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Representation spaces of genus zero Fuchsian groups and Hamiltonian torus action

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Abstract. We construct explicitly a Hamiltonian torus action on the SU_2-representation space of a cocompact genus zero Fuchsian group and study the topology of this space.

§ 1. Introduction

A symplectic structure on the representation space of the fundamental group of a compact orientable 2-dimensional manifold is constructed and applied by Goldman ([G1], [G2] see also [A-B]). On the other hand, the importance of a Hamiltonian group action (i.e. a symplectic group action with a moment map) on a symplectic manifold and Morse theory of its moment map is recognized recently by many mathematicians.

In this paper we construct a Hamiltonian torus action on the representation space of a cocompact genus zero Fuchsian group and study the structure of this space.

Let $\Gamma=\Gamma(0:a_1,\cdots,a_n)\subset PSL_2\boldsymbol{R}$ be a cocompact genus zero Fuchsian group with ramification indices $a_1,\cdots,a_n$. Throughout this paper we assume that $a_1,\cdots,a_n$ are pairwise coprime. The orbit space $\boldsymbol{H}/\Gamma$ of the linear fractional action on the upper half plane is a 2-dimensional orbifold with underlying topological space S^2, and Γ is viewed as its orbifold fundamental group. The object we consider is the space of SU_2-representations of Γ

$$X(\Gamma, SU_2)=Hom(\Gamma, SU_2)/SU_2.$$

This space can be interpreted as the moduli space of flat SU_2-V-bundles over 2-orbifold $\boldsymbol{H}/\Gamma$, which turns out to admit a symplectic structure.

Let $\alpha=x_ix_j$ denote a simple loop on the orbifold $M=\boldsymbol{H}/\Gamma$ around presicely two ramification points of ramification indices a_i and a_j. To

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this simple loop α we associate a function f_α and an S^1-action $A_\alpha : S^1 \times X(\Gamma, SU_2) \to X(\Gamma, SU_2)$ on the space $X(\Gamma, SU_2)$. The function f_α is defined by

$$f_\alpha(\rho) := \cos^{-1}\left(\frac{1}{2} \operatorname{tr} \rho(\alpha)\right)$$

for $\rho \in Hom(\Gamma, G)$. For the presice difinition of S^1-action A_α, see §3. In case of the representation space of the fundamental group of a compact orientable 2-dimensional manifold, similar constructions are given by Goldman [G2].

Our main result is the following.

THEOREM. *The representation space $X(\Gamma, SU_2)$ has a Hamiltonian torus action. More explicitly, for $\alpha = x_1x_2, x_3x_4, \cdots, x_{2[n/2]-1}x_{2[n/2]}$ the function f_α is the moment map of the S^1-action A_α on $X(\Gamma, SU_2)$ and S^1-actions A_α's commute so that they define the Hamiltonian torus action.*

Our result is closely related to the study of Kirk-Klassen [K-K] on on the SU_2-representation spaces of the Seifert fibered homology 3-sphere $\sum(a_1, a_2, \cdots, a_n)$, as we will see in §3 and §4. Note that although $\pi_1(\Sigma(a_1, a_2, \cdots, a_n))$ is a central extension of Γ, SU_2-representation spaces of this two groups are essentially the same ([B-O], [Bo]).

We comment upon another approach to the study of $X(\Gamma, SU_2)$. By the theorem of Mehta-Seshadri [M-S], the space $X(\Gamma, SU_2)$ can be identified with the moduli space of rank 2-stable parabolic bundles (with certain conditions on weights) over $\boldsymbol{P}^1$. The latter space is studied by Furuta-Steer [Fur-St], Bauer-Okonek [B-O], Bauer [Bau], Boden [Bo], etc. For example the cohomology groups are completely determined in [Fur-St], [Bau] and [B].

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§2. Symplectic structure of $X(\Gamma, SU_2)$

In this section we construct a symplectic structure on $X(\Gamma, SU_2)$. Our construction is a generalization of the one given by [A-B] [G1] on $X(\pi_1(\Sigma_g), SU_2)$, where Σ_g denotes a compact orientable surface of genus $g(\geq 2)$. First, we recall some fundamental results of $X(\Gamma, SU_2)$ from [F-S] and [K-K].

