—$W^{m,q}$=Sobolev space $W^{m,q}(\Omega)(m=1,2,3,\cdots;\ 1\leq q\leq\infty)$ of function from L^q, whose generalized derivatives up to the order m are in L^q, with usual norm denoted by $\|\cdot\|_{m,q}$;

—$W^{1-1/q,q}=W^{1-1/q,q}(\partial\Omega)$-space of traces on $\partial\Omega$ of functions from $W^{1,q}$, with usual norm denoted by $[\cdot]_{1-1/q}$;

—$C^0=C^0(\bar{\Omega})$-space of continuous functions on $\bar{\Omega}$, with horm $|\cdot|_\infty$.

For an open subset ω of Ω we define:

—$C^{0,\alpha}_\omega=C^{0,\alpha}(\bar{\omega})$-space of Hölder continuous functions on $\bar{\omega}$, with usual norm, denoted by $|\cdot|_{0,\alpha,\omega}$, $(0<\alpha\leq 1)$;

—$C^{0,\alpha}=C^{0,\alpha}(\bar{\Omega})$.

For basic properties of the above function spaces see [1], [5], [7], [14].

In what follows we assume that:

$$f,g\in L^q \quad\text{for an arbitrary but fixed}\quad q>3, \tag{A1}$$

$$K,D:R\longrightarrow R \quad\text{are continuous functions satisfying}\quad (1.6),(1.7), \tag{A2}$$

$$h\in W^{1/2,2},\quad 0\leq h\leq 1 \quad\text{on}\quad \partial\Omega. \tag{A3}$$

For convenience, we assume also that the density of the mixture equals one.

In this paper we prove the following theorems:

THEOREM 1.1. (*existence*). *Let the assumptions* (A1)-(A3) *hold, and let* m *in* (1.6) *be large enough with respect to the* L^q*-norms of* f *and* g. *Then there exists a triple of functions*

$$(u,p,c)\in W^{2,q}\times W^{1,q}\times W^{1,2},\quad (c \text{ is continuous in } \Omega) \tag{1.13}$$

such that

$$0\leq c\leq 1 \qquad \text{in } \Omega, \tag{1.14}$$

$$-\nu\Delta u+(u\cdot\nabla)u+\nabla p=f+cg \qquad \text{a.e. in } \Omega, \tag{1.15}$$

$$\operatorname{div} u=0 \qquad \text{in } \Omega, \tag{1.16}$$

and

$$\int_\Omega D(c)\nabla c\cdot\nabla\varphi+\int_\Omega u\cdot\nabla c\varphi=-\int_\Omega K(c)\nabla p\cdot\nabla\varphi \quad \text{for each}\quad \varphi\in H^1_0, \tag{1.17}$$

with

$$u=0 \qquad \text{on } \partial\Omega, \tag{1.18}$$

$$c=h \qquad \text{on } \partial\Omega, \tag{1.19}$$

THEOREM 1.2. (*existence*). *Let the assumptions* (A1)-(A3) *hold, and let K be Lipschitz continuous, that is*

$$|K(t)-K(s)|\leq L_K|t-s| \quad \text{for all} \quad t, s\in R, \tag{1.20}$$

for some positive constant L_K. Then there exists a triple of functions (u, p, c) satisfying conditions (1.13)-(1.19).

THEOREM 1.3. (*uniqueness*). *Let the assumptions* (A1)-(A3) *hold, the diffusion coefficient D be a positive constant, and K satisfy* (1.20). *Let $X=|f|+|g|_q$. Then there exists a continuous, increasing and positive function F of $X>0$, with $F(X)\to 0$ as $X\to 0$, such that the solution (u, p, c) of problem* (1.13)-(1.19) (*guaranteed by Theorem* 1.2) *is unique, provided $F(X)<D$.*

THEOREM 1.4. (*uniqueness*). *Let the assumptions of Theorem* 1.2 *hold. Moreover, let D be Lipshitz continuous* (*with Lipshitz constant L_D*) *and let $h\in W^{1-1/r,r}$ for some $r, 3<r\leq\min\{6, q\}$, Let $\tilde{X}=|f|_q+|g|_q+|h|_{1-1/r}$. Then there exists a continuous, increasing and positive function $\tilde{F}$ of $\tilde{X}>0$, with $\tilde{F}(\tilde{X})\to 0$ as $\tilde{X}\to 0$, such that the solution (u, p, c) of problem* (1.13)-(1.19) (*guaranteed by Theorem* 1.2) *is unique, provided $\tilde{F}(\tilde{X})<1$. Furthermore, $c\in W^{1,r}$.*

Observe that by (1.11), (1.12) each classical and smooth up to the boundary solution of problem (1.1)-(1.5) is also a weak solution of the problem in the sense of definition (1.13)-(1.19); conversely, each sufficiently smooth weak solution satisfying (1.13)-(1.19) is a classical solution of problem (1.1)-(1.5).

