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Continuous variation of the discrete Godbillion-Vey invariant

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Abstract. For each pair of integers (s, m) satisfying that $s\geq 2$ and $|m|\leq 2s-2$, we construct a continuous family $\{\phi_t: \pi_1(\Sigma_s)\rightarrow PL_+(S^1)\}_{t\in \boldsymbol{R}}$ of homomorphism from the fundamental group of the closed oriented surface of genus s to the group of orientation preserving piecewise linear homoeomorphisms of the circle such that the Euler number of the associated S^1-bundler is equal to m and the discrete Godbillon-Vey invariant of ϕ_t is equal to t.

1. Introduction.

The discrete Godbillon-Vey invariant $\overline{GV}$ is a 2-cocycle of $PL_+(S^1)$ defined in [2] and [4]. Here $PL_+(S^1)$ denotes the group of all orientation preserving homeomorphisms of the circle whose lifts via the projection $\boldsymbol{R}\rightarrow S^1=\boldsymbol{R}/\boldsymbol{Z}$ are piecewise linear homeomorphisms of the real line $\boldsymbol{R}$. Let Σ_s be a closed oriented surface of genus $s\geq 2$. For any homomorphism $\phi: \pi_1(\Sigma_s)\rightarrow PL_+(S^1)$, the discrete Godbillon-Vey invariant $\overline{GV}(\phi)$ of ϕ is defined by $\overline{GV}(\phi)=\overline{GV}(\phi_*[\Sigma_s])$, where $[\Sigma_s]$ denotes the fundamental class of Σ_s. On the other hand, any homomorphism $\phi: \pi_1(\Sigma_s)\rightarrow G_+^0$ determines a foliated S^1-bundle E_ϕ whose total holonomy is ϕ. Here G_+^0 is the group of all orientation preserving homeomorphisms of S^1. So the Euler number $eu(\phi)$ of ϕ is defined by $eu(\phi)=\langle e(E_\phi), [\Sigma_s]\rangle$, where $e(E_\phi)$ is the Euler class of the S^1-bundle E_ϕ. Then we have the Milnor-Wood inequality $|eu(\phi)|\leq|\chi(\Sigma_s)|=2s-2$ (see [8]). As for the discrete Godbillon-Vey invariant, in [2] Ghys shows that there exists a continuous family $\{\phi_t^0: \pi_1(\Sigma_1)\rightarrow PL_+(S^1)\}_{t\in \boldsymbol{R}}$ of homomorphisms with respect to the C^0 topology of $PL_+(S^1)$ such that $\overline{GV}(\phi_t^0)=t$. In this paper, we consider the existence of such families in the case where $s\geq 2$ and prove the following theorem.

THEOREM 1.1. *For any integers $s\geq 2$ and m with $|m|\leq 2s-2$, there exists a continuous family $\{\phi_t: \pi_1(\Sigma_s)\rightarrow PL_+(S^1)\}_{t\in \boldsymbol{R}}$ of homomorphisms*

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which satisfies the following conditions.

(1) $eu(\phi_t)=m$,

(2) $\overline{GV}(\phi_t)=t \quad (t\in \boldsymbol{R})$.

The continuous variation of $\overline{GV}(\phi)$ is caused by the continuous variation of right derivatives, left derivatives and bending points of elements of $\mathrm{Im}(\phi)\subset PL_+(S^1)$. In Ghys' example, $\overline{GV}$ varies as the right derivatives and the left derivatives of the holonomy of the two toral leaves vary. On the other hand, when the Euler number is maximal, our example is a cylinder-plane foliation (see [3] and [7]), and it is finite cylinder leaves that contain its bending points.

We can use Ghys' example to prove the case where $m=0$. There exists a branched covering b: $\Sigma_s\to\Sigma_1$ such that $b_*[\Sigma_s]=s[\Sigma_1]$. So, $\{\phi_t^0\circ b_\sharp: \pi_1(\Sigma_s)\to PL_+(S^1)\}_{t\in \boldsymbol{R}}$ satisfies $eu(\phi_t^0\circ b_\sharp)=0$ and $\overline{GV}(\phi_t^0\circ b_\sharp)=st$. Hence, we will prove Theorem 1.1 in the case where $m\neq 0$.

In the differentiable case, there is a next conjecture;

Is a C^∞ foliated S^1-bundle which has the maximal Euler number diffierentiably conjugate to the Anosov foliation?

