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Singular hyperbolic systems, VIII. On the well-posedness in Gevrey classes for Fuchsian hyperbolic equations

Dedicated to Professor Raymond GÉRARD on his sixtieth birthday

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Abstract. The paper discusses the well-posedness problem in Gevrey classes for Fuchsian type partial differential equations $Pu=f$ with P being of the form $P=(t\partial_t)^m+\Sigma_{j+|\alpha|\leq m, j<m} t^{l(j,\alpha)} a_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha$. The main subject is to investigate the difference between the following two assertions: (A) $Pu=f$ is well-posed in $C^\infty([0,T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$, and (B) $Pu=f$ is well-posed in $\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$, where $\mathcal{E}^{\{s\}}(\Omega)$ denotes the space of all Gevrey functions of class $\{s\}$ on Ω. The author's motivation comes from the following example: in the case $P=(t\partial_t+1)^2-t\partial_x^2$, (A) is true for all $s>1$, but (B) is not true for any $s>1$.

Introduction

The Cauchy problem for Fuchsian hyperbolic operators P in C^∞ or Gevrey classes was investigated by Tahara [8, 9, 10], Uryu [11] and Itoh-Uryu [3] in the following form:

(A) $\qquad Pu=f$ is well-posed in $C^\infty([0,T], \mathcal{E}^*(\boldsymbol{R}^n))$,

where $*=\{s\}$ or (s), and $\mathcal{E}^*(\boldsymbol{R}^n)$ is the space of functions of the Gevrey class $*$ on $\boldsymbol{R}^n$ (see § 1).

Recently, the author noticed the following example in the study on Maillet's type theorem with R. Gérard (see Gérard-Tahara [1, 2]):

Example. Let $*=\{s\}$ or (s), and let A be of the form

$$A=(t\partial_t+1)^2-t\partial_x^2,$$

where $(t,x)\in[0,T]\times\boldsymbol{R}$. Then, we have:

(1) $Au=f$ is well-posed in $C^\infty([0,T], \mathcal{E}^*(\boldsymbol{R}))$ for any $s>1$.

(2) For any $s>1$, $Au=f$ is not well-posed in $\mathcal{E}^*([0,T]\times\boldsymbol{R})$.

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This implies that the well-posedness in $\mathcal{E}^*([0,T]\times \boldsymbol{R}^n)$ is somewhat different from the well-posedness in $C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$, and that to get the well-posedness in $\mathcal{E}^*([0,T]\times \boldsymbol{R}^n)$ from the well-posedness in $C^\infty([0,T],$ $\mathcal{E}^*(\boldsymbol{R}^n))$ we need some additional condition.

Thus, motivated by this example, in this paper the author will discuss the following problem:

PROBLEM. Under what additional condition does the assertion

(B) $\qquad Pu=f$ is well-posed in $\mathcal{E}^*([0,T]\times \boldsymbol{R}^n)$

follow from the assertion (A)?

Throughout this paper, we will use the following notations (see Tahara [10], Komatsu [5]): $\boldsymbol{N}=\{1,2,3,\cdots\}$, $\boldsymbol{Z}_+=\{0,1,2,\cdots\}$, $*=\emptyset$, $\{s\}$ or (s), $\mathcal{E}^*$ with $*=\emptyset$ means $\mathcal{E}$, $\mathcal{E}^*$ with $*=\{s\}$ means $\mathcal{E}^{\{s\}}$, $\mathcal{E}^*$ with $*=(s)$ means $\mathcal{E}^{(s)}$, $\mathcal{E}^*(\boldsymbol{R}^n)$ [resp. $\mathcal{E}^*([0,T]\times \boldsymbol{R}^n)$] denotes the set of functions of the Gevrey class $*$ on $\boldsymbol{R}^n$ [resp. $[0,T]\times \boldsymbol{R}^n$], and $C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$ denotes the set of infinitely differentiable functions on $[0,T]$ with values in $\mathcal{E}^*(\boldsymbol{R}^n)$ equipped with the usual topology.

§ 1. Main Results

First, let us state our main results of this paper.

Let $(t,x)\in[0,T]\times \boldsymbol{R}^n$ and let us consider

$$P=(t\partial_t)^m+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}} t^{l(j,\alpha)}a_{j,\alpha}(t,x)(t\partial_t)^j\partial_x^\alpha, \tag{1.1}$$

where $m\in \boldsymbol{N}$, $x=(x_1,\cdots,x_n)$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in \boldsymbol{Z}_+^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\partial_t=\partial/\partial t$, $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$, $l(j,\alpha)\in \boldsymbol{Z}_+$ (for $j+|\alpha|\leqq m$ and $j<m$) and $a_{j,\alpha}(t,x)\in C^\infty([0,T]\times \boldsymbol{R}^n)$ (for $j+|\alpha|\leqq m$ and $j<m$).

Impose the following condition on P:

$$(\mathrm{C}_1) \qquad l(j,\alpha)>0, \quad \text{if} \quad |\alpha|>0.$$

This implies that the operator P is of Fuchsian type with respect to t.

DEFINITION. We say that the equation $Pu=f$ is well-posed in $C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$ [resp. $\mathcal{E}^*([0,T]\times \boldsymbol{R}^n)$], if the following condition is satisfied: For any $f(t,x)\in C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$ [resp. $f(t,x)\in\mathcal{E}^*([0,T]\times \boldsymbol{R}^n)$] there exists a unique $u(t,x)\in C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$ [resp. $u(t,x)\in\mathcal{E}^*([0,T]\times \boldsymbol{R}^n)$] satisfying $Pu=f$ on $[0,T]\times \boldsymbol{R}^n$.

Put

$$(1.2)\qquad \mathcal{S}=\{(j,\alpha);\ a_{j,\alpha}\not\equiv 0 \ \text{ and } \ l(j,\alpha)<|\alpha|\}.$$

Then, we can state our main theorem as follows.

THEOREM 1. *Let* $s>1$, *let* P *be the operator in* (1.1) *satisfying* (C_1), *and assume the following*:

i) $*=\{s\}$ *or* (s);
ii) $a_{j,\alpha}(t,x)\in\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$ $(j+|\alpha|\leqq m$ *and* $j<m)$;
iii) $Pu=f$ *is well-posed in* $C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$.

Then, if s *satisfies*

$$(1.3)\qquad 1<s\leqq\min\left[\infty,\ \min_{(j,\alpha)\in\mathcal{S}}\left(\frac{m-j-l(j,\alpha)}{|\alpha|-l(j,\alpha)}\right)\right],$$

the equation $Pu=f$ *is well-posed also in* $\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$.

The proof will be given in §2∼§4. If the conditions i), ii) and iii) in Theorem 1 are supposed, it is easy to see that the following two assertions are equivalent:

(B) $Pu=f$ is well-posed in $\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$,
(R) $u\in C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$ and $Pu\in\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$ imply $u\in\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$.

Hence, in §2∼§4 we will study the assertion (R) instead of (B). We note that in the proof of (R) we do not use any hyperbolicity condition. Hyperbolicity lies only behind the condition iii).

REMARK 1. (1) When $*=\{s\}$, Theorem 1 is also true for $s=1$ (see Proposition 2 in §2).

(2) When $\mathcal{S}=\emptyset$, the condition (1.3) is trivially satisfied. In this case, the well-posedness in $C^\infty([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$ implies the well-posedness in $\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$.

(3) When R is a non-characteristic operator, by putting $P=t^mR$ we can apply Theorem 1 with $\mathcal{S}=\emptyset$.

(4) When $\mathcal{S}\cap\{(j,\alpha);\ j+|\alpha|=m\}\neq\emptyset$, the condition (1.3) means $1<s\leqq 1$. In this case there are no s satisfying (1.3).

REMARK 2. For $s>1$, the condition (1.3) is equivalent to

$$(1.4)\qquad s\geqq 1+\max\left[0,\ \max_{\substack{j+|\alpha|\leqq m\\ |\alpha|>0}}\left(\frac{j+s|\alpha|-m}{l(j,\alpha)}\right)\right].$$

To see this, we have only to solve the inequality (1.4) with respect to s. The right hand side of (1.4) corresponds to the formal Gevrey index for P introduced by Gérard-Tahara [1, 2] and Miyake [7].

As to the necessity of the condition (1.3), the author believes that (1.3) is a necessary condition for $Pu=f$ to be well-posed in $\mathcal{E}^*([0, T]\times \boldsymbol{R}^n)$; though he has not yet succeeded in proving it. The following Proposition 1 will support this conjecture.

Let us consider

$$L=(t\partial_t)^m+b_{m-1}(t\partial_t)^{m-1}+\cdots+b_0-\sum_{\substack{j+|\alpha|\leqq m\\ j<m}} t^{l(j,\alpha)}c_{j,\alpha}(t\partial_t)^j\partial_x^\alpha, \tag{1.5}$$

where

d$_1$) $b_i\in \boldsymbol{R}$ $(i=0, 1, \cdots, m-1)$;
d$_2$) $\lambda^m+b_{m-1}\lambda^{m-1}+\cdots+b_0\neq 0$ for any $\lambda\in \boldsymbol{Z}_+$;
d$_3$) $c_{j,\alpha}\in \boldsymbol{R}$ and $c_{j,\alpha}\geqq 0$ $(j+|\alpha|\leqq m$ and $j<m)$;
d$_4$) $l(j, \alpha)\in \boldsymbol{Z}_+$ and $l(j, \alpha)>0$ $(j+|\alpha|\leqq m$ and $j<m)$.

Put

$$S_L=\{(j, \alpha);\ c_{j,\alpha}\neq 0 \quad \text{and} \quad l(j, \alpha)<|\alpha|\}.$$

Then we have

PROPOSITION 1. *Let* $s>1$, *let* L *be the operator in* (1.5) *satisfying* d$_1$)$\sim$d$_4$), *and assume the following*:
i) $*=\{s\}$ *or* (s);
ii) $Lu=f$ *is well-posed in* $\mathcal{E}^*([0, T]\times \boldsymbol{R}^n)$.
Then, s must satisfy

$$s\leqq \min\left[\infty,\ \min_{(j,\alpha)\in S_L}\left(\frac{m-j-l(j, \alpha)}{|\alpha|-l(j, \alpha)}\right)\right]. \tag{1.6}$$

The proof will be given in § 5. The idea of the proof is that by the formal Taylor expansion at the origin we reduce the problem to the one in formal Gevrey classes.