The plan of the remaining sections of the paper is as follows. In Section 2 we study linearized problem (1.15), (1.16), (1.18) in (u, p). In Section 3 we consider problem (1.17), (1.19) in c, with given u and p. In Section 4 we prove existence Theorems 1.1 and 1.2, by using Schauder's fixed point theorem and estimates obtained in Sections 2 and 3. Section 5 presents proofs of Theorems 1.3 and 1.4.

For convenience, we denote several universal numeric constants by the letter C without bothering to distinguish them with subscripts.

2. Linearized Navier-Stokes system

In this section we consider the boundary-value problem in (u, p):

$$-\nu\Delta u+(v\cdot\nabla)u+\nabla p=f+bg \qquad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} u=0 \qquad \text{in } \Omega, \tag{2.2}$$

$$u=0 \qquad \text{on } \partial\Omega, \tag{2.3}$$

where f, g, b and v are given functions.

We define, for $q\geq 1$, $f, g\in L^q$, and $b\in C^0$:

$$R=|f|_q+|b|_\infty|g|_q, \qquad R_0=R+R^2, \qquad R_1=R+R_0^2. \tag{2.4}$$

LEMMA 2.1. *Let q be an arbitrary real>3, $f, g\in L^q$, and $b\in C^0$. There exist positive reals r, r_0 and r_1 such that if $v\in A$,*

$$A=\{v\in V\cap C^0: \|v\|_1\leq rR, \quad |v|_q\leq r_0R_0, \quad |v|_\infty\leq r_1R_1\}$$

then problem (2.1)-(2.3) *has a unique solution* $(u, p)\in(W^{2,q}\cap A)\times W^{1,q}$, *with* $\int_\Omega p(x)dx=0$. *Moreover,*

$$\|u\|_{2,q}+\|p\|_{1,q}\leq C(R+R_1^2). \tag{2.5}$$

PROOF. Fix $f, g\in L^q$, $b\in C^0$ and $v\in V\cap C^0$. We shall show at first the existence of a unique pair $(u, p)\in V\times L^2$, $\int_\Omega p(x)dx=0$, satisfying (2.1) in the distribution sense. We multiply both sides of (2.1) by some function w in $\tilde{V}$ and integrate over Ω. After integration by parts we obtain

$$\nu\int_\Omega \nabla u\cdot\nabla w+\int_\Omega(v\cdot\nabla)uw=\int_\Omega(f+bg)w. \tag{2.6}$$

We notice that [7], [14]

$$\int_\Omega(v\cdot\nabla)uw=-\int_\Omega(v\cdot\nabla)wu, \quad \text{for all} \quad u, v, w \quad \text{in} \quad V. \tag{2.7}$$

Then it is easy to see that the left-hand side of (2.6) defines a continuous and coercive bilinear form in (u, w) on $V\times V$, and the right-hand side defines a continuous linear functional in w on V. Thus, by the Lax-Milgram lemma [5], there exists a unique $u\in V$ such that (2.6) holds for each $w\in V$. Now, $\tilde{f}=-\nu\Delta u+(v\cdot\nabla)u+\nabla p-f-bg$ belongs to H^{-1}, the dual space to H_0^1, and $\langle\tilde{f}, w\rangle=0$ for each w in $\tilde{V}$, which implies

[14] that $\tilde{f}=\nabla p$ in the distribution sense in Ω, for some $p\in L^2$. We normalize p so that $\int_\Omega p(x)dx=0$. Thus (u, p) is the unique pair in $V\times L^2$ satisfying (2.1) in the distribution sense.

Now we shall show that the obtained solution belongs to $W^{2,q}\times W^{1,q}$, and determine constants r, r_0 and r_1 in the definition of the set A in such a way that $u\in A$ if only $v\in A$, cf. [2], [9]. Our main tool is the well known estimate

$$\|u\|_{k+2,s}+\|p\|_{k+1,s}\leq C\|\tilde{f}\|_{k,s}, \tag{2.8}$$

($s>1, k$ is an integer ≥ -1, $\|\cdot\|_{-1,s}$ is the norm in the dual space to $W_0^{1,s'}(\Omega)$, $1/s+1/s'=1$) belonging to Cattabriga [4] (see also [14]) of the solution (u, p), $\int_\Omega p(x)dx=0$ of the Stokes problem

$$\begin{aligned} -\nu\Delta u+\nabla p&=\tilde{f} && \text{in } \Omega,\\ \operatorname{div} u&=0 && \text{in } \Omega,\\ u&=0 && \text{on } \partial\Omega. \end{aligned}$$

Setting $w=u$ in (2.6) and using (2.7) we obtain

$$\nu\|u\|_1^2=\int_\Omega (f+bg)u. \tag{2.9}$$

Using Hölder's inequality and Poincaré's inequality: $|u|_2\leq C\|u\|_1$, we estimate the right-hand side of (2.9) by $CR\|u\|_1$. Now, we define r as C/ν, so that

$$\|u\|_1\leq rR. \tag{2.10}$$

We shall assume that $\|v\|_1\leq rR$ (cf. the definition of the set A).