As a corollary of Theorem 1.1, we know this conjecture is false in the *PL* case.

COROLLARY 1.2. *There are uncountably many PL conjugacy classes of PL foliated S^1-bundles whose Euler numbers are maximal.*

2. Construction of a continuous family $\{\phi_{\beta,n}^{(\sigma)}\}_{\beta>0}$.

A one punctured torus Σ_1' is developed as follows.

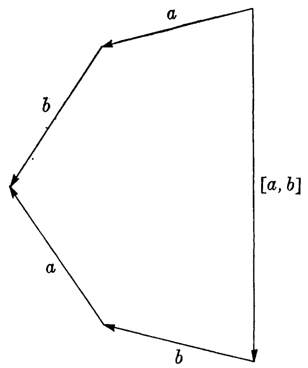

Fig. 1

The fundamental group $\pi_1(\Sigma_1')$ is the free group generated by the homotopy classes of the two closed loops a and b. That is, $\pi_1(\Sigma_1')\cong\langle a, b\rangle$. If we define a 2-chain $Z_{a,b}$ of $\langle a, b\rangle$ by $Z_{a,b}=(a, b)-(b, a)-([a, b], ba)$, then $Z_{a,b}$ represents the fundamental class in $H_2(\langle a, b\rangle, \langle[a, b]\rangle; \boldsymbol{Z})=H_2(\Sigma_1', \partial\Sigma_1'; \boldsymbol{Z})$.

LEMMA 2.1. *Let G be a group and α, β, γ, elements of G. If $\beta\gamma=\gamma\beta$, then the chain $Z_{\alpha\gamma,\beta}$ is homologous to the chain $Z_{\alpha,\beta}+(\gamma, \beta)-(\beta, \gamma)$.*

PROOF. $\partial\{(\alpha, \beta, \gamma)-(\alpha, \gamma, \beta)-([\alpha, \beta], \beta\alpha, \gamma)-(\beta, \alpha, \gamma)\}=Z_{\alpha\gamma,\beta}-Z_{\alpha,\beta}-\{(\gamma, \beta)-(\beta, \gamma)\}$. ■

We define two-parameter families $\tilde{f}_{\alpha,\beta}$, $\tilde{g}_{\alpha,\beta}$ and $\tilde{h}_{\alpha,\beta}(\alpha>1, \beta>0)$ of piecewise linear homeomorphisms of $\boldsymbol{R}$ which commute with the translation $T_{l_{\alpha,\beta}}$ by $l_{\alpha,\beta}=(\alpha+1)(\alpha+\beta)$. To do so, we determine such homeomorphisms on an interval of the length $l_{\alpha,\beta}$.

$$\tilde{f}_{\alpha,\beta}(x)=\begin{cases}\alpha x & \text{if } x\in[-\alpha, 1]\\ \dfrac{\beta}{\alpha+\beta-1}x+\dfrac{(\alpha-1)(\alpha+\beta)}{\alpha+\beta-1} & \text{if } x\in[1, l_{\alpha,\beta}-\alpha]\end{cases}$$

$$\tilde{g}_{\alpha,\beta}(x)=\begin{cases}\alpha^{-1}x & \text{if } x\in[-\alpha(\alpha+\beta), 0]\\ \alpha x & \text{if } x\in[0, \alpha+\beta]\end{cases}$$

$$\tilde{h}_{\alpha,\beta}(x)=\begin{cases}\tilde{f}_{\alpha,\beta}(x) & \text{if } x\in[0, \alpha+\beta]\\ x & \text{if } x\in[\alpha+\beta, l_{\alpha,\beta}].\end{cases}$$

Let M_a: $\boldsymbol{R}\rightarrow\boldsymbol{R}$ be the multiplication map $M_a(x)=ax$ and p: $\tilde{G}_+^0\rightarrow G_+^0$ the universal covering projection. $\tilde{G}_+^0$ is the naturally identified with the group of all orientation preserving homeomorphisms of $\boldsymbol{R}$ which commute with the translation T_1. For any real numbers $\alpha>1$, $\beta>0$, we define homomorphisms $\phi_{\alpha,\beta,n}^{(\sigma)}$: $\pi_1(\Sigma_1')\rightarrow PL_+(S^1)$ $(\sigma=\pm1)$ by

$$\phi_{\alpha,\beta,n}^{(1)}(a)=\phi_{\alpha,\beta,n}^{(-1)}(b)=p(M_{l_{\alpha,\beta}}^{-1}\circ T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}\circ(\tilde{h}_{\alpha,\beta})^n\circ M_{l_{\alpha,\beta}}),$$

$$\phi_{\alpha,\beta,n}^{(1)}(b)=\phi_{\alpha,\beta,n}^{(-1)}(a)=p(M_{l_{\alpha,\beta}}^{-1}\circ\tilde{f}_{\alpha,\beta}\circ M_{l_{\alpha,\beta}}).$$

