J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
39 (1992), 185-205.

Bound states for momentum and asymptotic completeness in $L^2(R^n)$: II. Mourre's theory of local conjugacy for $n\geq 2$

By PL. MUTHURAMALINGAM

§ 1. Introduction.

Let $\mathcal{H}=L^2(R^n)$ be the Hilbert space of complex valued square integrable functions on R^n w.r.t the Lebesgue measure. Let $Q=(Q_1, Q_2, \cdots, Q_n)$ and $P=(P_1, P_2, \cdots, P_n)$ be the position and momentum operators on $L^2(R^n)$ given by $(Q_jf)(x)=x_jf(x)$, $(P_jf)(x)=-i(D_jf)(x)$ where $D_j=\partial/\partial x_j$. Let $h(\zeta)=\zeta^2$ so that $H_0=h(P)=P^2$. Let $\langle x\rangle=(1+x^2)^{1/2}$ for x in R^n. Let W be the sum of a real valued short range potential and a real valued smooth long range potential; i.e. $W(x)=W_S(x)+W_L(x)$ where apart from local singularities, $W_S(x)$ decays like $\langle x\rangle^{-1-\varepsilon}$ at ∞ for some $\varepsilon>0$ and W_L is (say) C^∞ and $(D^\alpha W_L)(x)$ decays like $\langle x\rangle^{-|\alpha|-\delta}$ at ∞ for all multi-indices α and some δ (independent of α) in $(0,1]$. Then for the self adjoint operator $H=H_0+W(Q)=P^2+W(Q)$, scattering theory and spectral theory are developed in a very great detail.

For any self adjoint operator A, let $\mathcal{H}_p(A)$, $\mathcal{H}_c(A)$, $\mathcal{H}_{ac}(A)$, $\mathcal{H}_{sc}(A)$ denote respectively the point subspace, the continuous subspace, the absolutely continuous subspace and the singularly continuous subspace for A. Various parts of the following theorem is known from [E2, H1, JMP, KY, L, Mo2].

THEOREM 1.1. a) *Each nonzero eigen value of H is of finite multipli city. The nonzero eigen values of H can accumulate only at* 0.

b) $\mathcal{H}_{ac}(H)=\mathcal{H}_c(H)$ *so that* $\mathcal{H}_{sc}(H)=(0)$

c) *Let $X: R\times R^n\to R$ be the solution of the Hamilton-Jacobi equation given by*

$$\frac{\partial}{\partial t}X(t, \zeta)=h(\zeta)+W_L[\nabla_\zeta X(t, \zeta)]$$

Put

$$U_t = \exp[-it\,H_0] \text{ and } V_t = \exp[-it\,H]$$

Then the wave operators

$$\Omega_{\pm} = s - \lim_{t\to\pm\infty} V_t^* \exp[-i\,X(t,\,P)]$$

exist, are isometries satisfying the intertwining relation

$$V_t \Omega_{\pm} = \Omega_{\pm} U_t \quad \text{for all real } t.$$

Also

$$\text{Range } \Omega_{\pm} \subseteq \mathcal{H}_{ac}(H)$$

d) *Range* $\Omega_{\pm} = \mathcal{H}_{ac}(H)$. *Consequently* $\Omega_{\pm} : L^2(R^n) \to \mathcal{H}_{ac}(H)$ *is unitary and* $H|_{\mathcal{H}_{ac}(H)}$ *is unitarily equivalent to* H_0.

e) *Range* $\Omega_{\pm} = \mathcal{H}_{ac}(H) = \mathcal{H}_c(H)$

f) *There exists a dense subspace* D *of* $\mathcal{H}_{ac}(H)$ *such that for each* $\rho > 0$ *and each* g *in* D, $\|\langle Q\rangle^{-\rho} V_t g\|$ *decays like* $\langle t\rangle^{-\sigma}$ *for suitable* $\sigma < \rho$; σ *depends on* ρ.

In Theorem 1.1 parts (a) and (b) are results in spectral theory; part (c) is a result in scattering theory; (d) and (e) connect spectral and scattering theory; (f) is a deeper result about the total evolution V_t.

Note that (e) = (b) + (d). Part (b) can be proved by showing the existence of the boundary values of the resolvent $(H-x-iy)^{-1}$ as y converges to 0, in suitable weighted spaces [L, Mo2, RS3, RS4]. In [JMP, MS] the result (f) is proved first and (f) is used to prove (d). However one can prove (d) and (e) directly without using (f) e.g. [D, E1, E2, KY, Mo1, Pe].

Now one wants to know the answers for $H = h(P) + W(Q)$ when h is any real valued polynomial on R^n instead of $h(\zeta) = \zeta^2$. For (c) we refer to [H1]. The article [Mu2] proves (d) when $n = 1$, for a general h, viz the local compactness assumption A4 of § 2.

For results on a class of $h(P) + W(Q)$ when h is different from $h(\zeta) = \zeta^2$, the reader is referred to [H2, H4, Mu1, Mu3, Pa1, Pa2, Si. U1, U2].

In this article we prove for a large class of $h(P) + W(Q)$, the results (a), (b) and (f*) a very weak form of (f), viz

$$(f^*) \lim_{r\to\infty} r^2 \int dt \|\langle Q\rangle^{-\sigma} \langle P\rangle^{1/2} (P^2 + r^2)^{-1} V_t g\|^2 = 0$$

for each $\sigma > \frac{1}{2}$ and g in a dense subspace of $\mathscr{H}_{ac}(H)$. Using (f*) we prove in § 2

$$\lim_{r\to\infty} \sup_t \|\chi(|P|\geq r) V_t g\| = 0 \qquad \text{for each } g \text{ in } \mathscr{H}_{ac}(H).$$

Here χ stands for the characteristic (indicator) function. In other words, each element of $\mathscr{H}_{ac}(H)$ is a bound state for the momentum operator P under the total evolution V_t [AJS]. Now using the results of [Mu1] we get the part (d) of Theorem 1.1.

We choose $A = Q\cdot\psi(P) + \psi(P)\cdot Q$ for a suitable vector field ψ on R^n so that A is "locally conjugate to H" according to Mourre [JMP, Mo2]. Using [Mo2] and local smoothness [RS3] we prove (f*). Sufficient conditions on h, W are stated in § 2, in the assumptions A1 to A9 so that Theorem 1.1 holds for H. In § 3 we give examples of h, W for which the assumptions of § 2 hold. In the rest of the article we prove the result.

§ 2. Statement of the result.

Let $\mathscr{P}$ be the class of functions behaving like a polynomial given by

$\mathscr{P} = \{f : R^n \to R$, f is C^∞, f and all its derivatives have atmost polynomial growth$\}$

On h and W we assume A1, A2, $\cdots$, A9 given by

A1: $h : R^n \to R$ is in $\mathscr{P}$

A2: $\{\zeta$ in $R^n : \nabla h(\zeta) = 0\}$ is a set with Lebesgue measure zero.

A3: (On critical values) If $C_V = \{h(\zeta) : \nabla h(\zeta) = 0\}$ is the set of critical values for h, then $\bar{C}_V$, the closure of C_V is a countable set.

A4: (Local compactness) For each $r > 0$, the operator $\chi(|Q| \leq r)\ [h(P) + i]^{-1}$ is a compact operator.

A5: (Long range potential) $W : R^n \to R$ is C^∞ and there exists some δ in $(0, 1]$ such that

$$|D^\alpha W(x)| \leq K(\alpha) \langle x \rangle^{-|\alpha| - \delta}$$

for all multi indices α. Here $K(\alpha)$ are suitable constants. In this article all constants will be denoted by the same letter K.

