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L^2-theory of singular perturbation of hyperbolic equations I
A priori estimates with parameter ε

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Abstract. We consider Cauchy problems for a linear strictly hyperbolic equation of order l with a small parameter ε:

(1.1) $(i\varepsilon)^{l-m}L(t, x, D_t, D_x)+M(t, x, D_t, D_x)u(t, x; \varepsilon)=f(t, x) \in (0, T)\times \boldsymbol{R}_x{}^n$

(1.2) $D_t{}^j u(0, x; \varepsilon)=g_j(x)$ $j=0, 1, 2, \cdots l-1$

where L and M are linear strictly hyperbolic operators of order l and m ($l=m+1$ or $m+2$). The aim of this paper is to give L^2 theoretical foundation of asymptotic expansion of solutions to singularly perturbed Cauchy problems of this type. We investigate a priori L^2 and higher order Sobolev estimates of the solution to (1.1) and (1.2) under various separation conditions of characteristic roots of L and M.

The point is to make explicit the contribution of ε in L^2 estimate of the solution and to make clear the role of the separation conditions. We prove the classical Leray-Gårding inequality for pseudo-differential operators (§2 Theorem 2.1 and 2.2), transplanting the Euclidian algorithm between symbols of L and M into pseudo-differential operators. Then, the separation conditions assure L^2 estimates (Theorem 3.1 and its variation Theorems 3.2, 3.4, 3.6). They lead to higher order estimates (Theorem 3.3, 3.5, 3.7).

§1. Introduction.

There are many works on singular perturbation of linear partial differential equations from various points of view. It seems, however, that there are few general results, in case a hyperbolic equation with small parameter ε in the highest term reduces to another hyperbolic equation of inferior order, as ε tends to 0. This type is called hyperbolic-hyperbolic (in [Li]) and is related to some mathematical model equations of wave phenomena (e.g. G.B. Whitham [Wh2]).

The aim of this series of papers is to give a mathematical foundation of linear theory of singularly perturbed Cauchy problems of hyperbolic-hyperbolic type.

We consider Cauchy problems for linear strictly hyperbolic equations

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of order l:

(1.1) $((i\varepsilon)^{l-m}L(t, x, D_t, D_x)+M(t, x, D_t, D_x))u(t, x; \varepsilon)=f(t, x)\ (t, x)\in(0, T)\times R_x^n$

(1.2) $$D_t^j u(0, x; \varepsilon)=g_j(x)\quad j=0, 1, 2, \cdots, l-1$$

where L and M are linear strictly hyperbolic operators of order l and m $(l>m)$. When we fix the positive parameter ε, we know well a priori estimates for the equations (1.1) and (1.2), from which we have existence, uniqueness and regularity of the solution.

When ε tends to $+0$, the equation is reduced to

(1.3) $$M(t, x, D_t, D_x)v_0(t, x)=f(t, x).$$

We impose to (1.3) initial conditions

(1.4) $$D_t^j v_0(0, x)=g_j(x)\quad j=0, 1, 2, \cdots, m-1.$$

Roughly speaking, $v_0(t, x)$ is an approximate solution of (1.1) for small ε with Cauchy data up to order $m-1$ under a certain condition on the pair L and M. In order to construct an approximate solution satisfying the Cauchy data approximately up to order m, we add a so called boundary layer term (or rather initial layer term) $\varepsilon^{r_0}w_0(t, x; \varepsilon)$ such that

(1.5) $$((i\varepsilon)^{l-m}L+M)(v_0+\varepsilon^{r_0}w_0)(t, x; \varepsilon)=f(t, x)+O(\varepsilon)$$

(1.6) $$D_t^j(v_0+\varepsilon^{r_0}w_0)(0, x; \varepsilon)=g_j(x)+O(\varepsilon)$$

where $\varepsilon^{j+|\alpha|}D_t^j D_x^\alpha w_0(t, x; \varepsilon)$ is bounded or decreasing in a function space on $(0, T]\times R_x^n$ as ε tends to $+0$.

In a forthcoming paper, we construct a formal expansion of $u(t, x; \varepsilon)$ in the form

(1.7) $$u(t, x; \varepsilon)\sim\sum_{n=0}^{\infty}\varepsilon^n v_n(t, x)+\sum_{n=n_0}^{\infty}\varepsilon^n w_n(t, x; \varepsilon)$$

where

(1.8) $$((i\varepsilon)^{l-m}L+M)\Big(\sum_{n=0}^{\infty}\varepsilon^n v_n(t, x)\Big)\sim f(t, x),$$

(1.9) $$((i\varepsilon)^{l-m}L+M)\Big(\sum_{n=n_0}^{\infty}\varepsilon^n w_n(t, x; \varepsilon)\Big)\sim 0,$$

(1.10) $$D_t^j\Big(\sum_{n=0}^{\infty}\varepsilon^n v_n(t, x)+\sum_{n=n_0}^{\infty}\varepsilon^n w_n(t, x; \varepsilon)\Big)\sim g_j(x)\quad j=0, 1, 2, \cdots, l-1.$$

We will show that for any fixed Sobolev norm on the half space with the initial plane, a partial sum up to a sufficiently late term of the right hand side of (1.7) converges in the sense of the Sobolev norm to the true solution $u(t, x; \varepsilon)$ as ε tend to $+0$.

To give remainder estimate of the expansion, we need the separation conditions on L and M.

We investigate in this paper a priori L^2 and higher order Sobolev estimates of the solution to (1.1) and (1.2).

Assumptions. We assume the following conditions on the operators L and M.

(H0) $L(t, x, D_t, D_x)=D_t^l+\cdots$ is a regularly hyperbolic operator with respect to t of order l,

(H1) $M(t, x, D_t, D_x)=M_0(t, x)D_t^m+\cdots$ is a regularly hyperbolic operator with respect to t of order m.

We also assume that

(S) the characteristic roots $\{\varphi_j(t, x, \xi);\ j=1, \cdots, l\}$ of the principal symbol $\sigma_l(L)(t, x, \tau, \xi)$ and $\{\phi_j(t, x, \xi);\ j=1, \cdots, m\}$ of the principal symbol $\sigma_m(M)(t, x, \tau, \xi)$ satisfy one of the following separation conditions from (S1) to (S4).

(S1) (D1) $l=m+1$,

(E) $\operatorname{Re} M_0(t, x)\geqq\delta>0$

(S0) $\varphi_1(t, x, \xi)<\phi_1(t, x, \xi)<\varphi_2(t, x, \xi)<\cdots$
$<\phi_m(t, x, \xi)<\varphi_{m+1}(t, x, \xi)$;

(S2) (D1) $l=m+1$,

(S^+) $\operatorname{Re} M_0(t, x)=0$ identically and there exists a positive constant δ such that

$$\operatorname{Im} M_0(t, x)\geqq\delta>0,$$

(WS^+) $\varphi_1(t, x, \xi)<\{\phi_1(t, x, \xi), \varphi_2(t, x, \xi)\}<\cdots$
$<\{\phi_{m-1}(t, x, \xi), \varphi_m(t, x, \xi)\}<\{\phi_m(t, x, \xi), \varphi_{m+1}(t, x, \xi)\}$;

(S3) (D1) $l=m+1$,

(S^-) $\operatorname{Re} M_0(t, x)=0$ identically and there exists a positive constant δ such that

$$\operatorname{Im} M_0(t, x)\leqq-\delta<0$$

(WS^-) $\{\phi_1(t, x, \xi), \varphi_1(t, x, \xi)\}<\{\phi_2(t, x, \xi), \varphi_2(t, x, \xi)\}<$
$<\{\phi_3(t, x, \xi), \varphi_3(t, x, \xi)\}<\cdots$
$<\{\phi_m(t, x, \xi), \varphi_m(t, x, \xi)\}<\varphi_{m+1}(t, x, \xi)$;

(S4) (D2) $l=m+2$,

(P) $M_0(t, x)\geqq\delta>0$,

(WS) $\varphi_1(t, x, \xi)<\{\psi_1(t, x, \xi), \varphi_2(t, x, \xi)\}<\cdots$
$<\{\psi_m(t, x, \xi), \varphi_{m+1}(t, x, \xi)\}<\varphi_{m+2}(t, x, \xi)$,

where $\{a, b\}<\{c, d\}$ means $\max\{a, b\}<\min\{c, d\}$ and inequalities in the above conditions are assumed to hold uniformly with respect to $(t, x, \xi)\in[0, T]\times R_x^n\times\{|\xi|=1\}$.

REMARK. We will call the condition (S1) dissipative and the others (S2), (S3), (S4) dispersive. This is due to the difference of nature of correction terms $w_n(t, x; \varepsilon)$. In fact, the magnitude of $w_n(t, x; \varepsilon)$ is exponentially decaying in the dissipative case and highly oscillating in the dispersive cases as ε tends to $+0$. In the following estimates, the difference of the cases appears as loss of the power of ε.

Main results. *Under these assumptions, there exist positive constants* γ_0 *and* ε_0 *such that for any solution* $u(t, x; \varepsilon)$ *to* (1.1) *and* (1.2), *any* $\varepsilon\in(0,\varepsilon_0]$, *any* $\gamma>\gamma_0$ *and any natural number* p, *we have in case*
(S1)

$$
\begin{aligned}
(1.11)\qquad & C\Big\{\int_0^T e^{-2\gamma t}\,\varepsilon^{2p}\|D^p f(t)\|^2 dt \\
&+\varepsilon\sum_{j=0}^{p}\varepsilon^{2j}\|D^m u(0)\|_j^2+\gamma\sum_{j=1}^{p}\varepsilon^{2j}\|D^m u(0)\|_{j-1}^2 \\
&+\varepsilon\sum_{j=0}^{p-1}\varepsilon^{2j}\|D^j f(0)\|^2+\gamma\sum_{j=0}^{p-1}\varepsilon^{2j}\|D^{j-1}f(0)\|^2\Big\} \\
&\geqq\gamma\int_0^T e^{-2\gamma t}\,\varepsilon^{2p}(\varepsilon\|D^{m+p}\,u(t)\|^2+\gamma\|D^{m+p-1}\,u(t)\|^2)dt \\
&+e^{-2\gamma T}\,\varepsilon^{2p}(\varepsilon\|D^{m+p}\,u(T)\|^2+\gamma\|D^{m+p-1}\,u(T)\|^2).
\end{aligned}
$$

We have in case (S2) *or* (S3)

$$
\begin{aligned}
(1.12)\qquad & C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j\|D^j f(t)\|^2 dt+\|D^{m-1}\,u(0)\|_{1/2}^2 \\
&+\gamma^p\Big\{\varepsilon\sum_{j=0}^{p}\varepsilon^{2j}\|D^m u(0)\|_j^2+\sum_{j=1}^{p}\varepsilon^{2j}\|D^m u(0)\|_{j-1/2}^2 \\
&+\varepsilon\sum_{j=0}^{p-1}\varepsilon^{2j}\|D^j f(0)\|^2+\sum_{j=0}^{p-1}\varepsilon^{2j}\|D^{j-1}f(0)\|_{1/2}^2\Big\} \\
&\geqq\gamma\int_0^T e^{-2\gamma t}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}\,u(t)\|^2+\|D^{m+j-1}\,u(t)\|_{1/2}^2)dt
\end{aligned}
$$

$$+e^{-2\gamma T}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(T)\|^2+\|D^{m+j-1}u(T)\|_{1/2}^2).$$

We have in case (S4)

$$C\left\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon^2}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j\|D^jf(t)\|^2dt\right.$$
$$\left.+\gamma^p\Big(\|D^m u(0)\|^2+\sum_{k=0}^{p}\varepsilon^{2k+2}\|D^{m+1}u(0)\|_k^2+\sum_{k=0}^{p-1}\varepsilon^{2k}\|D^kf(0)\|^2\right\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+1}u(t)\|^2+\|D^{m+j}u(t)\|^2)dt$$
$$+e^{-2\gamma T}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+1}u(T)\|^2+\gamma\|D^{m+j}u(T)\|^2). \tag{1.13}$$

(In the left hand side of inequalities, $\|D^k u(0)\|_q^2=\sum_{j=0}^{k}\|g_j\|_{k-j+q}^2$ by the difinition in § 2).

The point is to make explicit the contribution of ε in L^2 estimate of the solution and to make clear the role of the separation conditions. We prove the classical Leray-Gårding inequality for pseudo-differential operators (§ 2 Theorem 2.1 and 2.2). We translate the Euclidian algorithm between the symbols of L and M into that of pseudo-differential operators (R. Sakamoto [Sa]). Then, the separation conditions assure L^2 estimates (Theorem 3.1 and its variation Theorems 3.2 for (S1), Theorem 3.4 for (S2) and (S3), and Theorem 3.6 for (S4)). They lead to higher order estimates Theorem 3.3, for (S1), Theorem 3.5 for (S2) and (S3), and Theorem 3.7 for (S4). The main results except the higher order estimates were announced in [U]. Construction of the asymptotic expansions and the error estimates of the remainder will be treated in a forthcoming paper.

We will give some historical remarks. The separation condition (S0) was introduced by J. Leray (e.g. in [Le]) in the study of L^2 energy integral method for strictly hyperbolic differential operators. In different context, G.B. Whitham pointed out the condition (S0) for 2-dimensional constant coeffificient operators and T.T. Wu completed the other three (WS^+), (WS^-) and (WS). They derived them through the stability of plane wave solutions as ε tends to 0. Singular perturbation of general hyperbolic operators with constant coefficients in distribution category is studied by J. Chaillou in [Ch]. R. Ashino recently analysed singular perturbation of Cauchy problems in view point of Cauchy-Kowalevskaya theorem and explained the separation conditions ([As]).

L^2 estimate of singularly perturbed hyperbolic equation with variable coefficients in a higher dimensional space for (S1) was given by M.G. Dzavadov ([Dz]). R.X. Gao gave L^2 estimates and asymptotic expansion of the solutions for the cases (S1) and (S4) under the assumption $M_0(t, x) = M_0(t)$. Both of them used directly the classical result of Leray-Gårding for differential operators. Our method by pseudo-differential operators will be useful for the boundary value problems for singular perturbation of hyperbolic equations.