Let $\Gamma=\Gamma(0:a_1, \cdots, a_n)$ be as in § 1. This group Γ has a presentation

$$\Gamma \cong \langle x_1, \cdots, x_n \mid x_i^{a_i}=1, x_1 x_2 \cdots x_n=1 \rangle. \tag{2.1}$$

When we consider Γ as the orbifold fundamental group of $M=H/\Gamma$, the generator x_i corresponds to the circle around the ramification point p_i. Set

$$\begin{aligned} G &= SU_2, \\ R(\Gamma, G) &= Hom(\Gamma, G), \\ R^*(\Gamma, G) &= Hom^*(\Gamma, G) \\ &= \{\text{irreducible representations}\}. \end{aligned}$$

Since $a_1, \cdots, a_n$ are pairwise coprime,

$$R^*(\Gamma, G)=R(\Gamma, G) \setminus \{\text{the trivial representation}\}.$$

Let $X(\Gamma, G)$ and $X^*(\Gamma, G)$ denote the quotients by the adjoint action of G on $R(\Gamma, G)$ and $R^*(\Gamma, G)$ respectively.

LEMMA 2.2 ([K-K], [F-S]). *For $\rho \in R(\Gamma, G)$, we set*

$$m := \{i \mid \rho(x_i) \neq \pm 1 \in SU_2\}.$$

Then the component of $R(\Gamma, G)$ (resp. $X(\Gamma, G)$) which contains ρ (resp. $[\rho]$) is $2m-3$ (resp. $2m-6$) dimensional closed smooth manifold.

For $\rho \in R(\Gamma, G)$ and $i=1, \cdots, n$, we have

$$\rho(x_1) \simeq \begin{pmatrix} e^{\sqrt{-1}\theta_i} & 0 \\ 0 & e^{-\sqrt{-1}\theta_i} \end{pmatrix} \tag{2.3}$$

where $\theta_i=\frac{2\pi l_i}{a_i}$ for some $l_i=0, \cdots, \left[\frac{a_i}{2}\right]$. According to [K-K], the *rotation number* $(l_1, \cdots, l_n)$ characterizes the component of $R(\Gamma, G)$ (resp. $X(\Gamma, G)$) which contains ρ (resp. $[\rho]$).
Let $R(l_1, \cdots, l_n)$ (resp. $X(l_1, \cdots, l_n)$) denote the corresponding component. Then

$$\begin{aligned} R(\Gamma, G) &= \amalg_{(l_1, \cdots, l_n)} R(l_1, \cdots, l_n) \\ X(\Gamma, G) &= \amalg_{(l_1, \cdots, l_n)} X(l_1, \cdots, l_n). \end{aligned}$$

We use additional notations. Let $R_n(\Gamma, G)$ (resp. $X_n(\Gamma, G)$) denote the union of top dimentional components of $R(\Gamma, G)$ (resp. $X(\Gamma, G)$). Since the lower dimentional components of $R(\Gamma, G)$ (resp. $X(\Gamma, G)$) can be viewed as contained in the representation spaces of $\Gamma'=\Gamma(0: a_1', \cdots, a_{n-1}')$ for some $a_1', \cdots, a_{n-1}'$, our main objects are components of $R_n(\Gamma, G)$ and $X_n(\Gamma, G)$.

Now consider the symplectic structure on $X(\Gamma, G)$. For this purpose it is natural to identify $X(\Gamma, G)$ with the moduli space of flat SU_2-V-bundles over the 2-orbifold $M=\boldsymbol{H}/\Gamma$. An element $\rho \in R(\Gamma, G)$ defines the principal G-bundle $P(\rho)=\boldsymbol{H}\times_{\Gamma} G$ with the flat connection $A(\rho)$ which comes from the trivial flat connection on $\boldsymbol{H}\times G$. For a principal G-V-bundle $P\to M$, set

$$R(\Gamma, G)_P=\{\rho \in R(\Gamma, G) \mid P(\rho)\cong P\},$$
$$X(\Gamma, G)_P=R(\Gamma, G)_P/G.$$

Let $\mathcal{F}(P)$ denote the space of all flat V-connections on P, and $\mathcal{G}(P)$ denote the group of all gauge transformations on P. Then by the holonomy representations, we have a bijection (diffeomorphism)

$$\text{(2.4)} \qquad hol: \mathcal{F}(P)/\mathcal{G}(P) \xrightarrow{\cong} X(\Gamma, G)_P.$$