Since $\pi_1(\Sigma_1')$ is the free group of rank 2, any homomorphism ϕ: $\pi_1(\Sigma_1')\rightarrow G_+^0$ has a lift $\tilde{\phi}$ of ϕ. That is, there is a homomorphism $\tilde{\phi}$: $\pi_1(\Sigma_1')\rightarrow\tilde{G}_+^0$ such that $p\circ\tilde{\phi}=\phi$. For any lift $\tilde{\phi}_{\alpha,\beta,n}^{(\sigma)}$ of $\phi_{\alpha,\beta,n}^{(\sigma)}$, we have

$$\begin{aligned}[\tilde{\phi}_{\alpha,\beta,n}^{(\sigma)}(a), \tilde{\phi}_{\alpha,\beta,n}^{(\sigma)}(b)]&=(M_{l_{\alpha,\beta}}^{-1}\circ[T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}\circ(\tilde{h}_{\alpha,\beta})^n, \tilde{f}_{\alpha,\beta}]\circ M_{l_{\alpha,\beta}})^\sigma\\ &=(M_{l_{\alpha,\beta}}^{-1}\circ[T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}, \tilde{f}_{\alpha,\beta}]\circ M_{l_{\alpha,\beta}})^\sigma\\ &=(T_{((\alpha-1)^2/l_{\alpha,\beta})})^\sigma=T_{\sigma((\alpha-1)^2/l_{\alpha,\beta})}\ (\sigma=\pm1).\end{aligned}$$

For a real number $\lambda(0\leq\lambda<1)$, put $\alpha=\alpha(\beta,\lambda)=\dfrac{\beta\lambda+\lambda+2+\sqrt{D}}{2(1-\lambda)}$, where $D=(\beta\lambda+\lambda+2)^2+4(1-\lambda)(\beta\lambda-1)$, then we have $\dfrac{(\alpha-1)^2}{l_{\alpha,\beta}}=\lambda$.

Given an integer $s\geq 2$, there exists a $2s-1$ fold covering map π_s: $\Sigma'_s\to\Sigma'_1$ satisfying the following properties, where Σ'_s is a one punctured orientable surface of genus s. We can take generators $a_1, b_1, \cdots, a_s, b_s$ of $\pi_1(\Sigma'_s)$ such that $[a_1, b_1]\# \cdots \#[a_s, b_s]$ is represented by the oriented boundary loop ∂_s and $(\pi_s)_\#(\partial_s)=[a, b]^{2s-1}$. We introduced an orientation of Σ'_s such that π_s is an orientation preserving map. That is, $(\pi_s)_*([\Sigma'_s, \partial\Sigma'_s])=(2s-1)[\Sigma'_1, \partial\Sigma'_1]$. By the consideration above and Milnor's algorithm to calculate the Euler number, for any integers $s\geq 2$ and $0<m\leq 2s-2$, the families $\{\phi^{(\sigma)}_{\alpha(\beta,m/2s-1)\beta,n}\circ(\pi_s)_\#\}_{\beta>0,n\in Z}$ naturally induce families $\{\psi^{(\sigma)}_{\beta,n}$: $\pi_1(\Sigma_s)\to PL_+(S^1)\}_{\beta>0,n\in Z}$ such that $eu(\psi^{(\sigma)}_{\beta,n})=\sigma m(\sigma=\pm 1)$.

3. The discrete Godbillon-Vey invariant of $\psi^{(\sigma)}_{\beta,n}$.

The discrete Godbillion-Vey invariant $\overline{GV}$ is the 2-cocycle of $PL_+(S^1)$

$$\overline{GV}(f, g)=\frac{1}{2}\sum_{x\in S^1}\begin{vmatrix}\log g'(x) & \log(f\circ g)'(x)\\ \Delta(\log g')(x) & \Delta(\log(f\circ g)')(x)\end{vmatrix},$$

where $g'(x)$ is the right derivative at x and $\Delta(h)(x)=h(x+0)-h(x-0)$.