Now, by Hölder's inequality, the inequality $|v|_6\leq C\|v\|_1$, and by (2.10) we have

$$|(v\cdot\nabla)u|_{3/2}\leq |v|_6\|u\|_1\leq C\|v\|_1\|u\|_1\leq CR^2.$$

Cattabriga's estimate (2.8) applied to problem (2.1)-(2.3) with $\tilde{f}=f+bg-(v\cdot\nabla)u$, $k=0$ and $s=3/2$ gives

$$\|u\|_{2,3/2}+\|p\|_{1,3/2}\leq C(R^2+R)=CR_0.$$

Since $W^{2,3/2}\hookrightarrow W^{1,3}\hookrightarrow L^q$ for each $q>3$, there exists a positive constant r_0 such that $|u|_q\leq r_0R_0$. We shall assume that $|v|_q\leq r_0R_0$. Now, let $1/t=1/q+1/3$. By Hölder's inequality and the above imbeddings we have

$$|(v\cdot\nabla)u|_t \le C|v|_q|\nabla u|_3 \le CR_0^2.$$

We apply (2.8) again to obtain

$$\|u\|_{2,t}+\|p\|_{1,t}\le C(R_0^2+R)=CR_1.$$

Since $W^{2,t}\hookrightarrow W^{1,q}\hookrightarrow C^0$, there exists a constant r_1 such that $|u|_\infty \le r_1R_1$. We shall assume that $|v|_\infty \le r_1R_1$. In the end

$$|(v\cdot\nabla)u|_q \le C|v|_\infty|\nabla u|_q \le CR_1^2,$$

and using Cattabriga's estimate again we obtain inequality (2.5). Moreover, if $v\in A$ then $u\in A$. ■

Now we shall show that the map $(v, b)\to(u, p)$, where (u, p) is the unique solution of the boundary-value problem (2.1)-(2.3) from Lemma 2.1, is continuous in certain topologies. More precisely, let M_1 be a positive real and

$$B=\{b\in C^0 : |b|_\infty \le M_1\}.$$

In view of further applications, we consider the map $(v, b)\to(u, p)$ on the product $A\times B$, where A is the set from Lemma 2.1.

LEMMA 2.2. *The map*

$$\Phi : C^0\times C^0\supset A\times B\ni(v, b)\longrightarrow(u, p)\in C^0\times W^{1,2}$$

is continuous.

PROOF. Let (v, b), (v_n, b_n), $n=1, 2, 3, \dots$ be in $A\times B$, and let $\Phi(v_n, b_n)=(u_n, p_n)$, $\Phi(v, b)=(u, p)$. Then $(u-u_n, p-p_n)$ is the solution of the problem

$$-\nu\Delta(u-u_n)+\nabla(p-p_n)=\tilde{f}_n \qquad \text{in } \Omega, \quad (2.11)$$

$$\operatorname{div}(u-u_n)=0 \qquad \text{in } \Omega, \quad (2.12)$$

$$u-u_n=0 \qquad \text{on } \partial\Omega, \quad (2.13)$$

where $\tilde{f}=(b-b_n)g+((v_n-v)\cdot\nabla)u_n+(v\cdot\nabla)(u_n-u)$. Multiplying both sides of (2.11) by $u-u_n$ and integrating over Ω we obtain

$$\nu\|u-u_n\|_1^2=\int_\Omega(b-b_n)g(u-u_n)+\int_\Omega((v_n-v)\cdot\nabla)u_n(u-u_n).$$

Using Hölder's and Poincaré's inequalities we estimate the right-hand side by

$$C\{|b-b_n|_\infty|g|_2\|u-u_n\|_1+|v-v_n|_\infty\|u_n\|_1\|u-u_n\|_1\}.$$

Hence

$$\|u-u_n\|_1\leq(C/\nu)\{|b-b_n|_\infty|g|_2+|v-v_n|_\infty\|u_n\|_1\}$$

Let $(v_n, b_n)\to(v, b)$ in $C^0\times C^0$, as $n\to\infty$. Since $\|u_n\|_1\leq rR$ by (2.10), we conclude that $u_n\to u$ in H_0^1, as $n\to\infty$, and, in consequence, $\tilde{f}_n\to 0$ in L^2. We apply Cattabriga's estimate (2.8) to problem (2.11)-(2.13), obtaining $u_n\to u$ in $W^{2,2}$ (hence uniformly), and $p_n\to p$ in $W^{1,2}$. This proves the continuity of Φ. ∎

We shall use Lemmas 2.1 and 2.2 in Section 4.

3. Equation of Diffusion

In this section we consider the following problem in c:

$$c\in W^{1,2}, \tag{3.1}$$

$$\int_\Omega D(\tilde{b})\nabla c\cdot\nabla\varphi+\int_\Omega u\cdot\nabla c\varphi=-\int_\Omega K(c)\nabla p\cdot\nabla\varphi \quad \text{for each} \quad \varphi\in H_0^1, \tag{3.2}$$

$$c-\xi\in H_0^1, \tag{3.3}$$

where

$$\xi\in W^{1,2}, \quad \text{with} \quad \xi=h \quad \text{on} \quad \partial\Omega, \tag{3.4}$$

$$u\in V\cap C^0;\ p\in W^{1,q} \quad \text{for some} \quad q>3;\ \tilde{b}\in C^0, \tag{3.5}$$

and K and D are as in (A2).

