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On the solvability of ordinary differential equations in the space of distributions

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0. Introduction

In this paper, we study the solvability of a system of ordinary differential equations with real analytic coefficients in the space of ultra-distributions.

It is well known that systems of ordinary differential equations are always surjective on the space of hyperfunctions (Sato [S]). Moreover we can easily calculate the dimension of the hyperfunction solutions of a homogeneous equation (Kashiwara [K 1], Komatsu [Ko 2]). On the other hand, although the structure of distribution solutions is more comlicated and depends not only on the irregularity of the equation but on its Stokes lines, Malgrange showed, in his paper [Ma 1], the solvability in the distributions always holds. He proved this fact using the existence theorem of asymptotic expansion solutions in C^∞ category. Here we shall show that the solvability for ultra-distributions also holds by constructing holomorphic solutions satisfying suitable growth conditions in the complex domain and taking their boundary values.

Let $X=\boldsymbol{C}$ and $M=\boldsymbol{R}$ with a coordinate $z=x+\sqrt{-1}y$, $Z=\overline{\boldsymbol{R}^+}$, $p=(0;\sqrt{-1}dz)\in T_M^*X$ and $q=0\in M$. We denote by $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients.

THEOREM 0.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module in a neighborhood of q. Then we have*

$$H^1\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_Z\mathcal{F})_q=0$$

where $\mathcal{F}$ denotes $\mathcal{D}'_M$ or $\mathcal{D}_M^{(s)\prime}$ $(s\in(1,\infty))$.

For the notations of the theorem, refer to the next section. This theorem implies that a system of ordinary differential equations is solvable on the space of (ultra-)distributions with support in the half line.

COROLLARY 0.2. *Let $P\neq 0$ be a differential operator with real analy-*

tic coefficients. Then P is surjective on the space of distributions and ultradistributions of Beurling class $(*)$ $(*\in(1,\infty))$.

We remark again that Malgrange showed this corollary in the case of distributions by different method. We also obtain microlocal version of this result.

THEOREM 0.3. *Let $\mathcal{M}$ be a coherent $\mathcal{E}_X$ module at p. Then we have*

$$H^1\boldsymbol{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{F})_p=0$$

where $\mathcal{F}$ denotes $\mathcal{C}^{R}_{\{0\}|X}$, $\mathcal{C}^{R,f}_{\{0\}|X}$, $\mathcal{C}^{f}_{M}$ or $\mathcal{C}^{(s)}_{M}$ $(s\in(1,\infty))$ *(Refer to Section 1 for the definition of these sheaves.)*

The plan of our paper is as follows. In Section 1, we give a review of the several sheaves which appear as solution sheaves. In Section 2, we construct solutions of a system on small sectors and give estimates of their growth order. In Section 3, the solutions constructed in Section 2 are connected with each other, and we obtain a holomorphic function which represents an ultra-distribution solution as boundary value.

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1. Preliminary

In this section, we briefly recall the definitions of several sheaves which appear in this paper. Let M be a real analytic manifold of dimension n and X its complexification. The sheaf $\mathcal{E}_X$ of micro-differential operators on the cotangent bundle T^*X of X was first constructed by Sato-Kashiwara-Kawai (see [S-K-K]). Sato also introduced the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions on M and $\mathcal{C}_M$ of microfunctions on the conormal bundle T^*_MX of M. These sheaves are defined as follows.

$$\mathcal{B}_M := \boldsymbol{R}\Gamma_M(\mathcal{O}_X)[n]\otimes\omega_M,$$
$$\mathcal{C}_M := \mu_M(\mathcal{O}_X)[n]\otimes\omega_M$$

where ω_M is the orientation sheaf of M and $\mu_M(\cdot)$ is Sato's microlocalization functor (see Kashiwara-Schapira [K-Sc 1] and [K-Sc 2]). There exist the exact sequence and the spectrum map

$$0 \to \mathcal{A}_M \to \mathcal{B}_M \to \pi_* \mathcal{C}_M \to 0, \tag{1.0}$$
$$\mathrm{sp} : \pi^{-1}\mathcal{B}_M \to \mathcal{C}_M \to 0$$

where $\pi : T^*_M X \to M$ is the canonical projection and $\mathcal{A}_M$ is the sheaf of real analytic functions. Remark that $\mathcal{C}_M$ is an $\mathcal{E}_X$ module. For the properties of $\mathcal{B}_M$, $\mathcal{C}_M$ and $\mathcal{E}_X$, refer to [S-K-K] and Schapira [Sc].

We denote by $\mathcal{D}'_M$ and $\mathcal{D}^{(s)\prime}_M$ the sheaf of distributions and that of ultra-distributions of Beurling class (s) $(s \in (1, \infty))$. $\mathcal{D}'_M$ and $\mathcal{D}^{(s)\prime}_M$ can be regarded as subsheaves of $\mathcal{B}_M$.

Next we recall the definitions of tempered microfunctions $\mathcal{C}^f_M$ and $\mathcal{C}^{(s)}_M$ the microfunctions of Beurling class (s). The sheaf $\mathcal{C}^f_M$ was first introduced by Martineau [M] and functorially constructed by Andronikof [A] (see also [Be-Sc]). These are subsheaves of $\mathcal{C}_M$ on $T^*_M X$ and defined by

$$\mathcal{C}^f_M := \mathrm{sp}(\pi^{-1}(\mathcal{D}'_M)), \tag{1.1}$$
$$\mathcal{C}^{(s)}_M := \mathrm{sp}(\pi^{-1}(\mathcal{D}^{(s)\prime}_M))$$

where sp is the spectrum map defined in (1.0). We have the exact sequences

$$0 \to \mathcal{A}_M \to \mathcal{D}'_M \to \pi_* \mathcal{C}^f_M \to 0, \tag{1.2}$$
$$0 \to \mathcal{A}_M \to \mathcal{D}^{(s)\prime}_M \to \pi_* \mathcal{C}^{(s)}_M \to 0.$$

From now on, we consider the one dimensional case and assume $M = \boldsymbol{R}$ and $X = \boldsymbol{C}$ with a coordinate $z = x + \sqrt{-1}y$. Let $s \in (1, \infty)$, and set $\sigma := \dfrac{1}{s-1}$. Since hyperfunctions are expressed as boundary values of holomorphic functions, $\mathcal{D}'_M$ and $\mathcal{D}^{(s)\prime}_M$ are also represented by holomorphic functions satisfying suitable growth conditions. To describe them, we make several preparations. Let U be an open subset in $\boldsymbol{C}$. We define the norms $|\cdot|^{l,s}_U$ and $|\cdot|^N_U$ as follows.

$$|f|^{l,s}_U = \sup_{z \in U} |\mathrm{Exp}(-l(\mathrm{dist}(z, \complement U))^{-\sigma}) f(z)|, \tag{1.3}$$
$$|f|^N_U = \sup_{z \in U} |(\mathrm{dist}(z, \complement U))^N f(z)|.$$

Let B_ϵ be an open ball with radius ϵ and center at 0, and $Z = \overline{\boldsymbol{R}^+}$. We introduce the spaces $O^{l,(s)}(U)$ and $O^N(U)$ as

$$O^{l,(s)}(U) = \{f \in \mathcal{O}(U);\ |f|^{l,s}_U < \infty\}, \tag{1.4}$$
$$O^N(U) = \{f \in \mathcal{O}(U);\ |f|^U_N < \infty\}.$$

Now we give alternative definitions of $\Gamma_Z\mathcal{D}'_M$ and $\Gamma_Z\mathcal{D}^{(s)\prime}_M$ as the boundary values of holomorphic functions.

$$(1.5)\qquad \begin{aligned}(\Gamma_Z\mathcal{D}'_M)_0 &= \frac{\lim\limits_{\epsilon\to 0, N\to\infty} O^N(B_\epsilon\backslash Z)}{\mathcal{O}_0}\\ (\Gamma_Z\mathcal{D}^{(s)\prime}_M)_0 &= \frac{\lim\limits_{\epsilon\to 0, l\to\infty} O^{l,(s)}(B_\epsilon\backslash Z)}{\mathcal{O}_0}\end{aligned}$$

Here $\mathcal{O}_0$ denotes the stalk of $\mathcal{O}_X$ at the origin. Finally we review the sheaves $\mathcal{C}^R_{\{0\}|X}$ and $\mathcal{C}^{R,f}_{\{0\}|X}$ which were introduced by Sato-Kashiwara-Kawai [S-K-K] and Andronikof [A]. These are the sheaves on $T^*_{\{0\}}X$ and functorially defined by

$$(1.6)\qquad \begin{aligned}\mathcal{C}^R_{\{0\}|X} &= \mu_{\{0\}}(\mathcal{O}_X)[1],\\ \mathcal{C}^{R,f}_{\{0\}|X} &= T-\mu_{\{0\}}(\mathcal{O}_X)[1].\end{aligned}$$

For the definition of the functor $T-\mu(\cdot)$, refer to [A]. These sheaves are also represented by the boundary values of holomorphic functions. Set

$$T_\epsilon = \{z \in B_\epsilon;\ \epsilon|\Im z| > \Re z\}.$$

Then we have

$$(1.7)\qquad (\mathcal{C}^{R,f}_{\{0\}|X})_{(0;dx)} = \frac{\lim\limits_{\epsilon\to 0, N\to\infty} O^N(T_\epsilon)}{\mathcal{O}_0}$$

2. Construction of solutions on small sectors

Throughout this and next sections, we consider the following system of ordinary differential equations.

$$(2.0)\qquad Pu = \left(z^d\frac{d}{dz} + A(z)\right)u = f(z).$$

Here $d\in N$, $A(z)\in gl(m, \mathcal{O}_0)$, and $f(z)$ is a column vector of holomorphic functions of size m, which represent ultra-distributions with support in the half line. The aim of this section is to give the estimates of the integrals which appear in the solutions of the system (2.0). There are many works for the estimates by the distance from the origin (that is the singular point of the system), which assure existence of solutions satisfying desired asymptotic expansions (Hukuhara [H 1], [H 2], Iwano [I], Wasow [W] and etc.). We need, however, the estimates of the integrals by the distance from the real axis to show the surjectivity of the system (2.0) in the space of (ultra-)distributions.