A6: There exists $\phi: R^n \to R^n$, $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ such that each ϕ_j is in $\mathcal{P}$. Further they satisfy the conditions (a), (b) and (c) below.

a) $\sup\left\{\langle\zeta\rangle^{-1} \sum_j |\phi_j(\zeta)| : \zeta \text{ in } R^n\right\} < \infty$

b) $\sup\left\{\sum_{j,k} |D_j\phi_k(\zeta)| : \zeta \text{ in } R^n\right\} < \infty$

c) $\sup\left\{\langle\zeta\rangle^{-1} \sum_{j,k,m} |D_jD_k\phi_m(\zeta)| : \zeta \text{ in } R^n\right\} < \infty$

Note that (b) implies (a). For quick reference later on, we have stated the condition (a) explicitly.

A7: $\sup\left\{\sum_{k,m} |\sum_j \phi_j(\zeta)D_jD_k\phi_m(\zeta)| : \zeta \text{ in } R^n\right\} < \infty$

A8: a) $\sup\left\{|\sum_j \phi_j(\zeta)D_jh(\zeta)|\langle h(\zeta)\rangle^{-1} : \zeta \text{ in } R^n\right\} < \infty$

b) $\sup\left\{|\sum_j \phi_j(\zeta)D_j \sum_k \phi_k(\zeta)D_kh(\zeta)|\langle h(\zeta)\rangle^{-1} : \zeta \text{ in } R^n\right\} < \infty$

A9: There exists a countable closed subset N (N for negligible) of R with $\bar{C}_V \subset N$ such that for each closed interval $I \subset R \backslash N$ we have either

$$\inf\left\{\sum_j \phi_j(\zeta)D_jh(\zeta) : h(\zeta) \text{ in } I\right\} > 0$$

or

$$\inf\left\{-\sum_j \phi_j(\zeta)D_jh(\zeta) : h(\zeta) \text{ in } I\right\} > 0$$

The Fourier transform $\mathcal{F}$ is given by

$$(\mathcal{F}f)(\zeta) = \hat{f}(\zeta) = (2\Pi)^{-n/2} \int dx \exp[-ix\zeta] f(x)$$

so that $\mathcal{F}$ is unitary on $L^2(R^n)$.

With the above (possible) assumptions we have our technical Theorem 2.1 which will be proved in later sections.

THEOREM 2.1. *Let h, W be as in* A1, A4 *and* A5. *Let*

$$H = h(P) + W(Q) = H_0 + W(Q).$$

(i) *If* A1, A3, A4, A5, A6, A8(a) *and* A9 *are satisfied, then each compact interval of* $R\backslash N$ *contains* (*atmost*) *a finite number of eigenvalues of* H. *Let* $N_0=N\cup C$ *where* C *is the set of eigen values for* H.
(ii) *Let in addition* A7 *and* A8(b) *hold. Then* $\mathcal{H}_{ac}(H)=\mathcal{H}_c(H)$.
(iii) *Let* A1, A3 *to* A9 *hold. Assume further that the vector field* ϕ *of* A6 *satisfies the condition* (a) (*see below*) *or the function* h *satisfies the conditions* (b) *and* (c)

a) *There exists some* ρ *in* $[0,1)$ *such that*

$$\sup\left\{\langle\zeta\rangle^{-\rho}\sum_j|\phi_j(\zeta)|:\zeta \text{ in } R^n\right\}<\infty$$

<u>*or*</u>

b) *There exists* $k\geq 1$ *such that*

$$\sum_{|\beta|=k+1}|D^\beta h(\zeta)|\leq K\left\{1+\sum_{|\alpha|\leq k}|D^\alpha h(\zeta)|\right\} \qquad \textit{for all } \zeta,$$

for some constant K. *Also*

$$\lim_{|\zeta|\to\infty}\sum_{|\alpha|\leq k}|D^\alpha h(\zeta)|=\infty$$

c) *There exist constants* $K>0$ *and* $M>0$ *such that*

$$\sum_{|\alpha|\leq k}|D^\alpha h(\zeta)|\geq K\langle\zeta\rangle^M$$

Let D *be the dense subspace of* $\mathcal{H}_{ac}(H)$ *given by*

$$D=\{f \text{ in } \mathcal{H}_{ac}(H): E_H(I)f=f \text{ for some compact subset } I \text{ of } R\backslash N_0\}$$

$E_H(dt)$ *is the spectral measure for the self adjoint operator* H. *Then for each* f *in* D *and* $\sigma>1/2$ *we get*

$$\lim_{r\to\infty} r^2\int dt\|\langle Q\rangle^{-\sigma}\langle P\rangle^{1/2}(P^2+r^2)^{-1}\exp[-it H]f\|^2=0$$

PROOF: i) and (ii) We refer to Theorem 5.1 of § 5. iii) We refer to Theorem 5.4. Q.E.D.

REMARK 2.2. If h satisfies the assumption (b) of Theorem 2.1 (iii) then, by Theorems 9 and A1 of [DM], the local compactness assumption A4 holds for h. If h is any real valued polynomial on R^n with $\lim_{|\zeta|\to\infty}\sum_\alpha|D^\alpha h(\zeta)|^2=\infty$, then h satisfies the assumption (c) of Theorem 2.1

(iii). A proof can be given by using the Tarski-Seidenberg Theorem on algebraic sets [H2] and the techniques of example A.2.7 of [H2].

THEOREM 2.3. Let the assumptions be as in Theorem 2.1 (iii). Then

$$\lim_{r\to\infty}\sup_t \|\chi(|P|\geq r)V_t f\|=0 \qquad \text{for each } f \text{ in } \mathcal{H}_{ac}(H)$$

PROOF. Since

$$\|\chi(|P|\geq r)g\|\leq 2\||P|(P^2+r^2)^{-1/2}g\|$$

for all g and r, it suffices to show that

$$\lim_{r\to\infty}\sup_t \||P|(P^2+r^2)^{-1/2}V_t f\|^2=0 \tag{2.1}$$

for each f in $\mathcal{H}_{ac}(H)$. By density arguments we have to prove (2.1) for f in D of Theorem 2.1 (iii).

Let f be in D. Then with $[X, Y]=XY-YX$, by the fundamental theorem of calculus,

$$\begin{aligned}&\||P|(P^2+r^2)^{-1/2}V_t f\|^2\\&=\langle P^2(P^2+r^2)^{-1}f, f\rangle-i\int_0^t ds\langle[P^2(P^2+r^2)^{-1}, W(Q)]V_s f, V_s f\rangle\end{aligned} \tag{2.2}$$

Since $P^2(P^2+r^2)^{-1}=1-r^2(P^2+r^2)^{-1}$ we have

$$\begin{aligned}&[P^2(P^2+r^2)^{-1}, W(Q)]\\&=r^2(P^2+r^2)^{-1}\sum_j[P_j^2, W(Q)]\,(P^2+r^2)^{-1}\\&=-i\sum_j r^2(P^2+r^2)^{-1}\{P_j(D_jW)(Q)+(D_jW)(Q)P_j\}(P^2+r^2)^{-1}\end{aligned} \tag{2.3}$$

For δ as in the assumption A5, choose σ such that $1<2\sigma\leq 1+\delta$. Define the bounded operators A_j and B_j by

$$A_j=\langle Q\rangle^{\sigma}\langle P\rangle^{-1/2}P_j(D_jW)(Q)\langle P\rangle^{-1/2}\langle Q\rangle^{\sigma}, \qquad B_j=A_j^*$$

Now from (2.2) and (2.3) we get

$$\begin{aligned}&\sup\||P|(P^2+r^2)^{-1/2}V_t f\|^2\\&\leq\||P|(P^2+r^2)^{-1/2}f\|^2+\Big\{2\sum_j\|A_j\|\Big\}r^2\int ds\|\langle Q\rangle^{-\sigma}\langle P\rangle^{1/2}(P^2+r^2)^{-1}V_s f\|^2\end{aligned}$$

Now (2.1) easily follows from Theorem 2.1 (iii). Q.E.D.

The next Theorem 2.4, from [Mu1], is an important input for proving asymptotic completeness in Theorem 2.5.