In 2-dimensional (t, x)-space, L^2 and maximum norm estimates are given by E.W. de Jager ([deJa]). Linear and nonlinear hyperbolic singular perturbation problems for second order equations in 2-dimensional space are systematically studied by R. Geel ([Ge]).

Various types of singular perturbation including hyperbolic-hyperbolic type are studied in higher dimensional spaces in [Li].

After this work was done, a comprehensive book on singular perturbation [Fr] has appeared, where elliptic coercive problems are mainly studied by pseudo-differential operators.

§2. Basic inequality.

In this section, we will give a pseudo-differential operator version of the classical Leray-Gårding's inequality. Our method of the proof is inspired by R. Sakamoto's work on hyperbolic boudary value problems [Sa].

We will introduce pseudo-differential operators on $\boldsymbol{R}_x^n$ with smooth parameters $(t, \varepsilon) \in [0, +\infty) \times [0, \varepsilon_0]$, where ε_0 is a fixed positive constant.

DEFINITION 2.1. Let m be a real number. S^m is the set of all C^∞ functions $a(t, x, \xi; \varepsilon)$ in $\boldsymbol{R} \times \boldsymbol{R}^n \times \boldsymbol{R}^n \times [0, \varepsilon_0]$ such that for all j, k, α, β the derivative $\partial_\varepsilon^j \partial_\xi^\alpha \partial_t^k \partial_x^\beta a$ has the bound

$$(2.1) \qquad \sup\{|\partial_\varepsilon^j \partial_\xi^\alpha \partial_t^k \partial_x^\beta a(t, x, \xi; \varepsilon)|;\ 0 \leqq \varepsilon \leqq \varepsilon_0,\ t \geqq 0,\ x \in \boldsymbol{R}^n,\ \xi \in \boldsymbol{R}^n\} \leqq C(1+|\xi|)^{m-|\alpha|}$$

where C depends on j, k, α, β. We will call S^m the space of *symbols* of order m. $a(t, x, \xi; \varepsilon) \in S^m$ is said to be *uniformly positive*, if there esist positive constants C and c such that it has the bound

$$(2.2) \qquad \inf\{a(t, x, \xi; \varepsilon);\ 0 \leqq \varepsilon \leqq \varepsilon_0,\ t \geqq 0,\ x \in \boldsymbol{R}^n\} \geqq c|\xi|^m$$

when $|\xi|\geqq C$. The homogeneous symbols of order m are defined as usual and the total set is denoted by S^m/S^{m-1}. We restrict ourselves to the symbols having the homogeneous principal part in S^m/S^{m-1}.

DEFINITION 2.2. L^m is the set of pseudo-differential operators on $\boldsymbol{R}_x^n$ with parameters t, ε associated to the symbols $a(t, x, \xi; \varepsilon)\in S^m$:

$$a(t, x, D_x; \varepsilon)v(x)=(2\pi)^{-n}\int e^{ix\xi}a(t, x, \xi; \varepsilon)\hat{v}(\xi)d\xi$$

where $\hat{v}(\xi)$ is the Fourier transform $\int e^{-ix\xi}v(x)dx$ for $v\in C_0^\infty(\boldsymbol{R}_x^n)$. For $P\in L^m$, its principal symbol is denoted by $\sigma_m(P)(t, x, \tau, \xi; \varepsilon)$. $P\in L^m$ is said *uniformly positive* when its symbol is so.

DEFINITION 2.3. $S(m; r)$ is the set of polynomials in τ of degree m whose coefficient at τ^j is a symbol in S^{m-j+r}. Let $L(m; r)$ be the set of operators of the form:

$$P(t, x, D_t, D_x; \varepsilon)=\sum_{j=0}^{m}P_j(t, x, D_x; \varepsilon)D_t^{m-j}$$

where $P_j(t, x, D_x; \varepsilon)\in L^{j+r}$.

We consider two operators $L\in L(l; 0)$ and $M\in L(m; r)$:

$$L(t, x, D_t, D_x; \varepsilon)=D_t^l+\sum_{j=1}^{l}L_j(t, x, D_x; \varepsilon)D_t^{l-j} \tag{2.3}$$

$$M(t, x, D_t, D_x; \varepsilon)=M_0(t, x, D_x; \varepsilon)D_t^m+\sum_{j=1}^{m}M_j(t, x, D_x; \varepsilon)D_t^{m-j} \tag{2.4}$$

with their principal symbols

$$l(t, x, \tau, \xi; \varepsilon)=\tau^l+\sum_{j=1}^{l}l_j(t, x, \xi; \varepsilon)\tau^{l-j} \tag{2.5}$$

$$m(t, x, \tau, \xi; \varepsilon)=m_0(t, x, \xi; \varepsilon)\tau^m+\sum_{j=1}^{m}m_j(t, x, \xi; \varepsilon)\tau^{m-j} \tag{2.6}$$

where $l_j\in S^j$ and $m_j\in S^{j+r}$.

We assume the following assumptions:

(H0) Regular Hyperbolicity of L: $l(t, x, \tau, \xi; \varepsilon)$ has the decomposition

$$l(t, x, \tau, \xi; \varepsilon)=\prod_{j=1}^{l}(\tau-\varphi_j(t, x, \xi; \varepsilon)) \tag{2.7}$$

where $\varphi_j(t, x, \xi; \varepsilon)$ are real distinct elements in S^1/S^0 such that

$$\varphi_1(t, x, \xi; \varepsilon)<\varphi_2(t, x, \xi; \varepsilon)<\cdots<\varphi_l(t, x, \xi; \varepsilon) \text{ uniformly:} \tag{2.8}$$

that is, $\varphi_{j+1}(t, x, \xi; \varepsilon)-\varphi_j(t, x, \xi; \varepsilon)$ is uniformly positive for $j=1, \cdots, l-1$.

(P0) Positivity: The operator $M_0(t, x, D_x; \varepsilon) \in L^r$ is formally self-adjoint in $L^2(\boldsymbol{R}_x^n)$ and uniformly positive, having the inverse $M_0^{-1} \in L^{-r}$. (The setting $M_0(t, x, D_x; \varepsilon) \in L^r$ is made for generality of Theorem 2.1 itself. In later application, we consider the only case where $M_0(t, x, D_x; \varepsilon)=M_0(t, x; \varepsilon)$.)

(H1) Regular Hyperbolicity of M: $m(t, x, \tau, \xi; \varepsilon)$ has the decomposition

$$m(t, x, \tau, \xi; \varepsilon)=m_0(t, x, \xi; \varepsilon) \prod_{j=1}^{m} (\tau-\phi_j(t, x, \xi; \varepsilon)) \tag{2.9}$$

where $\phi_j(t, x, \xi)$ are real distinct elements in S^1/S^0 such that

$$\phi_1(t, x, \xi; \varepsilon)<\phi_2(t, x, \xi; \varepsilon)<\cdots<\phi_m(t, x, \xi; \varepsilon) \text{ uniformly.} \tag{2.10}$$

If k is a nonnegative integer and q is real, we define for $u(t, x) \in C_0^\infty(\boldsymbol{R}^{n+1})$

$$\|D^k u(t)\|_q=\sum_{j=0}^{k} \|D_t^j u(t, \cdot)\|_{k+q-j}^2$$

where $\|\cdot\|_s$ in the right hand side is the Sobolev norm in $\boldsymbol{R}_x^n$.

THEOREM 2.1. (**Basic inequality**). *Let* $L \in L(m+1; 0)$ *and* $M \in L(m; r)$. *Assume* (H0), (P0), (H1), *and*

(S0) *Separation: The roots* $\{\phi_j(t, x, \xi; \varepsilon)\}$ *uniformly separate* $\{\varphi_j(t, x, \xi; \varepsilon)\}$, that is by definition,

$$\begin{aligned}\varphi_1(t, x, \xi; \varepsilon)&<\phi_1(t, x, \xi; \varepsilon)<\varphi_2(t, x, \xi; \varepsilon)<\cdots \\ &<\phi_m(t, x, \xi; \varepsilon)<\varphi_{m+1}(t, x, \xi; \varepsilon) \text{ uniformly.}\end{aligned} \tag{2.11}$$

Then, there exist positive constants c_0 *and* C *such that for any positive* s *and any* $u(t) \in C^\infty([0, s]; C_0^\infty(\boldsymbol{R}_x^n))$

$$\begin{aligned}&-\mathrm{Im}\int_0^s (Lu(t), Mu(t))dt \\ &\geqq c_0\|D^m u(s)\|_{r/2}^2-C\Big\{\{\|D^{m-1} u(s)\|_{r/2}^2+\|D^m u(0)\|_{r/2}^2 \\ &\quad+\int_0^s\|D^m u(t)\|_{r/2}^2\, dt\Big\}\end{aligned} \tag{2.12}$$

PROOF. We put

$$a^{(0)}(\tau)=\sigma_{m+1}(L)(t, x, \tau, \xi; \varepsilon) \tag{2.13}$$

$$=\tau^{m+1}+a_1^{(0)}(t,\ x,\ \xi;\ \varepsilon)\tau^m+\cdots+a_{m+1}^{(0)}(t,\ x,\ \xi;\ \varepsilon),$$

where $a_j^{(0)}(t, x, \xi; \varepsilon)$ is the principal symbol of $L_j(t, x, D_x; \varepsilon)$ of (2.3), and

$$\begin{aligned}a^{(1)}(\tau)&=\sigma_{m+r}(M)(t,\ x,\ \tau,\ \xi;\ \varepsilon)\\&=a_0^{(1)}(t,\ x,\ \xi;\ \varepsilon)\tau^m+\cdots+a_m^{(1)}(t,\ x,\ \xi;\ \varepsilon),\end{aligned}\tag{2.14}$$

where $a_j^{(1)}(t, x, \xi; \varepsilon)$ is the principal symbol of $M_j(t, x, D_x; \varepsilon)$ of (2.4).

Dividing $a^{(0)}(\tau)$ by $a^{(1)}(\tau)$ as polynomials of τ, we define the homogeneous symbols

$$q^{(1)}(\tau)=q_0^{(1)}(t,\ x,\ \xi;\ \varepsilon)\tau+q_1^{(1)}(t,\ x,\ \xi;\ \varepsilon)\tag{2.15}$$

and

$$a^{(2)}(\tau)=a_0^{(2)}(t,\ x,\ \xi;\ \varepsilon)\tau^{m-1}+\cdots+a_{m-1}^{(2)}(t,\ x,\ \xi;\ \varepsilon)\tag{2.16}$$

such that

$$a^{(2)}(\tau)=q^{(1)}(\tau)a^{(1)}(\tau)-a^{(0)}(\tau).\tag{2.17}$$

We claim that

(i) $q_0^{(1)}(t, x, \xi; \varepsilon)$ is a real and uniformly positive homogeneous symbol of order $-r$, and $q_1^{(1)}(t, x, \xi; \varepsilon)$ is a real homogeneous symbol of order $1-r$,

(ii) $a_0^{(2)}(t, x, \xi; \varepsilon)$ is a real and uniformly positive homogeneous symbol of order 2 and

(iii) the equation $a^{(2)}(\tau)=0$ has $m-1$ real distinct roots $\{\tau_j^{(2)}\}$ such that

$$\tau_1^{(2)}(t,\ x,\ \xi;\ \varepsilon)<\tau_2^{(2)}(t,\ x,\ \xi;\ \varepsilon)<\cdots<\tau_{m-1}^{(2)}(t,\ x,\ \xi;\ \varepsilon)\ \text{uniformly},\tag{2.18}$$

which uniformly separate $\{\tau_j^{(1)}\}(=\{\phi_j\})$.

We shall check carefully uniform positivity in (ii) and (iii), since the rest is easy. In fact, omitting the variables $(t, x, \xi; \varepsilon)$, we have

$$\begin{aligned}a_0^{(2)}&=(a_1^{(0)}-(a_0^{(1)})^{-1}a_1^{(1)})(a_1^{(1)})^{-1}a_1^{(1)}-(a_2^{(0)}-(a_0^{(1)})^{-1}a_2^{(1)})\\&=\Big(\sum_{j=1}^{m+1}(-\varphi_j)-\sum_{j=1}^{m}(-\phi)\Big)\Big(-\sum_{k=1}^{m}\phi_k\Big)\\&\quad-\Big(\sum_{1\leqq j<k\leqq m+1}\varphi_j\varphi_k-\sum_{1\leqq j<k\leqq m}\phi_j\phi_k\Big)\\&=\sum_{j=1}^{m+1}\sum_{k=1}^{m}\varphi_j\phi_k-\sum_{j,k=1}^{m}\phi_j\phi_k-\sum_{1\leqq j<k\leqq m+1}\varphi_j\varphi_k\\&\quad+\sum_{1\leqq k<j\leqq m}\phi_j\phi_k\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{m} \psi_j^2 - \sum_{j=1}^{m-1} \psi_j \psi_{j+1} - \cdots - \psi_1 \psi_m \\
&\quad - \sum_{j=1}^{m} \varphi_j \varphi_{j+1} - \sum_{j=1}^{m-1} \varphi_j \varphi_{j+2} - \cdots - \varphi_1 \varphi_{m+1} \\
&\quad + \sum_{j=1}^{m} (\psi_j \varphi_j + \psi_j \varphi_{j+1}) + \sum_{j=1}^{m-1} (\psi_{j+1} \varphi_j + \psi_j \varphi_{j+2}) + \cdots \\
&\quad \cdots + (\psi_m \varphi_1 + \psi_1 \varphi_{m+1}) \\
&= -\Big\{ \sum_{j=1}^{m} (\psi_j - \varphi_j)(\psi_j - \varphi_{j+1}) + \sum_{j=1}^{m-1} (\psi_j - \varphi_j)(\psi_{j+1} - \varphi_{j+2}) + \cdots \\
&\quad \cdots + (\psi_1 - \varphi_1)(\psi_m - \varphi_{m+1}) \Big\} .
\end{aligned}
$$