The following examples will illustrate our theory.

Example 1. Let us consider

$$P=(t\partial_t+1)^2-t^{2k}\partial_x^2-t^l\partial_x,$$

where $(t, x)\in[0, T]\times\boldsymbol{R}$, $2k\in N$ and $l\in N$. Then:

(1) $Pu=f$ is well-posed in $C^{\infty}([0, T], \mathcal{E}^{*}(\boldsymbol{R}))$, if and only if s satisfies

$$(1.7)\qquad \begin{cases} 1<s<\sigma/(\sigma-1), & \text{when } *=\{s\}, \\ 1<s\leqq\sigma/(\sigma-1), & \text{when } *=(s) \end{cases}$$

with $\sigma=\max\{1, (2k-l)/k\}$ (see § 6 and [4]).

(2) The condition (1.3) is

$$\begin{cases} 1<s\leqq 1, & \text{when } k=\dfrac{1}{2}, \\ 1<s\leqq\infty, & \text{when } k\geqq 1. \end{cases}$$

Hence, by Theorem 1 and Proposition 1 we see the following: (i) When $k=1/2$, $Pu=f$ is not well-posed in $\mathcal{E}^{*}([0, T]\times\boldsymbol{R})$ for any $s>1$; (ii) When $k\geqq 1$, $Pu=f$ is well-posed in $\mathcal{E}^{*}([0, T]\times\boldsymbol{R})$ for s satisfying (1.7).

Example 2. Let us consider

$$P=(t\partial_t+1)((t\partial_t+1)^2-t^m\partial_x^2)-t\partial_x^2,$$

where $(t, x)\in[0, T]\times\boldsymbol{R}$ and $m\in N$. Then:

(1) $Pu=f$ is well-posed in $C^{\infty}([0, T], \mathcal{E}^{*}(\boldsymbol{R}))$, if and only if s satisfies

$$(1.8)\qquad \begin{cases} 1<s<\dfrac{3m-2}{2m-2}, & \text{when } *=\{s\}, \\ 1<s\leqq\dfrac{3m-2}{2m-2}, & \text{when } *=(s) \end{cases}$$

(see § 6 and [4]).

(2) The condition (1.3) is

$$\begin{cases} 1<s\leqq 1, & \text{when } \mathrm{m}=1, \\ 1<s\leqq 2, & \text{when } \mathrm{m}\geqq 2. \end{cases}$$

Hence, by Theorem 1 and Proposition 1 we see the following: (i) When $m=1$, $Pu=f$ is not well-posed in $\mathcal{E}^{*}([0, T]\times\boldsymbol{R})$ for any $s>1$; (ii) When $m\geqq 2$, $Pu=f$ is well-posed in $\mathcal{E}^{*}([0, T]\times\boldsymbol{R})$ for s satisfying (1.8).

For the reader's convenience, we will give some definitions of Gevrey classes (for details, see Komatsu [5]).

Let $s\geqq 1$. A function $f(x)\in C^{\infty}(\boldsymbol{R}^n)$ is said to belong to the Gevrey class $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ [resp. $\mathcal{E}^{(s)}(\boldsymbol{R}^n)$], if $f(x)$ satisfies the following: for any compact subset K of $\boldsymbol{R}^n$, there are $C>0$ and $h>0$ [resp. for any $h>0$ and any compact subset K of $\boldsymbol{R}^n$, there is a $C>0$] such that

$$\sup_{x\in K}|\partial_x^{\alpha}f(x)|\leqq Ch^{|\alpha|}(|\alpha|!)^s \quad \text{for any } \alpha\in \boldsymbol{Z}_+^n. \tag{1.9}$$

We denote by $C^{\infty}([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ [resp. $C^{\infty}([0,T],\mathcal{E}^{(s)}(\boldsymbol{R}^n))$] the set of all infinitely differentiable functions on $[0,T]$ with values in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ [resp. $\mathcal{E}^{(s)}(\boldsymbol{R}^n)$] equipped with the locally convex topology in Komatsu [5].

In other words, $C^{\infty}([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ [resp. $C^{\infty}([0,T],\mathcal{E}^{(s)}(\boldsymbol{R}^n))$] is the set of all functions $u(t,x)\in C^{\infty}([0,T]\times\boldsymbol{R}^n)$ satisfying the following: for any $i\in\boldsymbol{Z}_+$ and any compact subset K of $\boldsymbol{R}^n$, there are $C>0$ and $h>0$ [resp. for any $i\in\boldsymbol{Z}_+$, $h>0$ and any compact subset K of $\boldsymbol{R}^n$, there is a $C>0$] such that

$$\sup_{[0,T]\times K}|\partial_t^i\partial_x^{\alpha}u(t,x)|\leqq Ch^{|\alpha|}(|\alpha|!)^s \quad \text{for any } \alpha\in\boldsymbol{Z}_+^n. \tag{1.10}$$

By $\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$ [resp. $\mathcal{E}^{(s)}([0,T]\times\boldsymbol{R}^n)$] we denote the set of all functions $v(t,x)\in C^{\infty}([0,T]\times\boldsymbol{R}^n)$ satisfying the following: for any compact subset K of $\boldsymbol{R}^n$, there are $C>0$ and $h>0$ [resp. for any $h>0$ and any compact subset K of $\boldsymbol{R}^n$, there is a $C>0$] such that

$$\sup_{[0,T]\times K}|\partial_t^i\partial_x^{\alpha}v(t,x)|\leqq Ch^{i+|\alpha|}[(i+|\alpha|)!]^s \quad \text{for any } (i,\alpha)\in\boldsymbol{Z}_+\times\boldsymbol{Z}_+^n. \tag{1.11}$$

Note that the difference between $C^{\infty}([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ and $\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$ [resp. $C^{\infty}([0,T],\mathcal{E}^{(s)}(\boldsymbol{R}^n))$ and $\mathcal{E}^{(s)}([0,T]\times\boldsymbol{R}^n)$] lies in the following: in (1.10) the constants C and h depend on i; while in (1.11) the constants C and h do not depend on i [resp. in (1.10) the constant C depends on i; while in (1.11) the constant C does not depend on i]. Obviously, we have $\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)\subset C^{\infty}([0,T],\mathcal{E}^*(\boldsymbol{R}^n))$.

§ 2. Basic Estimates

As preparations for the proof of Theorem 1, in this section we will establish some basic estimates for the solution of the ordinary differential equation with a parameter $r\in\boldsymbol{R}$:

$$(t\partial_t+r)^m u+\sum_{j<m}a_j(x)(t\partial_t+r)^j u=f. \tag{2.1}$$

For simplicity, we introduce the following formal norms: for a compact subset K of $\boldsymbol{R}^n$ and $u(t,x)\in C^\infty([0,T]\times K)$ we define $\|\nabla^\infty u(t)\|_K$ by

$$\|\nabla^\infty u(t)\|_K=\sum_{q=0}^{\infty}\sum_{|\beta|=q}\Big(\max_{x\in K}|\partial_x^\beta u(t,x)|\Big)\frac{\rho^q}{q!} \tag{2.2}$$

(which is a formal power series in ρ whose coefficients are functions in t), and define $\|\nabla^\infty u\|_{0,K}$ by

$$\|\nabla^\infty u\|_{0,K}=\sum_{q=0}^{\infty}\sum_{|\beta|=q}\Big(\max_{[0,T]\times K}|\partial_x^\beta u(t,x)|\Big)\frac{\rho^q}{q!} \tag{2.3}$$

(which is a formal power series in ρ). The convenience of these formal norms lies in the following formulae:

$$\|\nabla^\infty au(t)\|_K\ll\|\nabla^\infty a\|_{0,K}\|\nabla^\infty u(t)\|_K, \tag{2.4}$$

$$\|\nabla^\infty\partial_x^\alpha u(t)\|_K\ll\partial_\rho^{|\alpha|}\|\nabla^\infty u(t)\|_K \tag{2.5}$$

for $a,\ u\in C^\infty([0,T]\times K)$ and $\alpha\in \boldsymbol{Z}_+^n$ (see Tahara [9, Lemma 3] and Leray-Ohya [6, formulae (10.1) and (10.3)]). Here $\sum_{q=0}^\infty a_q\rho^q\ll\sum_{q=0}^\infty b_q\rho^q$ means that $|a_q|\leqq b_q$ holds for any $q\in \boldsymbol{Z}_+$.