THEOREM 2.4. Let the assumptions A1 to A5 hold. Then there exists a C^∞ function $X: R\times R^n \to R$ such that (i),$\cdots$(v) are valid.

i) $\Omega_\pm = s-\lim_{t\to\pm\infty} V_t^* \exp[-iX(t, P)]$ exists

ii) $\Omega_\pm$ is an isometry

iii) $V_t\Omega_\pm = \Omega_\pm U_t$ where $U_t = \exp[-it H_0]$

iv) Range $\Omega_\pm \subseteq \mathcal{H}_{ac}(H)$

v) Let $G=(\zeta$ in $R^n : \nabla h(\zeta)\neq 0)$. Then

$$\mathcal{H}_{ac}(H)\theta \text{ Range } \Omega_\pm = \left\{ f \text{ in } \mathcal{H}_{ac}(H) : \lim_{T\to\infty} \frac{1}{T}\int_0^{\pm T} \|\gamma(P)V_t f\| dt = 0 \right.$$
$$\left. \text{for each } \gamma \text{ in } C_0^\infty(G) \right\}$$

Here θ stands for the orthogonal difference.

PROOF. The proof follows from Theorems 2.1 and 2.3 of [Mu1] and the proof of Theorem 2.2 (ii) of [Mu1]. Q.E.D.

THEOREM 2.5. Let the assumptions A1 to A9 and the assumptions of Theorem 2.1 (iii) hold. Then Range $\Omega_\pm = \mathcal{H}_{ac}(H)$.

PROOF. Let $f\in \mathcal{H}_{ac}(H)\theta$ Range Ω_+. Let $\varphi \in C_0^\infty(R\backslash \bar{C}_V)$. Choose ϕ in $C_0^\infty(R^n)$ so that $0\leq\phi\leq 1$, $\phi(\zeta)=1$ for $|\zeta|\leq 1$ and $\phi(\zeta)=0$ for $|\zeta|\geq 2$. Then no correction in this line. Every thing is correct for all $r\geq 1$ we have

$$\|\varphi(H)f\| \leq \lim_{t\to\infty} \|[\varphi(H)-\varphi(H_0)]V_t f\| + \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \|\varphi(H_0)\phi(P/r)V_t f\| +$$
$$\|\varphi\|_\infty \sup \|[1-\phi(P/r)]V_t f\| \tag{2.4}$$

The first term of R.H.S. of (2.4) is 0 since $\varphi(H)-\varphi(H_0)$ is compact and f is in $\mathcal{H}_{ac}(H)$; second term is 0 by Theorem 2.4 (v). Now using Theorem 2.3 for the third term, we see that $\varphi(H)f=0$ for each φ in $C_0^\infty(R\backslash\bar{C}_V)$. Since f is in $\mathcal{H}_{ac}(H)$ and $\bar{C}_V$ is countable we get $f=0$. Thus $\mathcal{H}_{ac}(H)=$ Range Ω_+. Similarly $\mathcal{H}_{ac}(H)=$ Range Ω_-. Q.E.D.

§ 3. Examples.

Through out this section we assume that A1 to A5 hold. We put $H=H_0+W=h(P)+W(Q)$.

THEOREM 3.1. (*Simply characteristic operators* [H2], [Mu3], [Pa1], [Pa2]). *Let* $h: R^n \to R$ *be simply characteristic i.e. in addition to* A1 *through* A4, *the function* h *satisfies the assumption* (b) *of Theorem* 2.1 (iii) *with* $k=1$. *Take* $N=\bar{C}_V$. *Then the conclusions of Theorems* 2.1 *and* 2.5 *are valid.*

PROOF. Define $\phi_j(\zeta)=[D_j h(\zeta)]\{1+|h(\zeta)|^2+|\nabla h(\zeta)|^2\}^{-1}$ Q.E.D.

THEOREM 3.2. (*Paraboblic operators*) *Let* $h: R^n \to R$ *be given by* $h(\zeta_1, \zeta')=\zeta_1-g(\zeta')$ *for* (ζ_1, ζ') in $R \times R^{n-1}$, *Then Theorems* 2.1 *and* 2.5 *are valid with* $N=C_V=$ empty.

PROOF. Take $\phi_1(\zeta)=1$, $\phi_2(\zeta)=0=\phi_3(\zeta)=\cdots=\phi_n(\zeta)$. Q.E.D.

THEOREM 3.3. (*Non-negative linear combination of non negative monomials* [Mu3]) *Let* $h(\zeta)=\sum a_\beta \zeta^\beta$ (*finite sum*) *where* $\beta=(\beta_1, \beta_2, \cdots, \beta_n)$, β_j *are nonnegative even integers*, $a_\beta \geq 0$ *and* $\sum_\alpha |D^\alpha h(\zeta)| \to \infty$. *Take* $N=(0)$. *Then Theorems* 2.1 *and* 2.5 *hold for* H.

PROOF. Take $\phi_j(\zeta)=\zeta_j \exp[-\zeta_j^2]$ Q.E.D.

THEOREM 3.4. (*Twisted? homogeneous*) *Let* $h: R^n \to R$ *be any polynomial such that for some real* $r_1, r_2, \cdots r_n$ *and* $m \neq 0$ *we get* $h(t^{r_1}\zeta_1, t^{r_2}\zeta_2, \cdots, t^{r_n}\zeta_n)=t^m\ h(\zeta_1, \zeta_2, \cdots, \zeta_n)$ *for all* $t>0$. *Also let* $\sum_\alpha |D^\alpha h(\zeta)| \to \infty$. *Then with* $N=\{0\}$, *Theorems* 2.1 *and* 2.5 *are valid.*

PROOF. Take $\phi_j(\zeta)=r_j \zeta_j$ Q.E.D.

REMARK 3.5. Consider the example $h(\zeta_1, \zeta_2, \zeta_3)=\zeta_3\ \zeta_1(2-\zeta_1\zeta_2)$. Then Theorems 2.1 and 2.5 are valid for H using Theorem 3.4 by noting $r_3=1=m$, $r_1=0=r_2$. One easily verifies the following properties (i), (ii) and (iii).

(i) 0 is the only critical value for h.

(ii) for any $\lambda \neq 0$ we have $\lim_{r\to\infty} |h(r^{-1}, r, \lambda r)-\lambda|+|\nabla h(r^{-1}, r, \lambda r)|=0$

(iii) $\lim_{r\to\infty} \sum_{|\alpha|=2} |D^\alpha h(r^{-1}, r, \lambda r)|=\infty=\lim_{r\to\infty} \sum_{|\alpha|=3} |D^\alpha h(r^{-1}, r, \lambda r)|$.

So from (ii) and (iii) we easily conclude that h is "far away" from being simply characteristic.

Using Theorems 3.1 to 3.4 and the results of [Mu2], [Mu3] and [Pa2] we make our following.

CONJECTURE: Let h be as in A1 to A4 and W be the sum of a real valued short range potential and a real valued long range potential. Then Theorem 1.1 (a) and (e) are valid for $H=h(P)+W(Q)$.

§ 4. Study of A, [A, W], [A, [A, W]].