PROPOSITION 3.1. *Let $\tilde{f}, \tilde{g}$ be lifts of $f, g\in PL_+(S^1)$. We have*

$$\begin{aligned}\overline{GV}(f, g)&=\frac{1}{2}\sum_{0\leq x<1}\begin{vmatrix}\log\tilde{g}'(x) & \log(\tilde{f}\circ\tilde{g})'(x)\\ \Delta(\log\tilde{g}')(x) & \Delta(\log(\tilde{f}\circ\tilde{g})')(x)\end{vmatrix}\\ &=\frac{1}{2}\sum_{0\leq x<\alpha}\begin{vmatrix}\log(M_\alpha\circ\tilde{g}\circ M_\alpha^{-1})'(x) & \log(M_\alpha\circ\tilde{f}\circ\tilde{g}\circ M_\alpha^{-1})'(x)\\ \Delta(\log(M_\alpha\circ\tilde{g}\circ M_\alpha^{-1})')(x) & \Delta(\log((M_\alpha\circ\tilde{f}\circ\tilde{g}\circ M_\alpha^{-1})')(x)\end{vmatrix}.\end{aligned}$$

This proposition means that the discrete Godbillion-Vey invariant does not depend on the choice of a length of S^1.

PROPOSITION 3.2. *If either f or g is an element of $SO(2)$, i.e., a rotation of S^1, then $\overline{GV}(f, g)=0$.*

So $\overline{GV}$ is a cocycle of a pair of groups $(PL_+(S^1), SO(2))$. On the other hand, we have the following commutative diagram between the second integral homologies of pairs of groups.

$$\begin{array}{ccc}
H_2(\Sigma'_s, \partial\Sigma'_s) & \xrightarrow{(\phi^{(\sigma)}_{\alpha(\beta,m/2s-1),\beta,n})_*(\pi_s)_*} & H_2(PL_+(S^1), SO(2)) \\
\pi_* \downarrow & & \uparrow \iota_* \\
H_2(\Sigma_s) & \xrightarrow{(\psi^{(\sigma)}_{\beta,n})_*} & H_2(PL_+(S^1)).
\end{array}$$

Here, π: $\Sigma'_s \to \Sigma_s \cong \Sigma'_s/\partial\Sigma'_s$ is the natural quotient map and ι: $(PL_+(S^1), \varnothing) \to (PL_+(S^1), SO(2))$ is the natural inclusion. Since Σ_s (resp. Σ'_s) is a $K(\pi_1(\Sigma_s), 1)$ (resp. $K(\pi_1(\Sigma'_s), 1)$) space, the second homology $H_2(\Sigma_s)$ (resp. $H_2(\Sigma'_s, \partial\Sigma'_s)$) is naturally isomorphic to $H_2(\pi_1(\Sigma_s))$ (resp. $H_2(\pi_1(\Sigma'_s), \langle\partial_s\rangle)$). So we have

$$\begin{aligned}
\overline{GV}(\psi^{(\sigma)}_{\beta,n}) &= \overline{GV}((\psi^{(\sigma)}_{\beta,n})_*[\Sigma_s]) = \overline{GV}(\iota_*(\psi^{(\sigma)}_{\beta,n})_*\pi_*[\Sigma'_s, \partial\Sigma'_s]) \\
&= (2s-1)\overline{GV}((\phi^{(\sigma)}_{\alpha(\beta,m/2s-1),\beta,n})_*[\Sigma'_1, \partial\Sigma'_1]) \\
&= ((2s-1)\overline{GV}(Z_{\phi^{(\sigma)}_{\alpha(\beta,m/2s-1),\beta,n}(a),\, \phi^{(\sigma)}_{\alpha(\beta,m/2s-1),\beta,n}(b)}).
\end{aligned}$$