Then, our basic estimate for (2.1) is stated as follows. Let $B(\rho)=\sum_{q=0}^\infty B_q\rho^q$ be a formal power series in ρ and let $\varphi(t,\rho)=\sum_{q=0}^\infty\varphi_q(t)\rho^q$ be a formal power series in ρ whose coefficients $\varphi_q(t)(q=0,1,2,\cdots)$ belong to $C^0([0,T])\cap C^1((0,T])$. Then, we have

LEMMA 1. *Assume that* $u,\ f\in C^\infty([0,T]\times K)$ *satisfy* (2.1), *that the estimates*

$$\|\nabla^\infty a_j\|_{0,K}\ll B(\rho),\ j=0,1,\cdots,m-1, \tag{2.6}$$

$$\|\nabla^\infty f(t)\|_K\ll(t\partial_t+r-1-B(\rho))\varphi(t,\rho)\quad \text{on } (0,T] \tag{2.7}$$

hold, and that $r>1+B_0$ *holds. Then, we have*

$$\sum_{j=0}^{m-1}\|\nabla^\infty(t\partial_t+r)^ju(t)\|_K\ll\varphi(t,\rho)\quad \text{on } [0,T]. \tag{2.8}$$

PROOF. Since

$$(t\partial_t+r)^ju(t)=t^{-r}\int_0^t\tau^{r-1}(\tau\partial_\tau+r)^{j+1}u(\tau)d\tau$$

holds for any j, we have

$$\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K \ll t^{-r}\int_0^t \tau^{r-1}\|\nabla^{\infty}(\tau\partial_\tau+r)^{j+1}u(\tau)\|_K d\tau \tag{2.9}$$

for any j and hence

$$\sum_{j=0}^{m-1}\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K \ll \sum_{j=0}^{m-1} t^{-r}\int_0^t \tau^{r-1}\|\nabla^{\infty}(\tau\partial_\tau+r)^{j+1}u(\tau)\|_K d\tau. \tag{2.10}$$

On the other hand, by (2.1), (2.4) and (2.6) we have

$$\begin{aligned}(2.11)\qquad &\|\nabla^{\infty}(t\partial_t+r)^m u(t)\|_K \\ &\ll \sum_{j=0}^{m-1}\|\nabla^{\infty}a_j\|_{0,K}\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K+\|\nabla^{\infty}f(t)\|_K \\ &\ll B(\rho)\sum_{j=0}^{m-1}\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K+\|\nabla^{\infty}f(t)\|_K.\end{aligned}$$

Therefore, by (2.10) and (2.11) we obtain

$$\begin{aligned}(2.12)\qquad &\sum_{j=0}^{m-1}\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K \\ &\ll t^{-r}\int_0^t \tau^{r-1}\Big\{(1+B(\rho))\sum_{j=0}^{m-1}\|\nabla^{\infty}(\tau\partial_\tau+r)^j u(\tau)\|_K+\|\nabla^{\infty}f(\tau)\|_K\Big\}d\tau.\end{aligned}$$

Denote by $\Phi(t,\rho)$ the right hand side of (2.12). Then, by (2.12) and (2.7) we have $\sum_{j=0}^{m-1}\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K \ll \Phi(t,\rho)$ and

$$\begin{aligned}(2.13)\qquad &0\ll(t\partial_t+r)\Phi(t,\rho) \\ &=(1+B(\rho))\sum_{j=0}^{m-1}\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K+\|\nabla^{\infty}f(t)\|_K \\ &\ll(1+B(\rho))\Phi(t,\rho)+(t\partial_t+r-1-B(\rho))\varphi(t,\rho).\end{aligned}$$

Hence, to obtain Lemma 1 it is sufficient to show that (2.13) implies $\Phi(t,\rho)\ll\varphi(t,\rho)$ on $[0,T]$. We will show this now.

Put $\Phi(t,\rho)=\sum_{q=0}^{\infty}\Phi_q(t)\rho^q$ and $\varphi(t,\rho)=\sum_{q=0}^{\infty}\varphi_q(t)\rho^q$. Then, (2.13) is equivalent to

$$\begin{aligned}(2.14)_q\qquad &(t\partial_t+r-1)\Phi_q(t)-\sum_{i=0}^{q}B_i\Phi_{q-i}(t)\leqq(t\partial_t+r-1)\varphi_q(t)-\sum_{i=0}^{q}B_i\varphi_{q-i}(t), \\ &q=0,1,2,\cdots.\end{aligned}$$

Our aim is to show that $\Phi_q(t)\leqq\varphi_q(t)$ on $[0,T]$ for $q=0,1,2,\cdots$.

$\Phi_0(t)\leqq\varphi_0(t)$ is proved as follows. By $(2.14)_0$ we have

$$(t\partial_t+r-1-B_0)\Phi_0(t)\leqq(t\partial_t+r-1-B_0)\varphi_0(t). \tag{2.15}$$

Multiplying both sides of (2.15) by t^{r-1-B_0} we have

$$\partial_t(t^{r-1-B_0}\Phi_0(t))\leqq\partial_t(t^{r-1-B_0}\varphi_0(t)).$$

Therefore, by integrating from 0 to t we obtain

$$t^{r-1-B_0}\Phi_0(t)\leqq t^{r-1-B_0}\varphi_0(t).$$

Thus, $\Phi_0(t)\leqq\varphi_0(t)$ is proved.

Let $q\geqq 1$. Then, $\Phi_q(t)\leqq\varphi_q(t)$ is proved by induction on q as follows. Assume that $\Phi_j(t)\leqq\varphi_j(t)$ is already proved for $j=0, 1, \cdots, q-1$. Then, we have

$$\sum_{i=1}^{q} B_i\Phi_{q-i}(t)\leqq\sum_{i=1}^{q} B_i\varphi_{q-i}(t). \tag{2.16}$$

Therefore, by $(2.14)_q$ and (2.16) we obtain

$$(t\partial_t+r-1-B_0)\Phi_q(t)\leqq(t\partial_t+r-1-B_0)\varphi_q(t).$$

Hence, we can prove $\Phi_q(t)\leqq\varphi_q(t)$ by the same argument as in the case $q=0$. Thus, $\Phi_q(t)\leqq\varphi_q(t)$ is proved for any q. Q.E.D.

Now, let us apply Lemma 1 to our situation in Gevrey classes. Let $1\leqq s<\infty$ and put

$$\theta_s(\rho)=\sum_{q=0}^{\infty}(q!)^s\frac{\rho^q}{q!}. \tag{2.17}$$

Then, we can easily see the following:

i) $a(x)\in C^\infty(K)$ belongs to $\mathcal{E}^{\{s\}}(K)$, if and only if $\|\nabla^\infty a\|_{0,K}\ll A\theta_s(k\rho)$ holds for some $A>0$ and $k>0$;

ii) If $0<k<h$, then

$$\theta_s(k\rho)\theta_s(h\rho)\ll(1-k/h)^{-1}\theta_s(h\rho). \tag{2.18}$$

LEMMA 2. *Assume that* $u, f\in C^\infty([0, T]\times K)$ *satisfy* (2.1), *that the estimates*

$$\|\nabla^\infty a_j\|_{0,K}\ll A\theta_s(k\rho),\quad j=0, 1, \cdots, m-1, \tag{2.19}$$

$$\|\nabla^\infty f(t)\|_K\ll B\theta_s(h\rho)\quad \text{on } [0, T] \tag{2.20}$$

hold for some $s\geqq 1$, $A>0$, $B>0$ *and* $h>2k>0$, *and that* $r\geqq 2+2A$ *holds. Then, we have*

$$\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K \ll \frac{(1+2mA)}{(r-1-2A)^{m-j}} B\theta_s(h\rho) \tag{2.21}$$

on $[0, T]$ *for* $j=0, 1, \cdots, m$.

PROOF. Put $\varphi(t, \rho)=(r-1-2A)^{-1}B\theta_s(h\rho)$. Then, by (2.18) we have

$$\theta_s(k\rho)\varphi(t, \rho)\ll 2\varphi(t, \rho)$$

and therefore

$$\begin{aligned}\|\nabla^{\infty}f(t)\|_K &\ll B\theta_s(h\rho)\\ &=(t\partial_t+r-1-2A)\varphi(t, \rho)\\ &\ll (t\partial_t+r-1-A\theta_s(k\rho))\varphi(t, \rho).\end{aligned}$$

Hence, by Lemma 1 we obtain

$$\begin{aligned}&\|\nabla^{\infty}(t\partial_t+r)^{m-1}u(t)\|_K\\ &\ll \sum_{j=0}^{m-1}\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K \ll \frac{1}{(r-1-2A)} B\theta_s(h\rho).\end{aligned} \tag{2.22}$$

This implies (2.21) for $j=m-1$. By (2.9) (with $j=m-2$) and (2.22) we have

$$\begin{aligned}\|\nabla^{\infty}(t\partial_t+r)^{m-2}u(t)\|_K &\ll \frac{1}{r(r-1-2A)} B\theta_s(h\rho)\\ &\ll \frac{1}{(r-1-2A)^2} B\theta_s(h\rho).\end{aligned} \tag{2.23}$$

This implies (2.21) for $j=m-2$. Thus, by induction on j, we can prove

$$\|\nabla^{\infty}(t\partial_t+r)^j u(t)\|_K \ll \frac{1}{(r-1-2A)^{m-j}} B\theta_s(h\rho) \tag{2.24}$$

for $j=m-3, \cdots, 1, 0$

in the same way as (2.23). (2.21) for $j=m$ is proved as follows: by (2.11), (2.19), (2.20), (2.22)$\sim$(2.24), (2.18) and $r-1-2A\geqq 1$ we have

$$\begin{aligned}&\|\nabla^{\infty}(t\partial_t+r)^m u(t)\|_K\\ &\ll A\theta_s(k\rho)\sum_{j=0}^{m-1}\frac{1}{(r-1-2A)^{m-j}} B\theta_s(h\rho)+B\theta_s(h\rho)\end{aligned}$$

$$\ll(2mA+1)B\theta_s(h\rho). \qquad \text{Q.E.D.}$$

For simplicity we write $\partial_\rho^i\theta_s(h\rho)=\partial_\rho^i(\theta_s(h\rho))$.

Lemma 2 is generalized to

LEMMA 3. *Assume that* $u, f\in C^\infty([0, T]\times K)$ *satisfy* (2.1), *that the estimates*

$$\|\nabla^\infty a_j\|_{0,K}\ll A\theta_s(k\rho), \quad j=0, 1, \cdots, m-1,$$
$$\|\nabla^\infty f(t)\|_K\ll\sum_{i=1}^d B_i\partial_\rho^i\theta_s(h\rho)$$

hold for some $s\geqq 1$, $A>0$, $d\in N$, $B_i>0$ *and* $h>2k>0$, *and that* $r\geqq 2+2A$ *holds. Then, we have*

$$\|\nabla^\infty(t\partial_t+r)^j u(t)\|_K\ll\frac{(1+2mA)}{(r-1-2A)^{m-j}}\sum_{i=1}^d B_i\partial_\rho^i\theta_s(h\rho)$$

on $[0, T]$ *for* $j=0, 1, \cdots, m$.

PROOF. Since

$$\theta_s(k\rho)\partial_\rho^i\theta_s(h\rho)\ll(1-k/h)^{-1}\partial_\rho^i\theta_s(h\rho)\ll 2\partial_\rho^i\theta_s(h\rho),$$

we can prove this in the same way as Lemma 2. Q.E.D.

The following lemma also plays an important role in the proof of Theorem 1.