We have the well known formulas F1, F2, F3 for the commutators which will be repeatedly used.

$$XY=[X, Y]+YX \tag{F1}$$
$$[X, YZ]=[X, Y]Z+Y[X, Z] \tag{F2}$$
$$[X,[Y, Z]]+[Y,[Z, X]]+[Z,[X, Y]]=0 \tag{F3}$$

PROPOSITION 4.1. *Let $\phi: R^n \to R^n$ be as in the assumption* A6. *Define $A=Q.\phi(P)+\phi(P).Q=\sum_j Q_j\phi_j(P)+\phi_j(P)Q_j$ on $\mathcal{S}=\mathcal{S}(R^n)$, the Schwartz space of rapidly decreasing smooth functions. Then*

(i) *A is self adjoint and Dom $A \supset \mathcal{S}$.*

(ii) *$\mathcal{S}$ is a core for A; infact any core for $N=1+Q^2+P^2$ is a core for A.*

PROOF. The proof is by Nelson's commutator Theorem [RS2]. Clearly $N \geq 1$. Denoting any bounded operator by the same letter B we easily have

$$\langle Q\rangle N^{-1/2}=B, \quad \langle P\rangle N^{-1/2}=B, \quad N^{-1/2}\langle P\rangle=B, \quad N^{-1/2}\langle Q\rangle=B \tag{4.1}$$

Using (4.1) and the assumption A6(a) we define $a(f, g)$ for f, g in $D(N^{1/2})$ by

$$a(f, g)=\sum_j \left\{\langle\phi_j(P)f, Q_j g\rangle+\langle Q_j f, \phi_j(P)g\rangle\right\}$$

It is obvious by (4.1) and the assumption A6(a) that

$$|a(f, g)| \leq K\|N^{1/2}f\|\,\|N^{1/2}g\| \qquad \text{for } f, g \text{ in } D(N^{1/2}) \tag{4.2}$$

Clearly $a(f, g)=\langle Af, g\rangle$ for all f, g in $\mathcal{S}$. Let f, g be in $\mathcal{S}$. Then

$$a(Nf, g)-a(f, Ng)=\langle[A, N]f, g\rangle$$

$$=\sum_{j,k}\langle[\psi_j(P),\ Q_k^2]f, Q_jg\rangle-\langle\psi_j(P)f, [Q_j, P_k^2]g\rangle-\langle Q_jf, [\psi_j(P), Q_k^2]g\rangle$$
$$+\langle[Q_j, P_k^2]f, \psi_j(P)g\rangle \tag{4.3}$$

Since $[\psi_j(P), Q_k^2]=[Q_k, [\psi_j(P), Q_k]]+2[\psi_j(P), Q_k]Q_k$, using the assumptions A6(b), (c) and (4.1) we get

$$|\text{first term of RHS of (4.3)}|$$
$$\leq K\sum_{j,k}\|\langle P\rangle f\|\|Q_jg\|+2\|Q_kf\|\|Q_jg\|\leq K\|N^{1/2}f\|\|N^{1/2}g\|.$$

Using the assumption A6(a) and (4.1) we get

$$|\text{second term of (4.3)}|\leq K\|N^{1/2}f\|\|N^{1/2}g\|.$$

Doing similar analysis for the third and fourth terms we finally get

$$|a(Nf, g)-a(f, Ng)|\leq K\|N^{1/2}f\|\|N^{1/2}g\| \qquad \text{for } f, g \text{ in } \mathcal{S} \tag{4.4}$$

$\mathcal{S}$ is a core for $N^{3/2}$. So from (4.2) and (4.4) we see that (4.4) holds for all f, g in $D(N^{3/2})$. Now the result follows from Nelson's Commutator Theorem. Q.E.D.

On $\mathcal{S}$ we have

$$[A, W(Q)]=\sum_j 2[\psi_j(P),\ W(Q)Q_j]+i\{D_j\psi_j(P)\}W(Q)+iW(Q)D_j\psi_j(P) \tag{4.5}$$

So we study $[\psi_j(P),\ Q_jW(Q)]$ in proposition 4.4.

LEMMA 4.2. *Let $n\geq 2$. Let $V: R^n\to C$ be in C^{n+1} and assume that for some δ in $(0, 1]$ we have*

$$|D^\alpha V(x)|\leq K(\alpha)\langle x\rangle^{-|\alpha|-\delta} \qquad \textit{for all } |\alpha|\leq n+1$$

Then $\hat{V}\in L^1(R^n)$ and for q satisfying $\min\{2, n/(n-1)\}>q>\max\{1, n/(n-1+\delta)\}$ *we get*

$$\|\hat{V}\|_1\leq K(q)\sum_j\|D_j^{n+1}V\|_1+\|D_j^{n-1}V\|_q$$

PROOF. Clearly in the sense of (Schwartz) distributions $[R]$,

$$\hat{V}(\zeta)\zeta_j^{n+1}=(-i)^{n+1}(2\pi)^{-n/2}\int dx\, e^{-ix\zeta}(D_j^{n+1}V)(x). \tag{4.6}$$

So by the decay assumption on V we see that $\hat{V}$ is given by a function on $R^n\backslash\{0\}$ and

$$\sup_{\zeta\neq 0} |\zeta|^{n+1}|\hat{V}(\zeta)| \leq K \sum_j \|D_j^{n+1}V\|_1.$$

So we conclude

$$\int d\zeta\, \chi(|\zeta|\geq 1)|\hat{V}(\zeta)| \leq K \sum_j \|D_j^{n+1}\|_1 \tag{4.7}$$

For q as in the assumption, choose p with $p^{-1}+q^{-1}=1$. Then by Hölder inequality and Hausdorff-Young Theorem for Fourier transform [RS2] we get

$$\begin{aligned} &\int d\zeta\, \chi(0<|\zeta|\leq 1)|\hat{V}(\zeta)| \\ &\leq \| |\zeta|^{1-n} \chi(0<|\zeta|\leq 1)\|_q \| |\zeta|^{n-1}\hat{V}(\zeta)\|_p \\ &\leq K(q) \sum_j \|D_j^{n-1}V\|_q. \end{aligned} \tag{4.8}$$

Let $g=\{\chi(\zeta\neq 0)\hat{V}(\zeta)\}^{\vee}$. Then we easily have $V=g+P$ for some polynomial P. By (4.7), (4.8) and Riemann-Lebesque Lemma, the function g vanishes at ∞. So P vanishes at ∞ forcing $P\equiv 0$. Now the result follows from (4.7) and (4.8). Q.E.D.

LEMMA 4.3. *Let J be an integral operator on $L^2(R^n)$ with integral kernel $J(x, y)$ i.e. $(Jf)(x)=\int dy\ J(x, y)f(y)$. Further let $|J(x, y)|\leq F(x-y)$ for all x, y where F is in $L^1(R^n)$. Then $\|J\|\leq\|F\|_1$.*

PROOF. For f, g in $L^1(R^n)$, clearly $|\langle Jf, g\rangle|\leq\int dx\, dy\, F(x-y)|f(x)|\,|g(y)|$. Now apply Schur test [H3, HS, W]. Q.E.D.

PROPOSITION 4.4. *Let $n\geq 2$, W as in* A5, *q as in Lemma* 4.2 *and h as in* A4. *Let ψ be in $\mathcal{P}$ and assume that $\|\nabla\psi\|_\infty<\infty$. Then*

(i) $\|[\psi(P), W(Q)]\| \leq K(q)\|\nabla\psi\|_\infty \sum_{j,k} \|D_j^{n+1} D_k W\|_1 + \|D_j^{n-1} D_k W\|_q$.

(ii) $\|[\psi(P), Q_j W(Q)]\|$

$$\leq K(q)\|\nabla\psi\|_\infty\Big\{\sum_{|\beta|+p=n+2}\|\langle x\rangle^{1-p}D^\beta W\|_1 + \sum_{|\beta|+p=n+1}\|\langle x\rangle^{-p}D^\beta W\|_1 + \sum_{|\beta|+p=n}\|\langle x\rangle^{1-p}D^\beta W\|_q + \sum_{|\beta|+p=n-1}\|\langle x\rangle^{-p}D^\beta W\|_q\Big\}$$

(iii) *$[\psi(P), Q_j W(Q)]\langle h(P)\rangle^{-1}$ is a compact operator for each j.*

PROOF. (i) Follows from Lemmas 4.3 and 4.2 since.

$$[\phi(Q),\ W(P)](x, y)=i\int_0^1 dt \sum_k (D_k\phi)[tx+(1-t)y](D_k W)^\wedge(y-x)$$

(ii) Similar to (i) if we replace $W(x)$ by $x_j W(x)$. The calculation of the integral kernel for $[\phi(Q), P_j W(P)]$ can be jusatified by the theory of pseudo differential operators [H3]

(iii) Let χ in $C_0^\infty(R^n)$ be such that $0\leq\chi\leq 1$, $\chi(x)=1$ for $|x|\leq 1$ and $\chi(x)=0$ for $|x|\geq 2$. Define $W_r(x)=W(x)\ \chi(x/r)$. Then by (ii) we get

$$0=\lim_{r\to\infty}\|[\phi(Q),\ P_j\{W(P)-W_r(P)\}]\| \tag{4.9}$$

Since $x_j W_r(x)$ is in $C_0^\infty(R^n)$ the operator $[\phi(Q), P_j W_r(P)]\ (P_k+i)$ is bounded for each k by (i) proving that $[\phi(Q),\ P_j W_r(P)]\langle P\rangle$ is bounded. So the operator $[\phi(Q),\ P_j W_r(P)]\ \langle h(Q)\rangle^{-1}$ is compact. Now the result follows by (4.9). Q.E.D.