Let $X=C$ with a coordinate $z=x+\sqrt{-1}y$, U an open subset of C, l a positive constant and $s\in(1,\infty)$. Set $\sigma:=\frac{1}{s-1}$. We introduce a new norm $\|\cdot\|_U^{l,s}$ which is slightly different from the norm $|\cdot|_U^{l,s}$ defined by (1.3).

$$\|f\|_U^{l,s}=\sup_{z\in U}|\operatorname{Exp}(-l(\operatorname{dist}(z,\overline{R^+}))^{-\sigma})f(z)| \qquad \text{for } f\in\mathcal{O}(U). \tag{2.1}$$

We abbreviate $\|\cdot\|_U^{l,s}$ to $\|\cdot\|^{l,s}$, if there is no risk of confusion. Now we define the space of holomorphic functions $\tilde{O}^{l,(s)}(U)$ which is also slightly different from the space $O^{l,(s)}(U)$ in Section 1.

DEFINITION 2.1. *The space $\tilde{O}^{l,(s)}(U)$ is defined by*

$$\tilde{O}^{l,(s)}(U):=\{f\in\mathcal{O}_X(U)\cap C^0(\overline{U}\setminus\overline{R^+});\ \|f\|_U^{l,s}<\infty\}.$$

Let $\lambda=\lambda_1>\cdots>\lambda_n=0$ be positive real numbers $(n\geq 1)$, and

$$\Lambda(z)=\sum_{k=1}^{n}a_k z^{-\lambda_k} \tag{2.2}$$

where $a_k\in C$, $a_1\neq 0$ and $a_n=1$. Remark that we consider $z^{-\lambda_k}$ as a holomorphic function on Riemann domain, and always choose its branch which has positive real values on $\arg z=0$. Set $\bar{\Lambda}(z):=a_1 z^{-\lambda}$ and $\omega_k=\arg(a_k)$. Then we have

$$\Re(\Lambda(\rho e^{i\theta}))=\sum_{k=1}^{n}|a_k|\rho^{-\lambda_k}\cos(-\lambda_k\theta+\omega_k). \tag{2.3}$$

Let L be an open half line starting from the origin.

DEFINITION 2.2. (i) We say L is a separate line of $\Lambda(z)$ if the real part of $\bar{\Lambda}(z)$ vanishes on L.

(ii) An open subset $\{\Re\bar{\Lambda}(z)<0$ (resp. >0), $0<\arg z<2\pi\}$ is said to be a negative (resp. positive) region of Λ.

It is easy to see that $L=R^+e^{i\theta}$ is a separate line of Λ if and only if $-\lambda\theta+\omega_1-\frac{\pi}{2}\in\pi\boldsymbol{Z}$.

Let θ_0 and θ_1 be real numbers satisfying $2\pi\geq\theta_1>\theta_0\geq 0$, and $R>0$. Set.

$$S(\theta_0,\theta_1,R)=\{z\in\boldsymbol{C};\ \theta_0<\arg z<\theta_1,\ 0<|z|<R\}. \tag{2.4}$$

If $\theta_0=0$ (resp. $\theta_0=0$ and $\theta_1=2\pi$), we denote $S(\theta_0,\theta_1,R)$ by $S(\theta_1,R)$ (resp. $S(R)$). Now we consider the integral

$$I_{z_0}(f)(z):=\operatorname{Exp}(-\Lambda(z))\int_{z_0}^{z}\operatorname{Exp}(\Lambda(z))f(z)dz \tag{2.5}$$

where $f\in\mathcal{O}(S(\theta_0,\theta_1,R))$ and z_0 is a point in $\overline{S(\theta_0,\theta_1,R)}$ which will be determined later on.

The first step is to show the following proposition.

PROPOSITION 2.3. *There exist* $\theta_1\in\left(0,\frac{\pi}{2}\right]$ *and* $l_0>0$ *with the following property; for any* $l\geq l_0$, *there exist positive constants* l' *and* C_l *such that we have*

$$\|I_{z_0}(f)\|_{S(\theta_1,R)}^{l',s}\leq C_l\|f\|_{S(\theta_1,R)}^{l,s}\qquad for\quad f\in\tilde{\mathcal{O}}^{l,(s)}(S(\theta_1,R)), \tag{2.6}$$

if we take $z_0\in\overline{S(\theta_1,R)}$ *and the path of the integral* (2.5) *in a suitable way which will be shown in the proof of this proposition.*

We consider the problem in the following three cases. The first case is that $\sigma\geq\lambda$, which is trivial. The second is that neither $\boldsymbol{R}^+$ nor $\boldsymbol{R}^+e^{i\theta_1}$ is a separate line of Λ, and $S(\theta_1,R)$ intersects with at most one component of a negative region of Λ. The last is the most important case in which the positive part of the real axis is a separate line.

(I) The first case.

We consider the problem under the condition $\sigma\geq\lambda$. Since $|z|\geq|y|$, we can easily show Proposition 2.3. In this case, we can choose an arbitrary point in $S(\theta_1,R)$ as z_0. Thus from now on, we always assume $\lambda>\sigma>0$.

(II) The second case.

We consider the problem in the following situation.

(2.7) $\lambda>\sigma>0$ and $\theta_1\leq\frac{\pi}{2}$.

(2.8) The sector $S(\theta_1, R)$ intersects with at most one separate line of Λ, and neither $\boldsymbol{R}^+e^{i\theta_1}$ nor $\boldsymbol{R}^+$ is a separate line.

Set

$$R^{l,(s)}(z)=\Re\Lambda(z)+l\left(\frac{1}{\Im z}\right)^{\sigma}. \tag{2.9}$$

Let $z_1=\rho_1 e^{i\phi_1}$ be a point in $S(\theta_1, R)$. First we choose the point z_0 in the sector $S(\theta_1, R)$, real numbers $\phi_0\in(0, \theta_1]$ and $\epsilon\in(0, R]$ to determine the path of (2.5) as follows.

(II. A): Assume the sector $S(\theta_1, R)$ intersects with a negative region of Λ. Then we take ϕ_0 and ϵ so that a half line $\boldsymbol{R}^+e^{i\phi_0}$ intersects with a negative region of Λ, and $R^{l,(s)}(te^{i\phi_0})$ is an increasing function of t on $(0, \epsilon]$. Moreover we take the origin as z_0.

(II. B): Assume the sector $S(\theta_1, R)$ does not intersect with a negative region of Λ. Then ϕ_0 is arbitrary, and ϵ is chosen so that $R^{l,(s)}(te^{i\phi_0})$ is a decreasing function of t on $(0, \epsilon]$. Moreover we set $z_0=\epsilon e^{i\phi_0}$.

Let $a(\phi)$ be a real valued piecewise continuous function on $[0, \theta_1]$ with values $(0, \pi)$. We define the functions $\tilde{\rho}: S(\theta_1, R)\to\boldsymbol{R}$ and $\tilde{z}: S(\theta_1, R)\to\boldsymbol{C}$ by

$$\begin{aligned}\tilde{\rho}(z=\rho e^{i\phi}) &:=\rho\,\mathrm{Exp}\left(-\int_{\phi_0}^{\phi}\operatorname{cotan} a(\phi)d\phi\right).\\ \tilde{z}(z) &:=\tilde{\rho}(z)e^{i\phi_0}.\end{aligned} \tag{2.10}$$

The path Γ_{z_1} from z_0 to z_1 of the integral (2.5) consists of two parts Γ_{1,z_1} and Γ_{2,z_1} (see fig 2.1): the path Γ_{1,z_1} is the segment from z_0 to $\tilde{z}(z_1)$, and the path Γ_{2,z_1} is

$$\Gamma_{2,z_1}:\phi\in[\phi_0, \phi_1]\to z=\left(\tilde{\rho}(z_1)\mathrm{Exp}\left(\int_{\phi_0}^{\phi}\operatorname{cotan} a(\phi)d\phi\right)\right)e^{i\phi}.$$

We take the path Γ_{2,z_1} inspired by the papers [H 1], [H 2] and [I].