LEMMA 4. *Let* $s\geqq 1$, $0<k<h$ *and* $i, j, l\in Z_+$. *Then, we have*

$$\partial_\rho^i\theta_s(k\rho)\times\partial_\rho^{j+l}\theta_s(h\rho)\ll\frac{(1-k/h)^{-1}}{\binom{i+j}{i}}\partial_\rho^{i+j+l}\theta_s(h\rho). \tag{2.25}$$

PROOF.

$$\begin{aligned}\partial_\rho^i\theta_s(k\rho)&\times\partial_\rho^{j+l}\theta_s(h\rho) \\ &=\sum_{p=0}^\infty k^{i+p}(i+p)!^s\frac{\rho^p}{p!}\times\sum_{q=0}^\infty h^{j+l+q}(j+l+q)!^s\frac{\rho^q}{q!} \\ &=\sum_{r=0}^\infty C_r h^{i+j+l+r}(i+j+l+r)!^s\frac{\rho^r}{r!},\end{aligned} \tag{2.26}$$

where

$$C_r=\sum_{p=0}^{r}\left(\frac{k}{h}\right)^{i+p}\frac{\binom{r}{p}}{\binom{i+j+l+r}{i+p}^s}.$$

Since

$$\binom{r}{p}\binom{i+j}{i}\leqq\binom{i+j+r}{i+p}\leqq\binom{i+j+l+r}{i+p}^s,$$

we have

$$C_r\leqq\frac{\sum_{p=0}^{r}\left(\frac{k}{h}\right)^{i+p}}{\binom{i+j}{i}}\leqq\frac{(1-k/h)^{-1}}{\binom{i+j}{i}}.$$

Hence, by applying this to (2.26) we obtain (2.25). Q.E.D.

§ 3. Proof of Theorem 1 under the condition $*=\{s\}$

Since $\mathcal{E}^{\{s\}}([0, T]\times \boldsymbol{R}^n)\subset C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ holds, to show Theorem 1 under the condition $*=\{s\}$ it is sufficient to prove the following:

PROPOSITION 2. *Let* $s\geqq 1$ *and let* P *be the operator in* (1.1) *satisfying* (C_1). *Assume* $a_{j,\alpha}(t, x)\in\mathcal{E}^{\{s\}}([0, T]\times \boldsymbol{R}^n)$ ($j+|\alpha|\leqq m$ *and* $j<m$) *and*

$$1\leqq s\leqq \min\left[\infty, \min_{(j,\alpha)\in S}\left(\frac{m-j-l(j,\alpha)}{|\alpha|-l(j,\alpha)}\right)\right]. \tag{3.1}$$

Then, if $u\in C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ *satisfies* $Pu\in\mathcal{E}^{\{s\}}([0, T]\times \boldsymbol{R}^n)$, *we have* $u\in\mathcal{E}^{\{s\}}([0, T]\times \boldsymbol{R}^n)$.

PROOF. It is easy to see that $u\in\mathcal{E}^{\{s\}}([0, T]\times \boldsymbol{R}^n)$ is equivalent to the following: for any compact subset K of $\boldsymbol{R}^n$, there are $C_0>0$ and $H_0>0$ such that

$$\|\nabla^\infty\partial_t^i u\|_{0,K}\ll C_0^{1+i}\partial_\rho^i\theta_s(H_0\rho) \quad \text{for any } i\in \boldsymbol{Z}_+. \tag{3.2}$$

Let us prove this from now.

Let $u\in C^\infty([0, T], \mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ and assume that $Pu\in\mathcal{E}^{\{s\}}([0, T]\times \boldsymbol{R}^n)$ holds. Take any compact subset K of $\boldsymbol{R}^n$ and fix it hereafter.
Put

$$a_j(x)=[t^{l(j,0)}a_{j,0}(t,x)]|_{t=0},\quad j=0,1,\cdots,m-1$$

and define $C(\rho,x)$ by

$$C(\rho,x)=\rho^m+\sum_{j<m}a_j(x)\rho^j. \tag{3.3}$$

Then, $a_j(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ $(j<m)$ and therefore

$$\|\nabla^\infty a_j\|_{0,K}\ll A_0\theta_s(k_0\rho),\quad j=0,1,\cdots,m-1 \tag{3.4}$$

for some $A_0>0$ and $k_0>0$. Hence, if $r\geqq 2+2A_0$ holds, we can apply Lemma 3 to the equation of the form $C(t\partial_t+r,x)U=F$.

Take an $r\in N$ satisfying $r\geqq 2+2A_0$ and fix it hereafter. Since $u\in C^\infty([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ holds, $u(t,x)$ is expressed in the form

$$u(t,x)=\sum_{i=0}^{r-1}u_i(x)t^i+t^rU(t,x) \tag{3.5}$$

for some $u_i(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ and $U(t,x)\in C^\infty([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$.
Therefore, to prove (3.2) it is sufficient to show the following:

$$\|\nabla^\infty\partial_t^iU\|_{0,K}\ll C_1^{1+i}\partial_\rho^i\theta_s(H_1\rho)\qquad\text{for any } i\in\boldsymbol{Z}_+ \tag{3.6}$$

for some $C_1>0$ and $H_1>0$. Since $Pu\in\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$ is assumed, by simple calculation we obtain the following equation

$$(t\partial_t+r)^mU+\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}t^{l(j,\alpha)}a_{j,\alpha}(t,x)(t\partial_t+r)^j\partial_x^\alpha U=F \tag{3.7}$$

for some $F\in\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$ (note that F is determined by Pu and u_i $(i=0,1,\cdots,r-1)$). Since $F\in\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$, we may assume that

$$\|\nabla^\infty\partial_t^iF(t)\|_K\ll B^{1+i}\partial_\rho^i\theta_s(h\rho)\qquad\text{for any } i\in\boldsymbol{Z}_+ \tag{3.8}$$

holds for some $B>0$ and $h>0$.

Define $p(j,\alpha)\in N$ $(j+|\alpha|\leqq m$ and $j<m)$ by

$$p(j,\alpha)=\begin{cases}1, & \text{when } |\alpha|=0,\\ \min\{|\alpha|,l(j,\alpha)\}, & \text{when } |\alpha|>0\end{cases} \tag{3.9}$$

and put

$$d=\max\left\{\frac{|\alpha|-p(j,\alpha)}{p(j,\alpha)};\ j+|\alpha|\leqq m \text{ and } j<m\right\}. \tag{3.10}$$

Note that $d\geqq 0$ holds. Then, we have

LEMMA 5. *Let l, $\mu\in Z_+$ such that $l\geqq p(j,\alpha)$ and $\mu\leqq d(l-p(j,\alpha))$. Then*:

$$(3.11)\qquad (1+l)^{j+p(j,\alpha)}\leqq\begin{cases}(1+l)^m, & \text{when } |\alpha|=0,\\ (1+d)^{ms}(1+l)^m\dfrac{\mu!^s}{(\mu+|\alpha|-p(j,\alpha))!^s}, & \text{when } |\alpha|>0.\end{cases}$$

PROOF OF LEMMA 5. When $|\alpha|=0$, we have $p(j,\alpha)=1$ (by (3.9)) and $j<m$, and hence $j+p(j,\alpha)\leqq m$. This leads us to (3.11).

When $|\alpha|>0$, by (3.9) and (3.1) we have

$$m-j-p(j,\alpha)\geqq s(|\alpha|-p(j,\alpha))\geqq 0$$

and therefore

$$(3.12)\qquad (1+l)^{j+p(j,\alpha)}\leqq(1+l)^m\left\{\frac{1}{(1+l)^{|\alpha|-p(j,\alpha)}}\right\}^s.$$

Since $\mu\leqq d(l-p(j,\alpha))$ and $|\alpha|-p(j,\alpha)\leqq dp(j,\alpha)$ hold, we have $\mu+|\alpha|-p(j,\alpha)\leqq dl\leqq(1+d)(1+l)$ and therefore

$$(1+l)\geqq\left(\frac{1}{1+d}\right)(\mu+|\alpha|-p(j,\alpha)).$$

Hence, we have

$$(3.13)\qquad \begin{aligned}&(1+l)^{|\alpha|-p(j,\alpha)}\\ &\geqq\left(\frac{1}{1+d}\right)^{|\alpha|-p(j,\alpha)}(\mu+|\alpha|-p(j,\alpha))^{|\alpha|-p(j,\alpha)}\\ &\geqq\left(\frac{1}{1+d}\right)^m\frac{(\mu+|\alpha|-p(j,\alpha))!}{\mu!}.\end{aligned}$$

Thus, by (3.12) and (3.13), we obtain (3.11). Q.E.D.

Since $t^{l(j,\alpha)}a_{j,\alpha}(t,x)\in\mathcal{E}^{\{s\}}([0,T]\times R^n)$ is expressed in the form

$$(3.14)\qquad t^{l(j,\alpha)}a_{j,\alpha}(t,x)=\begin{cases}a_j(x)-t^{p(j,\alpha)}b_{j,\alpha}(t,x), & \text{when } |\alpha|=0,\\ -t^{p(j,\alpha)}b_{j,\alpha}(t,x), & \text{when } |\alpha|>0\end{cases}$$

for $a_j(x)$ in (3.3) and for some $b_{j,\alpha}(t,x)\in\mathcal{E}^{\{s\}}([0,T]\times R^n)$, by (3.7) we have

$$(3.15)\qquad C(t\partial_t+r,x)U=\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}t^{p(j,\alpha)}b_{j,\alpha}(t,x)(t\partial_t+r)^j\partial_x^\alpha U+F.$$

It is easy to see that the equality

$$\partial_t^{p(j,\alpha)}(t^{p(j,\alpha)}b_{j,\alpha}(t,x)(t\partial_t+r)^j\partial_x^\alpha U)=\sum_{\nu=0}^{p(j,\alpha)}b_{j,\alpha,\nu}(t,x)(t\partial_t+r)^{j+\nu}\partial_x^\alpha U \tag{3.16}$$

holds for some $b_{j,\alpha,\nu}(t,x)\in\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$. Therefore, by (3.15) and (3.16) we see that for any $l\geqq\max\{p(j,\alpha);\,j+|\alpha|\leqq m$ and $j<m\}$

$$\begin{aligned}
C(t\partial_t+r+l,x)\partial_t^l U
&=\partial_t^l\Big[\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}t^{p(j,\alpha)}b_{j,\alpha}(t,x)(t\partial_t+r)^j\partial_x^\alpha U\Big]+\partial_t^l F\\
&=\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}\partial_t^{l-p(j,\alpha)}\Big[\sum_{\nu=0}^{p(j,\alpha)}b_{j,\alpha,\nu}(t,x)(t\partial_t+r)^{j+\nu}\partial_x^\alpha U\Big]+\partial_t^l F\\
&=\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}\sum_{\nu=0}^{p(j,\alpha)}\sum_{i=0}^{l-p(j,\alpha)}\binom{l-p(j,\alpha)}{i}\partial_t^{l-p(j,\alpha)-i}(b_{j,\alpha,\nu})\\
&\qquad\times(t\partial_t+r+i)^{j+\nu}\partial_t^i\partial_x^\alpha U+\partial_t^l F.
\end{aligned} \tag{3.17}$$

Since $b_{j,\alpha,\nu}(t,x)\in\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$, we may assume that

$$\|\nabla^\infty\partial_t^i b_{j,\alpha,\nu}\|_{0,K}\ll A^{1+i}\partial_\rho^i\theta_s(k\rho)\quad\text{for any } i\in\boldsymbol{Z}_+ \tag{3.18}$$

holds for some $A>0$ and $k>0$.