Therefore, $a_0^{(2)}$ has an estimate from below such that

$$a_0^{(2)}(t, x, \xi; \varepsilon) \geqq c_0 > 0 \quad \text{for} \quad |\xi| = 1 \tag{2.19}$$

by virtue of the separation condition (S0). The rest of (ii) is clear. We shall show (iii). By the condition (S0), $a^{(2)}(\tau_j^{(1)})$ changes the signature alternatively, when j goes from 1 to m. Hence, there exist $m-1$ roots $\{\tau_j^{(2)}\}$ separating roots $\{\tau_j^{(1)}\}$. To show that the separation is uniform, we claim that

$$R(a^{(0)}(\tau), a^{(1)}(\tau)) = (-1)^m (a_0^{(1)})^2 R(a^{(1)}(\tau), a^{(2)}(\tau))$$

where $R(f(\tau), g(\tau))$ is the resultant of two polynomials $f(\tau)$ and $g(\tau)$. In fact,

$$
R(a^{(0)}(\tau), a^{(1)}(\tau)) = \left| \begin{array}{cccccccccc}
a_0^{(0)} & a_1^{(0)} & \cdot & \cdot & \cdot & a_m^{(0)} & a_{m+1}^{(0)} & & & \\
& a_0^{(0)} & & & \cdot & & & a_{m+1}^{(0)} & & \\
& & \cdot & & & & & & \cdot & \\
& & & \cdot & & & & & & \\
& & & & a_0^{(0)} & a_1^{(0)} & \cdot & \cdot & \cdot & a_{m+1}^{(0)} \\
a_0^{(1)} & a_1^{(1)} & \cdot & \cdot & \cdot & a_m^{(1)} & & & & \\
& \cdot & & & \cdot & & \cdot & & & \\
& & \cdot & & \cdot & & & \cdot & & \\
& & & \cdot & & & & & & \\
& & & & \cdot & & & & & \\
& & & & & a_0^{(1)} & \cdot & \cdot & \cdot & a_m^{(1)}
\end{array} \right|
$$

$$
= \begin{vmatrix}
0 & 0 & -a_0^{(2)} & \cdots & -a_{m-1}^{(2)} & & & \\
 & \ddots & & \ddots & & \ddots & & \\
 & & & & & & \ddots & \\
 & & & 0 & 0 & -a_2^{(2)} & \cdots & -a_{m-1}^{(2)} \\
a_0^{(1)} & a_1^{(1)} & \cdots & a_m^{(1)} & & & & \\
 & a_0^{(1)} & & \cdots & a_m^{(1)} & & & \\
 & & a_0^{(1)} & & & a_m^{(1)} & & \\
 & & & & & & \ddots & \\
 & & & \ddots & & & & \\
 & & & & a_0^{(1)} & \cdots & \cdots & a_m^{(1)}
\end{vmatrix}
$$

(by (2.17))

$$
= (-1)^m a_0^{(1)} \begin{vmatrix}
0 & -a_0^{(2)} & -a_1^{(2)} & \cdots & -a_{m-1}^{(2)} & & & \\
 & \ddots & & \ddots & & \ddots & & \\
 & & & & & & \ddots & \\
 & & & 0 & -a_0^{(2)} & & \cdots & -a_{m-1}^{(2)} \\
a_0^{(1)} & a_1^{(1)} & \cdots & a_{m-1}^{(1)} & a_m^{(1)} & & & \\
 & a_0^{(1)} & a_1^{(1)} & \cdots & a_m^{(1)} & & & \\
 & & & & & \ddots & & \\
 & & & & & & \ddots & \\
 & & & a_0^{(1)} & a_1^{(1)} & \cdots & \cdots & a_m^{(1)}
\end{vmatrix}
$$

$$
= (a_0^{(1)})^2 \begin{vmatrix}
-a_0^{(2)} & \cdots & \cdots & -a_{m-1}^{(2)} & & & & \\
 & \ddots & & & \ddots & & & \\
 & & \ddots & & & \ddots & & \\
 & & & \ddots & & & \ddots & \\
 & & & -a_0^{(2)} & -a_1^{(2)} & \cdots & \cdots & -a_{m-1}^{(2)} \\
a_0^{(1)} & a_1^{(1)} & \cdots & \cdots & a_m^{(1)} & & & \\
 & \ddots & & & & \ddots & & \\
 & & \ddots & & & & \ddots & \\
 & & & a_0^{(1)} & a_1^{(1)} & \cdots & \cdots & a_m^{(1)}
\end{vmatrix}
$$

$$=(a_0^{(1)})^2(-1)^m\begin{vmatrix} a_0^{(1)} & a_1^{(1)} & \cdots & a_{m-1}^{(1)} & a_m^{(1)} & & & \\ & \ddots & & & & \ddots & & \\ & & & a_0^{(1)} & a_1^{(1)} & \cdots & \cdots & a_m^{(1)} \\ a_0^{(2)} & \cdots & \cdots & a_{m-1}^{(2)} & & & & \\ & \ddots & & & \ddots & & & \\ & & \ddots & & & \ddots & & \\ & & & a_0^{(2)} & a_1^{(2)} & \cdots & \cdots & a_{m-1}^{(2)} \end{vmatrix}$$

$$=(a_0^{(1)})^2(-1)^m R(a^{(1)}(\tau),\ a^{(2)}(\tau)).$$

Combining this with the definition of the resultant, we have

$$\prod_{j,k}(\tau_j^{(1)}-\tau_k^{(2)})=(-a_0^{(0)}/a_0^{(2)})^m\prod_{i,j}(\tau_i^{(0)}-\tau_j^{(1)}).$$

Therefore, uniform separation in (iii) follows from uniform positivity of $a_0^{(2)}(t,\ x,\ \xi;\ \varepsilon)$ and the condition (S0).

Repeating inductively the division (2.17), we define

$$q^{(j)}(\tau)=q_0^{(j)}(t,\ x,\ \xi;\ \varepsilon)\tau+q_1^{(j)}(t,\ x,\ \xi;\ \varepsilon) \quad \text{for } j=2,\cdots,m+1 \tag{2.20}$$

and

$$\begin{cases} a^{(j)}(\tau)=a_0^{(j)}(t,\ x,\ \xi;\ \varepsilon)\tau^{m+1-j}+\cdots+a_{m+1-j}^{(j)}(t,\ x,\ \xi;\ \varepsilon) \\ \text{for } j=3,\cdots,m \\ a^{(m+1)}(\tau)=a_0^{(m+1)}(t,\ x,\ \xi;\ \varepsilon) \end{cases} \tag{2.21}$$

such that

$$\begin{cases} a^{(j)}(\tau)=q^{(j-1)}(\tau)a^{(j-1)}(\tau)-a^{(j-2)}(\tau) \quad \text{for } j=3,\cdots,m+1, \\ 0=q^{(m+1)}(\tau)a^{(m+1)}-a^{(m)}(\tau). \end{cases} \tag{2.22}$$

By induction, we have

(i)$_j$ $q_0^{(j)}(t,\ x,\ \xi;\ \varepsilon)$ is a real and uniformly positive homogeneous symbol of order

$$\begin{cases} r-2, \text{ if } j \text{ is even} \\ -r, \text{ if } j \text{ is odd;} \end{cases}$$

$q_1^{(j)}(t,\ x,\ \xi;\ \varepsilon)$ is a real homogeneous symbol of order

$$\begin{cases} r-1, & \text{if } j \text{ is even} \\ -r+1, & \text{if } j \text{ is odd.} \end{cases}$$

(ii)$_j$ $a_k^{(j)}(t, x, \xi; \varepsilon)$ is a real homogeneous symbol of order

$$\begin{cases} j+k, & \text{if } j \text{ is even} \\ r+j-1+k, & \text{if } j \text{ is odd,} \end{cases}$$

moreover, $a_0^{(j)}(t, x, \xi; \varepsilon)$ is uniformly positive.
(iii)$_j$ $a^{(j)}(\tau)=0$ has $m+1-j$ real distinct roots $\{\tau_p^{(j)}\}$ such that

(2.23) $$\tau_1^{(j)}(t, x, \xi; \varepsilon)<\cdots<\tau_{m+1-j}^{(j)}(t, x, \xi; \varepsilon) \text{ uniformly,}$$

which uniformly separate $\{\tau_p^{(j-1)}\}_{p=1,\cdots,m+2-j}$.

Changing notations for (2.3) and (2.4), we define operators

(2.24) $$A^{(0)}=L=\sum_{k=0}^{m-1} A_k^{(0)} D_t^{m+1-k},$$

that is, $A_0^{(0)}=1$, $A_k^{(0)}=L_k(t, x, D_x; \varepsilon)$ for $k\geqq 1$ and

(2.25) $$A^{(1)}=M=\sum_{k=0}^{m} A_k^{(1)} D_k^{m-k},$$

that is, $A_k^{(1)}=M_k(t, x, D_x; \varepsilon)$ for $k\geqq 0$.
Starting from (2.24) and (2.25), we shall construct operators associated with (2.20), (2.21) and (2.22). First of all, there exist $Q_0^{(1)}\in L^{-r}$ and $Q_1^{(1)}\in L^{-r+1}$ such that

(2.26) $$Q^{(1)}A^{(1)}-A^{(0)}\in L(m-1, 2),$$

where we define

(2.27) $$Q^{(1)}=Q_0^{(1)}(t, x, D_x; \varepsilon)D_t+Q_1^{(1)}(t, x, D_x; \varepsilon).$$

Since $A_0^{(1)}$ is invertible, $Q_0^{(1)}$ and $Q_1^{(1)}$ are uniquely determined by the condition (2.26) Notice that $Q_0^{(1)}=(A_0^{(1)})^{-1}=M_0^{-1}$, which is also formally self-adjoint. We put

(2.28) $$A^{(2)}=Q^{(1)}A^{(1)}-A^{(0)}=\sum_{k=0}^{m-1} A_k^{(2)} D_t^{m-1-k}.$$

Then,

$$\sigma_{k+2}(A_k^{(2)})=a_k^{(2)}(t, x, \xi; \varepsilon) \; k=0, 1, \cdots, m-1,$$

$$\sigma_{-r+k}(Q_k^{(1)})=q_k^{(1)}(t, x, \xi; \varepsilon) \; k=0, 1.$$

We use a known fact in the theory of pseudo-differential operators that there exists an invertible pseudo-differential operator $\tilde{A}_0^{(j)}$ whose principal symbol is $a_0^{(j)}(t, x, \xi; \varepsilon)$ $(2\leqq j\leqq m+1)$, since it is uniformly positive (See e.g. D. Fujiwara [Fu] or H. Kumano-go[Ku]).

We define operators uniquely

$$Q^{(j)}=Q_0^{(j)}(t, x, D_x; \varepsilon)D_t+Q_1^{(j)}(t, x, D_x; \varepsilon) \tag{2.29}$$

and

$$A^{(j+1)}=\sum_{k=0}^{m-j} A_k^{(j+1)}(t, x, D_x; \varepsilon)D_t^{m-j-k}, \tag{2.30}$$

succesively for $j=2, \cdots, m+1$ as follows:
Modifying $A^{(j)}$ by

$$\tilde{A}^{(j)}=\tilde{A}_0^{(j)}D_t^{m+1-j}+\sum_{k=1}^{m+1-j} A_k^{(j)}(t, x, D_x; \varepsilon)D_t^{m+1-j-k},$$

we define

$$A^{(j+1)}=Q^{(j)}\tilde{A}^{(j)}-A^{(j-1)},\ j=2, \cdots, m+1, \tag{2.31}$$

by the condition that $A^{(j+1)}$ is in $L(m-j, j+1)$ if $j+1$ is even and in $L(m-j, j+r)$ if $j+1$ is odd and that $A^{(m+2)}$ is zero.

If we take the principal symbols in (2.31), we obtain (2.22). In the sequel, C stands for various constants.

$$\begin{aligned}
&-2\,\mathrm{Im}\int_0^s (Lu(t), Mu(t))dt\\
&=-2\,\mathrm{Im}\int_0^s (Q^{(1)}A^{(1)}u(t), A^{(1)}u(t))dt-2\,\mathrm{Im}\int_0^s (A^{(1)}u(t), A^{(2)}u(t))dt.
\end{aligned}$$

The first integral of the right hand side is

$$\begin{aligned}
&=-2\,\mathrm{Im}\int_0^s (Q_0^{(1)}D_tA^{(1)}u(t), A^{(1)}u(t))dt-2\,\mathrm{Im}\int_0^s (Q_1^{(1)}A^{(1)}u(t), A^{(1)}u(t))dt\\
&=i\int_0^s \{(D_tA^{(1)}u(t), Q_0^{(1)*}A^{(1)}u(t))-(Q_0^{(1)*}A^{(1)}, D_tA^{(1)}u(t))\}dt\\
&+i\int_0^s ((Q_1^{(1)}-Q_1^{(1)*})A^{(1)}u(t), A^{(1)}u(t))dt\\
&\geqq[(Q_0^{(1)}A^{(1)}u(t), A^{(1)}u(t))]_{t=0}^{t=s}-\left|\int_0^s (A^{(1)}u(t), (D_tQ_0^{(1)})A^{(1)}u(t))dt\right|\\
&-C\int_0^s \|D^m u(t)\|_{r/2}^2 dt\\
&\geqq[(Q_0^{(1)}A^{(1)}u(t), A^{(1)}u(t))]_{t=0}^{t=s}-C\int_0^s \|D^m u(t)\|_{r/2}^2 dt.
\end{aligned}$$

For $1\leqq j\leqq m-1$,

$$-2\,\mathrm{Im}\int_0^s (A^{(j)}u(t),\ A^{(j+1)}u(t))dt$$
$$=-2\,\mathrm{Im}\int_0^s (Q^{(j+1)}\tilde{A}^{(j+1)}u(t),\ A^{(j+1)}u(t))dt$$
$$-2\,\mathrm{Im}\int_0^s (A^{(j+1)}u(t),\ A^{(j+2)}u(t))dt.$$

We integrate by parts the first term of the right hand side.