As for the tangent spaces, there are natural identifications

$$T_{[A]}\mathcal{F}(P)/\mathcal{G}(P)=H_A^1(M, adP),$$
$$T_{[\rho]}X(\Gamma, G)_P=H^1(\Gamma, \mathfrak{g}_{Ad\rho}).$$

Here $H_A^1(M, adP)$ is the de Rham cohomology defined by the covariant exterior derivative d_A on the space of adP-valued V-forms on M, and $H^1(\Gamma, \mathfrak{g}_{Ad\rho})$ is the group cohomology of Γ with coefficient in the Γ-module $\mathfrak{g}_{Ad\rho}$ (i.e. the vector space $\mathfrak{g}$ with Γ-action defined by $Ad\circ\rho$). As is well known, there exists an isomorphism

$$\text{(2.5)} \qquad H_A^*(M, adP) \xrightarrow{\cong} H^*(\Gamma, \mathfrak{g}_{Ad\rho})$$

And it is easy to verify that for $*=1$ the isomorphism

$$H_A^1(M, adP) \xrightarrow{\cong} H^1(\Gamma, \mathfrak{g}_{Ad\rho})$$

is given by the differential of the map hol (2.4). Note that the isomorphism (2.5) is natural with respect to the cup products of $H^*(M, adP)$ and $H^*(\Gamma, \mathfrak{g}_{Ad\rho})$.

Let $\mathcal{C}(P)$ denote the space of all V-connections on P. This is an $\Omega^1(M, adP)$-affine space. Here $\Omega^1(M, adP)$ is the space of all adP-valued V-1-forms on M. Let $B: \mathfrak{g}\times\mathfrak{g}\to\boldsymbol{R}$ denote an AdG-invariant nondegenerate inner product on $\mathfrak{g}=su_2$ (e.g. $B(X, Y)=-\operatorname{tr} XY$). This defines the metric B on the V-vector bundle adP. Now we have a symplectic structure ω on $\mathcal{C}(P)$ defined by

$$\omega_A(\xi, \eta)=\int_M B(\xi\wedge\eta)$$

for $A\in\mathcal{C}(P)$ and $\xi, \eta\in\Omega^1(M, adP)=T_A\mathcal{C}(P)$.

LEMMA 2.6. *The symplectic structure ω on $\mathcal{C}(P)$ reduces to define the symplectic structure on $\mathcal{F}(P)/\mathcal{G}(P)$.*

This fact is well known when M is a compact orientable surface ([A-B], [G-1]). Althouth we are concerned with an orbifold H/Γ, the proof is the same.

By the identification (2.4) and (2.5), we have the following.

COROLLARY 2.7. *The representation space $X(\Gamma, G)$ has a symplectic structure. Explicitely, the symplectic 2-form ω is given by*

$$\omega_{[\rho]}([u], [v])=(B_*([u]\cup[v]))\cap[z],$$

for $[\rho]\in X(\Gamma, G)$ and $[u], [v]\in H^1(\Gamma, \mathfrak{g}_{Ad\rho})=T_{[\rho]}X(\Gamma, G)$, where $[z]\in H_2(\Gamma, \boldsymbol{R})\cong\boldsymbol{R}$ is a generator, and $B_*: H^2(\Gamma, \mathfrak{g}_{Ad\rho}\otimes\mathfrak{g}_{Ad\rho})\to H^2(\Gamma, \boldsymbol{R})$ is a natural map defined by the AdG-invariant inner product B on $\mathfrak{g}$.

In § 3 we will need the explicit description of a generator $[z]$ of $H_2(\Gamma, \boldsymbol{R})$. So we take the homology class $[z]\in H_2(\Gamma, \boldsymbol{R})$, represented by

$$z=\sum_{i=1}^{n}\left\{-\frac{1}{a_i}\sum_{\nu=0}^{a_i-1}(x_i, x_i^{\nu})+(x_i, x_{i+i}x_{i+2}\cdots x_n)\right\}\in Z_2(\Gamma, \boldsymbol{R}). \tag{2.8}$$

§ 3. Hamiltonian S^1-actions on $X(\Gamma, SU_2)$

In this section we define a Hamiltonian S^1-action on $X(\Gamma, G)$ associated to a simple loop α on the 2-orbifold $M=H/\Gamma$ with certain conditions. We give applications later. As is noted in § 1, in case of $X(\pi_1(\Sigma_g), G)$ a similar construction of an $\boldsymbol{R}$-action is given by [G2].