Under the above situation, we have

LEMMA 6. *There are $C>0$ and $H>0$ such that for any $i\in\boldsymbol{Z}_+$ we have*

$$\|\nabla^\infty(t\partial_t+r+i)^j\partial_t^i U(t)\|_K\ll(1+i)^jC^{1+i}\sum_{\mu=0}^{[di]}\frac{1}{\mu!^s}\partial_\rho^{i+\mu}\theta_s(H\rho) \tag{3.19}$$
$$\text{on } [0,T] \text{ for } j=0,1,\cdots,m,$$

where d is the same as in (3.10) *and $[di]$ means the integer-part of di.*

PROOF OF LEMMA 6. Let A_0, k_0 be as in (3.4), let B, h be as in (3.8) and let A, k be as in (3.18). Choose $C>0$ and $H>0$ sufficiently large so that

$$\begin{aligned}
&C\geqq\max\{1,2A,((n+2)^m4A(1+m)(1+d)^{ms}+B)(1+2mA_0)\},\\
&H\geqq\max\{1,2k_0,2k,h\}.
\end{aligned} \tag{3.20}$$

Since $U\in C^\infty([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ is known and C, H are sufficiently large, we may assume that (3.19) is satisfied for $i=0,1,\cdots,m$.

Under these conditions, we will show by induction that (3.19) is valid for any $i\geqq m+1$.

Now, let $l\geqq m+1$ and assume that (3.19) is already known for $i=0, 1, \cdots, l-1$. Our aim is to show that (3.19) is true also for $i=l$.

Denote by G_l the right hand side of (3.17). Then,

$$C(t\partial_t+r+l, x)\partial_t^l U=G_l. \tag{3.21}$$

If we know

$$\|\nabla^\infty G_l(t)\|_K \ll \frac{(1+l)^m}{(1+2mA_0)} C^{1+l}\sum_{\mu=0}^{[dl]}\frac{1}{\mu!^s}\partial_\rho^{l+\mu}\theta_s(H\rho), \tag{3.22}$$

by applying Lemma 3 to (3.21) we can easily see that (3.19) is true for $i=l$. Thus, to prove (3.19) for $i=l$ it is sufficient to prove (3.22). Let us show this from now.

Since G_l is the right hand side of (3.17), by using (2.4) and (2.5) we have

$$\begin{aligned}&\|\nabla^\infty G_l(t)\|_K \\ &\ll \sum_{\substack{j+|\alpha|\leqq m\\ j<m}}\sum_{\nu=0}^{p(j,\alpha)}\sum_{i=0}^{l-p(j,\alpha)}\left\{\binom{l-p(j,\alpha)}{i}\|\nabla^\infty\partial_t^{l-p(j,\alpha)-i}b_{j,\alpha,\nu}\|_{0,K}\right.\\ &\left.\times\partial_\rho^{|\alpha|}\|\nabla^\infty(t\partial_t+r+i)^{j+\nu}\partial_t^i U(t)\|_K\right\}+\|\nabla^\infty\partial_t^l F(t)\|_K.\end{aligned} \tag{3.23}$$

Note that in (3.23) we have $j+\nu\leqq j+p(j,\alpha)\leqq j+|\alpha|\leqq m$ and $i\leqq l-p(j,\alpha)\leqq l-1$. Therefore, we can estimate the right hand side of (3.23) by using (3.8), (3.18) and the induction hypothesis (that is, (3.19) for $i=0, 1, \cdots, l-1$). Hence,

$$\begin{aligned}&\|\nabla^\infty G_l(t)\|_K \\ &\ll \sum_{\substack{j+|\alpha|\leqq m\\ j<m}}\sum_{\nu=0}^{p(j,\alpha)}\sum_{i=0}^{l-p(j,\alpha)}\left\{\binom{l-p(j,\alpha)}{i}A^{1+l-p(j,\alpha)-i}\partial_\rho^{l-p(j,\alpha)-i}\theta_s(k\rho)\right.\\ &\left.\times(1+i)^{j+\nu}C^{1+i}\sum_{\mu=0}^{[di]}\frac{1}{\mu!^s}\partial_\rho^{i+\mu+|\alpha|}\theta_s(H\rho)\right\}+B^{1+l}\partial_\rho^l\theta_s(h\rho).\end{aligned} \tag{3.24}$$

Note that the estimates

$$(1+i)^{j+\nu}\leqq(1+i)^{j+p(j,\alpha)}\leqq(1+l)^{j+p(j,\alpha)},$$
$$[di]\leqq[d(l-p(j,\alpha))]$$

hold in (3.24) and that by Lemma 4 we have

$$\binom{l-p(j,\alpha)}{i}\partial_\rho^{l-p(j,\alpha)-i}\theta_s(k\rho)\times\partial_\rho^{i+\mu+|\alpha|}\theta_s(H\rho)$$
$$\ll(1-k/H)^{-1}\partial_\rho^{l-p(j,\alpha)+\mu+|\alpha|}\theta_s(H\rho)$$
$$\ll 2\partial_\rho^{l-p(j,\alpha)+\mu+|\alpha|}\theta_s(H\rho).$$

Applying them to (3.24) we have

(3.25)
$$\|\nabla^\infty G_l(t)\|_K \ll \sum_{\substack{j+|\alpha|\leqq m\\ j<m}}\sum_{\nu=0}^{p(j,\alpha)}\sum_{i=0}^{l-p(j,\alpha)}\left\{2A\left(\frac{A}{C}\right)^{l-p(j,\alpha)-i}(1+l)^{j+p(j,\alpha)}C^{1+l-p(j,\alpha)}\right.$$
$$\left.\times\sum_{\mu=0}^{[d(l-p(j,\alpha))]}\frac{1}{\mu!^s}\partial_\rho^{l-p(j,\alpha)+\mu+|\alpha|}\theta_s(H\rho)\right\}+B^{1+l}\partial_\rho^l\theta_s(h\rho).$$

Since $p(j,\alpha)\leqq|\alpha|\leqq m$, $A/C\leqq 1/2$ and $C\geqq 1$ hold, we have

$$\sum_{\nu=0}^{p(j,\alpha)}1\leqq(1+p(j,\alpha))\leqq(1+m),$$
$$\sum_{i=0}^{l-p(j,\alpha)}\left(\frac{A}{C}\right)^{l-p(j,\alpha)-i}\leqq(1-A/C)^{-1}\leqq 2,$$
$$C^{1+l-p(j,\alpha)}\leqq C^l.$$

Hence, by (3.25) we obtain

(3.26)
$$\|\nabla^\infty G_l(t)\|_K\ll\sum_{\substack{j+|\alpha|\leqq m\\ j<m}}\{4A(1+m)C^l\Phi_{j,\alpha}\}+B^{1+l}\partial_\rho^l\theta_s(h\rho),$$

where

(3.27)
$$\Phi_{j,\alpha}=(1+l)^{j+p(j,\alpha)}\sum_{\mu=0}^{[d(l-p(j,\alpha))]}\frac{1}{\mu!^s}\partial_\rho^{l+\mu+|\alpha|-p(j,\alpha)}\theta_s(H\rho).$$

Note the following lemma.

LEMMA 7. *In the above situation, we have*

(3.28)
$$\Phi_{j,\alpha}\ll(1+d)^{ms}(1+l)^m\sum_{\mu=0}^{[dl]}\frac{1}{\mu!^s}\partial_\rho^{l+\mu}\theta_s(H\rho).$$

The proof of this will be given later. By applying Lemma 7 to (3.26) we have

$$\|\nabla^{\infty}G_l(t)\|_K$$
$$\ll \sum_{\substack{j+|\alpha|\leqq m \\ j<m}} \left\{4A(1+m)C^l(1+d)^{ms}(1+l)^m \sum_{\mu=0}^{[dl]} \frac{1}{\mu!^s}\partial_\rho^{l+\mu}\theta_s(H\rho)\right\} + B^{1+l}\partial_\rho^l\theta_s(h\rho).$$

Hence, by using the assumptions $C\geqq B$ and $H\geqq h$ we obtain

$$\|\nabla^{\infty}G_l(t)\|_K \tag{3.29}$$
$$\ll((n+2)^m 4A(1+m)(1+d)^{ms}+B)C^l(1+l)^m \sum_{\mu=0}^{[dl]}\frac{1}{\mu!^s}\partial_\rho^{l+\mu}\theta_s(H\rho).$$

Thus, by (3.20) and (3.29) we obtain (3.22) which completes the induction.

Q.E.D.