$$-2\,\mathrm{Im}\int_0^s (Q^{(j+1)}\tilde{A}^{(j+1)}u(t),\ A^{(j+1)}u(t))dt$$
$$\geqq -2\,\mathrm{Im}\int_0^s (Q_0^{(j+1)}D_tA^{(j+1)}u(t),\ A^{(j+1)}u(t))dt$$
$$-\left|\int_0^s ((Q_1^{(j+1)*}-Q_1^{(j+1)})A^{(j+1)}u(t),\ A^{(j+1)}u(t))dt\right|-C\int_0^s \|D^m u(t)\|_{r/2}^2 dt$$
$$\geqq \mathrm{Re}\,[(Q_0^{(j+1)}A^{(j+1)}u(t),\ A^{(j+1)}u(t))]_{t=0}^{t=s}$$
$$-\left|\mathrm{Im}\int_0^s ((Q_0^{(j+1)*}-Q_0^{(j+1)})A^{(j+1)}u(t),\ D_tA^{(j+1)}u(t))dt\right|$$
$$-\left|\mathrm{Im}\int_0^s (A^{(j+1)}u(t),\ (D_tQ_0^{(j+1)*})A^{(j+1)}u(t)dt\right|$$
$$-\left|\int_0^s ((Q_1^{(j+1)*}-Q_1^{(j+1)})A^{(j+1)}u(t),\ A^{(j+1)}u(t))dt\right|-C\int_0^s \|D^m u(t)\|_{r/2}^2 dt$$
$$\geqq \mathrm{Re}\,[(Q_0^{(j+1)}A^{(j+1)}u(t),\ A^{(j+1)}u(t))]_{t=0}^{t=s}-C\int_0^s \|D^m u(t)\|_{r/2}^2 dt.$$

At last, in the same way, we have

$$-2\,\mathrm{Im}\int_0^s (A^{(m)}u(t),\ A^{(m+1)}u(t)dt$$
$$=-2\,\mathrm{Im}\int_0^s (Q^{(m+1)}A^{(m+1)}u(t),\ A^{(m+1)}u(t)dt$$
$$-2\,\mathrm{Im}\int_0^s (Q^{(m+1)}(\tilde{A}^{(m+1)}-A^{(m+1)})u(t),\ A^{(m+1)}u(t))dt$$
$$\geqq \mathrm{Re}\,[(Q_0^{(m+1)}A^{(m+1)}u(t),\ A^{(m+1)}u(t))]_{t=0}^{t=s}-C\int_0^s \|D^m u(t)\|_{r/2}^2 dt.$$

Combining these results, we obtain

$$(2.32)\qquad -2\,\mathrm{Im}\int_0^s (Lu(t),\ Mu(t))dt$$
$$\geqq \sum_{j=1}^{m+1} \mathrm{Re}\,[(Q_0^{(j)}A^{(j)}u(t),\ A^{(j)}u(t))]_{t=0}^{t=s}-C\int_0^s \|D^m u(t)\|_{r/2}^2 dt$$

$$\geqq \sum_{j=1}^{m+1} \operatorname{Re}(Q_0^{(j)} A^{(j)} u(s),\ A^{(j)} u(s)) - C\|D^m u(0)\|_{r/2}^2 - C\int_0^s \|D^m u(t)\|_{r/2}^2 dt.$$

Applying Gårding's inequality for $j \geqq 2$, we have

$$\sum_{j=1}^{m+1} \operatorname{Re}(Q_0^{(j)} A^{(j)} u(s),\ A^{(j)} u(s)) \tag{2.33}$$

$$\geqq c_0 \sum_{j=1}^{m+1} \|A^{(j)} u(s)\|_{q_j/2}^2 - C \sum_{j=2}^{m+1} \|A^{(j)} u(s)\|_{(q_j/2)-1}^2,$$

where q_j denotes $\deg(Q_0^{(j)})$. q_j is $r-2$ or $-r$ according to the parity of j. The second term of the right hand side is estimated by

$$\sum_{j=2}^{m+1} \|A^{(j)} u(s)\|_{(q_j/2)-1}^2 \leqq C\|D^{m-1} u(s)\|_{r/2}^2. \tag{2.34}$$

On the other hand, the first term is estimated from below as follows. Since $A_0^{(1)}$ has its inverse $E_1^{(1)} \in L^{-r}$ and $A_0^{(j)}$ $(j \geqq 2)$ has a parametrix $E_j^{(j)} \in L^{-\deg(A_0^{(j)})}$, we have succesively $E_k^{(j)} \in L^{k-j-\deg(A_0^{(k)})}$ such that

$$D_t^m = E_1^{(1)} A^{(1)} + E_2^{(1)} A^{(2)} + \cdots + E_{m+1}^{(1)} A^{(m+1)} \bmod L(m-1, 0),$$

$$D_t^{m-j} = E_{j+1}^{(j+1)} A^{(j+1)} + E_{j+2}^{(j+1)} A^{(j+2)} + \cdots + E_{m+1}^{(j+1)} A^{(m+1)} \bmod L(m-j,\ -1)$$

for $j=1, 2, \cdots, m$.

Hence, we have for $j=1, 2, \cdots, m$,

$$\begin{aligned}
&\|D_t^{m-j} u(s)\|_{j+r/2}^2 \\
&\leqq \|E_{j+1}^{(j+1)} A^{(j+1)} u(s)\|_{j+r/2}^2 + \|E_{j+2}^{(j+1)} A^{(j+2)} u(s)\|_{j+r/2}^2 \\
&+ \cdots + \|E_{m+1}^{(j+1)} A^{(m+1)} u(s)\|_{j+r/2}^2 + C\|D^{m-j} u(s)\|_{j-1+r/2}^2 \\
&\leqq C_0(\|A^{(j+1)} u(s)\|_{q_{j+1}/2}^2 + \|A^{(j+2)} u(s)\|_{q_{j+2}/2} \\
&+ \cdots + \|A^{(m+1)} u(s)\|_{q_{m+1}/2}^2) + C\|D^{m-1} u(s)\|_{r/2}^2,
\end{aligned}$$

and we have

$$\begin{aligned}
&\|D_t^m u(s)\|_{r/2}^2 \leqq \|E_1^{(1)} A^{(1)} u(s)\|_{r/2}^2 + \|E_2^{(1)} A^{(2)} u(s)\|_{r/2}^2 \\
&+ \cdots + \|E_{m+1}^{(1)} A^{(m+1)} u(s)\|_{r/2}^2 + C\|D^{m-1} u(s)\|_{r/2}^2 \\
&\leqq C_0(\|A^{(1)} u(s)\|_{q_1/2}^2 + \|A^{(2)} u(s)\|_{q_2/2}^2 + \cdots \\
&+ \|A^{(m+1)} u(s)\|_{q_{m+1}/2}) + C\|D^{m-1} u(s)\|_{r/2}^2.
\end{aligned}$$

Using a parametrix of $A^{(m+1)}$, we have

$$\|u(s)\|_{m+r/2}^2 \leqq c_0 \|A^{(m+1)} u(s)\|_{q_{m+1}/2}^2 + C\|u(s)\|_{m-1+r/2}^2.$$

Therefore, we have

$$\|D^m u(s)\|_{r/2}^2 = \sum_{j=0}^{m} \|D_t^{m-j} u(s)\|_{j+r/2}^2 \tag{2.35}$$

$$\leqq C_0 \sum_{j=1}^{m+1} j \| A^{(j)} u(s) \|_{q_j/2}^2 + C \| D^{m-1} u(s) \|_{r/2}^2.$$

Combining (2.32), (2.33), (2.34) and (2.35), we obtain

$$-2 \operatorname{Im} \int_0^s (Lu(t), \ Mu(t)) dt$$

$$\geqq c \| D^m u(s) \|_{r/2}^2 - C \| D^m u(0) \|_{r/2}^2 - C \| D^{m-1} u(s) \|_{r/2}^2 - C \int_0^s \| D^m u(t) \|_{r/2}^2 dt.$$

Q.E.D.

REMARK. The conclusion of the theorem is true for $m=0$. In fact, if we assume (H0) and (P0) for L and M when $m=0$, then we have more directly

$$- \operatorname{Im} \int_0^s (Lu(t), \ Mu(t)) dt$$

$$\geqq c_0 \| u(s) \|_{r/2}^2 - C \left\{ \| u(0) \|_{r/2}^2 + \int_0^s \| u(t) \|_{r/2}^2 dt \right\}$$

for $u(t, \cdot) \in C^\infty([0, s]; \ C_0^\infty(\boldsymbol{R}_x^n))$.

THEOREM 2.2. *Let L and M satisfy the conditions* (H0), (P0), (H1) *and* (S0). *Let m be a positive integer.*

Then, there exist positive constants c_0, C, γ_0 such that for any positive T, any $\gamma \geqq \gamma_0$ and $u(t) \in C^\infty([0, T] C_0^\infty(\boldsymbol{R}_x^n))$

$$\text{(2.36)} \qquad - \operatorname{Im} \int_0^T e^{-2\gamma t} (Lu(t), \ Mu(t)) dt$$

$$\geqq c_0 \gamma \int_0^T e^{-2\gamma t} \| D^m u(t) \|_{r/2}^2 dt$$

$$+ c_0 e^{-2\gamma T} \| D^m u(T) \|_{r/2}^2 - C \| D^m u(0) \|_{r/2}^2.$$

LEMMA 2.1. *Let m be a positive integer and j be an integer in $\{0, 1, \cdots, m-1\}$. Then, for any positive constants T and γ.*

$$\text{(2.37)} \qquad \int_0^T e^{-2\gamma t} \| D_t^j u(t) \|_{m-1-j}^2 \, dt \leqq \frac{1}{\gamma^2} \int_0^T e^{-2\gamma t} \| D_t^{j+1} u(t) \|_{m-(j+1)}^2 dt$$

$$+ \frac{1}{\gamma} \| D_t^j u(0) \|_{m-1-j}^2 - \frac{1}{\gamma} e^{-2\gamma T} \| D_t^j u(T) \|_{m-1-j}^2 \quad \text{for } u \in C_0^\infty(\boldsymbol{R}_{t,x}^{n+1}).$$

Proof of Lemma 2.1. We put

$$I = \int_0^T e^{-2\gamma t} \| D_t^{j+1} u(t) \|_{m-(j+1)}^2 dt$$

$$=\int_0^T (e^{-\gamma t}D_t w(t),\ e^{-\gamma t}D_t w(t))dt$$

where $w(t)=(1-\Delta)^{(m-1-j)/2}D_t^j u(t)$. Introducing $v(t)=e^{-\gamma t}w(t)$, we have

$$e^{-\gamma t}D_t w(t)=(D_t-i\gamma)v(t).$$

Thus, we have

$$\begin{aligned}
I&=\int_0^T ((D_t-i\gamma)v(t),\ (D_t-i\gamma)v(t))dt\\
&=\int_0^T \|D_t v(t)\|^2dt+\gamma^2\int_0^T \|v(t)\|^2dt+i\gamma\int_0^T ((D_t v(t),\ v(t))-(v(t),\ D_t v(t))dt\\
&\geqq\gamma^2\int_0^T \|v(t)\|^2dt+\gamma\int_0^T \frac{d}{dt}\|v(t)\|^2dt\\
&=\gamma^2\int_0^T \|v(t)\|^2dt+\gamma(\|v(T)\|^2-\|v(0)\|^2)\\
&=\gamma^2\int_0^T e^{-2\gamma t}\|D_t^j u(t)\|_{m-1-j}^2dt+\gamma\{e^{-2\gamma T}\|D_t^j u(T)\|_{m-1-j}^2-\|D_t^j u(0)\|_{m-1-j}^2\}.
\end{aligned}$$

Q.E.D.

PROOF of the Theorem 2.2. Multiplying by $e^{-2\gamma s}$ both sides of the basic inequality (2.12), we integrate them with respect to s from 0 to T.

(2.38) $$\begin{aligned}
L.H.S.&=-\int_0^T e^{-2\gamma s}ds\,\mathrm{Im}\int_0^s (Lu(t),\ Mu(t))dt\\
&=\frac{e^{-2\gamma T}}{2\gamma}\mathrm{Im}\int_0^T (Lu(t),\ Mu(t))dt-\frac{1}{2\gamma}\mathrm{Im}\int_0^T e^{-2\gamma s}(Lu(s),\ Mu(s))ds.
\end{aligned}$$

(2.39) $$\begin{aligned}
R.H.S.&=c_0\int_0^T e^{-2\gamma s}\|D^m u(s)\|_{r/2}^2ds\\
&-C\Big\{\int_0^T e^{-2\gamma s}\|D^{m-1}u(s)\|_{r/2}^2ds+\frac{e^{-2\gamma T}-1}{-2\gamma}\|D^m u(0)\|_{r/2}^2\\
&+\frac{-e^{-2\gamma T}}{2\gamma}\int_0^T \|D^m u(t)\|_{r/2}^2dt+\frac{1}{2\gamma}\int_0^T e^{-2\gamma s}\|D^m u(s)\|_{r/2}^2ds\Big\}.
\end{aligned}$$

By the interpolation lemma 2.1, we have

(2.40) $$\begin{aligned}
&\int_0^T e^{-2\gamma s}\|D^{m-1}u(s)\|_{r/2}^2ds\\
&\leqq\frac{1}{\gamma^2}\int_0^T e^{-2\gamma s}\|D^m u(s)\|_{r/2}^2ds+\frac{1}{\gamma}\|D^{m-1}u(0)\|_{r/2}^2-\frac{e^{-2\gamma T}}{\gamma}\|D^{m-1}u(T)\|_{r/2}^2.
\end{aligned}$$

Replacing the both sides of the basic inequality (2.12) by (2.38) and (2.39), we obtain, through (2.40),

$$(2.41)\qquad \frac{e^{-2\gamma T}}{2\gamma}\mathrm{Im}\int_0^T(Lu(t),\ Mu(t))dt-\frac{1}{2\gamma}\mathrm{Im}\int_0^T e^{-2\gamma t}(Lu(t),\ Mu(t))dt$$
$$\geqq c_0\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt$$
$$-C\Big\{\frac{1}{\gamma^2}\int_0^T e^{-2\gamma t}\|Du(t)\|_{r/2}^2 dt+\frac{1}{\gamma}\|D^{m-1}u(0)\|_{r/2}^2$$
$$-\frac{e^{-2\gamma T}}{\gamma}\|D^{m-1}u(T)\|_{r/2}^2+\frac{e^{-2\gamma T}-1}{-2\gamma}\|D^m u(0)\|_{r/2}^2$$
$$+\frac{-e^{-2\gamma T}}{2\gamma}\int_0^T\|D^m u(t)\|_{r/2}^2 dt+\frac{1}{2\gamma}\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt\Big\}.$$