Since

$$\begin{aligned}
& Z_{\phi^{(1)}_{\alpha,\beta,n}(a),\, \phi^{(1)}_{\alpha,\beta,n}(b)} + Z_{\phi^{(-1)}_{\alpha,\beta,n}(a),\, \phi^{(-1)}_{\alpha,\beta,n}(b)} \\
&\quad = Z_{\phi^{(1)}_{\alpha,\beta,n}(a),\, \phi^{(1)}_{\alpha,\beta,n}(b)} + Z_{\phi^{(1)}_{\alpha,\beta,n}(b),\, \phi^{(1)}_{\alpha,\beta,n}(a)} \\
&\quad = -([\phi^{(1)}_{\alpha,\beta,n}(a), \phi^{(1)}_{\alpha,\beta,n}(b)], \phi^{(1)}_{\alpha,\beta,n}(b)\phi^{(1)}_{\alpha,\beta,n}(a)) \\
&\qquad -([\phi^{(1)}_{\alpha,\beta,n}(b), \phi^{(1)}_{\alpha,\beta,n}(a)], \phi^{(1)}_{\alpha,\beta,n}(a)\phi^{(1)}_{\alpha,\beta,n}(b)) \\
&\quad = -(p(T_{-(m/2s-1)}), \phi^{(1)}_{\alpha,\beta,n}(b)\phi^{(1)}_{\alpha,\beta,n}(a) - (p(T_{(m/2s-1)}), \phi^{(1)}_{\alpha,\beta,n}(a)\phi^{(1)}_{\alpha,\beta,n}(b)),
\end{aligned}$$

$\overline{GV}(\psi^{(-1)}_{\beta,n}) = -\overline{GV}(\psi^{(1)}_{\beta,n})$ by virtue of Lemma 2.1. Hence, it is sufficient to consider $\overline{GV}(\psi^{(1)}_{\beta,n})$, i.e., the case where $m>0$.

In the rest of this section, we suppose that $S^1 = \boldsymbol{R}/l_{\alpha,\beta}\boldsymbol{Z}$. By Lemma 2.1 we have

$$\begin{aligned}
\overline{GV}(\psi^{(1)}_{\beta,n}) &= (2s-1)\overline{GV}(Z_{R^{-1}_{\alpha-1}\circ g_{\alpha,\beta}\circ(h_{\alpha,\beta})^n,\, f_{\alpha,\beta}}) \\
&= (2s-1)\overline{GV}(Z_{R^{-1}_{\alpha-1}\circ g_{\alpha,\beta},\, f_{\alpha,\beta}} + ((h_{\alpha,\beta})^n, f_{\alpha,\beta}) - (f_{\alpha,\beta}, (h_{\alpha,\beta})^n)) \\
&= (2s-1)\{\overline{GV}(R^{-1}_{\alpha-1}\circ g_{\alpha,\beta}, f_{\alpha,\beta}) - \overline{GV}(f_{\alpha,\beta}, R^{-1}_{\alpha-1}\circ g_{\alpha,\beta}) \\
&\qquad + n\overline{GV}((h_{\alpha,\beta}, f_{\alpha,\beta}) - (f_{\alpha,\beta}, h_{\alpha,\beta}))\},
\end{aligned}$$

where $R_{\alpha-1}$, $f_{\alpha,\beta}$, $g_{\alpha,\beta}$ and $h_{\alpha,\beta}$ are the homeomorphisms of $S^1 = \boldsymbol{R}/l_{\alpha,\beta}\boldsymbol{Z}$ whose lifts are $T_{\alpha-1}$, $\tilde{f}_{\alpha,\beta}$, $\tilde{g}_{\alpha,\beta}$ and $\tilde{h}_{\alpha,\beta}$, respectively.

LEMMA 3.3.
(1) $\overline{GV}(R^{-1}_{\alpha-1}\circ g_{\alpha,\beta}, f_{\alpha,\beta}) = 2(\log\alpha)^2 + 2\log\alpha\log\alpha_0$
(2) $\overline{GV}(f_{\alpha,\beta}, R^{-1}_{\alpha-1}\circ g_{\alpha,\beta}) = -2(\log\alpha)^2 - 2\log\alpha\log\alpha_0$
(3) $\overline{GV}((h_{\alpha,\beta}, f_{\alpha,\beta}) - (f_{\alpha,\beta}, h_{\alpha,\beta})) = (\log\alpha)^2 - (\log\alpha_0)^2$.

Here $\alpha_0 = \dfrac{\alpha+\beta-1}{\beta}$.