Let r be a parameter of length along the curve Γ_{2,z_1}. By direct calculations, we obtain

$$\frac{d\Re\Lambda}{dr}(\rho e^{i\phi})=\pm\sum_{k=1}^{n}|a_k|\lambda_k\rho^{-\lambda_k-1}\cos(-\lambda_k+\omega_k+a(\phi))\qquad\text{if}\quad\pm\phi<\phi_0.$$

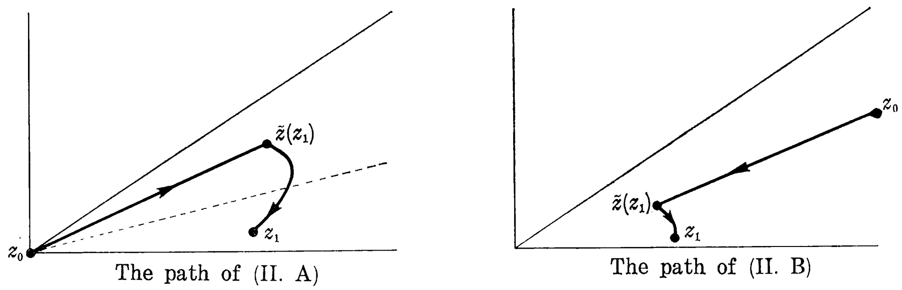

Fig. 2.1.

Now we give the proof of Proposition 2.3 in the case (II).

PROOF OF PROPOSITION 2.3 IN THE CASE (II). It is easy to see $I_{z_0}(f)\in\mathcal{O}(S(\theta_1, R))$. To obtain the estimate, we choose the function $a(\theta)$ so that $\operatorname{Exp}(\Re\Lambda)$ is an increasing function of r along Γ_{2,z_1}. It is possible under the condition (2.8) because the path Γ_{2,z_1} does necessarily pass form a negative region to a positive region when Γ_{2,z_1} intersects a separate line of Λ, and because the boundary lines of the sector are not separate lines. Remark that there exists a positive constant m_1 such that for any interval $[\theta', \theta'']\subset[0, \theta_1]$,

$$m_1^{-1}\leq\operatorname{Exp}\left(\int_{\theta'}^{\theta''}\operatorname{cotan} a(\theta)\,d\theta\right)\leq m_1.$$

For any $f\in\tilde{\mathcal{O}}^{l,(s)}(S(\theta_1, R))$ and any point $z_1\in S(\theta_1, R)$ satisfying $\tilde{\rho}(z_1)\in(0, \epsilon]$,

$$\begin{aligned}|I_{z_0}(f)(z_1)\leq&\|f\|^{l,s}\operatorname{Exp}(-\Re\Lambda(z_1))\int_{\Gamma_{1,z_1}}\operatorname{Exp}\left(\Re\Lambda(z)+l\left(\frac{1}{|\Im z|}\right)^{\sigma}\right)dz\\&+\|f\|^{l,s}\operatorname{Exp}(-\Re\Lambda(z_1))\int_{\Gamma_{2,z_1}}\operatorname{Exp}\left(\Re\Lambda(z)+l\left(\frac{1}{|\Im z|}\right)^{\sigma}\right)dz\\\leq&\|f\|^{l,s}\operatorname{Exp}\left(l\left(\frac{1}{|\Im\tilde{z}(z_1)|}\right)^{\sigma}\right)\operatorname{Exp}(\Re(-\Lambda(z_1)+\Lambda(\tilde{z}(z_1))))\\&+\|f\|^{l,s}\max_{z\in\Gamma_{2,z_1}}\left\{\operatorname{Exp}\left(l\left(\frac{1}{|\Im z|}\right)^{\sigma}\right)\right\}.\end{aligned}$$

We have for any $z\in\Gamma_{2,z_1}$,

$$|\Im z|=\rho_1\sin\phi\operatorname{Exp}\left(\int_{\phi_1}^{\phi}\operatorname{cotan} a(\phi)\,d\phi\right)\geq|\Im z_1|\frac{\sin\phi}{\sin\phi_1}m_1^{-1}.$$

Since $\dfrac{\sin\phi}{\sin\phi_1}\geq\sin\phi_0$, there exists $l'>0$ so that

$$\max_{z\in\Gamma_{2,z_1}}\left\{\mathrm{Exp}\left(l\left(\frac{1}{|\Im z|}\right)^{\sigma}\right)\right\}\leq\mathrm{Exp}\left(l'\left(\frac{1}{|\Im z_1|}\right)^{\sigma}\right).$$

Thus we obtain, for any point $z_1(\in S(\theta_1, R))$ satisfying $|\tilde{\rho}(z_1)|\leq\epsilon$,

$$|I_{z_0}(f)(z_1)|\leq C\,\mathrm{Exp}\left(l'\left(\frac{1}{|\Im z_1|}\right)^{\sigma}\right)\|f\|^{l,s}. \tag{2.11}$$

Moreover (2.11) is valid for all points in $S(\theta_1, R)$, if we replace l' and C larger. This completes the proof. ■

(III) The third case.

We assume the following conditions.

(2.12) $\lambda>\sigma>0$ and $\theta_1<\min\left\{\dfrac{\pi}{2}, \dfrac{\pi}{2\lambda}\right\}$.

(2.13) A positive real axis is a separate line of Λ, and the sector $\overline{S(\theta_1, R)\backslash \boldsymbol{R}}$ does not intersect with any separate line.

Set

$$k_0:=\min\{k;\ \Re a_k\neq 0,\ 1\leq k\leq n\}.$$

Then we have

$$\begin{aligned}\Re\Lambda(te^{i\phi})=&\sum_{k=1}^{k_0-1}\mathrm{sgn}(a_k)|a_k|t^{-\lambda_k}\sin(\lambda_k\phi)\\ &+\sum_{k=k_0}^{n}|a_k|t^{-\lambda_k}\cos(-\lambda_k\phi+\omega_k)\end{aligned} \tag{2.14}$$

with $\mathrm{sgn}(a_k)=\dfrac{\Im a_k}{|\Im a_k|}$. Under the conditions (2.12) and (2.13), it is sufficient to study, case by case, the following three cases (III. A), (III. B) and (III. C).

Case (III. A): Assume that $\Re\bar{\Lambda}<0$ on $S(\theta_1, R)$. In this case, we can apply the same argument as the second case to the problem. We choose the origin as z_0 and the path Γ_z as (II. A), and obtain the estimate (2.6) in the same way.

Case (III. B): Assume $\Re\bar{\Lambda}>0$ on $S(\theta_1, R)$ and $\Re a_{k_0}>0$. To choose the path, we prepare the following lemma.

LEMMA 2.4. *There exists a positiv constant r with the property that for any $\phi\in[0,\theta_1]$, $\Re A(te^{i\phi})$ is a decreasing function of t on $(0,r]$.*

PROOF. We have

$$\frac{d\Re A(te^{i\phi})}{dt}=-|a_1|\lambda t^{-\lambda-1}\sin(\lambda\phi)-\sum_{k=2}^{k_0-1}\operatorname{sgn}(a_k)|a_k|\lambda_k t^{-\lambda_k-1}\sin(\lambda_k\phi)$$
$$-\sum_{k=k_0}^{n}|a_k|\lambda_k t^{-\lambda_k-1}\cos(-\lambda_k\phi+\omega_k)$$
$$=-t^{-\lambda-1}\sin(\lambda\phi)\Big(|a_1|\lambda+\sum_{k=2}^{k_0-1}\operatorname{sgn}(a_k)|a_k|\lambda_k t^{\lambda-\lambda_k}\frac{\sin(\lambda_k\phi)}{\sin(\lambda\phi)}\Big)$$
$$-t^{-\lambda_{k_0}-1}\Big(\sum_{k=k_0}^{n}|a_k|\lambda_k t^{\lambda_{k_0}-\lambda_k}\cos(-\lambda_k\phi+\omega_k)\Big). \tag{2.15}$$

Since $\Re a_{k_0}>0$, there exists a positive constant $\chi\in(0,\theta_0]$ so that $\cos(-\lambda_{k_0}\phi+\omega_{k_0})\geq\epsilon>0$ on $[0,\chi]$. If $\phi\geq\chi$, since $\sin(\lambda\phi)\geq\sin(\lambda\chi)>0$, we obtain easily $\frac{d\Re A(te^{i\phi})}{dt}<0$ for sufficiently small t. If $\phi\leq\chi$, since $\frac{\sin(\lambda_k\phi)}{\sin(\lambda\phi)}\leq 1$, the both terms of the right hand side of (2.15) are less than or equal to 0 for small t. This completes the proof. ■

Let $z_1=\rho_1 e^{i\phi_1}$ be a point in $S(\theta_1,r)$ where r is given in Lemma 2.4. Now we determine the point z_0 and the path Γ_{z_1}. We set $z_0=Re^{i\theta_1}$, and the path Γ_{z_1} consists of three parts: Γ_1 is the segment from z_0 to $re^{i\theta_1}$, Γ_{2,z_1} is the arc

$$\Gamma_{2,z_1}:\phi\in[\theta_1,\phi_1]\rightarrow z=re^{i\phi},$$

and Γ_{3,z_1} is the segment from $re^{i\phi_1}$ to z_1 (see fig. 2.2).