On the other hand, multiplying by $\frac{e^{-2\gamma T}}{2\gamma}$ the basic inequality (2.12) with $s=T$, we have

$$(2.42)\qquad -\frac{e^{-2\gamma T}}{2\gamma}\mathrm{Im}\int_0^T(Lu(t),\ Mu(t))dt$$
$$\geqq\frac{c_0e^{-2\gamma T}}{2\gamma}\|D^m u(T)\|_{r/2}^2-C\Big\{\frac{e^{-2\gamma T}}{2\gamma}\|D^{m-1}u(T)\|_{r/2}^2$$
$$+\frac{e^{-2\gamma T}}{2\gamma}\|D^m u(0)\|_{r/2}^2+\frac{e^{-2\gamma T}}{2\gamma}\int_0^T\|D^m u(t)\|_{r/2}^2 dt\Big\}.$$

Adding the inequalities (2.41) and (2.42), we have

$$-\frac{1}{2\gamma}\mathrm{Im}\int_0^T e^{-2\gamma t}(Lu(t),\ Mu(t))dt$$
$$\geqq c_0\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt+\frac{c_0e^{-2\gamma T}}{2\gamma}\|D^m u(T)\|_{r/2}^2$$
$$-C\Big\{\frac{1}{\gamma^2}\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt+\frac{1}{\gamma}\|D^{m-1}u(0)\|_{r/2}^2$$
$$-\frac{e^{-2\gamma T}}{2\gamma}\|D^{m-1}u(T)\|_{r/2}^2+\frac{1}{2\gamma}\|D^m u(0)\|_{r/2}^2$$
$$+\frac{1}{2\gamma}\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt\Big\}.$$

Changing the constants c_0 and C if necessary, we have for sufficiently large γ the desired inequality:

$$-\mathrm{Im}\int_0^T e^{-2\gamma t}(Lu(t),\ Mu(t))dt$$

$$\geqq c_0\gamma\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt + c_0 e^{-2\gamma T}\|D^m u(T)\|_{r/2}^2 - C\|D^m u(0)\|_{r/2}^2.$$

Q.E.D.

REMARK. If $m=0$, the conclusion of Theorem 2.2 is true under the assumptions (H0) and (P0). The proof is parallel and more direct, since we need not the interpolation lemma.

By choosing M suitable for L, we reproduce a well-known result.

COROLLARY 2.1. *Let L satisfy the condition* (H0). *Then, for any real r there exist positive constants c_0, C, γ_0 such that for any T positive and $\gamma \geqq \gamma_0$,*

$$(2.43)\quad \int_0^T e^{-2\gamma t}\|Lu(t)\|_{r/2}^2 dt \geqq c_0\gamma^2\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt + c_0\gamma e^{-2\gamma T}\|D^m u(T)\|_{r/2}^2 - C\gamma\|D^m u(0)\|_{r/2}^2.$$

§ 3. A priori estimates for Cauchy problems with small parameter ε

We shall study the Cauchy problem in R^{n+1}

$$(3.1)\quad \begin{array}{l}(i\varepsilon)^{l-m}L(t, x, D_t, D_x; \varepsilon)u + M(t, x, D_t, D_x; \varepsilon)u = f,\ 0<t<T;\\ u(0, x; \varepsilon), \cdots, D_t^{l-1}u(0, x; \varepsilon) \text{ are given.}\end{array}$$

We assume that L is of type (2.3) and M of (2.4) satisfying a separation condition to be specified later. The aim of this section is to establish a priori estimates of the solution u to (3.1), which are applicable for an asymptotic expansion of the solution u with respect to ε. We divide the problems into two cases. One is called dissipative ($l=m+1$) and the other dispersive ($l=m+1$ or $m+2$). This classification is based on the work by G.B. Whitham [Wh1] and T.T. Wu [Wu] for the constant coefficient operators (See also R. Ashino [As]). This is related to conditions that each characteristic root of the principal symbol of an ε-pseudodifferential operator has its nonnegative real part.

3.1. Dissipative case.

We consider two operators $L\in L(m+1; 0)$ and $M\in L(m; r)$:

$$(3.2)\quad L(t, x, D_t, D_x; \varepsilon)D_t^{m+1} + \sum_{j=1}^{m+1} L_j(t, x, D_x; \varepsilon)D_t^{m+1-j}$$

$$(3.3)\quad M(t, x, D_t, D_x; \varepsilon) = M_0(t, x, D_x; \varepsilon)D_t^m + \sum_{j=1}^{m} M_j(t, x, D_x; \varepsilon)D_t^{m-j}$$

with their principal symbols

$$l(t, x, \tau, \xi; \varepsilon)=\tau^{m+1}+\sum_{j=1}^{m+1} l_j(t, x, \xi; \varepsilon)\tau^{m+1-j}$$

$$m(t, x, \tau, \xi; \varepsilon)=m_0(t, x, \xi; \varepsilon)\tau^m+\sum_{j=1}^{m} m_j(t, x, \xi; \varepsilon)\tau^{m-j}$$

where $l_j\in S^j$ and $m_j\in S^{j+r}$.

We assume the following assumptions (H0), (E), (H1) and (S0).

(H0) Regular hyperbolicity of L: $l(t, x, \tau, \xi; \varepsilon)$ has the decomposition

$$l(t, x, \tau, \xi; \varepsilon)=\prod_{j=1}^{m+1}(\tau-\varphi_j(t, x, \xi; \varepsilon))$$

where $\varphi_j(t, x, \xi; \varepsilon)$ are real distinct elements in S^1/S^0 such that

(3.4) $$\varphi_1(t, x, \xi; \varepsilon)<\varphi_2(t,x,\xi;\varepsilon)<\cdots<\varphi_{m+1}(t, x, \xi: \varepsilon) \text{ uniformly.}$$

(E) Strong ellipticity of M_0: $(M_0)^{-1}\in L^{-r}$ and

(3.5) $$\operatorname{Re}(M_0u, u)\geqq\mu\|u\|^2_{r/2} \text{ for } u\in C_0^\infty(\boldsymbol{R}^n_x),$$

where μ is a positive constant independent of ε.

(H1) Regular hyperbolicity of M: $m(t, x, \tau, \xi; \varepsilon)$ has the decomposition

$$m(t, x, \tau, \xi; \varepsilon)=m_0(t, x, \xi; \varepsilon)\prod_{j=1}^{m}(\tau-\phi_j(t, x, \xi; \varepsilon))$$

where $\phi_j(t, x, \xi; \varepsilon)$ are real distinct elements in S^1/S^0 such that

$$\phi_1(t, x, \xi; \varepsilon)<\phi_2(t, x, \xi; \varepsilon)<\cdots<\phi_m(t, x, \xi; \varepsilon) \text{ uniformly.}$$

(S0) Separation: The roots $\{\phi_j(t, x, \xi; \varepsilon)\}$ uniformly separate $\{\varphi_j(t, x, \xi; \varepsilon)\}$. (See the definition (2.11) in Theorem 2.1.)

We consider the equation under these assumptions

(3.6) $$Pu=(i\varepsilon)Lu+Mu=f$$

for $u(t), f(t)\in C^\infty([0, T]; C_0^\infty(\boldsymbol{R}^n_x))$.

REMARK. If $r>0$ and $\operatorname{Re} m_0(t, x, \xi; \varepsilon)\geqq c_0|\xi|^r$, then, $M_0(t, x, D_x; \varepsilon)+\lambda$ satisfies (E) for sufficiently large positive constant λ. If M_0 is a multiplication operator $m_0(t, x; \varepsilon)$ with uniformly positive real part, M_0 satisfies (E) with $r=0$. This is the case we consider later in the application to the asymtotic expansion of the solutions.

THEOREM 3.1. *We assume* (H0), (E), (H1), *and* (S0). *Then, there exist positive constants* C, γ_0 *such that for any* $\gamma\geqq\gamma_0$ *and for* $u(t)\in C^\infty$

$([0, T];\ C_0^\infty(\boldsymbol{R}_x^n))$

(3.7) $$C\left\{\int_0^T e^{-2\gamma t}\|Pu(t)\|_{-r/2}^2 dt+\varepsilon\|D^m u(0)\|^2+\gamma\|D^{m-1}u(0)\|_{r/2}^2\right\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}(\varepsilon\|D^m u(t)\|^2+\gamma\|D^{m-1}u(t)\|_{r/2}^2)dt$$
$$+e^{-2\gamma T}(\varepsilon\|D^m u(T)\|^2+\gamma\|D^{m-1}u(T)\|_{r/2}^2).$$

PROOF. We have from Theorem 2.2 and (3.5),

$$\mathrm{Re}\int_0^T e^{-2\gamma t}(Pu(t),\ M_0^{-1}Mu(t))dt$$
$$=-\varepsilon\int_0^T e^{-2\gamma t}\,\mathrm{Im}\,(Lu(t),\ M_0^{-1}Mu(t))dt+\int_0^T e^{-2\gamma t}\,\mathrm{Re}(Mu(t), M_0^{-1}Mu(t))dt$$
$$\geqq c_0\gamma\int_0^T e^{-2\gamma t}\varepsilon\|D^m u(t)\|^2 dt+c_0 e^{-2\gamma T}\varepsilon\|D^m u(T)\|^2-C_0\varepsilon\|D^m u(0)\|^2$$
$$+\mu\int_0^T e^{-2\gamma t}\|M_0^{-1}Mu(t)\|_{r/2}^2 dt.$$

From Schwarz's inequality and Corollary 2.1, we have

$$\frac{1}{2\mu}\int_0^T e^{-2\gamma t}\|Pu(t)\|_{-r/2}^2 dt+C_1\left\{\varepsilon\|D^m u(0)\|^2+\frac{\gamma\mu}{2}\|D^{m-1}u(0)\|_{r/2}^2\right\}$$
$$\geqq c_1\gamma\int_0^T e^{-2\gamma t}\left\{\varepsilon\|D^m u(t)\|^2+\frac{\gamma\mu}{2}\|D^{m-1}u(t)\|_{r/2}^2\right\}dt$$
$$+c_1 e^{-2\gamma T}\left\{\varepsilon\|D^m u(T)\|^2+\frac{\gamma\mu}{2}\|D^{m-1}u(T)\|_{r/2}^2\right\}.$$

Q.E.D.

We give a variation, when M_0 satisfies (P0).

THEOREM 3.2. *We assume* (H0), (E), (H1) *and* (P0). *Then, there exist positive constants C, γ_0 such that for any $\gamma\geqq\gamma_0$ and for $u\in C_0^\infty([0, T])\times\boldsymbol{R}_x^n)$*

(3.8) $$C\left\{\int_0^T e^{-2\gamma t}\|Pu(t)\|^2 dt+\varepsilon^2\gamma\|D^m u(0)\|^2+\varepsilon\|D^m u(0)\|_{r/2}^2+\gamma\|D^{m-1}u(0)\|_r^2\right\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}\{\varepsilon^2\gamma\|D^m u(t)\|^2+\varepsilon\|D^m u(t)\|_{r/2}^2+\gamma\|D^{m-1}u(t)\|_r^2\}dt$$
$$+e^{-2\gamma T}\{\varepsilon^2\gamma\|D^m u(t)\|^2+\varepsilon\|D^m u(T)\|_{r/2}^2+\gamma\|D^{m-1}u(T)\|_r^2\}.$$

PROOF.

$$\int_0^T e^{-2\gamma t}\|Pu(t)\|^2 dt=\varepsilon^2\int_0^T e^{-2\gamma t}\|Lu(t)\|^2 dt$$

$$+\int_0^T e^{-2\gamma t}\|Mu(t)\|^2 dt - 2\varepsilon\int_0^T e^{-2\gamma t}\,\mathrm{Im}\,(Lu(t),\ Mu(t))dt.$$

Since M_0 satisfies (P0), the right hand side is estimated by Theorem 2.2 and its Corollary 2.1. Q.E.D.

We will give the higher order estimates. In the following, we assume $\gamma=0$ and denote $Pu(t)$ by $f(t)$.

LEMMA 3.1. *There exists $C>0$ such that for any natural number p*

$$\|D^{m+p}u(0)\|_q^2 \leqq C\left\{\sum_{j=0}^{p}\varepsilon^{-2j}\|D^m u(0)\|_{p-j+q}^2 + \sum_{j=1}^{p}\varepsilon^{-2j}\|D^{p-j}f(0)\|_q^2\right\} \tag{3.9}$$

where

$$(i\varepsilon)Lu+Mu=f. \tag{3.10}$$

PROOF. It follows easily from (3.10) that

$$\|D_t^{m+1}u(0)\|_q^2 \leqq C\{\|D^m u(0)\|_{1+q}^2 + \varepsilon^{-2}\|D^m u(0)\|_q^2 + \varepsilon^{-2}\|f(0)\|_q^2\}.$$

This implies (3.9) for $p=1$. Suppose that (3.9) is true for $p=1, 2, \cdots, k$. We have from (3.10) that

$$D_t^{m+k+1}u = -D_t^k((L-D_t^{m+1})u) - \frac{1}{i\varepsilon}D_t^k(Mu) + \frac{1}{i\varepsilon}D_t^k f.$$

Then, we have

$$\begin{aligned}
&\|D_t^{m+k+1}u(0)\|_q^2 \\
&\leqq C\left\{\sum_{j=0}^{k}(\|D^m D_t^j u(0)\|_{1+q}^2 + \varepsilon^{-2}\|D^m D_t^j u(0)\|_q^2) + \varepsilon^{-2}\|D_t^k f(0)\|_q^2\right\} \\
&\leqq C\{\|D^m u(0)\|_{q+1}^2 + \varepsilon^{-2}\|D^m u(0)\|_q^2 + \varepsilon^{-2}\|D_t^k f(0)\|_q^2 \\
&+ \sum_{j=1}^{k} C\left(\sum_{h=0}^{j}\varepsilon^{-2h}\|D^m u(0)\|_{j-h+1+q}^2 + \sum_{h=1}^{j}\varepsilon^{-2h}\|D^{j-h}f(0)\|_{q+1}^2\right) \\
&+ \sum_{j=1}^{k} C\left(\sum_{h=0}^{j}\varepsilon^{-2h-2}\|D^m u(0)\|_{j-h+q}^2 + \sum_{h=1}^{j}\varepsilon^{-2h-2}\|D^{j-h}f(0)\|_q^2\right)\Bigg\} \\
&\leqq C_1\left\{\sum_{j=0}^{k+1}\varepsilon^{-2j}\|D^m u(0)\|_{k+1-j+q}^2 + \sum_{j=1}^{k+1}\varepsilon^{-2j}\|D^{k+1-j}f(0)\|_q^2\right\}.
\end{aligned}$$

Noticing that

$$\|D^{m+k+1}u(0)\|_q^2 = \|D_t^{m+k+1}u(0)\|_q^2 + \|D^{m+k}u(0)\|_{q+1}^2,$$

we have the desired inequality (3.9). Q.E.D.