PROPOSITION 4.5. *Let h, W, $\phi=(\phi_1, \cdots, \phi_n)$ be as in the assumptions* A4, A5, A6 *respectively. Then*

(i) *$[A, W(Q)]$ is a hounded operator*

(ii) *$[A, W(Q)]\langle h(P)\rangle^{-1}$ is a compact operator.*

PROOF. (i) Follows from (4.5) and Proposition 4.4 (ii). Using (4.5) and Proposition 4.4 (iii) we easily prove (ii). Q.E.D.

LEMMA 4.6. *Let W and $\phi=(\phi_1, \phi_2, \cdots, \phi_n)$ satisfy the assumptions* A5, A6 *and* A7. *Then for some bounded operator B we have*

$$[A,[A,\ W(Q)]]=B+4\sum_{j,k}[\phi_j(P),[\phi_k(P),\ Q_jQ_k\ W(Q)]]$$

PROOF. The proof is a lengthy calculation on commutators. It can be found in Lemma 6.7 of § 6. Q.E.D.

LEMMA 4.7. *Let W, ϕ be as in Lemma* 4.6. *Then the operators* (i) $[\phi_j(P),[\phi_k(P),\ Q_jQ_k\ W(Q)]]$ *and* (ii) $[A,[A,\ W(Q)]]$ *are bounded.*

PROOF. (i) Similar to the proof of Proposition 4.4 (ii) since $[\phi_j(Q),[\phi_k(Q),\ P_jP_k\ W(P)]]\ (x, y)$

$$=-\sum_{m,p}\int_0^1\int_0^1 dt\ ds(D_m\phi_j)[tx+(1-t)y](D_p\phi_k)[sx+(1-s)y]$$

$$\left\{D_pD_m[\zeta_j\zeta_k W(\zeta)]\right\}^{\wedge}(y-x)$$

(ii) Follows from (i) and Lemma 4.6. Q.E.D.

THEOREM 4.8. *Let h, W and ϕ be as in the assumptions* A1, A5, A6 *and* A8(a). *Then $(H-i)\ A(H\pm i)^{-1}\ (A+i)^{-1}$ is a bounded operator. Consequently $(H\pm i)^{-1}$ leaves Dom A invariant.*

PROOF. Since A and H leave $\mathcal{S}$ invariant, clearly for f in $\mathcal{S}$ we have $\|[A, H]f\|\leq K\|(H\pm i)f\|$. So we get

$$|\langle Af, (H-i)g\rangle-\langle(H+i)f, Ag\rangle|\leq K\|(H+i)f\|\,\|g\| \quad \text{for } f, g, \text{ in } \mathcal{S} \tag{4.10}$$

For f in $D(A)$, using $\mathcal{S}$ is a core for H, we can choose a sequence f_k in $\mathcal{S}$ such that $f_k\to(H+i)^{-1}f$ and $(H+i)f_k\to f$. So using (4.10) we get, for all g in $\mathcal{S}$ and f in $D(A)$

$$|\langle(H+i)^{-1}f, A(H-i)g\rangle-\langle f, Ag\rangle|\leq K\|f\|\,\|g\|$$

Replacing f by $(A+i)^{-1}x$ we get

$$|\langle(H+i)^{-1}(A+i)^{-1}x, A(H-i)g\rangle|\leq K\|x\|\,\|g\| \quad \text{for } g \text{ in } \mathcal{S}, x \text{ in } \mathcal{H}$$

Now the result follows. Q.E.D.

The following Lemma 4.9 and 4.10 are well known classical facts in the theory of pseudo differential operators.

LEMMA 4.9. *The operators $\langle P\rangle^a\langle Q\rangle^b\langle P\rangle^{-a}\langle Q\rangle^{-b}$ and $\langle Q\rangle^a\langle P\rangle^b\langle Q\rangle^{-a}\langle P\rangle^{-b}$ are bounded for all real a, b.*

PROOF. See page 284, definition 30.2.2 of [H4]. Q.E.D.

LEMMA 4.10. *Let λ be any diadic rational in $[-1, 1]$ i.e. $-1\leq\lambda\leq 1$ and $\lambda=j2^{-m}$ for some integers j and m. Let p be any diadic rational in $[0, 1]$. Define $L\geq 1$ by $L=\langle Q\rangle^{1/2}\langle P\rangle^p\langle Q\rangle^{1/2}$. Then $L^\lambda\langle Q\rangle^{-\lambda/2}\langle P\rangle^{-\lambda p}\langle Q\rangle^{-\lambda/2}$ is a bounded operator.*

PROOF. By Lemma 4.9, it is clear that $\langle Q\rangle^{a_1}\langle P\rangle^{b_1}\langle Q\rangle^{a_2}\langle P\rangle^{b_2}\cdots\langle Q\rangle^{a_k}\langle P\rangle^{b_k}$ is bounded when $a_1, b_1, a_2, b_2, \cdots a_k, b_k$ are reals with $a_1+a_2+\cdots+a_k\leq 0$ and $b_1+b_2+\cdots+b_k\leq 0$. Now by interpolation techniques [RS2], the Lemma can be proved by induction on m where $\lambda=j2^{-m}$. Q.E.D.

§ 5. Mourre's Theory

In this section we closely follow Mourre's theory from [Mo2]. Of the conditions (a), (b), (c), (d) and (e) in [Mo2], for a self adjoint operator A to be locally conjugate w.r.t. another self adjoint operator H, the condition (b) of [Mo2] was used to prove in [Mo2], essentially.

PROPOSITION II.2. *$(H-z)^{-1}$ leaves $D(A)$ invariant for z not in spec H.*

Our theorem 4.8 essentially proves this fact. With this remark, all the arguments, form now on, are seen to be valid by very closely following [Mo2].

THEOREM 5.1. (i) *Theorem* 2.1 (i) *is true*
(ii) *Theorem* 2.1 (ii) *is true.*

PROOF. (i) Let $I=[a, b]\subset R\backslash N$. If in A9 we have

$$\inf\{\phi(\zeta).\ \nabla h(\zeta) : h(\zeta) \text{ in } I\}>0 \quad \text{then let us take}$$
$$A=Q\cdot\phi(P)+\phi(P)\cdot Q. \quad \text{Otherwise we take } A=-\{Q\cdot\phi(P)+\phi(P)\cdot Q\}$$

By Proposition 4.1 (i) A is self adjoint and $D(A)\cap D(H)$ is a core for H. By the assumption A8(a) and Proposition 4.5 (i) the operator $[A, H](H+i)^{-1}$ is bounded. By Proposition 4.5 (ii) the operator $[A, W](H_0+i)^{-1}$ is compact. Now the proof of (e) of [Mo2] is similar to the proof given in page 396 of [Mo2]. Now (i) follows from part (i) of Theorem on page 392 of [Mo2].