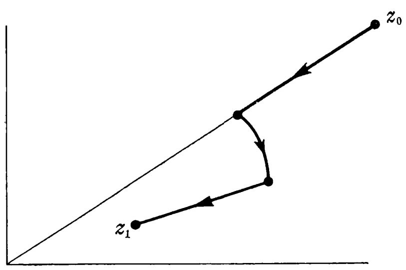

Fig. 2.2.

PROOF OF PROPOSITION 2.3 IN THE CASE (III. B). By Lemma 2.4, there exists a positive constant m_1 such that

$$\mathrm{Exp}(-\Re\Lambda(z))\leq m_1 \qquad (z\in S(\theta_1, R)).$$

Then we have

$$\begin{aligned}|I_{z_0}(f)(z_1)| &\leq \mathrm{Exp}(-\Re\Lambda(z_1))\int_{\Gamma_1+\Gamma_{2,z_1}}\mathrm{Exp}(\Re\Lambda(z))|f(z)|dz \\ &\quad +\mathrm{Exp}(-\Re\Lambda(z_1))\int_{\Gamma_{3,z_1}}\mathrm{Exp}(\Re\Lambda(z))|f(z)|dz \\ &\leq \|f\|^{l,s}m_1\,\mathrm{Exp}\Big(l\Big(\frac{1}{|r\sin(\phi_1)|}\Big)^{\sigma}\Big)\int_{\Gamma_1+\Gamma_{2,z_1}}\mathrm{Exp}(\Re\Lambda(z))dz \\ &\quad +\|f\|^{l,s}\,\mathrm{Exp}\Big(l\Big(\frac{1}{|\Im z_1|}\Big)^{\sigma}\Big).\end{aligned}$$

Thus we have the estimate (2.6). This completes the proof of the case (III. B). ■

Case (III. C): Assume $\Re\bar{\Lambda}(z)>0$ on $S(\theta_1, R)$ and $\Re a_{k_0}<0$. In this case, we need several lemmas to determine the path. Let $\alpha>1$. We define the function $w_\alpha: \boldsymbol{R}^+\to\boldsymbol{C}$ by

$$w_\alpha(x)=x+\sqrt{-1}x^{\alpha}. \tag{2.16}$$

LEMMA 2.5. *There exist l_0 and $x_0>0$ with the property that for any $l\geq l_0$ and any $x\in(0, x_0]$, $R^{l,(s)}(tw_\alpha(x))$ is a decreasing function of t on $(0, 1]$, if one of the following conditions is satisfied.*
(1) $\alpha\in(1, \lambda-\lambda_{k_0}+1)$.
(2) $\alpha=\lambda-\lambda_{k_0}+1$ *and* $\alpha\sigma\geq\lambda_{k_0}$.

PROOF. We may assume $x_0<1$. We have

$$\begin{aligned}\frac{dR^{l,(s)}(tw_\alpha(x))}{dt} &= -\sum_{k=1}^{k_0-1}\mathrm{sgn}(a_k)|a_k|\lambda_k t^{-\lambda_k-1}|w_\alpha(x)|^{-\lambda_k}\sin(\lambda_k\theta_\alpha(x)) \\ &\quad -\sum_{k=k_0}^{n}|a_k|\lambda_k t^{-\lambda_k-1}|w_\alpha(x)|^{-\lambda_k}\cos(-\lambda_k\theta_\alpha(x)+\omega_k)-l\sigma x^{-\alpha\sigma}t^{-\sigma-1}\end{aligned} \tag{2.17}$$

where $\theta_\alpha(x)=\arg(w_\alpha(x))$. We remark that for any μ_1, μ_2 with $\mu_1\geq\mu_2>0$ and $\mu_1\phi\leq\dfrac{\pi}{2}$

$$1\leq\frac{\sin(\mu_1\phi)}{\sin(\mu_2\phi)}\leq\frac{\mu_1}{\mu_2}. \tag{2.18}$$

If $0\leq x\leq 1$,

$$x\leq|w_\alpha(x)|\leq\sqrt{2}x. \tag{2.19}$$

Thereby there exist positive constants m_1 and m_2 so that

$$\frac{dR^{l,(s)}(tw_\alpha(x))}{dt}\leq -m_1x^{-\lambda}t^{-\lambda-1}\sin(\theta_\alpha(x))+m_2x^{-\lambda_{k_0}}t^{-\lambda_{k_0}-1}-l\sigma x^{-\alpha\sigma}t^{-\sigma-1}$$

for sufficiently small x. Since

$$\frac{x^{\alpha-1}}{\sqrt{2}}\leq \sin(\theta_\alpha(x))\leq x^{\alpha-1},$$

we obtain the estimate

$$\frac{dR^{l,(s)}(tw_\alpha(x))}{dt}\leq(tx)^{-\lambda-1}(-m_3x^\alpha+m_2t^{\lambda-\lambda_{k_0}}x^{\lambda-\lambda_{k_0}+1})-l\sigma x^{-\alpha\sigma}t^{-\sigma-1} \tag{2.20}$$

where m_3 is a positive constant. If $\alpha<\lambda-\lambda_{k_0}+1$, the first term of the right hand side of (2.20) is negative for small x. This proves the first assertion of the lemma. Now we assume $\alpha=\lambda-\lambda_{k_0}+1$. If we take $\epsilon'>0$ small enough, the first term of (2.20) has negative values for $t\in(0,\epsilon']$. Thus it is enough to show the lemma when $t\in[\epsilon',1]$, and we obtain

$$(tx)^{-\lambda-1}(-m_3x^\alpha+m_2t^{\lambda-\lambda_{k_0}}x^{\lambda-\lambda_{k_0}+1})\leq(\epsilon'x)^{-\lambda-1}(-m_3x^\alpha+m_2x^{\lambda-\lambda_{k_0}+1})\leq m_4x^{-\lambda_{k_0}}.$$

Thus we have the estimate

$$\frac{dR^{l,(s)}(tw_\alpha(x))}{dt}\leq m_4x^{-\lambda_{k_0}}-lx^{-\alpha\sigma}\qquad(t\in[\epsilon',1]),$$

and this has negative values for large l because of $\alpha\sigma\geq\lambda_{k_0}$. This completes the proof. ■

LEMMA 2.6. (i) *Assume* $\sigma(\lambda-\lambda_{k_0}+1)\geq\lambda_{k_0}$. *Then there exist positive constants* l_1 *and* x_1 *so that* $R^{l,(s)}(w_\alpha(x))$ *is a decreasing function of* x *on* $(0,x_1]$ *for any* $\alpha\geq\lambda-\lambda_{k_0}+1$ *and* $l\geq l_1$.

(ii) *Assume* $\sigma(\lambda-\lambda_{k_0}+1)<\lambda_{k_0}$. *Then for any* α *satisfying* $\sigma(\lambda-\lambda_{k_0}+1)<\alpha\sigma<\lambda_{k_0}$ *and any* $l>0$, *there exists a positive constant* $x_{1,\alpha,l}$ *such that* $R^{l,(s)}(w_\alpha(x))$ *is an increasing function of* x *on* $(0,x_{1,\alpha,l}]$.

FROOF. We have

$$\begin{aligned}\frac{dR^{l,(s)}(w_\alpha(x))}{dx}=&-\sum_{k=1}^{k_0-1}\operatorname{sgn}(\alpha_k)|a_k|\lambda_k\frac{d|w_\alpha|}{dx}(x)|w_\alpha(x)|^{-\lambda_k-1}\sin(\lambda_k\theta_\alpha(x))\\&+\sum_{k=1}^{k_0-1}\lambda_k\operatorname{sgn}(a_k)|a_k||w_\alpha(x)|^{-\lambda_k}\frac{d\theta_\alpha}{dx}(x)\cos(\lambda_k\theta_\alpha(x))\\&-\sum_{k=k_0}^{n}\operatorname{sgn}(a_k)|a_k|\lambda_k\frac{d|w_\alpha|}{dx}(x)|w_\alpha(x)|^{-\lambda_k-1}\cos(-\lambda_k\theta_\alpha(x)+w_k)\\&+\sum_{k=k_0}^{n}\lambda_k\operatorname{sgn}(a_k)|a_k||w_\alpha(x)|^{-\lambda_k}\frac{d\theta_\alpha}{dx}(x)\sin(-\lambda_k\theta_\alpha(x)+w_k)\\&-\alpha l\sigma x^{-\alpha\sigma-1}=(I)+(II)+(III)+(IV)+(V).\end{aligned}$$

From now on, we always assume $x\leq\frac{1}{2}$. There exists a positive constant m_1 which does not depend on α so that

$$1\leq\frac{d|w_\alpha|}{dx}=\frac{1+\alpha x^{2(\alpha-1)}}{\sqrt{1+x^{2(\alpha-1)}}}\leq m_1. \tag{2.21}$$

Moreover since $\sin(\theta_\alpha(x))=\frac{x^\alpha}{|w_\alpha(x)|}$ and $\alpha>1$,

$$\left|\frac{d\theta_\alpha}{dx}(x)\right|=\left|\frac{1}{\cos(\theta_\alpha(x))}\frac{d}{dx}\left(\frac{x^\alpha}{|w_\alpha(x)|}\right)\right|\leq(\alpha+m_1)x^{\alpha-2}. \tag{2.22}$$