First, we recall the fundamental notations from [G2]. Let $f: G\to\boldsymbol{R}$

denote an AdG-invarianat function. Then the AdG-equivariant function $F: G \to \mathfrak{g}$ (called *the variation function of* f) is defined by

$$\left.\frac{d}{dt}\right|_{t=0} f((\exp tX)A) = B(X, F(A)) \tag{3.1}$$

for $X \in \mathfrak{g}$, $A \in G$. Here $B: \mathfrak{g} \times \mathfrak{g} \to \boldsymbol{R}$ is an inner product of G. In the following, we set $G = SU_2$ and $B(X, Y) = -\operatorname{tr} XY$ for $X, Y \in \mathfrak{g} = su_2$. For example, if $f(A) = \operatorname{tr} A$, then $F(A) = -\frac{1}{2}(A - A^{-1})$. Note that since F is AdG-equivariant $\exp tF(A)$ commutes with A for any $t \in \boldsymbol{R}$.

DEFINITION 3.2. We define an AdG-invariant function $f: G \setminus \{\pm 1\} \to \boldsymbol{R}$ by $f(A) := \cos^{-1}\left(\frac{1}{2} \operatorname{tr} A\right)$.

That is to say, if $A \simeq \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix}$ for $0 < \theta < \pi$, then $f(A) = \theta$.

It is easy to verify the following from the definition of the variational function (3.1).

LEMMA 3.3. *Let* $f: G \setminus \{\pm 1\} \to \boldsymbol{R}$ *be as in* (3.2). *Then the variation function of* $f, F: G \setminus \{\pm 1\} \to \mathfrak{g}$, *is the AdG-equivariant function such that*

$$F\begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},$$

for $0 < \theta < \pi$.

Now we take a simple loop α on the orbifold $M = \boldsymbol{H}/\Gamma$. The element of $\Gamma = \pi_1^{orb}(M)$ associated to α is also written by α. We assume the simple loop α satisfies the following condition.

Assumption 3.4. *For any* $\rho \in R_n(\Gamma, G)$, $\rho(\alpha) \neq \pm 1$.

For example,

$$\alpha = x_i^{\pm 1},\ (x_i x_j)^{\pm 1} \qquad (\text{where } i \neq j)$$

satisfy (3.4). Note that $\alpha = x_i x_j x_k$ (where i, j, k are all distinct) does not necessarily satisfy the assumption (3.4), since $R(\Gamma(0: \alpha_i, \alpha_j, \alpha_k), G)$ is not empty in general. In the following we take $\alpha = (x_i x_j)^{\pm 1}$ for some $i, j (i \neq j)$.

For the simple loop α which satisfies the assumption (3.4), we decompose M as

$$M=M_1 \underset{\alpha}{\cup} M_2, \qquad \partial M_1=\alpha, \partial M_2=-\alpha.$$

Define

$$f_\alpha : R_n(\Gamma, G) \longrightarrow \boldsymbol{R} \quad \text{and} \quad F_\alpha : R_n(\Gamma, G) \longrightarrow \mathfrak{g}$$

by $f_\alpha(\rho) := f(\rho(\alpha))$ and $F_\alpha(\rho) := F(\rho(\alpha))$ for $\rho \in R_n(\Gamma, G)$.

These induce

$$f_\alpha : X_n(\Gamma, G) \longrightarrow \boldsymbol{R} \quad \text{and} \quad F_\alpha : X_n(\Gamma, G) \longrightarrow \mathfrak{g}.$$

Now we define an S^1-action on $R_n(\Gamma, G)$ and $X_n(\Gamma, G)$.

DEFINITION 3.5. Let α, M_1, M_2 be as above.

For $t \in \boldsymbol{R}/2\pi\boldsymbol{Z}$ and $\rho \in R_n(\Gamma, G)$ we define $t\cdot\rho \in R_n(\Gamma, G)$ by

$$(t\cdot\rho)(\gamma) := \begin{cases} \rho(\gamma) & \text{if} \quad \gamma \in \pi_1^{orb}(M_1) \\ z_t(\rho)\rho(\gamma)z_t(\rho)^{-1} & \text{if} \quad \gamma \in \pi_1^{orb}(M_2) \end{cases}$$

where $z_t(\rho) = \exp tF_\alpha(\rho) \in Z(\rho(\alpha))$ (the centralizer of $\rho(\alpha)$ in G). This left S^1-action on $R_n(\Gamma, G)$ induces a left S^1-action on $X_n(\Gamma, G)$.