Our aim is to prove existence of solutions of problem (3.1)-(3.3), estimate them in C^0 and $C^{0,\alpha}$ norms (under suitable assumptions about h), and prove the unique solvability of the problem, provided K is Lipschitz continuous.

We begin with the linearized problem

$$c\in W^{1,2}, \tag{3.6}$$

$$\int_\Omega D(\tilde{b})\nabla c\cdot\nabla\varphi+\int_\Omega u\cdot\nabla c\varphi=-\int_\Omega K(b)\nabla p\cdot\nabla\varphi \quad \text{for each} \quad \varphi\in H_0^1, \tag{3.7}$$

$$c-\xi\in H_0^1. \tag{3.8}$$

LEMMA 3.1. *Let* (3.4), (3.5) *and* (A2) *hold, and* $b\in C^0$. *Then there* *exists a unique solution of problem* (3.6)-(3.8). *Moreover*

$$\|c\|_1 \leq C\{[h]_{1/2} + |K(b)\nabla p|_2\}, \tag{3.9}$$

where C *depends on* $|u|_\infty$.

Proof. Since $u \in V$, we have

$$\int_\Omega u\cdot\nabla c\varphi = -\int_\Omega u\cdot\nabla\varphi c \quad \text{for all} \quad c, \varphi \in W^{1,2} \tag{3.10}$$

Thus, the left-hand side of equation (3.7) defines a continuous and coercive bilinear form on $H_0^1 \times H_0^1$. Introducing the new variable $\tilde{c} = c - \xi$ we reduce the problem to a homogeneous one, and then use the Lax-Milgram lemma. We omit the elementary details refering the reader to Chapter 8 in [5]. ∎

Lemma 3.2. *Let* c *be the solution of problem* (3.6)-(3.8). *Then* $c \in {}^{0,\beta}_{\omega}$ *for each subdomain* ω *of* Ω, *separated from* $\partial\Omega$, *and*

$$|c|_{0,\beta,\omega} \leq C_0\{|c|_2 + (1/m)|K(b)\nabla p|_q\}. \tag{3.11}$$

The constant C_0 *depends on* $\operatorname{dist}(\omega, \partial\Omega)$, $|u|_\infty$, q, *and* β *depends on* $\operatorname{dist}(\omega, \partial\Omega)$, q.

Moreover, if $h \in C^{0,1}(\partial\Omega)$ *then* $c \in C^{0,\alpha}$ *for some* α, $0<\alpha<1$, *and*

$$|c|_\infty \leq |h|_{\infty,\partial\Omega} + (C/m)|K(b)\nabla p|_q, \tag{3.12}$$

$$|c|_{0,\alpha,\Omega} \leq C\{|h|_{0,1,\partial\Omega} + (1/m)|K(b)\nabla p|_q\}, \tag{3.13}$$

where C *depends on* $|u|_\infty$, q.

Proof. Lemma 3.2 follows directly from general results concerning estimates of weak solutions of elliptic problems; see Theorems 8.16, 8.24, 8.29 in [5], for example. ∎

Lemma 3.3. *Let* (3.4), (3.5) *and* (A2) *hold, and let* $h \in C^{0,1}(\partial\Omega)$. *Then there exists a solution of problem* (3.1)-(3.3). *Moreover,* $c \in C^{0,\alpha}$ *for some* α, $0<\alpha<1$, *and*

$$|c|_{0,\alpha,\Omega} \leq C\{|h|_{0,1,\partial\Omega} + (1/m)|K|_\infty|\nabla p|_q\}. \tag{3.14}$$

Proof. To prove existence of the solution we apply Schauder's fixed point theorem [5] to the map $\Phi_1 : C^0 \supset B_1 \ni b \to c \in C^0$, where

$$B_1 = \{b \in C^0 : |b|_\infty \leq |h|_{\infty,\partial\Omega} + (C/m)|K|_\infty|\nabla p|_q\},$$

and c is the unique solution of problem (3.6)-(3.8) from Lemma 3.1.

From (3.12) it follows that $\Phi_1(B_1)\subset B_1$, and inequality (3.13) implies compactness of $\Phi_1(B_1)$ in C^0. We shall prove that the map Φ_1 is continuous. Let $b_n\to b$ in C^0, $b, b_n\in B_1$ for $n=1, 2, 3, \dots$, and $c_n=\Phi_1(b_n)$, $c=\Phi_1(b)$. From (3.9) and (3.13) it follows the existence of a subsequence (c_μ) such that

$$c_\mu \longrightarrow c_\mu \quad \text{uniformly on} \quad \bar{\Omega}, \quad \text{and weakly in } W^{1,2}.$$

Passing to the limit in

$$\int_\Omega D(\tilde{b})\nabla c_\mu\cdot\nabla\varphi+\int_\Omega u\cdot\nabla c_\mu\varphi=-\int_\Omega K(b_\mu)\nabla p\cdot\nabla\varphi \quad \text{for each} \quad \varphi\in H_0^1,$$

we conclude that $\tilde{c}=c$, and that the whole sequence (c_n) converges uniformly to c. Thus, Φ_1 is continuous in the uniform topology. Other conclusions of the lemma are obvious. ∎