Set $a=\max_{1\leq k\leq n}\{|a_k|\}$. Then we have

$$|(I)|\leqq a(k_0-1)\lambda m_1x^{\alpha-\lambda-2},\quad |(III)|\leq a(-k_0+n+1)\lambda_{k_0}m_1x^{-\lambda_{k_0}-1}.$$

and

$$|(II)|\leq a(\alpha+m_1)(k_0-1)\lambda x^{\alpha-\lambda-2},\quad |(IV)|\leq a(\alpha+m_1)(-k_0+n+1)\lambda x^{\alpha-\lambda_k-2}$$

We first consider the case (i) of the lemma. Since $\alpha-\lambda-2\geq-\lambda_{k_0}-1$, we have

$$\frac{dR^{l,(s)}(w_\alpha(x))}{dx}\leq\alpha(m_2x^{-\lambda_{k_0}-1}-l\sigma x^{-\alpha\sigma-1}), \tag{2.23}$$

where a positive constant m_2 does not depend on α and l. If l is sufficiently large, (2.23) is negative because of $\alpha\sigma\geq\lambda_{k_0}$. Next we consider the case (ii) of the lemma. Since $\Re a_{k_0}<0$, there exist constants $\chi(\in(0,\theta_1))$ and $m_3>0$ so that $\cos(-\lambda_{k_0}\phi+\omega_{k_0})<-m_3$ on $[0,\chi]$. Moreover we take x_1

so that $\arg(w_\alpha(x_1))\leq\chi$. Then we have for $x\in(0, x_1]$,

$$\text{the first term of } (III)\geq|a_{k_0}|\lambda_{k_0}m_3x^{-\lambda_{k_0}-1},$$

and

$$|\text{the rest term of } (III)|\leq a(-k_0+n)\lambda_{k_0+1}x^{-\lambda_{k_0+1}-1}.$$

Thus we obtain the estimate

$$\frac{dR^{l,(s)}(w_\alpha(x))}{dx}\geq m_4x^{-\lambda_{k_0}-1}-m_5x^{-\lambda_{k_0-1}-1}-\alpha m_6x^{\alpha-\lambda-2}-\alpha l\sigma x^{-\alpha\sigma-1} \tag{2.24}$$

where positive constants m_4, m_5 and m_6 do not depend on α and l. Since $-\alpha+\lambda+2<\lambda_{k_0}+1$ and $\alpha\sigma<\lambda_{k_0}$, the right hand side of (2.24) becomes positive for small x. This completes the proof. ∎

LEMMA 2.7. (i) *Assume* $\alpha\geq\dfrac{\lambda+1}{\sigma+1}$. *Then there exist positive constants* l_2 *and* x_2 *with the property that for any* $x\in(0, x_2]$ *and* $l\geq l_2$, $R^{l,(s)}(|w_\alpha(x)|e^{i\phi})$ *is a decreasing function of* ϕ *on* $[0, \arg(w_\alpha(x))]$.

(ii) *Assume* $\alpha<\dfrac{\lambda+1}{\sigma+1}$ *and* $\alpha\geq 1$. *Then for any* $l>0$, *there exists a positive constant* $x_{2,l}$ *with the property that for any* $x\in(0, x_{2,l}]$, $R^{l,(s)}(|w_\alpha(x)|e^{i\phi})$ *is an increasing function of* ϕ *on* $[\arg(w_\alpha(x)), \theta_1]$.

PROOF. We have

$$\begin{aligned}\frac{dR^{l,(s)}(|w_\alpha(x)|e^{i\phi})}{d\phi}&=\lambda|a_1||w_\alpha(x)|^{-\lambda}\cos(\lambda\phi)\\&\quad-\sum_{k=2}^{n}\lambda_k|a_k||w_\alpha(x)|^{-\lambda_k}\sin(-\lambda_k\phi+\omega_k)\\&\quad-l\sigma\cos\phi|w_\alpha(x)|(\sin\phi|w_\alpha(x)|)^{-(\sigma+1)}.\end{aligned} \tag{2.25}$$

Then if x is sufficiently small, we obtain the estimate

$$\frac{dR^{l,(s)}(|w_\alpha(x)|e^{i\phi})}{d\phi}\leq|w_\alpha(x)|(m_1x^{-\lambda-1}-l\sigma\cos(\theta_\alpha(x))x^{-\alpha(\sigma+1)})$$
$$\text{if}\quad \phi\leq\arg(w_\alpha(x)),$$
$$\frac{dR^{l,(s)}(|w_\alpha(x)|e^{i\phi})}{d\phi}\geq|w_\alpha(x)|(m_2x^{-\lambda-1}-l\sigma\cos(\theta_\alpha(x))x^{-\alpha(\sigma+1)})$$
$$\text{if}\quad \phi\geq\arg(w_\alpha(x))$$

where positive constants m_1 and m_2 are independent of α and l. Then we can easily show the lemma. ■

Now we choose the point z_0 and the path Γ which consists of two parts Γ_1 and Γ_2 in the case (III. C). Let l be a positive real number satisfying $l \geq \bar{l} = \max\{l_0, l_1, l_2\}$ where l_0, l_1 and l_2 are given in Lemma 2.5, Lemma 2.6 and Lemma 2.7. We denote by L_z the line through the origin and the point z, and by C_α the curve: $t > 0 \rightarrow w_\alpha(t)$. We consider this problem in the following two cases (see fig. 2.3).

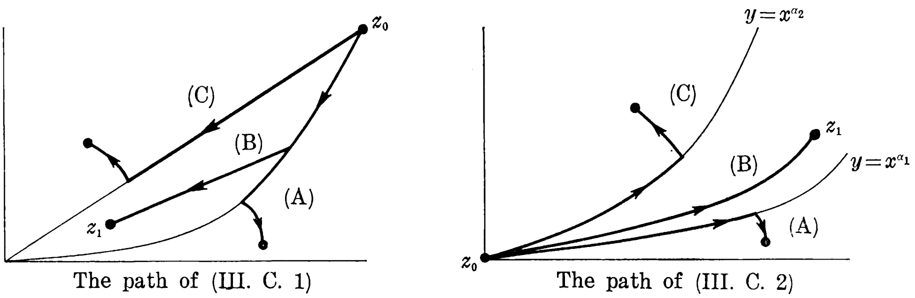

Fig. 2.3.

Case (III. C. 1): Assume $\sigma(\lambda - \lambda_{k_0} + 1) \geq \lambda_{k_0}$. Set $\alpha = \lambda - \lambda_{k_0} + 1$ and a real number $\tilde{x} = \min\{x_0, x_1, x_2\}$ where x_0, x_1 and x_2 are given in Lemma 2.5, Lemma 2.6 and Lemma 2.7. We choose the point $w_\alpha(\tilde{x})$ as z_0 of the integral (2.5). To determine the path, we divide the sector $S(\theta_1, R)$ into three regions.

Region (A) is $\{z \in S(\theta_1, R);\ \Im z \leq (\Re z)^\alpha\}$. For the point $z_1 = \rho_1 e^{i\phi_1}$ in the region (A), we take the point $\tilde{z}_1$ on the curve C_α which satisfies $|\tilde{z}_1| = \rho_1$. The path Γ_{1,z_1} joins the points z_0 to $\tilde{z}_1$ along the curve C_α, and the path Γ_{2,z_1} from $\tilde{z}_1$ to z_1 is the arc with radius ρ_1 and center at 0.

Region (B) is $\{z \in S(\theta_1, R);\ \Im z \geq (\Re z)^\alpha,\ \arg z \leq \theta_\alpha(\tilde{x})\}$. Let $\tilde{z}_1$ be an intersecting point between the curve C_α and the line L_{z_1}. Then Γ_{1,z_1} is the same as Region (A), and Γ_{2,z_1} is the segment from $\tilde{z}_1$ to z_1.

Region (C) is $\{z \in S(\theta_1, R);\ \Im z \geq (\Re z)^\alpha,\ \arg(z) \geq \theta_\alpha(\tilde{x})\}$. Let $\tilde{z}_1$ be a point on the line L_{z_0} which satisfies $|\tilde{z}_1| = \rho_1$. Then Γ_{1,z_1} is the segment from z_0 to $\tilde{z}_1$, and Γ_{2,z_1} is the arc from $\tilde{z}_1$ to z_1.

Case (III. C. 2): Assume $\sigma(\lambda - \lambda_{k_0} + 1) < \lambda_{k_0}$. Let α_1 and α_2 be positive real numbers satisfying

$$\lambda-\lambda_{k_0}+1<\alpha_2<\frac{\lambda+1}{\sigma+1}<\alpha_1<\frac{\lambda_{k_0}}{\sigma}.$$

We take the origin as z_0 of the integral (2.5). We divide the sector $S(\theta_1, R)$ into three regions.

Region (A) is $\{z\in S(\theta_1, R);\ \Im z\leq(\Re z)^{\alpha_1}\}$. For the point $z_1=\rho_1 e^{i\phi_1}$ in the region (A), we take the point $\tilde{z}_1$ on the curve C_{α_1} which satisfies $|\tilde{z}_1|=\rho_1$. Then the path Γ_{1,z_1} joints z_0 to $\tilde{z}_1$ along the curve C_{α_1}, and the path Γ_{2,z_1} from $\tilde{z}_1$ to z_1 is the arc with radius ρ_1 and center at 0.