We can describe this S^1-action using the *linkages* as in [F-S]. For example, let $\alpha = x_1 \cdots x_{n-2} (=(x_{n-1}x_n)^{-1})$. Under the identification

(3.6) $$R(\Gamma, G) = \{(X_1, \cdots, X_n) \in G^n \mid X_i^{a_i}=1,\ X_1 \cdots X_n = 1\}$$

where

$$X_i \simeq \begin{pmatrix} e^{\sqrt{-1}\theta_i} & 0 \\ 0 & e^{-\sqrt{-1}\theta_i} \end{pmatrix}$$

with $\theta_i = \dfrac{2\pi l_i}{a_i}$ for some $l_i = 0, \cdots, \left[\dfrac{a_i}{2}\right]$, the element of $R_n(\Gamma, G)$ is associated to the linkage in $G = SU_2 = S^3 = \boldsymbol{R}^3 \cup \{\infty\}$ spanned by the n-point

$$X_1,\ X_1X_2,\ X_1X_2X_3, \cdots, X_1 \cdots X_{n-1},\ X_1 \cdots X_n = 1.$$

Then the S^1-action on $R_n(\Gamma, G)$ is the rotation of the point $X_1 \cdots X_{n-1}$ prependicular to the axis through 1 and $X_1 \cdots X_{n-2}$. Note that the distance from 1 to $X_1 \cdots X_{n-2}$ is the value of the function f_α. See Figure 1.

THEOREM 3.7. *Let α be a simple loop on M satisfying the assumption* (3.4). *Then $f_\alpha : X_n(\Gamma, G) \to \boldsymbol{R}$ is the moment map of the S^1-action*

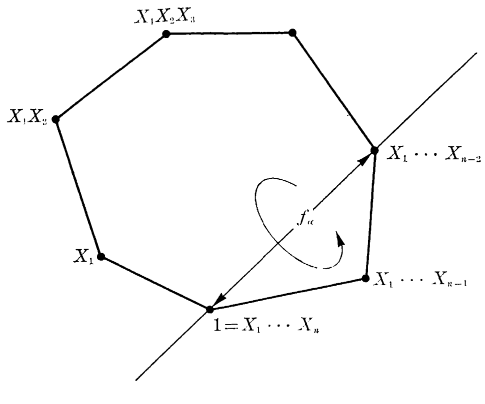

Figure 1

on the symplectic manifold $X_n(\Gamma, G)$ *associated to* α. *In particular this* S^1*-action preserves the symplectic structure* ω *on* $X_n(\Gamma, G)$.

REMARK 3.8. We need the assumption (3.4) for a simple loop α since for f as in (3.2) the variation function F of f is not well-defined on the whole of G. On the other hand, if we define $f: G \to \boldsymbol{R}$ by $f(A) = \operatorname{tr} A$, then as already seen, $F(A) = -\frac{1}{2}(A - A^{-1})$ which is defined globally on G. So we can associate to any simple loop α the $\boldsymbol{R}$-action on $X_n(\Gamma, G)$ with moment map f_α. But in general this $\boldsymbol{R}$-action can not be deformed to an S^1-action.

PROOF OF THEOREM 3.7. We assume that $\alpha = x_1x_2$. Otherwise the proof is similar. We have to show that for $[u] \in H^1(\Gamma, \mathfrak{g}_{Ad\rho}) = T_{[\rho]}X_n(\Gamma, G)$

$$(df_\alpha)_{[\rho]}[u] = \omega_{[\rho]}([u], [X]) \tag{3.9}$$

where $[X] \in H^1(\Gamma, \mathfrak{g}_{Ad\rho}) = T_{[\rho]}X_n(\Gamma, G)$ is the fundamental vector defined by the S^1-action on $X_n(\Gamma, G)$.
That is, the cocycle $X \in Z^1(\Gamma, \mathfrak{g}_{Ad\rho})$ is defined by

$$\begin{aligned} X(\gamma) &= \frac{d}{dt}\bigg|_{t=0} (t \cdot \rho)(\gamma)\rho(\gamma)^{-1} \\ &= \begin{cases} 0 & \text{if } \gamma = x_1, x_2, \\ F_\alpha(\rho) - Ad\rho(\gamma)F_\alpha(\rho) & \text{if } \gamma = x_3, \cdots, x_n. \end{cases} \end{aligned} \tag{3.10}$$