LEMMA 3.2. *Under the assumptions of Lemma 3.1, there exist $C>0$ and $\gamma_0>0$ such that for any integer $p\geqq 1$, any $\delta\geqq 0$ and any $\gamma\geqq\gamma_0$*

$$(3.11)\quad \sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(0)\|^2+\delta\|D^{m+j-1}u(0)\|_q^2)\}$$
$$\leqq C\gamma^p\Big\{\varepsilon\sum_{j=0}^{p}\varepsilon^{2j}\|D^m u(0)\|_j^2+\delta\sum_{j=1}^{p}\varepsilon^{2j}\|D^m u(0)\|_{j+q-1}^2$$
$$+\varepsilon\sum_{j=0}^{p-1}\varepsilon^{2j}\|D^j f(0)\|^2+\delta\sum_{j=1}^{p-1}\varepsilon^{2j}\|D^{j-1}f(0)\|_q^2\Big\}+C\delta\|D^{m-1}u(0)\|_q^2$$

(*Convention: the sum means* 0, *if the initial index in the sum is greater than the final one.*)

PROOF. By Lemma 3.1, we have, at first, (3.11) for $p=1$.
Let $p\geqq 2$. We will estimate the rest as follows:

$$\sum_{j=2}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(0)\|^?+\delta\|D^{m+j-1}u(0)\|^2)$$
$$\leqq\sum_{j=2}^{p}(\varepsilon^2\gamma)^j\Big\{\varepsilon\Big(\sum_{j=0}^{j}\varepsilon^{-2k}\|D^m u(0)\|_{j-k}^2+\sum_{k=1}^{j}\varepsilon^{-2k}\|D^{j-k}f(0)\|^2\Big)$$
$$+\delta\Big(\sum_{k=0}^{j-1}\varepsilon^{-2k}\|D^m u(0)\|_{j-k+q-1}^2+\sum_{k=1}^{j-1}\varepsilon^{-2k}\|D^{j-k-1}f(0)\|_q^2\Big).$$

Changing the index variables in the double sum by $r=j-k$, we see that the right hand side is estimated by the left hand side of (3.11). Q.E.D.

We will give a higher order estimate corresponding to Theorem 3.1.

PROPOSITION 3.1. *We assume the same assumptions as in Theorem* 3.1. *For sufficiently small* $\varepsilon>0$, *there exist positive constants* C, γ_0 *and* ε_0 *such that for any* $\gamma\geqq\gamma_0$, *any positive* $\varepsilon\leqq\varepsilon_0$ *and any* $p\geqq 0$

$$(3.12)\quad C\Big\{\int_0^T e^{-2\gamma t}\|D^p f(t)\|^2 dt+\varepsilon\|D^{m+p}u(0)\|^2+\gamma\|D^{m+p-1}u(0)\|^2\Big\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}(\varepsilon\|D^{m+p}u(t)\|^2+\gamma\|D^{m+p-1}u(t)\|^2)dt$$
$$+e^{-2\gamma T}(\varepsilon\|D^{m+p}u(T)\|^2+\gamma\|D^{m+p-1}u(T)\|^2).$$

PROOF. When $p=0$, (3.12) is given by Theorem 3.1. Suppose that (3.12) is true for $p=0,1,2,\cdots,k$. From the equation (3.10), we have

$$(i\varepsilon)LD_t u+MD_t u=D_t f+(i\varepsilon)[L,D_t]u+[M,D_t]u$$

and

$$(i\varepsilon)L\Lambda u+M\Lambda u=\Lambda f+(i\varepsilon)[L,\Lambda]u+[M,\Lambda]u,$$

for $\Lambda=(1-\Delta)^{1/2}$.

Noticing that

$$\|(i\varepsilon)D^k[L, D_t]u(t)\|^2+\|(i\varepsilon)D^k[L, \Lambda]u(t)\|^2$$
$$\leqq C\varepsilon^2\|D^{m+k}u(t)\|_1^2,$$

and that

$$\|D^k[M, D_t]u(t)\|^2+\|D^k[M, \Lambda]u(t)\|^2$$
$$\leqq C\|D^{m+k}u(t)\|^2,$$

we have

$$C\left\{\int_0^T e^{-2\gamma t}(\|D^{k+1}f(t)\|^2+\varepsilon^2\|D^{m+k}u(t)\|_1^2+\|D^{m+k}u(t)\|^2)dt\right.$$
$$\left.+\varepsilon\|D^{m+k+1}u(0)\|^2+\gamma\|D^{m+k}u(0)\|^2\right\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}(\varepsilon\|D^{m+k+1}u(t)\|^2+\gamma\|D^{m+k}u(t)\|^2)dt$$
$$+e^{-2\gamma t}(\varepsilon\|D^{m+k+1}u(T)\|^2+\gamma\|D^{m+k}u(T)\|^2).$$

The term $\varepsilon^2\|D^{m+k}u(t)\|_1^2+\|D^{m+k}u(t)\|^2$ in the left hand side is directly absorbed in the right hand side. Thus, we obtain (3.12) for $p=k+1$.

Q.E.D.

Estimating the higher order traces of u in (3.12) by Lemma 3.1, we have finally

THEOREM 3.3. *Assume the same conditions as in Theorem* 3.1 *with* $r=0$ *there exist* $C>0$, γ_0 *and* $\varepsilon_0>0$ *such that for any positive* $\varepsilon\leqq\varepsilon_0$, *any* $\gamma\geqq\gamma_0$ *and for any* $u(t)\in C^\infty([0, T]; C_0^\infty(R_x^n))$, *we have for any natural number* p

$$C\left\{\int_0^T e^{-2\gamma t}\,\varepsilon^{2p}\|D^pf(t)\|^2dt\right.\tag{3.13}$$
$$+\varepsilon\sum_{j=0}^{p}\varepsilon^{2j}\|D^mu(0)\|_j^2+\gamma\sum_{j=1}^{p}\varepsilon^{2j}\|D^mu(0)\|_{j-1}^2$$
$$\left.+\varepsilon\sum_{j=0}^{p-1}\varepsilon^{2j}\|D^jf(0)\|^2+\gamma\sum_{j=1}^{p-1}\varepsilon^{2j}\|D^{j-1}f(0)\|^2\right\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}\varepsilon^{2p}(\varepsilon\|D^{m+p}u(t)\|^2+\gamma\|D^{m+p-1}u(t)\|^2)dt$$
$$+\varepsilon^{-2\gamma T}\varepsilon^{2p}(\varepsilon\|D^{m+p}u(T)\|^2+\gamma\|D^{m+p-1}u(T)\|^2).$$

3.2. Dispersive case.

3.2a. Degeneration of order 1.

Let $L\in L(m+1;\ 0)$ and $M\in L(m;\ r)$ satisfy the conditions (H0), (H1) and (E). If we also assume that M_0 is skew-adjoint (i.e. $M_0=-M_0^*$ the case we call dispersive), we have a weaker estimate for the operator P under the separation condition (S0).

In this subsection 3.2a, dividing the equation (3.6) by the imaginary unit i, we consider equations

$$Pu=\varepsilon Lu+Mu=f \text{ or } Pu=\varepsilon Lu-Mu=f. \tag{3.14}$$

We assume a weak separation condition:
(WS^+) the roots $\{\psi_j(t,\ x,\ \xi;\ \varepsilon)\}$ uniformly separate $\{\varphi_j(t,\ x,\ \xi;\ \varepsilon)\}$ from above in a weak sense such that

$$\varphi_1(t,\ x,\ \xi;\ \varepsilon)<\min\{\varphi_2(t,\ x,\ \xi;\ \varepsilon),\ \psi_1(t,\ x,\ \xi;\ \varepsilon)\} \text{ uniformly}$$

and that

$$\max\{\varphi_{j-1}(t,\ x,\ \xi;\ \varepsilon),\ \psi_{j-2}(t,\ x,\ \xi;\ \varepsilon)\}<\min\{\varphi_j(t,\ x,\ \xi;\ \varepsilon),\ \psi_{j-1}(t,\ x,\ \xi;\ \varepsilon)\} \text{ uniformly for } j=3,\cdots,m+1.$$

We denote this condition by

$$\varphi_1<\{\varphi_2,\ \psi_1\}<\{\varphi_3,\ \psi_2\}<\cdots<\{\varphi_{m+1},\ \psi_m\} \text{ uniformly.}$$

Alternatively, we assume by the same notation

(WS^-) $\qquad \{\varphi_1,\ \psi_1\}<\{\varphi_2,\ \psi_2\}<\cdots<\{\varphi_m,\ \psi_m\}<\varphi_{m+1}$ uniformly.

We set

$$P=\varepsilon L(t,\ x,\ D_t,\ D_x;\ \varepsilon)+M(t,\ x,\ D_t,\ D_x;\ \varepsilon) \tag{3.15}$$

when (WS^+) is assumed. Alternatively, we set

$$P=\varepsilon L(t,\ x,\ D_t\ D_x;\ \varepsilon)-M(t,\ x,\ D_t,\ D_x;\ \varepsilon) \tag{3.16}$$

when (WS^-) is assumed.

THEOREM 3.4. *Under the assumptions* (H0), (H1), (P0) *and one of* ($WS^{\pm}$), *there exist positive constants* C, γ_0 *and* ε_0 *such that for any positive* $\varepsilon\leqq\varepsilon_0$, *for any* $\gamma\geqq\gamma_0$ *and for any* $u(t,\cdot)\in C^\infty([0,\ T];\ C_0^\infty(\boldsymbol{R}_x^n))$

$$C\left\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon}\|Pu(t)\|^2dt+\varepsilon\|D^m u(0)\|^2+\|D^{m-1}u(0)\|^2_{(r+1)/2}\right\} \tag{3.17}$$

$$\geqq \gamma \int_0^T e^{-2\gamma t}(\varepsilon \|D^m u(t)\|^2 + \|D^{m-1}u(t)\|^2_{(r+1)/2})dt$$
$$+e^{-2\gamma T}(\varepsilon \|D^m u(T)\|^2 + \|D^{m-1}u(T)\|^2_{(r+1)/2}),$$

where P is defined by (3.15) *or* (3.16).

PROOF. We assume (WS^+). Using the partition of unity, we can construct a set of homogeneous symbols $\{\chi_j(t, x, \xi; \varepsilon) \in S^1/S^0;\ j=1, \cdots, m\}$ such that $\{\chi_j\}$ separates $\{\varphi_j\}$ uniformly:

(3.18) $$\varphi_1 < \chi_1 < \cdots < \chi_m < \varphi_{m+1} \text{ uniformly}$$

and separates $\{\psi_j\}$ uniformly from below:

(3.19) $$\chi_1 < \psi_1 < \cdots < \chi_m < \psi_m \text{ uniformly.}$$

Let $B(t, x, D_t, D_x; \varepsilon) = \sum_{j=0}^{m} B_j(t, x, D_x; \varepsilon) D_t^{m-j}$ be of the class $L(m; 0)$ such that its principal symbol has the decomposition

(3.20) $$\sigma_m(B)(t, x, \tau, \xi; \varepsilon) = \prod_{j=1}^{m} (\tau - \chi_j(t, x, \xi; \varepsilon))$$

and that $B_0(t, x, D_x; \varepsilon) = 1$. B is a regularly hyperbolic operator of order m. Then, we set

(3.21) $$N = M_0 B - M,$$

where

(3.22) $$N = N(t, x, D_t, D_x; \varepsilon) = \sum_{j=0}^{m} N_j(t, x, D_x; \varepsilon) D_t^{m-1-j}.$$

Its principal symbol has the decomposition

$$\sigma_{m+r}(N)(t, x, \tau, \xi; \varepsilon) = \sigma_{r+1}(N_0)(t, x, \xi; \varepsilon) \prod_{j=1}^{m-1} (\tau - \kappa_j(t, x, \xi; \varepsilon))$$

where

$$\psi_j(t, x, \xi; \varepsilon) < \kappa_j(t, x, \xi; \varepsilon) < \chi_{j+1}(t, x, \xi; \varepsilon),\ j=1, \cdots, m-1.$$

N satisfies the condition of regular hyperbolicity of order $m-1$. It follows as in the proof of Theorem 2.1 in § 2 that $\{\kappa_j\}$ separates $\{\chi_j\}$ uniformly.