Region (B) is $\{z\in S(\theta_1, R);\ \Im z\geq(\Re z)^{\alpha_1},\ \Im z\leq(\Re z)^{\alpha_2}\}$. In this region, the path Γ_{z_1} consists of only one part which joins z_0 to z_1 along the curve C_β for some $\beta\in[\alpha_2, \alpha_1]$.

Region (C) is $\{z\in S(\theta_1, R);\ \Im z\geq(\Re z)^{\alpha_2}\}$. We choose the path in such a way as region (A).

PROOF OF THE PROPOSITION 2.3 IN THE CASE (III. C). If we take the point z_1 close to the origin, the function $R^{l,(s)}(z)$ is increasing along the path Γ_{z_1} by Lemma 2.5, Lemma 2.6 and Lemma 2.7. Thus we can easily show the estimate (2.6). This completes the long proof of Proposition 2.3. ■

Next we consider the same problem on the sector $S(\theta_0, \theta_1, R)$ when $0<\theta_0<\theta_1<2\pi$.

PROPOSITION 2.8. *Assume the sector $S(\theta_0, \theta_1, R)$ intersects with at most one separate line of Λ, and neither $Re^{i\theta_0}$ nor $Re^{i\theta_1}$ is a separate line. Then for large l, there exist positive constants l' and C such that*

$$\|I_{z_0}(f)\|^{l',s}_{S(\theta_0,\theta_1,R)}\leq C\|f\|^{l,s}_{S(\theta_0,\theta_1,R)}\qquad f\in\tilde{O}^{l,(s)}(S(\theta_0, \theta_1, R)) \tag{2.20}$$

where the point z_0 is chosen in such a way as the second case of the proof of Proposition 2.3.

Since the proof goes in the same way as the second case of Proposition 2.3, we omit it.

Now we construct solutions on small sectors of the system (2.0). It is known that there exists the formal fundamental solutions $P(z)z^A\,\mathrm{Exp}(-\Lambda)$ of (2.0) where

(1) $P(z)$ is the matrix of the formal power series of $z^{1/p}$,

(2) A is a constant matrix, and

(3) $\Lambda(z)$ is the diagonal matrix whose (j, j) component $\Lambda_j(z)$ is the polynomial of $z^{-1/p}$ and written as

$$\Lambda_j(z)=\sum_{k=1}^{n_j} a_{j,k}z^{-\lambda_{j,k}} \qquad (1\leq j\leq m). \tag{2.21}$$

Here $a_{j,k}\in C$, $a_{j,1}\neq 0$, $a_{j,n_j}=1$ and $\lambda_{j,k}\in Z\left[\frac{1}{p}\right]$ satisfying

$$\lambda_j:=\lambda_{j,1}>\cdots>\lambda_{j,n_j}=0.$$

Set

$$\begin{aligned}&\lambda=\max\{\lambda_1, \cdots, \lambda_m\}, \quad \text{and} \quad \bar{\lambda}_j=\lambda_{k_j}\\ &\text{where} \quad k_j=\min\{k; \Re a_{j,k}\neq 0, \ 1\leq k\leq n_j\}.\end{aligned} \tag{2.22}$$

Note that we choose the branch of $z^{1/p}$ which has positive real values on $\arg z=0$. We quote the existence theorem of the fundamental solutions due to Hukuhara ([H 1], [H 2]) and Wasow ([W]).

PROPOSITION 2.9 (CF. [W]). *Given any* ϕ. *Then for some constants* $R>0$, ϕ_0 *and* ϕ_1 *with* $\phi_0<\phi<\phi_1$, *the system* (2.0) *has the fundamental solution* $U(z)\mathrm{Exp}(-\Lambda(z))$ *on the sector* $S(\phi_0, \phi_1, R)$ *satisfying the conditions*;

(2.23) $U(z)\in GL(m, \mathcal{O}(S(\theta_0, \theta_1, R)))\cap C^0(\overline{S(\theta_0, \theta_1, R)}\backslash\{0\})$.

(2.24) *There exist positive constants* C *and* N *so that* $U(z)$ *has the estimate*

$$\begin{aligned}&|U(z)|\leq C|z|^{-N},\\ &|U^{-1}(z)|\leq C|z|^{-N} \qquad (z\in S(\phi_0, \phi_1, R)).\end{aligned}$$

We determine the positive constant R_0 and the partition $0=\theta_0<\theta_1<\cdots<\theta_q=2\pi$ so that on each sector $S(\theta_{i-1}, \theta_i, R_0)$ $(1\leq i\leq q)$, there exists the fundamental solution $U_i(z)\mathrm{Exp}(-\Lambda(z))$ of (2.0) which satisfies the conditions (2.23) and (2.24) of Proposition 2.9. Moreover dividing $[0, 2\pi]$ into smaller sectors, we may assume the following conditions.

(2.25) For $2\leq i\leq q-1$, the sector $S(\theta_{i-1}, \theta_i, R_0)$ intersects with at most one separate line of each Λ_j, and neither $R^+e^{i\theta_{i-1}}$ nor $R^+e^{i\theta_i}$ is a separate line of each Λ_j.

(2.26) For $i=1$ and $i=q$, $\theta_i-\theta_{i-1}<\min\left\{\frac{\pi}{2}, \frac{\pi}{2\lambda}\right\}$ and the open sector $S(\theta_{i-1}, \theta_i, R_0)$ contains no separate line of each Λ_j.

Let $l_{i,j}$ be a positive real number which is determined in Proposition

2.3 or Proposition 2.8 for Λ_j on $S(\theta_{i-1}, \theta_i, R_0)$. Set

$$\tilde{l}=\max_{1\leq i\leq q, 1\leq j\leq m}\{l_{i,j}\}.$$

Let $z_{0,i,j}$ $(1\leq i\leq q, 1\leq j\leq m)$ be a point in $\overline{S(\theta_{i-1}, \theta_i, R_0)}$ which is chosen in such a way that the integral (2.5) has the estimate (2.6) or (2.20) when we replace Λ of (2.5) with Λ_j. Let z be a point in $S(\theta_{i-1}, \theta_i, R_0)$. For any $l\geq\tilde{l}$ and $f\in\tilde{O}^{l,(s)}(S(\theta_{i-1}, \theta_i, R_0))^m$, we define the solution of (2.0) on $S(\theta_{i-1}, \theta_i, R_0)$ as

$$G_i(f)(z)=U_i(z)\mathrm{Exp}(-\Lambda(z))\int_{\Gamma_{i,z}}\mathrm{Exp}(\Lambda)U_i^{-1}z^{-d}f dz. \tag{2.27}$$

Here $\Gamma_{i,z}$ is the set of m paths $(\Gamma_{i,1,z}, \cdots, \Gamma_{i,m,z})$ with $\Gamma_{i,j,z}$ the path in $\overline{S(\theta_{i-1}, \theta_i, R_0)}$ which joins $z_{0,i,j}$ to z as described in the proof of Proposition 2.3 or Proposition 2.9. By the results of the same propositions, there exist positive constants l' and C such that we have $G_i(f)\in\tilde{O}^{l',(s)}(S(\theta_{i-1}, \theta_i, R_0))^m$ and

$$\|G_i(f)\|^{l',s}_{S(\theta_{i-1},\theta_i,R_0)}\leq C\|f\|^{l,s}_{S(\theta_{i-1},\theta_i,R_0)}. \tag{2.28}$$

REMARK. (1) Although $G_i(f)$ can be continued over $S(0, 2\pi, R_0)$ as the solution of (2.0) if $f\in\tilde{O}^{l,(s)}(S(0, 2\pi, R_0))^m$, we cannot expect the estimate (2.28) on whole domain.

(2) If $z_{0,i,j}\neq 0$, we can take the point $z_{0,i,j}$ close to the origin.