LEMMA 3.4. *Let* (3.4), (3.5) *and* (A2) *hold. Let us assume that* K *is Lipschitz continuous, as in* (1.20), *and that* $h\in C^{0,1}(\partial\Omega)$, *with* $0\leq h\leq 1$. *Then the solution of the problem* (3.1)-(3.3), *given by* Lemma 3.3, *is unique. Moreover*

$$0\leq c\leq 1 \quad in \quad \Omega. \tag{3.15}$$

PROOF. To prove that the solution is unique we assume, on the contrary, that there exist two different solutions c_1 and c_2. Let $\tau=c_1-c_2$. Using (3.10) we obtain

$$\int_\Omega D(\tilde{b})\nabla\tau\cdot\nabla\varphi=\int_\Omega F\cdot\nabla\varphi \quad \text{for each} \quad \varphi\in H_0^1, \tag{3.16}$$

where $F=\tau u+(K(c_2)-K(c_1))\nabla p$. By our assumptions

$$|F|\leq\{|u|+L_K|\nabla p|\}|\tau|\equiv|F_1|\cdot|\tau|, \tag{3.17}$$

with $F_1\in L^2$. From (3.16) and (3.17)

$$\int_\Omega D(\tilde{b})\nabla\tau\cdot\nabla\varphi\leq\int_\Omega |F_1|\cdot|\tau|\cdot|\nabla\varphi| \quad \text{for each} \quad \varphi\in H_0^1. \tag{3.18}$$

Let $\varphi^+=\max\{\varphi, 0\}$. For $\delta>0$ we set (cf. [3]) $\varphi=(\tau-\delta)^+/\tau$ in (3.18) ($\varphi\in H_0^1$, and $\nabla\varphi=\delta\cdot\nabla\tau/\tau^2$ on the set $\Omega_\delta=\{x\in\Omega : \tau(x)>\delta\}$), and obtain

$$\int_{\Omega_\delta}\frac{D(\tilde{b})|\nabla\tau|^2}{\tau_2}\leq\int_{\Omega_\delta}\frac{|F_1|\cdot|\nabla\tau|}{|\tau|}$$

Since $D(c_1)\geq m>0$, by Schwarz' inequality we obtain

$$\int_{\Omega_\delta}\frac{|\nabla\tau|^2}{\tau^2}\leq L, \qquad L=\frac{2}{m^2}\int_\Omega|F_1|^2,$$

hence

$$\int_\Omega\left|\nabla\ln\left(1+\frac{(\tau-\delta)^+}{\delta}\right)\right|^2\leq L.$$

Now, by Poincaré's inequality

$$\int_\Omega\left|\ln\left(1+\frac{(\tau-\delta)^+}{\delta}\right)\right|^2\leq CL,$$

independently of $\delta>0$. Let $\delta\to 0$ and we conclude that $\tau\leq 0$ *a.e.* in Ω. Thus $c_1\leq c_2$ *a.e.* in Ω. Similarly we show that $c_2\leq c_1$ *a.e.* in Ω, so that $c_1=c_2$ *a.e.* in Ω. We have come to the contradiction with our assumption $c_1\neq c_2$. This proves the uniqueness.

To prove (3.15), set $\varphi=c^-=\min\{c,0\}$ in (3.2) ($\varphi\in H_0^1$, as $c\geq 0$ on $\partial\Omega$). We obtain

$$\int_\Omega D(\tilde{b})|\nabla c^-|^2=0, \tag{3.19}$$

as $\int_\Omega u\cdot\nabla c^- c^-=0$ by (3.10) and $K(c)\nabla c^-=0$ by (1.7). As $c^-=0$ on $\partial\Omega$, and $D(\tilde{b})$ is positive, (3.19) implies that $c^-=0$ in Ω. Hence $c\geq 0$ in Ω. Similarly, taking $\varphi=(c-1)^+$ we obtain $c\leq 1$ in Ω. This proves (3.15). ■

Now we are in a position to prove the existence of a solution of the main problem in the paper.

4. Existence Theorems

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let the assumptions of Theorem 1.1 hold. For convenience, we shall keep the temporary assumption

$$h\in C^{0,1}(\partial\Omega), \tag{4.1}$$

and release from it at the end of the section. We use Schauder's fixed point theorem. Let $A\subset C^0$ be the set defined in Section 2 (see Lemma 2.1). Let $B=\{b\in C^0: |b|_\infty\leq M_1\}$ for an arbitrary number M_1 such that $|h|_{\infty,\partial\Omega}<M_1\}$. We consider the map

$$\Psi : C_0\times C^0\supset A\times B\ni (v,b)\longrightarrow (u,c)\in C^0\times C^0,$$

constructed as follows. For $(v,b)\in A\times B$, $\Psi(v,b)=(u,c)$, where (u,p,c) is the unique solution of the problem

$$\begin{aligned} &-\nu\Delta u+(v\cdot\nabla)u+\nabla p=f+bg && \text{in } \Omega,\\ &\operatorname{div} u=0 && \text{in } \Omega,\\ &u=0 && \text{on } \partial\Omega, \end{aligned}$$

$$\int_\Omega D(b)\nabla c\cdot\nabla\varphi+\int_\Omega v\cdot\nabla c\varphi=-\int_\Omega K(b)\nabla p\cdot\nabla\varphi \quad \text{for each} \quad \varphi\in H_0^1,$$
$$c-\xi\in H_0^1 \quad \xi=h \quad \text{on} \quad \partial\Omega.$$