We consider the case where

(3.23) $$N_0 = M_0 B_1 - M_1 \in L^{r+1}$$

satisfies the same assumption as (P0) for M_0 in § 2. If it is not the case, we will modify the lower part of B later. By Theorem 2.2 for the pairs (L, B) and (B, N), we have

$$\begin{aligned}
&-\operatorname{Im}\int_0^T e^{-2\gamma t}(Pu(t),\ Bu(t))dt \\
&=-\varepsilon \operatorname{Im}\int_0^T e^{-2\gamma t}(Lu(t),\ Bu(t))dt-\operatorname{Im}\int_0^T e^{-2\gamma t}(M_0Bu(t),\ Bu(t))dt \\
&-\operatorname{Im}\int_0^T e^{-2\gamma t}(Bu(t),\ Nu(t))dt \\
&\geqq c\varepsilon\gamma\int_0^T e^{-2\gamma t}\|D^m u(t)\|^2 dt+c\varepsilon e^{-2\gamma T}\|D^m u(T)\|^2-C\varepsilon\|D^m u(0)\|^2 \\
&+c\gamma\int_0^T e^{-2\gamma t}\|D^{m-1}u(t)\|^2_{(r+1)/2}+ce^{-2\gamma T}\|D^{m-1}u(T)\|^2_{(r+1)/2}-C\|D^{m-1}u(0)\|^2_{(r+1)/2}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&C\left\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon}\|Pu(t)\|^2 dt+\varepsilon\|D^m u(0)\|^2+\|D^{m-1}u(0)\|^2_{(r+1)/2}\right\} \\
&\geqq c\left\{\gamma\int_0^T e^{-2\gamma t}(\varepsilon\|D^m u(t)\|^2+\|D^{m-1}u(t)\|^2_{(r+1)/2})dt\right. \\
&\left.+e^{-2\gamma T}(\varepsilon\|D^m u(T)\|^2+\|D^{m-1}u(T)\|^2_{(r+1)/2})\right\}.
\end{aligned}$$

We have only to check the assumption (P0) for N_0 in the general case.

Since

$$\begin{aligned}
&\sigma_{r+1}(N_0)=\sigma_r(M_0)\sigma_1(B_1)-\sigma_{r+1}(M_1) \\
&=m_0(t,\ x,\ \xi;\ \varepsilon)\left\{-\sum_{j=1}^m \chi_j(t,\ x,\ \xi;\ \varepsilon)+\sum_{j=1}^m \phi_j(t,\ x,\ \xi;\ \varepsilon)\right\},
\end{aligned}$$

we have $N_0^*-N_0\in L^r$. Then, putting

$$\begin{aligned}
&R=M_0^{-1}(N_0^*-N_0)/2\in L^0 \\
&B_1'=R+B_1\in L^1 \\
&N_0'=M_0B_1'-M_1,
\end{aligned}$$

we have $R\in L^0$ and $N_0'=(B_1^*M_0+M_0B_1)/2-(M_1^*+M_0)/2$, which is self-adjoint. Since $B_1'-M_0^{-1}M_1$ is uniformly positive of order 1, $B_1''-M_0^{-1}M_1$ is invertible as pseudo-differential operator, where $B_1''=B_1'+\lambda$ with sufcicently large positive constant λ. Then, $N_0''=M_0(B_1''-M_0^{-1}M_1)$ satisfies the assumption (P0). Thus, we have only to replace B_1 by B_1''.

The alternative case (WS^-) is treated in the same way. In fact, there exists $\{\chi_j(t,\ x,\ \xi;\ \varepsilon)\in S^1/S^0,\ j=1,\cdots,m\}$ such that

(3.24) $$\varphi_1<\chi_1<\cdots<\chi_m<\varphi_{m+1} \text{ uniformly}$$

and

(3.25) $$\psi_1<\chi_1<\cdots<\psi_m<\chi_m \text{ uniformly.}$$

Let B be defined by (3.20) as before. The rest is similar, if we put $M=M_0B+N$ instead of (3.21). Q.E.D.

REMARK. The separation condition (S0) is equivalent to both conditions (WS^+) and (WS^-). For differential operators the weak condition (WS^+) for any ξ with $|\xi|=1$ means (WS^-) and vice versa. Hence, the separation conditons $(WS^\pm)$ are meaningful for pseudo-differential operators P.

We will give a higher order estimate of the solutions to (3.14) for the case $r=0$.

PROPOSITION 3.2. *Under the same assumptions as in Theorem* 3.4 *with* $r=0$, *there exist* $C>0$, γ_0 *and* $\varepsilon_0>0$ *such that for any positive* $\varepsilon\leqq\varepsilon_0$, *any* $\gamma\geqq\gamma_0$, *we have for any integer* $p\geqq 0$.

(3.26)
$$C\left\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j\|D^jf(t)\|^2dt\right.$$
$$\left.+\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(0)\|+\|D^{m+j-1}u(0)\|_{1/2}^2)\right\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(t)\|^2+\|D^{m+j-1}u(t)\|_{1/2}^2)dt$$
$$+e^{-2\gamma T}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(T)\|^2+\|D^{m+j-1}u(T)\|_{1/2}^2).$$

PROOF. Theorem 3.4 implies the inequality (3.26) for $p=0$. Assume (3.26) for $p=0, 1, \cdots, k$. We consider the case under the condition (WS^+). From the equation, we have

$$\varepsilon LD_tu+MD_tu=D_tf+\varepsilon[L, D_t]u+[M, D_t]u$$

and

$$\varepsilon L\Lambda u+M\Lambda u=\Lambda f+\varepsilon[L, \Lambda]u+[M, \Lambda]u,$$

for $\Lambda=(1-\Delta)^{1/2}$.

Noticing that for $j=0, 1, \cdots, k$

(3.27)
$$\|\varepsilon D^j[L, D_t]u(t)\|^2+\|\varepsilon D^j[L, \Lambda]u(t)\|^2$$
$$\leqq C\varepsilon^2\|D^{m+j}u(t)\|_1^2,$$

and that

$$\|D^j[M,\ D_t]u(t)\|^2+\|D^j[M,\ \Lambda]u(t)\|^2 \leqq C\|D^{m+j}u(t)\|^2, \tag{3.28}$$

we have

$$\begin{aligned}C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\gamma}\sum_{j=0}^k(\varepsilon^2\gamma)^j(\|D^{j+1}f(t)\|^2\\ +\varepsilon^2\|D^{m+j}u(t)\|_1^2+\|D^{m+j}u(t)\|^2)dt\\ +\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j+1}u(0)\|^2+\|D^{m+j}u(0)\|_{1/2}^2)\Big\}\\ \geqq\gamma\int_0^T e^{-2\gamma t}\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j+1}u(t)\|^2+\|D^{m+j}u(t)\|_{1/2}^2)dt\\ +e^{-2\gamma T}\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j+1}u(T)\|^2+\|D^{m+j}u(T)\|_{1/2}^2).\end{aligned} \tag{3.29}$$

Multiplying (3.29) by $\varepsilon^2\gamma$ to add the result to the inequality (3.26) for $p=k$, we have

$$\begin{aligned}C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon}\Big(\sum_{j=0}^k(\varepsilon^2\gamma)^{j+1}(\|D^{j+1}f(t)\|^2\\ +\varepsilon^2\|D^{m+j}u(t)\|_1^2+\|D^{m+j}u(t)\|^2)+\sum_{j=0}^k(\varepsilon^2\gamma)^j\|D^jf(t)\|^2\Big)dt\\ +\sum_{j=0}^k(\varepsilon^2\gamma)^{j+1}(\varepsilon\|D^{m+j+1}u(0)\|^2+\|D^{m+j}u(0)\|_{1/2}^2)\\ +\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(0)\|^2+\|D^{m+j-1}u(0)\|_{1/2}^2)\Big\}\\ \geqq\gamma\int_0^T e^{-2\gamma t}\Big(\sum_{j=0}^k(\varepsilon^2\gamma)^{j+1}(\varepsilon\|D^{m+j+1}u(t)\|^2+\|D^{m+j}u(t)\|_{1/2}^2)\\ +\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(t)\|^2+\|D^{m+j-1}u(t)\|_{1/2}^2)\Big)dt\\ +e^{-2\gamma T}\Big(\sum_{j=0}^k(\varepsilon^2\gamma)^{j+1}(\varepsilon\|D^{m+j+1}u(T)\|^2+\|D^{m+j}u(T)\|_{1/2}^2)\\ +\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(T)\|^2+\|D^{m+j-1}u(T)\|_{1/2}^2)\Big).\end{aligned}$$

Since the terms $\gamma^j\varepsilon^{2j+3}\|D^{m+j}u(t)\|_1^2$ and $\gamma^j\varepsilon^{2j+1}\|D^{m+j}u(t)\|^2$ are absorbed in the right hand side, we obtain the desired result. Q.E.D.

From Lemma 3.2 with $\delta=1$ and $q=1/2$, we have

THEOREM 3.5. *Under the same assumptions as in Theorem* 3.4, *there exist* $C>0$, γ_0 *and* $\varepsilon_0>0$ *such that for any positive* $\varepsilon\leqq\varepsilon_0$, *any* $\gamma\geqq\gamma_0$, *and* *for any* $u(t)\in C^\infty([0,\ T];\ C_0^\infty(\boldsymbol{R}_x^n))$ *we have for any natural number* p

$$C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j\|D^jf(t)\|^2dt+\|D^{m-1}u(0)\|_{1/2}^2 \tag{3.30}$$
$$+\gamma^p\Big\{\varepsilon\sum_{j=0}^{p}\varepsilon^{2j}\|D^mu(0)\|_j^2+\sum_{j=1}^{p}\varepsilon^{2j}\|D^mu(0)\|_{j-1/2}$$
$$+\varepsilon\sum_{j=0}^{p-1}\varepsilon^{2j}\|D^jf(0)\|^2+\sum_{j=1}^{p-1}\varepsilon^{2j}\|D^{j-1}f(0)\|_{1/2}^2\Big\}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+j}u(t)\|_{1/2}^2+\|D^{m+j-1}u(t)\|_{1/2}^2)dt$$
$$+e^{-2\gamma T}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon\|D^{m+}u^j(T)\|^2+\|D^{m+j-1}u(T)\|_{1/2}^2).$$

3.2b. Degeneration of order 2.

Let $L\in L(m+2;\ 0)$ and $M\in L(m;\ r)$. We assume (H0) for L, (H1) for M and (P0) for M_0. We assume a weak separation condition

$$\varphi_1<\{\varphi_2,\ \phi_1\}<\cdots<\{\varphi_{m+1},\ \phi_m\}<\varphi_{m+2}\ \text{uniformly.} \tag{WS}$$

Setting

$$P(t,x,D_t,D_x;\varepsilon)=\varepsilon^2L(t,x,D_t,D_x;\varepsilon)-M(t,x,D_t,D_x;\varepsilon), \tag{3.31}$$

we have the following

THEOREM 3.6. *Under the above assumptions* (H0), (H1), (P0) *and* (*WS*), *there exist positive constants* C, γ_0 *and* ε_0 *such that for any positive* $\varepsilon\leqq\varepsilon_0$, *for any* $\gamma\geqq\gamma_0$, *and for any* $u(t)\in C^\infty([0,T];\ C_0^\infty(\boldsymbol{R}_x^n))$, *we have for any natural number* p

$$C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon^2}\|Pu(t)\|^2dt+\varepsilon^2\|D^{m+1}u(0)\|^2+\|D^mu(0)\|_{r/2}^2\Big\} \tag{3.32}$$
$$\geqq\gamma\int_0^T e^{-2\gamma t}(\varepsilon^2\|D^{m+1}u(t)\|^2+\|D^mu(t)\|_{r/2}^2)dt$$
$$+e^{-2\gamma T}(\varepsilon^2\|D^{m+1}u(T)\|^2+\|D^mu(T)\|_{r/2}^2).$$

PROOF. Using the partition of unity, we can construct through (*WS*) a set of homogeneous symbols $\{\chi_j(t,x,\xi;\varepsilon)\in S^1/S^0;\ j=1,\cdots,m+1\}$ such that $\{\chi_j\}$ separates $\{\varphi_j\}$ uniformly i.e.

$$\varphi_1<\chi_1<\varphi_2<\cdots<\chi_{m+1}<\varphi_{m+2}\ \text{uniformly}$$

and that $\{\phi_j\}$ separates $\{\chi_j\}$ uniformly i.e.

$$\chi_1<\psi_1<\chi_2<\cdots<\psi_m<\chi_{m+1} \text{ uniformly.}$$

Let $B(t, x, D_t, D_x; \varepsilon)=\sum_{j=0}^{m+1} B_j(t, x, D_x; \varepsilon)D_t^{m+1-j}$ be an operator of the same type as in the proof of Theorem 3.4. Then, we have

$$\begin{aligned}
&-\operatorname{Im}\int_0^T e^{-2\gamma t}(Pu(t),\ Bu(t))dt \\
&=-\varepsilon^2\operatorname{Im}\int_0^T e^{-2\gamma t}(Lu(t),\ Bu(t))dt-\operatorname{Im}\int_0^T e^{-2\gamma t}(Bu(t),\ Mu(t))dt \\
&\geqq c\varepsilon^2\gamma\int_0^T e^{-2\gamma t}\|D^{m+1}u(t)\|^2 dt+c\varepsilon^2 e^{-2\gamma T}\|D^{m+1}u(T)\|^2 \\
&-C\varepsilon^2\|D^{m+1}u(0)\|^2 \\
&+c'\gamma\int_0^T e^{-2\gamma t}\|D^m u(t)\|_{r/2}^2 dt+c'e^{-2\gamma T}\|D^m u(T)\|_{r/2}^2 \\
&-C\|D^m u(0)\|_{r/2}^2.
\end{aligned}$$

Since

$$\begin{aligned}
&-\operatorname{Im}(Pu(t),\ Bu(t))\leqq\|Pu(t)\|\,\|Bu(t)\| \\
&\leqq\frac{c\varepsilon^2\gamma}{2}\|D^{m+1}u(t)\|^2+\frac{1}{2c\varepsilon^2\gamma}\|Pu(t)\|^2,
\end{aligned}$$

we obtain the desired inequality (3.32). Q.E.D.