Summing up, we have

THEOREM 2.10. *Assume $l\geq\tilde{l}$. Then there exists a constant l' such that for any $f\in O^{l,(s)}(S(0, 2\pi, R_0))^m$ and $i\in[1, q]$, the solution of* (2.0) *on $S(\theta_{i-1}, \theta_i, R_0)$ given by* (2.27) *belongs to $O^{l',(s)}(S(\theta_{i-1}, \theta_i, R_0))^m$.*

3. Construction of ultra-distribution solutions

In the previous section, we have constructed holomorphic solutions of (2.0) on small sectors which satisfy suitable growth conditions. The aim of this section is to prove the main theorems by connecting them. However we can not expect that they are directly connected with each other, and we need to consider this problem modulo holomorphic functions at the origin. To do this, we make several preparations. Throughout this section, we use the same notations as Section 2. Set

$$\alpha_b = \max_{j\in[1,m]} \left\{2, \frac{\bar{\lambda}_j}{\sigma}\right\} \tag{3.0}$$

where $\bar{\lambda}_j$'s are given in (2.22), and set

$$\Omega = C \backslash \{z \in C;\ \Re z \geq 0,\ |\Im z| \leq (|\Re z|)^{\alpha_b}\}.$$

Let $\tilde{z}_i$ $(i \in [1, q-1])$ be a point satisfying $\arg \tilde{z}_i = \theta_i$ and $|\tilde{z}_i| \in (0, R_0)$ where R_0 and θ_i are given after Proposition 2.9 in Section 2. For convenience, we set $\tilde{z}_0 = \tilde{z}_1$ and $\tilde{z}_q = \tilde{z}_{q-1}$. Now we define a C linear morphism $\Phi_i := (O^{l,(s)}(S(\theta_{i-1}, \theta_i, R_0)))^m \to C^{2m}$ $(i=1, \cdots, q)$ as follows.

$$\Phi_i(f) := \left\{\int_{\Gamma_{i,\tilde{z}_{i-1}}} \mathrm{Exp}(\Lambda) U_i^{-1} z^{-d} f dz,\ \int_{\Gamma_{i,\tilde{z}_i}} \mathrm{Exp}(\Lambda) U_i^{-1} z^{-d} f dz\right\} \tag{3.1}$$

where $\Gamma_{i,\tilde{z}_i}$ is the same set of paths as (2.27). Set

$$\Phi = \Phi_1 \oplus \cdots \oplus \Phi_q : O^{l,(s)}(S(R_0))^m \to C^{2mq}.$$

LEMMA 3.1. *Assume that $l > \bar{l}$ and $\tilde{z}_i$'s are contained in Ω. Then there exists a constant C with the property that the estimate*

$$|\Phi(f)| \leq C |f|^{l,s}_{S(R_0)\cap\Omega} \qquad f \in O^{l,(s)}(S(R_0))^m$$

where the norm $|\cdot|^{l,s}_{S(R_0)\cap\Omega}$ is defined in Section 1.

PROOF. We will show that Φ_1 is continuous. The problem is to estimate the integrals in (3.1) whose paths start from the origin and touch the real axis tangentially at the origin. For any $1 \leq \alpha \leq \alpha^b$ and $\beta > 0$, we obtain

$$\mathrm{dist}(\beta w_\alpha(t), C \backslash \Omega) \sim \mathrm{dist}(\beta w_\alpha(t), \overline{R^+}) \qquad (t \to +0).$$

Thus by Lemma 2.6 (ii), we can easily show the lemma. ∎

Let $P(D) = \sum_{k=0}^{\infty} c_k D^k$ be a differential operator of infinite order which satisfies the estimate

$$|c_k| \leq C l^k (k!)^{-s} \tag{3.2}$$

with positive constants C and l. The differential operator which satisfies the estimate (3.2) is said to be the differential operator of Gevrey class $\{l, (s)\}$. Let R and a be positive constants satisfying $a < R$.

PROPOSITION 3.2. *For any l, there exist a constant $l'\geq l$ and a differential operator $P(D)$ of Gevrey class $\{l', (s)\}$ which satisfy the following properties.*

For any $f\in O^{l,(s)}(S(R))$, there exist holomorphic functions $g(z)$ and $\tilde{g}(z)$ satisfying the conditions (i), (ii) *and* (iii);

(i) $f(z)=P(D)g(z)+\tilde{g}(z)$.

(ii) *g is holomorphic and bounded on $\boldsymbol{C}\backslash\overline{\boldsymbol{R}^{+}}\cap\{\Re z<a\}$.*

(iii) *$\tilde{g}$ is holomorphic on $B_R\cap\{\Re z<a\}$, and satisfies the estimate*

$$|\tilde{g}|^{2l,s}_{B_R\cap\{Rez<a\}}\leq C|f|^{l,s}_{S(R)}$$

with a positive constant C.

PROOF. We use the technique of [Ko 4]. Let δ be a positive real number such that $\dfrac{a+R}{2}+\delta i\in S(R)$. Set $t_{\pm}=\dfrac{a+R}{2}\pm\delta i$. We define the Fourier-Borel transformation and its inverse by

$$\hat{f}(\zeta):=\int_{\mathscr{H}_1}e^{z\zeta}f(z)dz,$$

$$\check{h}(z):=\frac{1}{2\pi i}\int_{-\infty}^{0}e^{-z\zeta}h(\zeta)d\zeta$$

where $\mathscr{H}_1$ is the path starting from t_-, turning around the origin and ending at t_+ (see fig. 3.1).

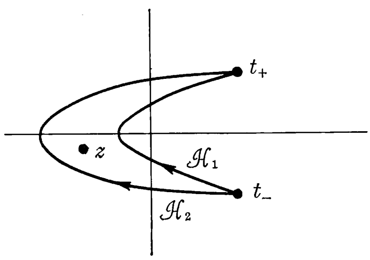

Fig. 3.1.

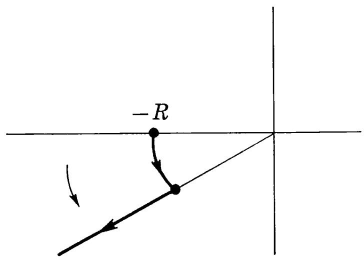

Fig. 3.2.

It is easily to see that for any $f\in\mathcal{O}(S(R))$,

(1) $\hat{f}(\zeta)$ is an entire function, and

(2) we have the estimate for any $\epsilon>0$,

$$|\hat{f}(\zeta)|\leq C_\epsilon e^{\epsilon|\Re\zeta|} \qquad (\zeta\in\overline{\boldsymbol{R}^-})$$

with a positive constant C_ϵ.

Thus we know that $(\hat{f})^\vee$ is well defined and holomorphic on $\Re z<0$. Moreover we have for any $\Re z<0$,

$$(3.3)\qquad \begin{aligned}(\hat{f})^\vee(z)&=\frac{1}{2\pi i}\int_{-\infty}^{0}\int_{\mathcal{H}_1}e^{\zeta(w-z)}f(w)dwd\zeta\\ &=\frac{1}{2\pi i}\int_{\mathcal{H}_1}\frac{f(w)}{w-z}dw=f(z)+\frac{1}{2\pi i}\int_{\mathcal{H}_2}\frac{f(w)}{w-z}dw.\end{aligned}$$

Here $\mathcal{H}_2$ is the path starting from t_-, turning around z and ending at t_+ (see fig. 3.1). By deforming the path $\mathcal{H}_2$, the second term of the right hand side of (3.3) is holomorphic on $B_R\cap\{\Re z<a\}$.

Let

$$P_l(\zeta)=\prod_{p=1}^{\infty}\Big(1-\frac{l\zeta}{p^s}\Big).$$

Then we have the estimate

$$|P_l(\zeta)^{-1}|\geq A_l\,\mathrm{Exp}\Big(\frac{1}{2}(l|\zeta|)^{1/s}\Big), \qquad (\Re\zeta\leq 0)$$

with a positive constant A_l (see [Ko 1]). We will show that $g_{l'}(z):=(P_{l'}(\zeta)^{-1}\hat{f}(\zeta))^\vee$ is holomorphic and bounded on $\boldsymbol{C}\backslash\overline{\boldsymbol{R}^+}\cap\{\Re\zeta<a\}$ for any $f\in O^{l,(s)}(S(R))$, if l' is sufficiently large. It is enough to show that for large $R_1>>0$,

$$\int_{-\infty}^{-R_1}P_{l'}^{-1}(\zeta)\hat{f}(\zeta)d\zeta$$

satisfies the above claim. Set

$$(3.4)\qquad \gamma_\pm(\zeta)=\frac{a+R}{2}\pm\sqrt{-1}(-\zeta)^{-1/s\sigma}.$$

We divide $\hat{f}(\zeta)$ into three parts as follows.

$$\hat{f}(\zeta)=\int_{t_-}^{\gamma_-(\zeta)} e^{z\zeta}f(z)dz+\int_{\mathscr{H}_3(\zeta)} e^{z\zeta}f(z)dz+\int_{\gamma_+(\zeta)}^{t_+} e^{z\zeta}f(z)dz$$
$$=(I)+(II)+(III).$$

Here $\mathscr{H}_3(\zeta)$ is the path starting from $\gamma_-(\zeta)$, turning around the origin and ending at $\gamma_+(\zeta)$. Then (I), (II) and (III) are holomorphic on $\left\{\frac{\pi(1-\sigma)}{2}<\arg\zeta<\frac{\pi(3+\sigma)}{2},\ |\zeta|>R_1\right\}$. Since we have

$$\cos\left(\frac{\pi}{2s\sigma}\right)|\zeta|^{-1/s\sigma}\leq|\Im(\gamma_\pm(\zeta))|\leq|\zeta|^{-1/s\sigma}$$

on $T=\left\{\frac{\pi}{2}<\arg\zeta<\frac{3\pi}{2},\ |\zeta|>R_1\right\}$, we obtain the estimate

$$\text{(3.5)}\qquad |(I)\text{ and }(III)|\leq C_1\operatorname{Exp}\left(\frac{a+R}{2}\Re\zeta\right)\qquad(\zeta\in\boldsymbol{R}^-\cap\{|\zeta|>R_1\}),$$
$$|(II)|\leq C_2\operatorname{Exp}\left(\left(2+l\cos\left(\frac{\pi}{2s\sigma}\right)^{-\sigma}\right)|\zeta|^{1/s}\right)\qquad(\zeta\in T).$$