PROOF OF LEMMA 7. When $|\alpha|=0$, by applying Lemma 5 to (3.27) we have

$$\Phi_{j,\alpha}\ll(1+l)^m \sum_{\mu=0}^{[d(l-1)]}\frac{1}{\mu!^s}\partial_\rho^{l+\mu-1}\theta_s(H\rho). \tag{3.30}$$

Since $H\geqq 1$, we have

$$\partial_\rho^{l+\mu-1}\theta_s(H\rho)\ll\partial_\rho^{l+\mu}\theta_s(H\rho). \tag{3.31}$$

Hence, by (3.30) and (3.31) we obtain

$$\Phi_{j,\alpha}\ll(1+l)^m \sum_{\mu=0}^{[d(l-1)]}\frac{1}{\mu!^s}\partial_\rho^{l+\mu}\theta_s(H\rho).$$

This leads us to (3.28).

When $|\alpha|>0$, by applying Lemma 5 to (3.27) we have

$$\Phi_{j,\alpha}\ll(1+d)^{ms}(1+l)^m \sum_{\mu=0}^{[d(l-p(j,\alpha))]}\frac{1}{(\mu+|\alpha|-p(j,\alpha))!^s}\partial_\rho^{l+\mu+|\alpha|-p(j,\alpha)}\theta_s(H\rho).$$

Since $0\leqq\mu\leqq[d(l-p(j,\alpha))]$ and $|\alpha|-p(j,\alpha)\leqq dp(j,\alpha)$ (by (3.10)) hold, by putting $\nu=\mu+|\alpha|-p(j,\alpha)$ we have $0\leqq\nu\leqq[dl]$. Hence, we obtain

$$\Phi_{j,\alpha}\ll(1+d)^{ms}(1+l)^m \sum_{\nu=0}^{[dl]}\frac{1}{\nu!^s}\partial_\rho^{l+\nu}\theta_s(H\rho).$$

Q.E.D.

Now, let us complete the proof of Proposition 2. By Lemma 6 we have

(3.32) $$\|\nabla^{\infty}\partial_t^i U\|_{0,K} \ll C^{1+i} \sum_{\mu=0}^{[di]} \frac{1}{\mu!^s} \partial_\rho^{i+\mu} \theta_s(H\rho) \quad \text{for any } i \in \boldsymbol{Z}_+.$$

Since

$$\partial_\rho^{i+\mu}\theta_s(H\rho) \ll (2^s H)^\mu (\mu!)^s \partial_\rho^i \theta_s(2^s H\rho),$$

$$\sum_{\mu=0}^{[di]} (2^s H)^\mu = \frac{(2^s H)^{[di]+1}-1}{2^s H-1} \leqq (2^s H)^{di+1} \leqq (2^s H)^{(1+d)(1+i)}$$

hold for any $i \in \boldsymbol{Z}_+$, by (3.32) we obtain

$$\|\nabla^{\infty}\partial_t^i U\|_{0,K} \ll ((2^s H)^{1+d} C)^{1+i} \partial_\rho^i \theta_s(2^s H\rho) \quad \text{for any } i \in \boldsymbol{Z}_+.$$

This leads us to (3.6) and hence (3.2). Thus, the proof of Proposition 2 is completed. Q.E.D.

§ 4. Proof of Theorem 1 under the condition $*=(s)$

As in § 3, it is sufficient to prove the following:

PROPOSITION 3. *Let* $s>1$ *and let* P *be the operator in* (1.1) *satisfying* (C_1). *Assume* $a_{j,\alpha}(t,x) \in \mathcal{E}^{(s)}([0,T]\times \boldsymbol{R}^n)$ $(j+|\alpha| \leqq m$ *and* $j<m)$ *and*

(4.1) $$1<s\leqq \min\left[\infty, \min_{(j,\alpha)\in S}\left(\frac{m-j-l(j,\alpha)}{|\alpha|-l(j,\alpha)}\right)\right].$$

Then, if $u \in C^{\infty}([0,T], \mathcal{E}^{(s)}(\boldsymbol{R}^n))$ *satisfies* $Pu \in \mathcal{E}^{(s)}([0,T]\times \boldsymbol{R}^n)$, *we have* $u \in \mathcal{E}^{(s)}([0,T]\times \boldsymbol{R}^n)$.

Before the proof, let us present some preliminaries. Throughout this section we assume $s>1$.

Let K be a compact subset of $\boldsymbol{R}^n$. It is easy to see that $w(t,x) \in \mathcal{E}^{(s)}([0,T]\times K)$ is equivalent to the following: for any $\varepsilon>0$ there is a $C_\varepsilon>0$ such that

$$\|\nabla^{\infty}\partial_t^i w\|_{0,K} \ll C_\varepsilon \partial_\rho^i \theta_s(\varepsilon\rho) \quad \text{for any } i \in \boldsymbol{Z}_+.$$

For a positive-valued function $C(\varepsilon)$ in $\varepsilon>0$, we write

$$\theta_{(s)}(\rho; C(\varepsilon)) = \sum_{q=0}^{\infty}\left(\inf_{\varepsilon>0}(C(\varepsilon)\varepsilon^q)\right)(q!)^s \frac{\rho^q}{q!}.$$

Then, $w(t,x) \in \mathcal{E}^{(s)}([0,T]\times K)$ is equivalent to the condition that

$$\|\nabla^\infty \partial_t^i w\|_{0,K} \ll \partial_\rho^i \theta_{(s)}(\rho; C(\varepsilon)) \quad \text{for any } i \in \boldsymbol{Z}_+$$

for some $C(\varepsilon)$. Note that by putting $A=C(1)$ and $C_0(\varepsilon)=C(\varepsilon)/A$ we have $\theta_{(s)}(\rho; C(\varepsilon))=A\theta_{(s)}(\rho; C_0(\varepsilon))$ and $C_0(1)=1$.

Note the following lemma:

LEMMA 8. *Let $s>1$. Let $C_0(\varepsilon)$ and $C(\varepsilon)$ be two positive-valued functions in $\varepsilon>0$ satisfying the following*:

$$(4.2)\quad \left(\inf_{\varepsilon>0}(C_0(\varepsilon)\varepsilon^p)\right)\frac{(p!)^s}{p!}\times\left(\inf_{\varepsilon>0}(C(\varepsilon)\varepsilon^q)\right)\frac{(q!)^s}{q!}\leqq\left(\inf_{\varepsilon>0}(C(\varepsilon)\varepsilon^{p+q})\right)\frac{(p+q)!^s}{(p+q)!}$$

for any $p, q \in \boldsymbol{Z}_+$.

Then we have for any $i, j, l \in \boldsymbol{Z}_+$

$$(4.3)\qquad \partial_\rho^i \theta_{(s)}(\rho; C_0(\varepsilon)) \times \partial_\rho^{j+l}\theta_{(s)}(2\rho; C(\varepsilon)) \ll \frac{2}{\binom{i+j}{i}}\partial_\rho^{i+j+l}\theta_{(s)}(2\rho; C(\varepsilon)).$$

This guarantees that the discussion in §2 with $\theta_s(k\rho)$ [resp. $\theta_s(h\rho)$] replaced by $\theta_{(s)}(\rho; C_0(\varepsilon))$ [resp. $\theta_{(s)}(2\rho; C(\varepsilon))$] is also valid in the case $*=(s)$.

PROOF OF PROPOSITION 3. Let $u \in C^\infty([0, T], \mathcal{E}^{(s)}(\boldsymbol{R}^n))$ and assume that $Pu \in \mathcal{E}^{(s)}([0, T]\times \boldsymbol{R}^n)$ holds. Let $a_j(x)$ $(j=0, 1, \cdots, m-1)$ be as in (3.3). Take any compact subset K of $\boldsymbol{R}^n$. Then, since $a_j(x) \in \mathcal{E}^{(s)}(\boldsymbol{R}^n)$ $(j=0, 1, \cdots, m-1)$, we can find $A_0>0$ and a positive-valued function $C_1(\varepsilon)$ in $\varepsilon>0$ such that $C_1(1)=1$ and

$$\|\nabla^\infty a_j\|_{0,K} \ll A_0\theta_{(s)}(\rho; C_1(\varepsilon)),\ j=0, 1, \cdots, m-1.$$

Take an $r \in N$ satisfying $r \geqq 2+2A_0$, and define $U(t, x)$, $F(t, x)$, $b_{j,\alpha,\nu}(t, x)$ by (3.5), (3.7), (3.16), respectively. Since $F \in \mathcal{E}^{(s)}([0, T]\times \boldsymbol{R}^n)$ and $b_{j,\alpha,\nu} \in \mathcal{E}^{(s)}([0, T]\times \boldsymbol{R}^n)$ hold, we have

$$\|\nabla^\infty \partial_t^i b_{j,\alpha,\nu}\|_{0,K} \ll A^{1+i}\partial_\rho^i\theta_{(s)}(\rho; C_2(\varepsilon)) \qquad \text{for any } i \in \boldsymbol{Z}_+,$$
$$\|\nabla^\infty \partial_t^i F(t)\|_K \ll B^{1+i}\partial_\rho^i\theta_{(s)}(\rho; C_2(\varepsilon)) \qquad \text{for any } i \in \boldsymbol{Z}_+$$

for some $A>0$, $B>0$ and some positive-valued function $C_2(\varepsilon)$ in $\varepsilon>0$ satisfying $C_2(1)=1$. Also, since $U \in C^\infty([0, T], \mathcal{E}^{(s)}(\boldsymbol{R}^n))$ is known,

$$\|\nabla^\infty \partial_t^i U(t)\|_K \ll \partial_\rho^i\theta_{(s)}(\rho; C_3(\varepsilon)) \qquad \text{for}\quad i=0, 1, \cdots, m$$

holds for some positive-valued function $C_3(\varepsilon)$ in $\varepsilon>0$.

Put

$$C_0(\varepsilon)=\max\{C_1(\varepsilon), C_2(\varepsilon)\}$$

and note the fact: $C_0(1)=1$. Then:

LEMMA 9. *For the above $C_0(\varepsilon)$ and $C_3(\varepsilon)$, we can find a positive-valued function $C(\varepsilon)$ in $\varepsilon>0$ satisfying the following*:

1) $\theta_{(s)}(\rho; C_i(\varepsilon))\ll\theta_{(s)}(\rho; C(\varepsilon))$ *for* $i=0, 3$.
2) $C(\varepsilon)$ *and* $C_0(\varepsilon)$ *satisfy* (4.2).