Since both the function f_α and the S^1-action on $X_n(\Gamma, G)$ are induced by those on $R_n(\Gamma, G)$, we lift those to $R_n(\Gamma, G)$ for calculations. The path ρ_s in $R_n(\Gamma, G)$ through ρ whic is tangent to $u \in Z^1(\Gamma, \mathfrak{g}_{Ad\rho}) = T_\rho R_n(\Gamma, G)$ is given by $\exp(su(\cdot)+O(s^2))\rho(\cdot)$. So the LHS of (3.9) is calculated as follows.

$$\begin{aligned}(df_\alpha)(u) &= \frac{d}{ds}\Big|_{s=0} f_\alpha(\rho_s)\\ &= \frac{d}{ds}\Big|_{s=0} f(\exp(su(\alpha)+O(s^2))\rho(\alpha))\\ &= B(F(\rho(\alpha)), u(\alpha))\\ &= B(F_\alpha(\rho), u(\alpha)).\end{aligned}$$

The 2-form on $R_n(\Gamma, G)$ which induces the symplectic 2-form ω on $X_n(\Gamma, G)$ is also denoted by ω. By (2.8) and (2.9), for $u, v \in Z^1(\Gamma, \mathfrak{g}_{Ad\rho})) = T_\rho R_n(\Gamma, G)$

$$\begin{aligned}\omega_\rho(u, v) &= B_*(u \cup v) \cap z\\ &= \sum_{i=1}^{n}\left\{\frac{1}{a_i}\sum_{\nu=0}^{a_i-1} B(u(x_i^{-1}), v(x_i^\nu)) - B(u(x_i^{-1}), v(x_{i+1}\cdots x_n))\right\}\end{aligned}$$

Here recall that the cup product $B_*(u \cup v) \in Z^2(\Gamma, \boldsymbol{R})$ is the map $\Gamma \times \Gamma \to \boldsymbol{R}$ defined by

$$\begin{aligned}(x, y) \longmapsto B(u(x), x\cdot v(y)) &= B(x^{-1}\cdot u(x), v(y))\\ &= -B(u(x^{-1}), v(y))\end{aligned}$$

So the RHS of (3.9) is

$$\begin{aligned}\omega_\rho(u, X) &= -\omega_\rho(X, u)\\ &= -\sum_{i=1}^{n}\left\{\frac{1}{a_i}\sum_{\nu=0}^{a_i-1} B(X(x_i^{-1}), u(x_i^\nu)) - B(X(x_i^{-1}), u(x_{i+1}\cdots x_n))\right\}.\end{aligned}$$

By (3.10), this equals (abbreviating $F_\alpha(\rho)$ to F)

$$\begin{aligned}&-\sum_{i=3}^{n}\left\{\frac{1}{a_i}\sum_{\nu=0}^{a_i-1} B(F - x_i^{-1}\cdot F, u(x_i^\nu)) - B(F - x_i^{-1}\cdot F, u(x_{i+1}\cdots x_n))\right\}\\ &= -\sum_{i=3}^{n}\left\{\frac{1}{a_i}\sum_{\nu=0}^{a_i-1}(B(F, u(x_i^\nu)) - B(Ad(x_i^{-1})\cdot F, u(x_i^\nu)))\right.\\ &\quad \left. -(B(F, u(x_{i+1}\cdots x_n)) - B(Ad(x_i^{-1})\cdot F, u(x_{i+1}\cdots x_n)))\right\}.\end{aligned}$$

Here note that

$$B(Ad(x_i^{-1})\cdot F,\, u(x_i^{\nu}))=B(F,\, u(x_i^{\nu+1}))-B(F,\, u(x_i))$$

and

$$B(Ad(x_i^{-1})\cdot F,\, u(x_{i+1}\cdots x_n))=B(F,\, u(x_i\cdots x_n))-B(F,\, u(x_i)).$$

Hence the last quantity equals

$$\begin{aligned}-B(F,\, u(x_3\cdots x_n)) &= -B(F_\alpha(\rho),\, u(\alpha^{-1}))\\ &= B(F_\alpha(\rho),\, u(\alpha)).\end{aligned}$$

This proves the theorem. ∎

By the theorem of Frankel ([F], [A-M-M]) an immediate consequeuce of Theorem (3.7) is the following.