Thus we obtain on T,

$$\text{(3.6)}\qquad |P_{l'}(\zeta)^{-1}*(III)|\leq C_{l'}\operatorname{Exp}\left(\left(-\frac{(l')^{1/s}}{2}+2+l\cos\left(\frac{\pi}{2s\sigma}\right)^{-\sigma}\right)|\zeta|^{1/s}\right)$$

with a positive constant $C_{l'}$. Since we have

$$\text{(3.7)}\qquad \int_{-\infty}^{-R_1} e^{-z\zeta}P_{l'}(\zeta)^{-1}\hat{f}(\zeta)d\zeta$$
$$=\int_{-\infty}^{-R_1} e^{-z\zeta}P_{l'}(\zeta)^{-1}((I)+(III))d\zeta+\int_{-\infty}^{-R_1} e^{-z\zeta}P_{l'}(\zeta)^{-1}(II)d\zeta,$$

the first term of the right hand side of (3.7) is holomorphic on $\Re z<\frac{a+R}{2}$, and by rotating the path of the second term as fig. 3.2, we easily find the second term is holomorphic and bounded on $\boldsymbol{C}\backslash\overline{\boldsymbol{R}^+}$ if we take l' sufficiently large. Finally we remark that

$$\text{(3.8)}\qquad P_{l'}(D)(P_{l'}(\zeta)^{-1}\hat{f}(\zeta))^\vee=(\hat{f}(\zeta))^\vee.$$

This completes the proof. ∎

PROPOSITION 3.3. *For any l, there exists l' so that $\mathcal{O}(\boldsymbol{C})$ is dense in*

$O^{l,(s)}(S(R))$ with respect to the norm $|\cdot|^{l',s}_{S(a)\cap\Omega}$ for any positive number $a<R$.

PROOF. First we quote the theorem of Mergelyan.

LEMMA 3.4 ([ME; THEOREM 1.4]). *If K is a compact set in $\boldsymbol{C}$ whose complement is connected, then $\mathcal{O}(\boldsymbol{C})$ is dense in $\mathcal{O}(\operatorname{int} K)\cap C^0(K)$ with respect to uniform convergence norm on K.*

Let $K=\overline{S(a)\cap\Omega}$. We take l' sufficiently large so that Proposition 3.2 holds. The holomorphic function $g(z)$ constructed in the same proposition is bounded on int K. By integrating $g(z)$, we may assume $g(z)\in\mathcal{O}(\operatorname{int} K)\cap C^0(K)$. Applying Lemma 3.4 to $g(z)$, we can find the uniformly convergence sequences $g_n(z)\to g(z)$ on K where $g_n(z)$'s are entire functions. Then it is easy to see that

$$P(D)g_n(z)\to P(D)g(z)$$

with respect to the norm $|\cdot|^{l'',s}_{S(a)\cap\Omega}$ for large l''. Note that $P(D)g_n(z)$ is an entire function. Applying Runge approximation theorem to $\tilde{g}(z)$ in Proposition 3.2, we obtain the desired result. ∎

Now we give the proof of Theorem 0.1 and Corollary 0.2.

PROOF. We first reduce the problem to the case that $\mathcal{M}$ has the form (2.0) in Section 2.

LEMMA 3.5. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module. Then there exist a coherent $\mathcal{D}_X$ module $\mathcal{N}$ which has the form* (2.0) *and an injective $\mathcal{D}_X$ morphism $\phi:\mathcal{M}\to\mathcal{N}$.*

PROOF. We may assume $char(\mathcal{M})\subset T^*_XX\cup T^*_{\{0\}}X$. Let $p=(0;\sqrt{-1}dz)$. We define the dual system $\mathcal{M}^*$ of $\mathcal{M}$ by

$$\mathcal{M}^*=\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}_X)\otimes\Omega_X^{-1}.$$

Since the dual functor is involutive in the category of holonomic systems, it is enough to show that there exists a surjective morphism $\tilde{\phi}:\mathcal{N}\to\mathcal{M}^*$ where $\mathcal{N}$ has the form (2.0). We endow a good filtration $F^k(\mathcal{M}^*)$ to $\mathcal{M}^*$ and consider the graded module $Gr(\mathcal{M}^*)$. Since $\operatorname{supp}(Gr(\mathcal{M}^*))\subset T^*_XX\cup T^*_{\{0\}}X$, we can find an integer N_0 such that $z^{N_0}Gr(\mathcal{M}^*)_p=0$. Since support of $Gr(\mathcal{M}^*)$ is conic, we have $\operatorname{supp}(z^{N_0}Gr(\mathcal{M}^*))\subset T^*_XX$, and $z^{N_0}Gr(\mathcal{M}^*)$ is a coherent $\mathcal{O}_X$ module. Thus the increasing sequence of coherent $\mathcal{O}_X$ modules $G^i := z^{N_0} \bigoplus_{k \leq i} Gr^k(\mathcal{M}^*)$ is locally stationary. This implies that there exists an integer k_0 such that $z^{N_0} Gr^k(\mathcal{M}^*) = 0$ for $k \geq k_0$. Choosing generators $u_1, \cdots, u_l$ of $F^{k_0}(\mathcal{M}^*)$, we can find the matrix $A(z) \in gl(l, \mathcal{O}_0)$ such that

$$z^{N_0} \frac{\partial}{\partial z} \vec{u} = A(z) \vec{u}$$

where $\vec{u} = (u_1, \cdots, u_l)^t$. This completes the proof. ∎

Continue to the proof of Theorem 0.1. Since we have the exact sequence

$$\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{N}, \Gamma_Z \mathcal{F}) \to \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}, \Gamma_Z \mathcal{F}) \to 0,$$

we may assume, from the beginning, $\mathcal{M}$ has the form (2.0).

We prove this theorem in the case $\mathcal{F} = \mathcal{D}^{(s)\prime}$. For the other cases, we need slight modifications in Section 2 and 3. The essential part of the proof, however, is the same, and we omit their proofs. Given a $h \in (\Gamma_Z \mathcal{D}^{(s)\prime})^m$. Then there exist $l, \epsilon > 0$ and $f \in (O^{l,(s)}(S(\epsilon)))^m$ which represents h as boundary value. By Theorem 2.10, we can find the solution u_i of the system (2.0) on the each sector $S(\theta_i, \theta_{i+1}, \epsilon')$ for a sufficiently small ϵ'. Here u_i's are given by the integral (2.27). We know $\Phi((O^{l,(s)}(S(\epsilon')))^m)$ is closed because of its finite dimensionality. Moreover $\mathcal{O}(\boldsymbol{C})$ is dense in $O^{l,(s)}(S(\epsilon'))$ with respect to the norm $|\cdot|^{l',s}_{S(\epsilon'') \cap \Omega}$ for small ϵ'' and large l' by Proposition 3.3. If we choose the point $z_{0,i,j}$ in (2.27) close to the origin, Φ is continuous with respect to the same norm by Lemma 3.1. Thus we have

$$\Phi((O^{l,(s)}(R(\epsilon')))^m) = \Phi((\mathcal{O}(\boldsymbol{C}))^m). \tag{3.9}$$

Then we can find $\tilde{f} \in (\mathcal{O}(\boldsymbol{C}))^m$ such that $\Phi(f) = \Phi(\tilde{f})$. We replace each u_i to

$$G_i(f - \tilde{f}) = U_i \operatorname{Exp}(-\Lambda) \int \operatorname{Exp}(\Lambda) U_i^{-1} z^{-d} (f - \tilde{f}) dz.$$

We have $u_i(\tilde{z}_i) = u_{i+1}(\tilde{z}_i)$ and $|\{u_i\}|^{l'',s}_{S(\epsilon')} < \infty$ for some l''. Thus $\{u_i\}$ give the holomorphic function on $S(\epsilon')$ which represents an ultradistribution of Beurling class (s), and we obtain

$$P\{u_i\} = f \qquad \text{mod} \quad \mathcal{O}(\boldsymbol{C}).$$

This completes the proof of Theorem 0.1.

To prove Corollary 0.2, we remark the exact sequence

$$(3.10) \qquad 0 \to \Gamma_{\{0\}}\mathcal{D}^{(s)\prime} \to \Gamma_{-Z}\mathcal{D}^{(s)\prime} \oplus \Gamma_{Z}\mathcal{D}^{(s)\prime} \to \mathcal{D}^{(s)\prime} \to 0.$$

■

PROOF OF THEOREM 0.3. We only show the case $\mathcal{F}=\mathcal{C}^{(s)}$. By the results of [Ma 2], we may assume $\mathcal{M}$ is a coherent $\mathcal{D}_X$ module. Then we remark the following exact sequence

$$(3.11) \qquad 0 \to \mathcal{A}_0 \to \mathcal{D}_0^{(s)\prime} \to (\mathcal{C}^{(s)})_{(0;-\sqrt{-1}dz)} \oplus (\mathcal{C}^{(s)})_{(0;\sqrt{-1}dz)} \to 0.$$

Applying the functor $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \cdot)$ to (3.11), we easily obtain the result. ■

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