Since $s>1$ and $C_0(1)=1$ are assumed, this is a corollary of Tahara [10, Lemma 3].

Now, by Lemma 9 we have

$$\begin{aligned}
&\|\nabla^\infty a_j\|_{0,K}\ll A_0\theta_{(s)}(\rho; C_0(\varepsilon)),\ j=0, 1, \cdots, m-1,\\
&\|\nabla^\infty\partial_t^i b_{j,\alpha,\nu}\|_{0,K}\ll A^{1+i}\partial_\rho^i\theta_{(s)}(\rho; C_0(\varepsilon)) \quad \text{for any } i\in \boldsymbol{Z}_+,\\
&\|\nabla^\infty\partial_t^i F(t)\|_K\ll B^{1+i}\partial_\rho^i\theta_{(s)}(2\rho; C(\varepsilon)) \quad \text{for any } i\in \boldsymbol{Z}_+,\\
&\|\nabla^\infty\partial_t^i U(t)\|_K\ll\partial_\rho^i\theta_{(s)}(2\rho; C(\varepsilon)) \quad \text{for } i=0, 1, \cdots, m.
\end{aligned}$$

Since $C(\varepsilon)$ and $C_0(\varepsilon)$ satisfy (4.2), by Lemma 8 we can see that the discussion in § 3 is also valid in this case, if we replace $\theta_s(k_0\rho)$, $\theta_s(k\rho)$, $\theta_s(h\rho)$ and $\theta_s(H\rho)$ by $\theta_{(s)}(\rho; C_0(\varepsilon))$, $\theta_{(s)}(\rho; C_0(\varepsilon))$, $\theta_{(s)}(2\rho; C(\varepsilon))$ and $\theta_{(s)}(2\rho; C(\varepsilon))$, respectively. Hence, by the argument quite parallel to that in § 3 we see as in (3.32) that for some $C_*>0$

$$\|\nabla^\infty\partial_t^i U\|_{0,K}\ll C_*^{1+i}\sum_{\mu=0}^{[di]}\frac{1}{\mu!^s}\partial_\rho^{1+\mu}\theta_{(s)}(2\rho; C(\varepsilon)) \qquad \text{for any } i\in \boldsymbol{Z}_+,$$

which gives

$$\begin{aligned}
&\|\nabla^\infty\partial_t^i U\|_{0,K}\ll 2C(\varepsilon)C_*^{1+i}\partial_\rho^i\theta_s(2^{s+1}\varepsilon\rho)\\
&\text{for any } i\in \boldsymbol{Z}_+ \text{ and } 0<\varepsilon<(1/2)^{s+2}.
\end{aligned}$$

This is equivalent to $U\in\mathcal{E}^{(s)}([0, T]\times K)$. Thus, we obtain $u\in\mathcal{E}^{(s)}([0, T]\times K)$. Q.E.D.

§ 5. Proof of Proposition 1

Put

$$C(\lambda)=\lambda^m+b_{m-1}\lambda^{m-1}+\cdots+b_0.$$

Since $b_i \in \boldsymbol{R}$ $(i=0, 1, \cdots, m-1)$ is assumed (by d_1)), we can choose $N \in \boldsymbol{N}$ sufficiently large so that $C(N+k)>0$ holds for any $k \in \boldsymbol{Z}_+$. By d_2), $C(\lambda) \neq 0$ is assumed for any $\lambda \in \boldsymbol{Z}_+$.

The following lemma is easy.

LEMMA 10. *Let us consider*

$$C(t\partial_t)w = ct^l(t\partial_t)^j\partial_x^\alpha w + t^N\varphi(x), \tag{5.1}$$

where $l \in \{1, 2, \cdots\}$, $j+|\alpha| \leqq m$, $c>0$ *and* $\varphi(x) \in C^\infty(\boldsymbol{R}^n)$. *Then,* (5.1) *has a unique solution* $w(t, x)$ *in* $(C^\infty(\boldsymbol{R}_x^n))[[t]]$ *and it is given by*

$$w = \sum_{q=0}^{\infty}\left(\frac{c^q N^j(N+l)^j \cdots (N+(q-1)l)^j}{C(N)C(N+l)\cdots C(N+ql)}\partial_x^{q\alpha}\varphi(x)\right)t^{N+ql}. \tag{5.2}$$

By using Lemma 10, let us give a proof of Proposition 1.

Let $s>1$, let L be the operator in (1.5) satisfying $d_1)\sim d_4)$, and assume the conditions i) and ii). Then, to show Proposition 1 it is sufficient to prove the following: if (1.6) does not hold, there appears a contradiction.

PROOF OF CASE: $*=\{s\}$. Take a $\varphi(x) \in \mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ such that $\partial_x^\alpha\varphi(0)>0$ for any $\alpha \in \boldsymbol{Z}_+^n$ and that

$$0 < \limsup_{|\alpha|\to\infty}\left[\left(\frac{\partial_x^\alpha\varphi(0)}{|\alpha|!^s}\right)^{1/|\alpha|}\right] < \infty. \tag{5.3}$$

Then, the well-posedness in $\mathcal{E}^{\{s\}}([0, T]\times\boldsymbol{R}^n)$ implies that the equation

$$Lu = t^N\varphi(x) \tag{5.4}$$

has a unique solution $u(t, x) \in \mathcal{E}^{\{s\}}([0, T]\times\boldsymbol{R}^n)$. Since $d_1)\sim d_3)$ are assumed and since $\partial_x^\alpha\varphi(0)>0$ for any $\alpha \in \boldsymbol{Z}_+^n$, it is easy to see that $\partial_t^k u(0, 0) \geqq 0$ for any $k \in \boldsymbol{Z}_+$. Moreover, since $u(t, x)$ belongs to $\mathcal{E}^{\{s\}}([0, T]\times\boldsymbol{R}^n)$, we obtain

$$\limsup_{k\to\infty}\left[\left(\frac{\partial_t^k u(0, 0)}{k!^s}\right)^{1/k}\right] < \infty. \tag{5.5}$$

Now, suppose that (1.6) does not hold. Then, there is a $(j, \alpha) \in \mathcal{S}_L$ such that

$$s > \frac{m-j-l(j, \alpha)}{|\alpha|-l(j, \alpha)}$$

which is equivalent to

$$s<1+\frac{j+s|\alpha|-m}{l(j,\alpha)}. \tag{5.6}$$

Take such a $(j,\alpha)\in\mathcal{S}_L$ and let us consider

$$C(t\partial_t)w=c_{j,\alpha}t^{l(j,\alpha)}(t\partial_t)^j\partial_x^\alpha w+t^N\varphi(x). \tag{5.7}$$

By Lemma 10 we have a unique formal solution $w(t,x)$, and by using (5.2) and (5.3) we easily see that

$$\limsup_{k\to\infty}\left[\left(\frac{\partial_t^k w(0,0)}{k!^\sigma}\right)^{1/k}\right]<\infty, \tag{5.8}$$

if and only if

$$\sigma\geqq 1+\frac{j+s|\alpha|-m}{l(j,\alpha)}. \tag{5.9}$$

Here, let us compare $u(t,0)$ with $w(t,0)$. Since $C(N+k)>0$ for any $k\in \boldsymbol{Z}_+$, since $c_{j,\alpha}\geqq 0$ for any (j,α), and since $\partial_x^\alpha\varphi(0)>0$ for any $\alpha\in \boldsymbol{Z}_+^n$, by comparing (5.4) with (5.7) it is easy to see that

$$0\ll w(t,0)\ll u(t,0) \quad \text{in} \quad \boldsymbol{C}[[t]] \tag{5.10}$$

as formal power series in t. Thus, by (5.5), (5.8), (5.9) and (5.10) we obtain

$$s\geqq 1+\frac{j+s|\alpha|-m}{l(j,\alpha)}$$

which contradicts the condition (5.6). Q.E.D.

PROOF OF CASE: $*=(s)$. Suppose that (1.6) does not hold. Then, there is a $(j,\alpha)\in\mathcal{S}_L$ such that

$$s>\frac{m-j-l(j,\alpha)}{|\alpha|-l(j,\alpha)}.$$

Take such a $(j,\alpha)\in\mathcal{S}_L$ and fix it. Since $|\alpha|>0$ and $|\alpha|-l(j,\alpha)>0$, by choosing s_1 sufficiently close to s we can find $s_1>1$ such that $s>s_1$ and

$$s-\frac{|\alpha|(s-s_1)}{|\alpha|-l(j,\alpha)}>\frac{m-j-l(j,\alpha)}{|\alpha|-l(j,\alpha)}$$

which is equivalent to

$$s<1+\frac{j+s_1|\alpha|-m}{l(j,\alpha)}. \tag{5.11}$$

Take a $\varphi(x)\in\mathcal{E}^{\{s_1\}}(\boldsymbol{R}^n)\subset\mathcal{E}^{(s)}(\boldsymbol{R}^n)$ such that $\partial_x^\alpha\varphi(0)>0$ for any $\alpha\in\boldsymbol{Z}_+^n$ and that (5.3) with s replaced by s_1 holds. Since the well-posedness in $\mathcal{E}^{(s)}([0,T]\times\boldsymbol{R}^n)$ is assumed, the equation $Lu=t^N\varphi(x)$ has a unique solution $u(t,x)\in\mathcal{E}^{(s)}([0,T]\times\boldsymbol{R}^n)$. Since $\mathcal{E}^{(s)}([0,T]\times\boldsymbol{R}^n)\subset\mathcal{E}^{\{s\}}([0,T]\times\boldsymbol{R}^n)$ holds, we also have the same kind of estimate as (5.5). Then, by the same argument as in (5.7), (5.8), (5.10) and (5.9) with s replaced by s_1 we can obtain

$$s\geqq 1+\frac{j+s_1|\alpha|-m}{l(j,\alpha)}.$$

This contradicts the condition (5.11). Q.E.D.