We will give a higher order estimate for the equation

$$Pu=\varepsilon^2 Lu-Mu=f. \tag{3.33}$$

PROPOSITION 3.3. *Under the same assumptions as in the Theorem 3.6 with $r=0$, there exist positive constants C, γ_0 and ε_0 such that for any positive $\varepsilon\leqq\varepsilon_0$, $\gamma\geqq\gamma_0$, we have for $p\geqq 0$*

$$\begin{aligned}
&C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon^2}\sum_{j=0}^p(\varepsilon^2\gamma)^j\|D^j f(t)\|^2 dt \\
&+\sum_{j=0}^p(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+1}u(0)\|^2+\|D^{m+j}u(0)\|^2)\Big\} \\
&\geqq\gamma\int_0^T e^{-2\gamma t}\sum_{j=0}^p(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+1}u(t)\|^2+\|D^{m+j}u(t)\|^2)dt \\
&+e^{-2\gamma T}\sum_{j=0}^p(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+1}u(T)\|^2+\gamma\|D^{m+j}u(T)\|^2).
\end{aligned} \tag{3.34}$$

PROOF. Theorem 3.6 implies the inequality (3.34) for $p=0$. Assume (3.34) for $p=0, 1, \cdots, k$. From the equation (3.33), we have

(3.35) $$\varepsilon^2 LD_t u - MD_t u = D_t f + \varepsilon^2[L, D_t]u - [M, D_t]u$$

and

(3.36) $$\varepsilon^2 L\Lambda u - M\Lambda u = \Lambda f + \varepsilon^2[L, \Lambda]u - [M, \Lambda]u$$

Noticing that

$$\|\varepsilon^2 D^j[L, D_t]u(t)\|^2 + \|\varepsilon^2 D^j[L, \Lambda]u(t)\|^2 \leqq C\varepsilon^4\|D^{m+j+1}u(t)\|_1^2,$$

and that

$$\|D^j[M, D_t]u(t)\|^2 + \|D^j[M, \Lambda]u(t)\|^2 \leqq C\|D^{m+j}u(t)\|^2,$$

we have, substituting (3.35) and (3.36) into the assumption of the induction,

(3.37) $$\begin{aligned} &C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon^2}\sum_{j=0}^k(\varepsilon^2\gamma)^j(\|D^{j+1}f(t)\|^2 \\ &+\varepsilon^4\|D^{m+j+1}u(t)\|_1^2 + \|D^{m+j}u(t)\|^2)dt \\ &+\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+2}u(0)\|^2 + \|D^{m+j+1}u(0)\|^2)\Big\} \\ &\geqq \gamma\int_0^T e^{-2\gamma t}\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+2}u(t)\| + \|D^{m+j+1}u(t)\|^2)dt \\ &+e^{-2\gamma T}\sum_{j=0}^k(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+2}u(T)\|^2 + \|D^{m+j+1}u(T)\|^2). \end{aligned}$$

Multiplying (3.37) by $\varepsilon^2\gamma$ to add it to the inequality (3.34) for $p=k$, we obtain the desired result as in the proof of Proposition 3.2. Q.E.D.

We will estimate the initial derivatives of higher order in (3.34) by the Cauchy data of u.

LEMMA 3.3. *Under the assumptions of the Proposition 3.3, there exists a positive constant C such that for any natural number p*

(3.38) $$\begin{aligned} \|D_t^{m+2p}u(0)\|^2 \leqq C\Big\{&\sum_{j=0}^{p-1}\varepsilon^{-4j}\|D^{m+1}u(0)\|_{2p-2j-1}^2 + \varepsilon^{-4p}\|D^m u(0)\|^2 \\ &+\sum_{j=1}^{p}\varepsilon^{-4j}\|D^{2p-2j}f(0)\|^2\Big\} \end{aligned}$$

(3.39) $$\begin{aligned} \|D_t^{m+2p+1}u(0)\|^2 \leqq C\Big\{&\sum_{j=0}^{p}\varepsilon^{-4j}\|D^{m+1}u(0)\|_{2p-2j}^2 \\ &+\sum_{j=1}^{p}\varepsilon^{-4j}\|D^{2p-2j+1}f(0)\|^2\Big\}, \end{aligned}$$

where $\varepsilon^2 Lu - Mu = f$.

PROOF. Since

$$D_t^{m+2} = (D_t^{m+2} - L)u + \varepsilon^{-2} Mu + \varepsilon^{-2} f,$$

we have

$$\|D_t^{m+2}u(0)\|^2 \leqq C\{\|D^{m+1}u(0)\|_1^2 + \varepsilon^{-4}\|D^m u(0)\|^2 + \varepsilon^{-4}\|f(0)\|^2\}. \tag{3.40}$$

This implies (3.38) for $p=1$. In the same way, from the relation

$$D_t^{m+3}u = D_t(D_t^{m+2} - L)u + \varepsilon^{-2} D_t(Mu) + \varepsilon^{-2} D_t f,$$

we have, using (3.40),

$$\begin{aligned}\|D_t^{m+3}u(0)\|^2 &\leqq C\{\|D_t^{m+2}u(0)\|_1^2 + \|D^{m+1}u(0)\|_2^2 \\ &\quad + \varepsilon^{-4}\|D^{m+1}u(0)\|^2 + \varepsilon^{-4}\|D_t f(0)\|^2\} \\ &\leqq C\{\|D^{m+1}u(0)\|_2^2 + \varepsilon^{-4}\|D^{m+1}u(0)\|^2 + \varepsilon^{-4}\|Df(0)\|^2\}.\end{aligned}$$

This implies (3.39) for $p=1$.

Suppose that the system of the inequalities (3.38) and (3.39) is true for $p=1, 2, \cdots, k$. Since

$$D_t^{m+2(k+1)}u = D_t^{2k}(D_t^{m+2} - L)u + \varepsilon^{-2} D_t^{2k}(Mu) + \varepsilon^{-2} D_t^{2k} f,$$

and

$$D_t^{m+2(k+1)+1}u = D_t^{2k+1}(D_t^{m+2} - L)u + \varepsilon^{-2} D_t^{2k+1}(Mu) + \varepsilon^{-2} D_t^{2k+1} f,$$

induction gives us firstly

$$\begin{aligned}\|D_t^{m+2(k+1)}u(0)\|^2 &\leqq C\{\|D_t^{m+2k+1}u(0)\|_1^2 + \|D^{m+2k}u(0)\|_2^2 \\ &\quad + \varepsilon^{-4}\|D^{m+2k}u(0)\|^2 + \varepsilon^{-4}\|D_t^{2k}f(0)\|^2\} \\ &\leqq C\left\{\sum_{j=0}^{(k+1)-1} \varepsilon^{-4j}\|D^{m+1}u(0)\|_{2(k+1)-2j-1} + \varepsilon^{-4(k+1)}\|D^m u(0)\|^2 \right. \\ &\quad \left. + \sum_{j=1}^{k+1} \varepsilon^{-4j}\|D^{2(k+1)-2j}f(0)\|^2\right\}\end{aligned}$$

and secondly

$$\begin{aligned}\|D_t^{m+2(k+1)+1}u(0)\|^2 &\leqq C\{\|D_t^{m+2k+2}u(0)\|_1^2 + \|D^{m+2k+1}u(0)\|_2^2 \\ &\quad + \varepsilon^{-4}\|D^{m+2k+1}u(0)\|^2 + \varepsilon^{-4}\|D_t^{2k+1}f(0)\|^2\} \\ &\leqq C\left\{\sum_{j=0}^{k+1} \varepsilon^{-4j}\|D^{m+1}u(0)\|_{2(k+1)-2j}^2 + \sum_{j=1}^{k+1} \varepsilon^{-4j}\|D^{2(k+1)-2j+1}f(0)\|^2\right\}.\end{aligned}$$

Q.E.D.

LEMMA 3.4. *Under the assumptions of Proposition* 3.3, *there exists a positive constant C such that for any natural number p and for any* $\gamma\geqq 1$.

$$\sum_{j=0}^{p}(\varepsilon^2\gamma)^j(\varepsilon^2\|D^{m+j+1}u(0)\|^2+\|D^{m+j}u(0)\|^2) \tag{3.41}$$
$$\leqq C\gamma^p\Big\{\|D^m u(0)\|^2+\sum_{k=0}^{p}\varepsilon^{2k+2}\|D^{m+1}u(0)\|_k^2$$
$$+\sum_{k=0}^{p-1}\varepsilon^{2k}\|D^k f(0)\|^2\Big\}.$$

PROOF. The estimate (3.41) for $p=1$ follows from (3.40). Assume (3.41) is true for $p=1, \cdots, 2k-1$. Then, we have from Lemma 3.3

$$(\varepsilon^2\gamma)^{2k}(\varepsilon^2\|D^{m+2k+1}u(0)\|^2+\|D^{m+2k}u(0)\|^2)$$
$$\leqq C\gamma^{2k}\Big\{\|D^m u(0)\|^2+\sum_{j=0}^{2k}\varepsilon^{2j+2}\|D^{m+1}u(0)\|_j^2$$
$$+\sum_{j=0}^{2k-1}\varepsilon^{2j}\|D^j f(0)\|^2\Big\}.$$

Alternatively, assume (3.41) is true for $p=1, \cdots, 2k$. Then, we have

$$(\varepsilon^2\gamma)^{2k+1}(\varepsilon^2\|D^{m+2k+2}u(0)\|^2+\|D^{m+2k+1}u(0)\|^2)$$
$$\leqq C\gamma^{2k+1}\Big\{\|D^m u(0)\|^2+\sum_{j=0}^{2k+1}\varepsilon^{2j+2}\|D^{m+1}u(0)\|_j^2$$
$$+\sum_{j=0}^{2k}\varepsilon^{2j}\|D^j f(0)\|^2\Big\}.$$

By the induction, we obtain the desired result. Q.E.D.

We finally obtain from Proposition 3.3 and Lemma 3.4

THEOREM 3.7. *Under the same assumptions as in Theorem* 3.6 *with* $r=0$, *there exist positive constants* C, γ_0 *and* ε_0 *such that for any positive* $\varepsilon\leqq\varepsilon_0$, *for any* $\gamma\geqq\gamma_0$, *for any* $u(t, \cdot)\in C^\infty([0, T]; C_0^\infty(\boldsymbol{R}_x^n))$, *we have for any natural number p.*

$$C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\frac{1}{\varepsilon^2}\sum_{j=0}^{p}(\varepsilon^2\gamma)^j\|D^j f(t)\|^2 dt \tag{3.42}$$
$$+\gamma^p\Big(\|D^m u(0)\|^2+\sum_{k=0}^{p}\varepsilon^{2k+2}\|D^{m+1}u(0)\|_k^2$$
$$+\sum_{k=0}^{p-1}\varepsilon^{2k}\|D^k f(0)\|^2\Big)\Big\}$$

$$\geqq \gamma \int_0^T e^{-2\gamma t} \sum_{j=0}^{p} (\varepsilon^2\gamma)^j (\varepsilon^2 \|D^{m+j+1}u(t)\|^2 + \|D^{m+j}u(t)\|^2) dt$$

$$+ e^{-2\gamma T} \sum_{j=0}^{p} (\varepsilon^2\gamma)^j (\varepsilon^2 \|D^{m+j+1}u(T)\|^2 + \gamma \|D^{m+j}u(T)\|^2).$$

References

[As] Ashino, R., On the admissibility of singular perturbations in Cauchy problems, Osaka J. Math. **26**, (1989), 387-398.

[Ch] Chaillou, J., Hyperbolic Differential Polynomials and their Singular Perturbations, D. Reidel, Dordrecht, 1979.

[Dz] Dzavadov, M. G., A mixed problem for a hyperbolic equation involving a small parameter with leading derivatives, Soviet Math. Dokl. 4″[151-153], (1963), 1400-1404.

[Fe] Fedoriuk, M. V,, Singularities of Fourier Integral Operators and Asymptotic Solutions of Mixed Problems, Russian Math. Surveys **32**: 6, (1977), 67-120.

[Fr] Frank, L. S., SINGULAR PERTURBATIONS I, Spaces and Singular Perturbations on Manifolds without Boundary, North-Holland, Amsterdam, 1990.

[Fu] Fujiwara, D., On the asymptotic behaviour of the Green operators for elliptic boundary problems and the pure imaginary powers of some second order operators, J. Math. Soc. Japan **21**, (1969), 481-522.

[Ga] Gao, R. X., SINGULAR PERTURBATION FOR HIGHER ORDER HYPERBOLIC EQUATION (I), (II), (in Chinese), Fudan Journal (Natural Science) **22**(3), (1983), 265-278, **23**(1), (1984), 85-94.

[Ge] Geel, R., Singular Pertubations of Hyperbolic Type, Mathematisch Centrum, Amsterdam, 1978.

[Ja] De Jager, E. M., Singular perturbations of hyperbolic type, Nieuw Archief voor Wiskunde **23**(3), (1975), 145-171.

[Ku] Kumano-go, H., Pseudo-differential operators, MIT Press, Cambridge, Mass., 1981.

[Le] Leray, J., LA THEORIE DE L. GÅRDING DES EQUATIONS HYPERBOLIQUES LINEAIRES, Roma, Instituto Matematico dell Univ., 1956.

[Ma] Maslov, V. P. and M. V. Fedoriuk, Semi-Classical Approximation in Quantum Mechanics (translated from Russian), D. Reidel, Dordrecht, 1981.

[Sa] Sakamoto, R., Hyperbolic boundary value problems (in Japanese), Iwanami Shoten, Tokyo, 1978, (in English) Cambridge Univ. Press, 1982.

[U] Uchiyama, K., L^2-theory of Singular Perturbation of Hyperbolic Equations. I A Priori Estimates with Parameter ε, Proc. Japan Acad. Ser. A Math. Sci. **63**, (1987), 386-389.

[Wh1] Whitham, G. B., Some comments on Wave Propagation and Shock Wave Structure With Application to Magnetohydrodynamics, Comm. Pure Appl. Math. **12**, (1959), 113-158.

[Wh2] Whitham, G. B., Linear and nonlinear waves, Chap. 10, Wiley, New York, 1974,

[Wu] Wu, T. T., A Note on the Stability Condition for Certain Wave Propagation Problems, Comm. Pure Appl. Math. **14**, (1961), 745-747.

[Yo] Yoshikawa, A., On a Canonical Standard Form of Second Order Linear Ordinary Differential Equations with a Small Parameter, Proc. Japan Acad. Ser. A Math.

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