Lemmas 2.1, 3.1 and 3.2 guarantee that the map Ψ is well defined. From Lemma 2.1 it follows that $u\in A$. By Lemma 3.2 $c\in B$, provided m is large enough (see inequality (3.12)). Hence, with m large enough, $\Psi(A\times B)\subset A\times B$. In view of (2.5) and (3.13), $\Psi(A\times B)$ is a compact subset of $C^0\times C^0$. We shall show that the map Ψ is continuous in the uniform topologies. Let $(v,b), (v_n,b_n), n=1,2,3,\dots$ be in $A\times B$, $(v_n,b_n)\to(v,b)$ in $C^0\times C^0$, as $n\to\infty$, $\Psi(v_n,b_n)=(u_n,c_n)$, $\Psi(v,b)=(u,c)$. We have to prove that $(u_n,c_n)\to(u,c)$ in $C^0\times C^0$, as $n\to\infty$. By Lemma 2.2 we have

$$u_n\longrightarrow u \quad \text{in} \quad C^0, \qquad \nabla p_n\longrightarrow\nabla p \quad \text{in} \quad W^{1,2}. \tag{4.2}$$

To show that $c_n\to c$ in C^0, consider the identities

$$\int_\Omega D(b_n)\nabla c_n\cdot\nabla\varphi+\int_\Omega v_n\cdot\nabla c_n\varphi=-\int_\Omega K(b_n)\nabla p_n\cdot\nabla\varphi \quad \text{for each} \quad \varphi\in H_0^1, \qquad (4.3)_n$$

$n=1,2,3,\dots.$ As (c_n) is a bounded sequence in $W^{1,2}$ and in $C^{0,\alpha}$, for some $\alpha, 0<\alpha<1$, there exists a subsequence (c_μ) such that

$$c_\mu\longrightarrow\tilde c \quad \text{in} \quad C^0, \quad \text{and weakly in} \quad W^{1,2}. \tag{4.4}$$

By our assumptions, (4.2) and (4.4), we can pass to the limit in $(4.3)_\mu$, obtaining

$$\int_\Omega D(b)\nabla\tilde c\cdot\nabla\varphi+\int_\Omega v\cdot\nabla\tilde c\varphi=-\int_\Omega K(b)\nabla p\cdot\nabla\varphi \quad \text{for each} \quad \varphi\in H_0^1. \tag{4.5}$$

Now, as (4.5) is uniquely solvable in $\tilde c$, we conclude that $\tilde c=c$, and $c_n\to c$ in C_0. This completes the proof of Theorem 1.1, under the additional assumption (4.1). ■

PROOF OF THEOREM 1.2. The proof is very similar to that of Theorem 1.1. Let the assumptions of Theorem 1.2 hold, together with (4.1). We consider the map

$$\Psi_1 : C^0 \times C^0 \supset A \times B \ni (v, b) \longrightarrow (u, c) \in C^0 \times C^0,$$

($B = \{b \in C_0 : |b|_\infty \leq 1\}$) defined as follows. For $(v, b) \in A \times B$ let $\Psi_1(v, b) = (u, c)$, where (u, p, c) is the unique solution of the problem

$$\begin{aligned} -\nu\Delta u + (v \cdot \nabla)u + \nabla p &= f + bg && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$$\int_\Omega D(b)\nabla c \cdot \nabla\varphi + \int_\Omega u \cdot \nabla c \varphi = -\int_\Omega K(c)\nabla p \cdot \nabla\varphi \quad \text{for each} \quad \varphi \in H_0^1, \tag{4.6}$$

$$c - \xi \in H_0^1, \qquad \xi = h \quad \text{on} \quad \partial\Omega. \tag{4.7}$$

In view of Lemmas 2.1 and 3.4, the map Ψ_1 is well defined, and $\Psi_1(A \times B) \subset A \times B$. By (2.5) and (3.14), $\Psi_1(A \times B)$ is compact in $C^0 \times C^0$. The continuity of Ψ_1 in the uniform topologies is obvious due to the unique solvability of problem (4.6), (4.7) in c (Lemma 3.4). We omit the details. ∎