§6. Application

In [8, 9, 10], the author investigated Fuchsian hyperbolic operators P in (1.1) under (C_1) and the following:

(C_2) There are $\kappa_1>0,\cdots,\kappa_n>0$ such that

$$l(j,\alpha)=\kappa_1\alpha_1+\cdots+\kappa_n\alpha_n,\quad \text{if } j+|\alpha|=m \text{ and } a_{j,\alpha}\not\equiv 0.$$

(C_3) All the roots of the equation in λ

$$\lambda^m+\sum_{\substack{j+|\alpha|=m\\ j<m}}a_{j,\alpha}(t,x)\lambda^j\xi^\alpha=0$$

are *real*, *simple* and *bounded* on $\{(t,x,\xi)\in[0,T]\times\boldsymbol{R}^n\times\boldsymbol{R}^n;\ |\xi|=1\}$.

In this section, we will apply Theorem 1 to the equation $Pu=f$ under the situation in [8, 9, 10].

The results on the well-posedness of $Pu=f$ obtained in [8, 9, 10] are summarized as follows. Put

$$a_j(x)=[t^{l(j,0)}a_{j,0}(t,x)]\Big|_{t=0},\quad j=0,1,\cdots,m-1$$

and let $\rho_1(x),\cdots,\rho_m(x)$ be the roots of the equation in ρ

$$\rho^m+\sum_{j<m}a_j(x)\rho^j=0.$$

Define the irregularity index $\sigma(\geqq 1)$ for P by the following:

$$\sigma=\max\left[1, \max_{\substack{j+|\alpha|<m \\ |\alpha|>0}}\left\{\min_{\tau\in\mathfrak{S}_n}\left(\max_{1\leqq r\leqq n} M_{j,\alpha}(\tau, r)\right)\right\}\right],$$

where $\mathfrak{S}_n$ is the permutation group of $\{1, \cdots, n\}$ and

$$M_{j,\alpha}(\tau, r)=\frac{\sum_{i=1}^{r}(\kappa_{\tau(i)}-\kappa_{\tau(r)})\alpha_{\tau(i)}+(m-j)\kappa_{\tau(r)}-l(j, \alpha)}{(m-j-|\alpha|)\kappa_{\tau(r)}}.$$

Then, under $(\mathrm{C}_1)\sim(\mathrm{C}_3)$ and

e$_1$) $a_{j,\alpha}(t, x)\in C^{\infty}([0, T], \mathcal{E}^*(\boldsymbol{R}^n))$ $(j+|\alpha|\leqq m$ and $j<m)$,
e$_2$) $\rho_i(x)\notin \boldsymbol{Z}_+$ for any $x\in \boldsymbol{R}^n$ and $i=1, \cdots, m$,

we already know the following.

(S$_1$)([8]). If $*=\emptyset$ and if $\sigma=1$, the equation $Pu=f$ is well-posed in $C^{\infty}([0, T], \mathcal{E}(\boldsymbol{R}^n))$ $(=C^{\infty}([0, T]\times \boldsymbol{R}^n))$.
(S$_2$)([9, 10]). If $*=\{s\}$ or (s), and if $1<s<\sigma/(\sigma-1)$, the equation $Pu=f$ is well-posed in $C^{\infty}([0, T], \mathcal{E}^*(\boldsymbol{R}^n))$.
(S$_3$)([10]). If $*=(s)$, if $s=\sigma/(\sigma-1)$ and if the condition $\Delta_P\cap S_P=\emptyset$ (given below) is satisfied, the equation $Pu=f$ is well-posed in $C^{\infty}([0, T], \mathcal{E}^{(s)}(\boldsymbol{R}^n))$.

Here, Δ_P and S_P are as follows. Put

$$\mathcal{J}=\{(j, \alpha)\in \boldsymbol{Z}_+\times \boldsymbol{Z}_+^n;\ j+|\alpha|<m \text{ and } |\alpha|>0\},$$
$$\sigma_{j,\alpha}=\max\left[1, \min_{\tau\in\mathfrak{S}_n}\left(\max_{1\leqq r\leqq n} M_{j,\alpha}(\tau, r)\right)\right].$$

It is clear that $\sigma=\max\{\sigma_{j,\alpha};\ (j, \alpha)\in \mathcal{J}\}$ holds. Then Δ_P is defined by

$$\Delta_P=\{(j, \alpha)\in \mathcal{J};\ \sigma_{j,\alpha}=\sigma\}.$$

Let $\kappa=(\kappa_1, \cdots, \kappa_n)\in \boldsymbol{R}^n$ be the one in (C_2). For $\alpha=(\alpha_1, \cdots, \alpha_n)\in \boldsymbol{Z}_+^n$, denote by $S_\kappa(\alpha)$ the set of all $l\in \boldsymbol{R}$ satisfying the following conditions (i) and (ii): (i) $0<l<\langle\kappa, \alpha\rangle$, and (ii) there are $\tau\in\mathfrak{S}_n$ and $p\in\{1, \cdots, n-1\}$ such that

$$\begin{cases} l=\kappa_{\tau(1)}\alpha_{\tau(1)}+\cdots+\kappa_{\tau(p)}\alpha_{(p)}, \\ \{\kappa_{\tau(1)}, \cdots, \kappa_{\tau(p)}\}<\{\kappa_{\tau(p+1)}, \cdots, \kappa_{\tau(n)}\}, \end{cases}$$

where $\langle\kappa, \alpha\rangle=\kappa_1\alpha_1+\cdots+\kappa_n\alpha_n$, and $\{a_1, \cdots, a_p\}<\{b_1, \cdots, b_q\}$ means that

$a_i<b_j$ holds for any i and j. Then S_P is defined by

$$S_P=\{(j,\alpha)\in\mathcal{J};\ l(j,\alpha)\in S_\kappa(\alpha)\}.$$

The following is a typical example of our theory.

Example 3. Let P_1 be of the form

$$P_1=(t\partial_t)^2-t^{2\kappa_1}\partial_{x_1}^2-\cdots-t^{2\kappa_n}\partial_{x_n}^2 \\ +t^{l_1}a_1(t,x)\partial_{x_1}+\cdots+t^{l_n}a_n(t,x)\partial_{x_n}+b(t,x)(t\partial_t)+c(t,x),$$

where $2\kappa_1,\cdots,2\kappa_n\in N$ and $l_1,\cdots,l_n\in N$. Then, P_1 satisfies $(C_1)\sim(C_3)$. In this case, $\rho_1(x)$, $\rho_2(x)$ are the roots of $\rho^2+b(0,x)\rho+c(0,x)=0$, and the irregularity index σ_1 for P_1 is given by

$$\sigma_1=\max\left\{1,\frac{2\kappa_1-l_1}{\kappa_1},\cdots,\frac{2\kappa_n-l_n}{\kappa_n}\right\}.$$

For $P_1(=P)$ the condition $\Delta_P\cap S_P=\emptyset$ is trivially satisfied (by [10, Remark (3) in §2]).

Now, let us apply Theorem 1 in §1 to our Fuchsian hyperbolic equation $Pu=f$. To do so, it is sufficient to see

LEMMA 11. *If $\kappa_1\geqq 1,\cdots,\kappa_n\geqq 1$ hold, we have*

$$\sigma/(\sigma-1)\leqq\min\left[\infty,\min_{(j,\alpha)\in S}\left(\frac{m-j-l(j,\alpha)}{|\alpha|-l(j,\alpha)}\right)\right] \tag{6.1}$$

and therefore the condition $1<s\leqq\sigma/(\sigma-1)$ implies (1.3).

REMARK 3. If $0<\kappa_i<1$ holds for some i, it may happen that there are no s satisfying (1.3). See Examples 1 and 2 in §1.

Hence, we obtain

THEOREM 2. *Under the situation in* (S_2) *or* (S_3), *we have the following: if the additional conditions*

i) $\kappa_1\geqq 1,\cdots,\kappa_n\geqq 1$,
ii) $a_{j,\alpha}(t,x)\in\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$ $(j+|\alpha|\leqq m$ *and* $j<m)$

hold, the equation $Pu=f$ is well-posed also in $\mathcal{E}^*([0,T]\times\boldsymbol{R}^n)$.

The following fact should be noted: in Theorem 2 we need the condition $\kappa_1 \geqq 1, \cdots, \kappa_n \geqq 1$; while in $(S_1) \sim (S_3)$ we used only the condition $\kappa_1 > 0, \cdots, \kappa_n > 0$.

PROOF OF LEMMA 11. Assume that $\kappa_1 \geqq 1, \cdots, \kappa_n \geqq 1$ hold. Take any $(j, \alpha) \in \mathcal{S}$ and fix it. Then, we have

1) $l(j, \alpha) < |\alpha|$,
2) $j + |\alpha| < m$ and $|\alpha| > 0$.

In fact, 1) is clear from the definition of $\mathcal{S}$ and 2) is verified by the following: if $j + |\alpha| = m$, then by (C_2) and the condition $\kappa_i \geqq 1$ $(i = 1, \cdots, n)$ we have $l(j, \alpha) = \kappa_1 \alpha_1 + \cdots + \kappa_n \alpha_n \geqq |\alpha|$ and hence $(j, \alpha) \notin \mathcal{S}$. Therefore, for any $\tau \in \mathfrak{S}_n$ we have

$$\max_{1 \leqq r \leqq n} M_{j,\alpha}(\tau, r) \geqq M_{j,\alpha}(\tau, 1) = \frac{m - j - l(j, \alpha)/\kappa_{\tau(1)}}{m - j - |\alpha|} \geqq \frac{m - j - l(j, \alpha)}{m - j - |\alpha|},$$

because $l(j, \alpha) \geqq l(j, \alpha)/\kappa_{\tau(1)}$. Hence, we obtain

$$\sigma \geqq \min_{\tau \in \mathfrak{S}_n} \left(\max_{1 \leqq r \leqq n} M_{j,\alpha}(\tau, r) \right) \geqq \frac{m - j - l(j, \alpha)}{m - j - |\alpha|},$$

which is equivalent to

$$\sigma/(\sigma - 1) \leqq \frac{m - j - l(j, \alpha)}{|\alpha| - l(j, \alpha)}.$$

This leads us to (6.1), since (j, α) is taken arbitrarily from $\mathcal{S}$. Q.E.D.

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