(ii) By the assumption A8(b) and Lemma 4.7 (ii) the operator $(H+i)^{-1}[[H, A], A](H+i)^{-1}$ is bounded. Now the result follows from part 2 of Theorem 2 on page 392 of [Mo2] and Theorem XIII.20 of [RS3] since N is a countable set. Q.E.D.

LEMMA 5.2. *Let X be a Banach space and $F_1, F_2, \cdots$, a sequence of X valued continuously norm differentiable functions on $(0, t_0)$. Assume that there exist constants $L, K_1, K_2, \cdots, 0\leq\beta<1, -\infty<\gamma<\infty$ such that*

(a) $\left\|\frac{d}{dt}F_j(t)\right\|\leq L\{\|F_j(t)\|+1\}t^{-\beta}$ for t in $(0, t_0)$

(b) $\|F_j(t)\|\leq K_j t^{-\gamma}$ *for t in $(0, t_0)$ and*

(c) $\lim_{j\to\infty} K_j=0$

Then $\lim_{t\downarrow 0} F_j(t)$ *exists in* X; *call it* $F_j(0)$. *Further more* $\lim_{j\to\infty}\|F_j(0)\|=0$.

PROOF. The proof is similar to the proof of Lemma 3.3 of [JMP]. Q.E.D.

LEMMA 5.3. *Let the assumptions* (b) *and* (c) *of Theorem* 2.1 (iii) *hold. If* $0<a_0\leq 1$ *and* $b_0<2a_0 M/[n(1+2k)]$ *then the operator* $\langle Q\rangle^{-a_0}\langle h(P)\rangle^{-1}\langle P\rangle^{b_0}$ *is bounded.*

PROOF. Using the ideas of the proof of Theorem A1 of [DM] we can find a partition of R^n into congruent parallel cubes C_j such that

$$\int_{C_j}\langle h(y)\rangle^{-2}dy\leq K_1\int_{C_j}\Big\{1+\sum_{|\alpha|\leq k}|D^{\alpha}h(y)|\Big\}^{-2/(1+2k)}dy \qquad (5.1)$$

where K_1 is independent of j. For a proof we refer to Lemma 4.4 of [Mu2]. By (5.1) and Theorem 2.1 of chapter 6 of [Sc] on Stummel class we see that $\langle Q\rangle^{-s}\langle h(P)\rangle^{-1}\langle P\rangle^{t_0}$ is bounded when $t_0=M/(1+2k)$ and $s>n/2$. Clearly $\langle Q\rangle^{-0}\langle h(P)\rangle^{-1}\langle P\rangle^{0}$ is bounded. Now the result follows by interpolation techniques [RS2]. Q.E.D.

THEOREM 5.4. *Let the assumption be as in Theorem* 2.1 (iii). *Then* (i) *for each* σ *in* (1/2, 1] *and* $r=1,2,3,\cdots$, *the operator* $r\langle Q\rangle^{-\sigma}\langle P\rangle^{1/2}(P^2+r^2)^{-1}$ *is* H *smooth on each compact subset* I *of* $R\backslash N_0$. *Furthermore*

$$0=\lim_{r\to\infty}\ \sup_{x\ in\ I,y\neq 0} r^2\|\langle Q\rangle^{-\sigma}\langle P\rangle^{1/2}(P^2+r^2)^{-1}(H-x-iy)^{-1}(P^2+r^2)^{-1}\langle P\rangle^{1/2}\langle Q\rangle^{-\sigma}\|$$

Consequently (ii) *Theorem* 2.1 (iii) *holds.*

PROOF. Clearly (ii) follows from (i) by using the proofs of Theorems XIII. 30 (a) (b), and XIII. 25 of [RS3].

For the proof of (i) we choose diadic rationals β and p with $0\leq\beta<1/2$, $\beta+\sigma>1$, and $0\leq p\leq 1$ as follow:

If the assumption (a) of Theorem 2.1 (iii) holds then A(a) $p<1$, and A(b) $2^{-1}(1-\beta)^{-1}>p>\beta^{-1}(\rho-2^{-1})$. If the assumptions (b), (c) of Theorem 2.1 (iii) hold then B(a) $p=1$ and B(b) $2^{-1}<\beta+\{2M(\beta+\sigma-1)\}[n(1+2k)]^{-1}$. It is clear that we can choose β and p satisfying these conditions.

Now define the operator $X(r,t)$ for t real, $r=1,2,\cdots$, by

$$X(r,t)=r\{1+|t|\langle Q\rangle^{1/2}\langle P\rangle^{p}\langle Q\rangle^{1/2}\}^{-\beta}\langle Q\rangle^{-\sigma}\langle P\rangle^{1/2}(P^2+r^2)^{-1}$$

For E_0 in $R\backslash N_0$, let A be as in the proof of. Theorem 5.1 Now as in the proof of Lemma on page 404 of [Mo2] we can choose a function f in $C_0^\infty(R)$ such that $0 \leq f \leq 1$, $f=1$ in $V_1=(E_0-\delta_1, E_0+\delta_1)$ and $f=0$ out side $V_2=(E_0-\delta_2, E_0+\delta_2)$ for some $\delta_1<\delta_2$ so that for some $\alpha>0$ we get

$$Y=f(H)i[H, A]f(H) \geq \alpha\{f(H)\}^2$$

As in the proof of Proposition II.5 of [Mo2] for x in V_1, y, t non-zero reals with $yt>0$ define the operator $G(x+iy, t)$ by

$$G(z, t)=G(x+iy, t)=[H-x-iy-it\ Y]^{-1}.$$

Now define $F(r, z, t)$ by

$$F(r, z, t)=X(r, t)G(z, t)\{X(r, t)\}^*.$$

We verify the hypothesis of Lemma 5.2 for $t_0=1$ and then clearly the result will follow. In what follows $z=x+iy$, x is in V_1, $yt>0$, $|t| \leq 1$ and various constants K are independent of y, t, r. From the proof of Lemma on page 404 of [Mo2] we easily see that

$$\|(H+i)f(H)G(z, t)\|+\|G(z, t)(H+i)f(H)\| \leq K|t|^{-1} \tag{5.2}$$

$$\|\{1-f(H)\}(H+i)G(z, t)\|+\|G(z, t)(H+i)\{1-f(H)\}\| \leq K \tag{5.3}$$

Clearly

$$\|\{X(r, t)\}^*\|=\|X(r, t)\| \leq r\|\langle P\rangle^{1/2}(P^2+r^2)^{-1}\| \leq Kr^{-1/2} \tag{5.4}$$

So by (5.2) (5.3) and (5.4) we easily get

$$\|F(r, z, t)\| \leq Kr^{-1}|t|^{-1} \tag{5.5}$$

Following the proof of part 2 of Proposition II.5 of [Mo2] closely we conclude

$$\|f(H)G(z, t)X^*(r, t)\|+\|X(r, t)G(z, t)f(H)\| \leq K|t|^{-1/2}\|F(r, z, t)\|^{1/2}. \tag{5.6}$$

From (5.6) and (5.3) we easily get

$$\|(H+i)G(z, t)X^*(r, t)\|+\|X(r, t)G(z, t)(H+i)\| \leq K\{|t|^{-1/2}\|F(r, z, t)\|^{1/2} + \|X(r, t)\|\} \tag{5.7}$$

Now closely following the Lemma on page 404 of [Mo2], using the identity

$$-f(H)[H, A]f(H)=-[\{G(z, t)\}^{-1}, A]-it[Y, A]+\{1-f(H)\}[H, A]f(H)$$

$$+f(H)[H, A]\{1-f(H)\}+\{1-f(H)\}[H, A]\{1-f(H)\}$$

and regrouping the terms, we get

$$\begin{aligned}
&\frac{d}{dt} F(r, z, t)=\frac{d}{dt} F \\
&=\left\{\frac{d}{dt} X(r, t)(H+i)^{-1}\right\}(H+i)GX^*+XG(H+i)\left\{(H+i)^{-1}\frac{d}{dt}X^*\right\} \\
&\quad+XG\{1-f(H)\}(H+i)\{(H+i)^{-1}[H, A]\}f(H)GX^* \\
&\quad+XGf(H)\{[H, A](H+i)^{-1}\}(H+i)\{1-f(H)GX^* \qquad (5.8)\\
&\quad+XG\{1-f(H)\}(H+i)\{(H+i)^{-1}[H, A](H+i)^{-1}\}(H+i)\{1-f(H)\}GX^* \\
&\quad-XA(H+i)^{-1}\{(H+i)GX^*\}+\{XG(H+i)\}(H+i)^{-1}AX^* \\
&\quad-it\, XG(H+i)(H+i)^{-1}[Y, A](H+i)^{-1}(H+i)GX^*
\end{aligned}$$