To complete the proofs of Theorems 1.1 and 1.2 we have to release from the additional assumption (4.1). Let $h \in W^{1/2,2}(\partial\Omega)$. We take a bounded sequence $(\xi_n) \subset W^{1,2}$ such that $\xi_n = h_n$ on $\partial\Omega$, with $h_n \in C^{0,1}$, $h_n \to h$ in $W^{1/2,2}(\partial\Omega)$, as $n \to \infty$. Let (u_n, p_n, c_n), $n = 1, 2, 3, \ldots$ be solutions as in Theorems 1.1 and 1.2, corresponding to boundary data $c_n = h_n$ on $\partial\Omega$. In view of estimates (2.5), (3.11) we can select a subsequence (u_μ, p_μ, c_μ) such that for some $(u, p, c) \in (W^{2,q} \cap V) \times W^{1,q} \times W^{1,2}$

$u_\mu \longrightarrow u$ uniformly on $\bar{\Omega}$, and in $W^{1,2}$,

$p_\mu \longrightarrow p$ weakly in $W^{1,2}$,

$c_\mu \longrightarrow c$ uniformly on compacts in Ω, pointwise in Ω, and weakly in $W^{1,2}$.

Now, by standard argument we show that (u, p, c) is a solution of problem (1.14)-(1.19). This completes the proofs of Theorems 1.1 and 1.2.

5. Uniqueness

In this section we prove Theorems 1.3 and 1.4.

PROOF OF THEOREM 1.3. Assume that (u_1, p_1, c_1) and (u_2, p_2, c_2) are two different solutions of problem (1.15)-(1.19), with $D=const>0$. Then the difference $(v_1-u_2,\ p_1-p_2,\ c_1-c_2)$ satisfies the following integral identities

$$\nu\int_\Omega \nabla(u_1-u_2)\cdot\nabla v+\int_\Omega (u_1\cdot\nabla)(u_1-u_2)v=\int_\Omega (c_1-c_2)gv+\int_\Omega ((u_2-u_1)\cdot\nabla)u_2 v, \quad (5.1)$$

for each $v\in V$, and

$$D\int_\Omega \nabla(c_1-c_2)\cdot\nabla\varphi=\int_\Omega G\cdot\nabla\varphi, \quad (5.2)$$

for each $\varphi\in H_0^1$, where

$$G=c_2(u_1-u_2)+(c_1-c_2)u_2+(K(c_2)-K(c_1))\nabla p_2+K(c_1)(\nabla p_2-\nabla p_1). \quad (5.3)$$

Since $c_1-c_2\in H_0^1$, from (5.2) and (3.12) we have

$$|c_1-c_2|_\infty\leq(C/D)|G|_r, \quad r>3 \text{ arbitrary}. \quad (5.4)$$

Our aim now is to estimate $|G|_r$ by $F_1(X)\cdot|c_1-c_2|_\infty$, where $X=|f|_q+|g|_q$, and F_1 has the same properties as the function F in Theorem 1.3. This, together with (5.4) would lead to a contradiction, provided $CF_1(X)<D$. Let $3<r\leq q$, where q is as in (A1). From (5.3) and our assumptions

$$|G|_r\leq|u_1-u_2|_r+|u_2|_r|c_1-c_2|_\infty+|\nabla p_2|_r L_K|c_1-c_2|_\infty+|K|_\infty|\nabla p_2-\nabla p_1|_r. \quad (5.5)$$

Now we shall estimate the first term on the right-hand side of (5.5). Let $r\leq 6$. Then $|u_1-u_2|_r\leq C|\nabla(u_1-u_2)|_2$. From (5.1) with $v=u_1-u_2$ we obtain

$$\nu\int_\Omega|\nabla(u_1-u_2)|^2=\int_\Omega (c_2-c_1)g(u_1-u_2)+\int_\Omega ((u_1-u_2)\cdot\nabla)u_2(u_1-u_2). \quad (5.6)$$

From (2.10) and (2.4) with $b=c$

$$\|u_2\|_1\leq CX. \quad (5.7)$$

From (5.6) and (5.7), by Hölder's and Poincaré's inequalities, we obtain

$$\nu|\nabla(u_1-u_2)|_2^2\leq CX|c_1-c_2|_\infty|\nabla(u_1-u_2)|_2+CX|\nabla(u_1-u_2)|_2^2\,.$$

If $CX \leq \nu/2$ we have

$$|\nabla(u_1-u_2)|_2 \leq \frac{2}{\nu} CX|c_2-c_1|_\infty,$$

hence

$$|u_1-u_2|_r \leq CX|c_2-c_1|_\infty, \qquad 3<r\leq \min\{6, q\}. \tag{5.8}$$

We assume that $CX \leq \nu/2$.

Now, we shall estimate the last term on the right-hand side of (5.5). We have

$$\begin{aligned} &-\nu\Delta(u_1-u_2)+\nabla(p_1-p_2)=S && \text{in } \Omega, \\ &\operatorname{div}(u_1-u_2)=0 && \text{in } \Omega, \\ &u_1-u_2=0 && \text{on } \partial\Omega, \end{aligned}$$

where $S=(c_1-c_2)g+(u_1\cdot\nabla)(u_2-u_1)+((u_2-u_1)\cdot\nabla)u_2$. From Cattabriga's estimate (2.8) we obtain

$$|\nabla(p_1-p_2)|_r \leq X|c_1-c_2|_\infty+|u_1|_\infty|\nabla(u_1-u_2)|_r+|((u_2-u_1)\cdot\nabla)u_2|_r.$$