COROLLARY 3.11. *Let α be as in* (3.7). *Then the function f_α is the Morse function* (*in the sense of Bott*) *with only even indices.*

In [K-K], Kirk-Klassen proved a conjecture of Fintushel-Stern which states that $X(\Gamma, G)$ has a Morse function (in the usual sense) with only even indices. (A similar proof is given in [T].) The main step of their proof is the claim of (3.11) for $\alpha=x_1x_2\cdots x_{n-2}$. Thus we obtain the alternative proof.

Now consider the Morse theory for the function f_α for $\alpha=x_1x_2\cdots x_{n-2}$ (cf. [K-K, §3]). Under the identification (3.6), the critical point of f_α in $X(\Gamma, G)$, which equals to the fixed point of the associated S^1-action, is represented by $(X_1, \cdots, X_n)\in R_n(\Gamma, G)$ of one of the next two types.

(A) $X_1, X_2, \cdots, X_{n-2}$ are all commutative (i.e. lie on the same maxlmal torus$\cong S^1$ of SU_2).

(B) X_{n-1} and X_n are commutative.

The critical points of type (A) are identified to the elements of $X(\Gamma(0:a, a_{n-1}, a_n), G)$ (where $a=a_1\cdots a_{n-2}$) and consists of finite number of isolated points. The critical points of type (B) are identified to the elements of $X(\Gamma(0:a_1, a_2, \cdots, a_{n-2}, b), G)$ (where $b=a_{n-1}a_n$) and consist to submanifolds of $X_n(\Gamma, G)$ of codimention 2.

By studying the critical point set of type (A) and type (B) in each component $X(l_1, \cdots, l_n)$ of $X_n(\Gamma, G)$ we will obtain the information about the topology of this component, but in general we need the inductive arguments (because of the critical submanifolds of type (B)), and it may be rather complicated.

But for some special components we can proceed further.

First, for example, assume that the rotation number $(l_1, \cdots, l_n)$ satisfies

$$(3.12)\qquad \theta_1+\cdots+(-\theta_i)+\cdots+\theta_{n-1}<\theta_n<\theta_1+\cdots+\theta_{n-1}<\pi$$

for each $i=1, \cdots, n-1$ where $\theta_i=\frac{2\pi l_i}{a_i}$. Then it is easy to see that $f_\alpha(\alpha=x_1\cdots x_{n-2})$ has only two critical values

$$\text{the maximum}=\theta_1+\cdots+\theta_{n-2}$$
$$\text{the minimum}=\theta_n-\theta_{n-1},$$

and the associated critical submanifolds consist of

a non degenerate critical point (type (A))

a (connected) submanifold of codimention 2 (type (B)).

Now the theorem of Delzant [D, Théorème 1.2], which is an analogue of the well-known theorem of Reeb [R], states that such a compact symplectic manifold with Hamiltonian S^1-action is just the complex projective space. Thus we have the following.

PROPOSITION 3.13 (cf. [B, LEMMA 2.13]). *For the rotation number* $(l_1, \cdots, l_n)$ *satisfying* (3.12), *the component* $X(l_1, \cdots, l_n)$ *is diffeomorphic to* $\boldsymbol{P}^{n-3}(\boldsymbol{C})$.

Secondly, the low dimentional (i.e. dim.$\leq$4) components can be completely determined ([K-K], [B-O]). We consider this problem in the next section.

§ 4. Hamiltonian torus action on $X(\Gamma, SU_2)$

In this section we construct a Hamiltonian torus action on $X(\Gamma, G)$ by applying the results of § 3 to some simple loops on M. We will see that this torus action is closely related to the method of [K-K] used to determine the 4-dimentional components of $X(\Gamma, G)$.

