

On the central limit theorem for the multiple point point range of random walk

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Abstract. Let $R_n^{(k)}$ be the number of the lattice points entered by a random walk in 5 or more dimensional integer lattice at least k times in the first n steps, where the random walk is defined by the sum of independent identically distributed random variables.

We prove the convergence of covariances, that is, for each $k, l \geq 1$, there exists $\lim_{n \rightarrow \infty} \text{Cov}(R_n^{(k)}, R_n^{(l)})/n$, and for a fixed integer $K \geq 1$, the random vector $(R_n^{(1)}, R_n^{(2)}, \dots, R_n^{(K)})$ obeys the central limit theorem.

§1. Introduction.

Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent identically distributed random variables defined on some probability space $(\Omega, \mathfrak{B}, P)$, which take values in d dimensional integer lattice \mathbb{Z}^d . Define $S_0 = 0$ and $S_n = \sum_{k=1}^n X_k$. Then $\{S_n\}_{n=0}^\infty$ is called a random walk. If $P(X_1 = e) = 1/2d$, where e is the unit vector in \mathbb{Z}^d , we call $\{S_n\}$ the simple random walk. The random walk may take place on some proper subgroup of \mathbb{Z}^d which is isomorphic to \mathbb{Z}^l for $l \leq d$. If $l < d$, then we may consider the random walk is in \mathbb{Z}^l from the beginning. The least such integer is called the genuine dimension of the random walk. Throughout this paper, we assume the genuine dimension coincides with the dimension of the lattice.

The random walk is called transient if $p > 0$ and recurrent otherwise, where $p = P(S_j \neq 0, j = 1, 2, \dots)$. An equivalent criterion for transience in $\sum_{n=1}^\infty P(S_n = 0) < \infty$. If $d \geq 3$, then the random walk is always transient.

The range of a random walk, denoted by R_n , means the number of distinct sites visited by the random walk in the first n steps. In 1951, Dvoretzky and Erdős [1] proved the strong law of large number for the range of the simple random walk in \mathbb{Z}^d for $d \geq 2$. Kesten, Spitzer, and Whitman proved for any random walk R_n/n converges to p almost surely by using the individual ergodic theorem (unpublished; c.f. [8]). From

these results, some questions arise naturally.

- (i) Is the central limit theorem obeyed?
- (ii) What about the law of the iterated logarithm and the large deviation?

These questions about the asymptotic behavior of R_n were almost solved. In 1971, Jain and Pruitt [4] proved that if $d \geq 4$ and $p < 1$, $\text{Var} R_n = \sigma^2 n + o(n)$ ($\sigma^2 > 0$) and R_n obeys the central limit theorem by using the Green function. Moreover, one year later, they established the law of the iterated logarithm [5] by employing a similar technique. If $p = 1$, it is not interesting because $R_n = n$ almost surely. In 1974, they showed that if $d = 3$ and $p < 1$, $\text{Var} R_n = n\phi(n) + o(n\phi(n))$ (ϕ is a nondecreasing slowly varying function) and the central limit theorem [6]. In 2 dimensional case, Jain and Pruitt¹⁾ calculated the positivity of the limit variance under a certain condition and in the same situation Le Gall²⁾ studied the fluctuation. However, the limit distribution of $(R_n - ER_n)/\sqrt{\text{Var} R_n}$ is not Gaussian. On the large deviation, Donsker and Varadhan³⁾ derived a partial result to obtain the rate function. However, we will leave it out in this paper.

Then our interest will turn to the more precise results. Let $R_n^{(k)}$ be the number of points entered at least k times up to time n and $Q_n^{(k)}$ be the number of points with multiplicity k up to time n . Each $R_n^{(k)}$ or $Q_n^{(k)}$ is called the multiple point range of the random walk. The studies of these numbers were begun in 1960. Erdős and Taylor [2] referred to the strong law of large number for the multiple point range of the simple random walk, but proved only in the case of $Q_n^{(1)}$ for $d = 2$. The proof in general was given by Pitt [7] in 1974. He showed that $R_n^{(k)}/n \rightarrow p(1-p)^{k-1}$ a.s. and $Q_n^{(k)}/n \rightarrow p^2(1-p)^{k-1}$ a.s. for a transient random walk in a discrete abelian group by employing the individual ergodic theorem.

Our object in this paper is to prove the central limit theorem for the multiple point range of the random walk (not necessarily simple random walk) with genuine dimension d .

In section 3, the convergence of the covariances is shown in 5 or more dimensions. Although Jain and Pruitt [4] gave the limit variance of R_n , which is equal to $R_n^{(1)}$, in a compact form, it is so complicated to

1) The range of random walk, Sixth Berkeley Symp. Vol. 3 (1973).

2) Propriétés d'intersection des marches aléatoires, Comm. Math. Phys. **104** (1986).