We have $|u_1|_\infty$, $|\nabla u_2|_\infty \leq F_1(X)$, by (2.5) and (2.4), so that by (5.8)

$$|\nabla(p_1-p_2)|_r \leq X|c_1-c_2|_\infty+F_1(X)|\nabla(u_1-u_2)|_r+F_1(X)|c_1-c_2|_\infty.$$

To estimate $|\nabla(u_1-u_2)|_r$, we use Cattabriga's estimate (2.8) with $k=-1$ We obtain

$$|\nabla(u_1-u_2)|_r \leq C\|S\|_{-1,r},$$

Since $S \in L^q$, $q \geq r > 3$, we have, with $s=r/(r-1)$

$$\begin{aligned} \|S\|_{-1,r} &= \sup\left\{\left|\int_\Omega S\varphi\right| : \|\varphi\|_{1,s} \leq 1\right\} \\ &\leq \sup\{|g|_r|c_1-c_2|_\infty|\varphi|_s+|u_1|_\infty|u_1-u_2|_r\|\varphi\|_{1,s} \\ &\quad +|u_1-u_2|_r|\nabla u_2|_\infty|\varphi|_s : \|\varphi\|_{1,s} \leq 1\} \\ &\leq F_1(X)|c_1-c_2|_\infty. \end{aligned}$$

Combining the above inequalities we conclude that

$$|\nabla(p_1-p_2)|_r \leq F_1(X)|c_1-c_2|_\infty,$$

which gives, together with (5.5) and (5.8)

$$|G|_r \leq F_1(X)|c_1-c_2|_\infty,$$

and by (5.4)

$$|c_1-c_2|_\infty \leq \frac{C}{D} F_1(X)|c_1-c_2|_\infty.$$

In conclusion, if $(C/D)F_1(X) \equiv F(X)/D < 1$, the considered problem is uniquely solvable. The proof of Theorem 1.3 is complete. ■

PROOF OF THEOREM 1.4. Let the assumptions of Theorem 1.4 hold. Let (u_1, p_1, c_1) and (u_2, p_2, c_2) be two different solutions of problem (1.15)-(1.19). Then the difference u_1-u_2 satisfies identity (5.1) for each $v \in V$, and c_1-c_2 satisfies

$$\int_\Omega D(c_1)\nabla(c_1-c_2)\cdot\nabla\varphi = \int_\Omega \tilde{G}\cdot\nabla\varphi$$

for each $\varphi \in H_0^1$, where $\tilde{G} = G + (D(c_2)-D(c_1))\nabla c_2$ with G as in (5.3). Now, let us assume that for some $r, 3 < r \leq \min\{6, q\}$, we have

$$\nabla c_2 \in L^r \quad \text{with} \quad |\nabla c_2|_r \leq C\cdot\tilde{X}. \tag{5.9}$$

Then

$$|\tilde{G}|_r \leq |G|_r + CL_D|c_1-c_2|\tilde{X},$$

and in the end

$$|c_1-c_2|_\infty \leq \tilde{F}(\tilde{X})|c_1-c_2|_\infty$$

just as in the proof of Theorem 1.3. Hence, to complete the proof of Theorem 1.4 we have to justify (5.9).

LEMMA 4.1. *Let the assumptions of Theorem* 1.4 *hold, and let* (u, p, c) *be a solution of problem* (1.13)-(1.19). *Then* $c \in W^{1,r}$ *and*

$$\|c\|_{1,r} \leq C\{|cu-K(c)\nabla p|_r + [h]_{1-1/r} + \|c\|_{1,2}\}. \tag{5.10}$$

PROOF. As $h \in W^{1-1/r,r}$, $r > 3$, three exists $\xi \in W^{1,r}$ such that $\xi = h$ on $\partial\Omega$. Let $\tilde{c} = c - \xi$. Then $\tilde{c} \in H_0^1$, and by (1.17) end (3.10)

$$\int_\Omega D(c)\nabla\tilde{c}\cdot\nabla\varphi = \int_\Omega G_1\cdot\nabla\varphi \quad \text{for each} \quad \varphi \in H_0^1, \tag{5.11}$$

where $G_1 = cu - K(c)\nabla p - D(c)\nabla\xi$ is in L^r. Moreover, since h is Hölder continuous on $\partial\Omega$, from Theorem 8.29 in [5] follows that $c \in C^{0,\alpha}$ for some $\alpha > 0$, and thus $D \circ c \in C^0$. Now, in view of Theorem 11.1 in [13], problem (5.11) has a unique solution $\tilde{c} \in W_0^{1,r}$, with

$$\|\tilde{c}\|_{1,r} \leq C\{|G_1|_r + |\tilde{c}|_r\}. \tag{5.12}$$

Since problem (5.11) is also uniquely solvable in H_0^1 we conclude that $c \in W^{1,r}$. Now, as $r \leq 6$ we have $|\tilde{c}|_r \leq C\|\tilde{c}\|_1$, and this and (5.12) yield (5.10).

As (5.10) implies (5.9), the proof of Theorem 1.4 is complete. ■

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