LEMMA 4.1. *For distinct* $i, j, k, l\in\{1, \cdots, n\}$, *two* S^1*-actions on* $X_n(\Gamma, G)$ *associated to* $\alpha=x_ix_j$ *and* $\beta=x_kx_l$ *are commutative.*

PROOF. We may assume $\alpha=x_1x_2$, $\beta=x_3x_4$ To distinguish the two S^1-action, let

$$A_\alpha : S^1 \times R_n(\Gamma, G) \longrightarrow R_n(\Gamma, G)$$
$$A_\beta : S^1 \times R_n(\Gamma, G) \longrightarrow R_n(\Gamma, G)$$

denote the S^1-actions on $R_n(\Gamma, G)$ defined by α and β respectively. What we have to show is

$$[A_\beta(s, A_\alpha(t, \rho))] = [A_\alpha(t, A_\beta(s, \rho))] \in X_n(\Gamma, G)$$

for $s, t \in S^1$ and $\rho \in R_n(\Gamma, G)$. An easy calculation shows that

$$A_\beta(s, A_\alpha(t, \rho)) = Ad[(z_\alpha)_t(\rho), (z_\beta)_s(\rho)](A_\alpha(t, A_\beta(s, \rho)))$$

where $(z_\alpha)_t(\rho) = \exp tF_\alpha(\rho)$ and $(z_\beta)_s(\rho) = \exp sF_\beta(\rho)$. This proves the lemma. ■

THEOREM 4.2. *For* $n \geq 5$ (resp. $n=4$), *the representation space* $X_n(\Gamma, G)$ *has an effective Hamiltonian* $T^{[n/2]}$*-action* (resp. S^1*-action*).

PROOF. For $n \geq 5$, the $\left[\frac{n}{2}\right]$'s S^1-actions associated to simple loops

$$x_1x_2,\ x_3x_4,\ \cdots,\ x_{2[n/2]-1}x_{2[n/2]}$$

are all commutative by (4.1), and define the desired Hamiltonian torus action. As for the case $n=4$, two S^1-actions associated to

$$x_1x_2 \quad \text{and} \quad x_3x_4 (= (x_1x_2)^{-1})$$

are essentially the same (more precisely, they are the inverse actions of each other). Hence in this case we must take one of them. ■

For example let $n=5$ and describe the Hamiltonian T^2-action associated to $\alpha = x_2x_3$ and $\beta = x_4x_5$ using *linkages* as in § 3. This action consists of the rotations of the point X_1X_2 and $X_1X_2X_3X_4$ around the axes from X_1 to $X_1X_2X_3$ and from $X_1X_2X_3$ to 1. See Figure 2.

Kirk-Klassen [K-K, § 4] determined the topological type of the 4-dimensional components of $X(\Gamma, G)$. They proved that a 4-dimensional component is homeomorphic to either $P^1 \times P^1$ or $P^2 \# n\overline{P^2}$ $(n=0, 1, \cdots, 5)$. Their method is to investigate the position of the point $X_1X_2X_3$. In our terminology, this is equivalent to study the image of the moment map of above T-action. That is, what they did is interpreted as the reconstruction of the 4-dimensional (symplectic) manifold from its image of the moment map of the T^2-action. In general Delzant [D] proved the

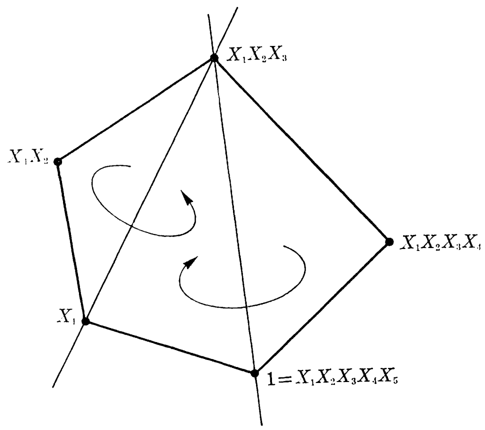

Figure 2

following result on the Hamiltonian T^n-action on a $2n$-dimentional symplectic manifold.

THEOREM ([D] see also [A, PROPOSITION 5.3.1]). *Let M be a $2n$-dimensional compact symplectic manifold with an effective Hamiltonian T^n-action. Then M is T^n-equivariantly symplectic diffeomorphic to a nonsingular toric variety.*

Here we mean by a toric variety a projective variety of complex dimension, say, N on which algebraic torus T_C^N acts algebraically and there exists an open dense orbit which is isomorphic to T_C^N. Toric varieties are constructed from *fans* (see [O]). Now by Theorem 4.2, for $n=4, 5, 6$, the compact symplectic manifold $X_n(\Gamma, G)$ has an effective Hamiltonian action of the real torus of the half dimension of it. Hence we have the following.

COROLLARY 4.3. *The* 2, 4, 6-*dimensional components of $X(\Gamma, G)$ admit a structure of nonsingular toric variety.*

This statement is previously noted in [K-K] without proof.

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