We have written X, G, F for $X(r, t)$, $G(z, t)$, $F(r, z, t)$ respectively. We estimate each term of R.H.S. of (5.8) separately. For the first term, if β, σ, p satisfy A(a), A(b) we choose a diadic rational λ_1 such that $0\leq\lambda_1<2^{-1}$ and $\lambda_1+\sigma\geq 1\geq 2p(1-\lambda_1)$; if β, σ, p satisfy B(a), B(b) choose the diadic rational λ_1 such that $0\leq\lambda_1<2^{-1}<\lambda_1+\{2M(\lambda_1+\sigma-1)\}[n(1+2k)]^{-1}$. Then using Lemmas 4.9, 4.10 and 5.3 we get

$$\begin{aligned}
\left\|\frac{d}{dt} X(r, t)\langle h(P)\rangle^{-1}\right\| &\leq Kr|t|^{-\lambda_1}\|\langle Q\rangle^{1-\lambda_1-\sigma}\langle P\rangle^{p(1-\lambda_1)}\langle P\rangle^{1/2}(P^2+r^2)^{-1} \\
&\qquad \langle h(P)\rangle^{-1}\|\leq K|t|^{-\lambda_1} \qquad (5.9)
\end{aligned}$$

Now from (5.9), (5.7) and (5.4) we get

$$\|\textit{first term of R.H.S. of } (5.8)\|\leq K|t|^{-\lambda_1}\{|t|^{-1/2}\|F\|^{1/2}+1\} \qquad (5.10)$$

Similarly we get

$$\|\textit{second term of R.H.S. of } (5.8)\|\leq K|t|^{-\lambda_1}\{|t|^{-1/2}\|F\|^{1/2}+1\} \qquad (5.11)$$

Using (5.4), (5.3), boundedness of $(H+i)^{-1}[H, A]$ and (5.6) we get

$$\|\textit{third term of R.H.S. of } (5.8)\|\leq K|t|^{-1/2}\|F\|^{1/2} \qquad (5.12)$$

Similarly we get

$$\|\textit{fourth term of R.H.S. } (5.8)\|\leq K|t|^{-1/2}\|F\|^{1/2} \qquad (5.13)$$

Using (5.4), (5.3) and boundedness of $(H+i)^{-1}[H, A](H+i)^{-1}$ we get

$$\|\textit{fifth term of R.H.S. of } (5.8)\| \leq K \tag{5.14}$$

By Lemma 5.5 (see below), (5.7) and (5.4) we get

$$\|\textit{sixth term of R.H.S. of } (5.8)\| \leq K|t|^{-\beta}\{|t|^{-1/2}\|F\|^{1/2}+1\} \tag{5.15}$$

Similarly we conclude

$$\|\textit{seventh term of R.H.S. of } (5.8)\| \leq K|t|^{-\beta}\{|t|^{-1/2}\|F\|^{1/2}+1\} \tag{5.16}$$

By part 2 of Proposition II.6 of [Mo2] the operator $(H+i)^{-1}[Y, A](H+i)^{-1}$ is bounded. So by (5.7) and (5.4) we conclude

$$\|\textit{last term of R.H.S. of } (5.8)\| \leq K|t|\{|t|^{-1/2}\|F\|^{1/2}+1\}^2 \tag{5.17}$$

Now from (5.10) to (5.17) we see that there exists λ with $0 \leq \lambda < 1$ such that

$$\left\|\frac{d}{dt} F(r, z, t)\right\| \leq K|t|^{-\lambda}\{\|F(r, z, t)\|+1\} \tag{5.18}$$

Now the result follows from Lemma 5.2, (5.5) and (5.18). Q.E.D.

LEMMA 5.5. $\|X(r, t)A\langle h(P)\rangle^{-1}\| \leq K|t|^{-\beta}$ *for* $|t| \leq 1$.

PROOF. Clearly, for $t>0$, by Lemmas 4.9 and 4.10 we have $\|X(x, t)A\langle h(P)\rangle^{-1}\|$

$$\leq Kr\, t^{-\beta} \sum_j \|\langle Q\rangle^{-\beta-\sigma}\langle P\rangle^{-\beta p+(1/2)}(P^2+r^2)^{-1}Q_j\phi_j(P)\langle h(P)\rangle^{-1}\| +$$
$$+Kr\, t^{-\beta} \sum_j \|\langle Q\rangle^{-\beta-\sigma}\langle P\rangle^{-\beta p+(1/2)}(P^2+r^2)^{-1}\{D_j\phi_j(P)\}\langle h(P)\rangle^{-1}\|.$$

Using $\|D_j\phi_j\|_\infty < \infty$ and commuting Q_j to the left, we get $\|X(r, t)A\langle h(P)\rangle^{-1}\|$

$$\leq Kr\, t^{-\beta} \sum_j \|\langle Q\rangle^{-\beta-\sigma}D_j\{\langle P\rangle^{-\beta p+(1/2)}(P^2+r^2)^{-1}\}\phi_j(P)\langle h(P\rangle^{-1}\| +$$
$$Kr\, t^{-\beta} \sum_j \|\langle Q\rangle^{-\beta-\sigma+1}\langle P\rangle^{-\beta p+(1/2)}(P^2+r^2)^{-1}\phi_j(P)\langle h(P)\rangle^{-1}\| + Kt^{-\beta}.$$

Now the result follows by using the conditions on $\phi_j(\zeta)$, β, σ, p. Q.E.D.

§6. Proof of Proposition 4.6

By B we denote any bounded operator. Also $W(Q), Q_kW(Q), \phi_j(P)$ will be denoted by W, Q_kW, ϕ_j.

LEMMA 6.1. $[Q_j\phi_k(P), W(Q)]=B$ for all j, k.

PROOF. $[Q_j\phi_k, W]=[\phi_k, Q_j W]-[\phi_k, Q_j]W$. Now use Proposition 4.4 (ii) and the assumption A6(b). Q.E.D.

LEMMA 6.2. $[A,[A, W(Q)]]=2\sum_j [Q_j\phi_j(P),[A, W(Q)]]+B$.

PROOF. Write $A=2\sum_j Q_j\phi_j(P)-i\sum_j D_j\phi_j(P)$. Use Proposition 4.5 (i) and the assumption A6(b). Q.E.D.

LEMMA 6.3. $\sum_j [Q_j\phi_j(P),[A, W(Q)]]=2\sum_{j,k} [Q_j\phi_j(P),[Q_k\phi_k(P), W(Q)]]+B$

PROOF. Write $A=2\sum_j Q_k\phi_k(P)-i\sum_k D_k\phi_k(P)$ and use Jacobi identify for $\sum_{j,k} [Q_j\phi_j(P),[D_k\phi_k(P), W(Q)]]$. Now the result follows by Lemma 6.1 and the assumptions A6(b), A7. Q.E.D.