PROOF. If we make the list of bending points, right derivatives and left derivatives at the bending points of the homeomorphisms $T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}$, $\tilde{f}_{\alpha,\beta}$, $\cdots$, etc., it is easy to calculate the value (1)~(3) of the discrete Godbillon-Vey cocycle. First we can show easily that

$$T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}(1)=1,$$
$$T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}(-\alpha)=-\alpha,$$
$$\tilde{f}_{\alpha,\beta}(0)=0$$

and

$$\tilde{f}_{\alpha,\beta}(\alpha+\beta)=\alpha+\beta.$$

By these facts, we see that the bending points of the homeomorphisms $T_{\alpha-1}\circ\tilde{g}_{\alpha,\beta}|_{[0,l_{\alpha,\beta})}$, $\tilde{f}_{\alpha,\beta}|_{[0,l_{\alpha,\beta})}$, $\cdots$, etc. are all contained in $\{0, 1, \alpha+\beta, l_{\alpha,\beta}-\alpha\}$. So we make the following table of such data (Table 3.1).

Table 3.1

	x
F	the right derivative of F at x
	the left derivative of F at x

	0	1	$\alpha+\beta$	$l_{\alpha,\beta}-\alpha$
$\tilde{f}_{\alpha,\beta}$	α	α_0^{-1}	α_0^{-1}	α
	α	α	α_0^{-1}	α_0^{-1}
$T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}\circ\tilde{f}_{\alpha,\beta}$	α^2	$\alpha\alpha_0^{-1}$	$\alpha^{-1}\alpha_0^{-1}$	1
	1	α_2	$\alpha\alpha_0^{-1}$	$\alpha^{-1}\alpha_0^{-1}$
$T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}$	α	α	α^{-1}	α^{-1}
	α^{-1}	α	α	α^{-1}
$\tilde{f}_{\alpha,\beta}\circ T_{\alpha-1}^{-1}\circ\tilde{g}_{\alpha,\beta}$	α^2	$\alpha\alpha_0^{-1}$	$\alpha^{-1}\alpha_0^{-1}$	1
	1	α^2	$\alpha\alpha_0^{-1}$	$\alpha^{-1}\alpha_0^{-1}$
$\tilde{h}_{\alpha,\beta}$	α	α_0^{-1}	1	1
	1	α	α_0^{-1}	1
$\tilde{f}_{\alpha,\beta}\circ\tilde{h}_{\alpha,\beta}^{-1}=\tilde{h}_{\alpha,\beta}^{-1}\circ\tilde{f}_{\alpha,\beta}$	1	1	α_0^{-1}	α
	α	1	1	α_0^{-1}

Now it is easy to see (1) and (2) by Table 3.1. Moreover it is well known that the 2-cycle $(h_{\alpha,\beta}, f_{\alpha,\beta})-(f_{\alpha,\beta}, h_{\alpha,\beta})$ is homologous to the 2-cycle $(h_{\alpha,\beta}, h_{\alpha,\beta}^{-1}\circ f_{\alpha,\beta})-(h_{\alpha,\beta}\circ f_{\alpha,\beta}, h_{\alpha,\beta})$ as an integral 2-chain of $PL_{+}(S^1)$. So it is easy to check (3). Hence, this lemma is proved. ■

COMPLETION OF THE PROOF OF THEOREM 1.1. By Lemma 3.3, we have

$$\overline{GV}(\phi_{\beta,n}^{(1)})=(2s-1)\{(n+4)(\log\alpha)^2+4\log\alpha\log\alpha_0-n(\log\alpha_0)^2\},$$

where $\alpha=\alpha\left(\beta, \frac{m}{2s-1}\right)$ and $\alpha_0=\frac{\alpha+\beta-1}{\beta}$. When the integers m and s are fixed, α and α_0 are positive continuous functions of $\beta>0$. They have the following limits.

$$\lim_{\beta\to 0}\alpha=\frac{\lambda+2+\sqrt{\lambda^2+8\lambda}}{2(1-\lambda)}, \quad \lim_{\beta\to 0}\alpha_0=\infty,$$

$$\lim_{\beta\to\infty}\alpha=\infty, \quad \lim_{\beta\to\infty}\alpha_0=\frac{1}{1-\lambda}.$$

If n is sufficiently large (for example $n>5s$), then $\overline{GV}$ $(\phi_{\beta,n}^{(1)})$ is a strictly monotone increasing function with respect to β. So, changing a parameter, we obtain a continuous family $\{\phi_t\colon \pi_1(\Sigma_s)\to PL_{+}(S^1)\}_{t\in R}$ satisfying that $eu(\phi_t)=m$ and $\overline{GV}(\phi_t)=t$. ■

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