LEMMA 6.4. $\sum_{j,k} [Q_j\phi_j(P),[Q_k\phi_k(P), W(Q)]]=\sum_{j,k} [Q_j\phi_j(P),[\phi_k(P), Q_k W(Q)]]$ $+B$

PROOF. Using $[Q_k\phi_k, W]=[\phi_k, Q_k W]+i\{D_k\phi_k\}W$ and expanding $[Q_j\phi_j, \{D_k\phi_k\}W]$ in the second variable we see that

$$\begin{aligned}&\sum_{j,k} [Q_j\phi_j,[Q_k\phi_k, W]]\\ &=\sum_{j,k} [Q_j\phi_j,[\phi_k, Q_k W]]+i\sum_{j,k} D_k\phi_k [Q_j\phi_j, W]-\sum_{j,k} \{\phi_j D_j D_k\phi_k\} W.\end{aligned}$$

Now the result follows by the assumption A6(b), Lemma 6.1 and the assumption A7. Q.E.D.

LEMMA 6.5. $\sum_{j,k} [Q_j\phi_j,[\phi_k, Q_k W]]=\sum_{j,k} Q_j[\phi_j,[\phi_k, Q_k W]]+B$.

PROOP. $\sum_{j,k} [Q_j\phi_j,[\phi_k, Q_k W]]-\sum_{j,k} Q_j[\phi_j,[\phi_k, Q_k W]]$

$$=-\sum_{j,k} [Q_k W,[Q_j, \phi_k]]\phi_j \quad \text{by Jacobi identity}$$

$$=-i\sum_k [Q_k W, \sum_j \phi_j D_j\phi_k]+i\sum_{j,k} D_j\phi_k[Q_k W, \phi_j]$$

For the first term of R.H.S. use Proposition 4.4 (ii) and the assumptions A6(b), A7; for the second term use Proposition 4.4 (ii) and the assumption A6(b). Q.E.D.

LEMMA 6.6. $\sum_{j,k} Q_j[\psi_j, [\psi_k, Q_k W]] = \sum_{j,k} [\psi_j, [\psi_k, Q_jQ_k W]] + B$

PROOF. $\sum_{j,k} Q_j[\psi_j, [\psi_k, Q_k W]]$

$$= \sum_{j,k} \{[\psi_j, [\psi_k, Q_jQ_k W]] - [\psi_k, Q_j][\psi_j, Q_k W] - [\psi_j, Q_j][\psi_k, Q_k W]\}.$$

Now the result follows by the assumption A6(b) and Proposition 4.4 (ii). Q.E.D.

LEMMA 6.7. $[A, [A, W]] = B + 4\sum_{j,k} [\psi_j, [\psi_k, Q_jQ_k W]]$.

PROOF. Follows from Lemma 6.2 to 6.6. Q.E.D.

Acknowledgement:

The author wishes to thank the referee for (i) his elegant and short proof of Theorem 4.8 and (ii) his suggestion to shorten the article.

References

[AJS] Amrein., W.O., Jauch, J.M. and K.B. Sinha: Scattering Theory in Quantum Mechanics, (Lecture notes and Suppl in Phys. 16) Benjamin, Reading, 1977.

[D] Davies., E.B.: On Enss' approach to scattering theory, Duke Math. J. **47** (1980), 171-185.

[DM] Davies, E.B. and Pl. Muthuramalingam, Trace properties of some highly anisotropic operators, J. London Math. Soc. (2), **31** (1985), 137-149.

[E1] Enss, V., Asymptotic completeness for quantum mechanical potential scattering I. Short range potentials, Comm. Math. Phys. **61** (1978), 285-291.

[E2] Enss V., Asymptotic completeness for quantum meahanical potential scattering-II. Singular and long range potentials, Ann. Physics **119** (1979), 117-132.

[H1] Hörmander, L., The existence of wave operators in scattering theory, Math. Z. **146** (1976), 69-91.

[H2] Hörmander, L., The Analysis of Linear Partial Differential Operators Vol. 2: Differential Operators with Constant Coefficients, Springer-Verlag, Berlin, 1983.

[H3] Hörmander, L., The Analysis of Linear Partial Differential Operators Vol. 3: Pseudo-Differential operators, Springer-Verlag, Berlin, 1985.

[H4] Hörmander, L., The Analysis of Linear Partial Differential Operators Vol. 4: Fourier Integral Operators, Springer-Verlag, Berlin, 1985.

[HS] Halmos, P.R. and V.S. Sunder, Bounded integral operators on L^2 spaces, Ergeb. Math. Grenzgeb. (3) 96, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[JMP] Jensen, A., Mourre, E. and P. Perry, Multiple Commutator estimates and resolvent smoothness in scattering theory, Ann. Inst. H. Poincaré, **41** (1984), 207-225.

[KY] Kitada, H. and K. Yajima, A scattering theory for time dependent long range potentials, Duke Math. J. **49** (1982), 341-376.

[L] Lavine, R., Absolute continuity of Positive spectrum for Schrodinger Operators with long range potentials, J. Funct. Anal. **12** (1973), 30-54.

[Mo1] Mourre, E., Link between the geometrical and the spectral transformation approaches in scattering theory, Comm. Math. Phys. **68** (1979), 91-94.

[Mo2] Mourre, E., Absence of singular continuous spectrum for certain self adjoint operators, Comm. Math. Phys. **78** (1981), 391-408.

[Mu1] Muthuramalingam, Pl., Spectral properties of vaguely elliptic pseudo-differential operators with momentum dependent long range potentials using time dependent scattering theory-II, Math. Scand. **58** (1986), 255-274.

[Mu2] Muthuramalingam, Pl., Bound states for momentum and asymptotic completeness in $L^2(R^n)$: I. Trace class commutators for $n=1$, (to appear in Rev. Roumaine Math. Pures Appl.)

[Mu3] Muthuramalingam, Pl., Asymptotic completeness for anisotropic operators by Mourre's theory, Quart. J. Math. Oxford Ser. (2) **37** (1986), 315-320.

[MS] Muthuramalingam, Pl. and K.B. Sinha, Asymptotic completeness in long range scattering II. Ann. Sci. Ecole Norm. Sup. (4) **18** (1985), 57-87.

[Pa1] Pascu, M., A time dependent proof of the asymptotic completeness for simply characteristic operators and general short range perturbations, Rev. Roumaine Math. Pures Appl. **34** (1989), 317-328.

[Pa2] Pascu, M., Asymptotic completeness of the wave operators for simply characteristic operators and long range potentials by Enss' method, J. Differential Equations **89** (1991), 154-175.

[Pe] Perry, P.A., Mellin transforms and scattering theory I. Short range potentials, Duke Math. J. **47** (1980), 187-193.

[R] Rudin, W., Functional Analysis, Tata McGraw-Hill publishing company ltd, New Delhi, 1977.

[RS1] Reed, M. and B. Simon, Methods of Modern Mathematical Physics, Vol. 1, Functional analysis, Academic Press, New York, 1972.

[RS2] Reed, M. and B. Simon, Methods of Modern of Modern Mathematical Physics, Vol. II: Fourier analysis, Self adjointness, Academic Press, New York, 1975.

[RS3] Reed, M. and B. Simon, Methods of Modern Mathematical Physics, Vol. III: Scattering theory, Academic Press, New York, 1979.

[RS4] Reed, M. and B. Simon, Methods of Modern Mathematical Physics, Vol. IV. Analysis of operators, Academic Press, New York, 1978.

[Sc] Schechter, M., Spectra of partial differential operators, North Holland publishing company, 1971.

[Si] Simon, B., Phase space analysis of simple scattering systems: extensions of some work of Enss, Duke Math. J. **46** (1979), 119-168.

[U1] Umeda, T., Scattering theory for pseudo-differential operators. I: The existence of wave operators, Osaka J. Math. **18** (1981), 361-392.

[U2] Umeda, T., Scattering theory for pseudo-differential operators, II: The completeness of wave operators, Osaka J. Math. **19** (1982), 511-526.

[W] Weidmann, J., Linear Operators in Hilbert spaces, Springer, New York-Heidelberg-Berlin, 1980.

(Received December 12, 1990)
(Revised January 7, 1992)

Indian Statistical Institute
R.V. College Post
Bangalore 560 059
India