3) On the number of distinct sites visited by a random walk, Comm. Pure and Appl. Math. **17** (1979).

derive the precise forms of the limit covariances of the multiple point ranges by using the same method as of theirs. We have, instead, put a certain subadditivity to good use and obtained that $Cov(R_n^{(k)}, R_n^{(l)})/n$ and $Cov(Q_n^{(k)}, Q_n^{(l)})/n$ converge to some constants respectively.

In section 4, the central limit theorem is proved for $d \geq 5$. The idea to prove this theorem is the same as that of Jain and Pruitt [4]. However, the estimate of the negligible term is more troublesome than that of Jain and Pruitt, although it is quite similar to the estimate in Section 3. Since we have not proved the convergence of the covariances of $R_n^{(k)}$ and $Q_n^{(k)}$ in 4 dimension in general, we can not refer to the central limit theorem.

Section 5 is devoted to the study of special cases. To establish the central limit theorem, the convergence of the variances is crucial. If the dimension $d \geq 4$, assuming $p < 1$, we can prove that

$$\begin{aligned} Var Q_n^{(1)} &= \theta_{(1)}^2 n + o(n), \\ Var R_n^{(2)} &= \sigma_{(2)}^2 n + o(n), \end{aligned}$$

and also their central limit theorems. We have $\theta_{(1)}^2 > 0$ always, however, the positivity of $\sigma_{(2)}^2$ is shown only in case $p \leq \frac{1}{2}$ or $\frac{2}{3} \leq p < 1$.

§2. Notations and preliminary results.

In this section we will give some notations and some lemmas used in [4], which are the main tool to estimate various probabilities. We will often regard the random walk as a Markov chain and use the terminology of general Markov chains. For $x \in \mathbb{Z}^d$, let $\{S_n\}_{n=0}^\infty$ be the random walk starting at x (i.e. $S_0 = x$, $S_n = x + \sum_{k=0}^{n-1} X_k$). The notation $P_x(\cdot)$ will be used to denote the probability measures of events related to this random walk, and when $x=0$, we will usually use $P(\cdot)$ instead of $P_0(\cdot)$. For $n \geq 0$ and $x, y \in \mathbb{Z}^d$, the notation $p^n(x, y)$, which we call the transition probability from x to y , means $P_x(S_n = y)$ and this is equal to $P(S_n = y - x)$ since the random walk is the sum of independent identically distributed random variables. The following lemma is essential.

LEMMA 2.1. *If $p < 1$ and d is the genuine dimension of the random walk, for $n \geq 1$ and $x \in \mathbb{Z}^d$, there is a positive constant A which is independent of x such that*

$$p^n(0, x) \leq A n^{-(1/2)^d}.$$

This is the best estimate since $p^n(0, 0) = 2(d/4n\pi)^{(1/2)^d} + o(n^{-(1/2)^d})$ when $\{S_n\}$ is the simple random walk. Let

$$G(x, y) = \sum_{n=0}^{\infty} p^n(x, y).$$

The two point function G is called the Green function. For a transient random walk, $G(x, y) \leq G(0, 0) < \infty$. By Lemma 2.1., we obtain useful bounds.

LEMMA 2.2. *If $p < 1$, then*

$$\begin{aligned} \sum_x p^n(0, x) \{G(u, x) + G(x, u)\} &\leq C n^{1-(1/2)^d} & \text{if } d \geq 3 \\ \sum_x p^n(0, x) G(u, x) G(x, v) &\leq \begin{cases} C n^{-(1/2)^d} & \text{if } d \geq 5 \\ C n^{-2} \log n & \text{if } d = 4 \end{cases} \end{aligned}$$

uniformly for $u, v \in \mathbb{Z}^d$.

For $x \in \mathbb{Z}^d$, τ_x will denote the first hitting time of x , i.e.

$$\tau_x = \inf\{n \geq 1; S_n = x\}.$$

If there are no positive integers with $S_n = x$, then $\tau_x = \infty$. The notation $p_z^n(x, y)$ (resp. $p_{zw}^n(x, y)$) means the taboo probability, i.e.

$$\begin{aligned} p_z^n(x, y) &= P_x(S_n = y, \tau_z \geq n) & \text{for } n \geq 1 \\ (\text{resp. } p_{zw}^n(x, y) &= P_x(S_n = y, \tau_z, \tau_w \geq n) & \text{for } n \geq 1). \end{aligned}$$

By Lemma 2.1., $P_x(n < \tau_y < \infty) = \sum_{i=n+1}^{\infty} p_y^i(x, y) \leq C n^{1-(1/2)^d}$ for all $x, y \in \mathbb{Z}^d$, which implies

LEMMA 2.3. *If $p < 1$ and $d \geq 3$, then*

$$P_x(n < \tau_x < \infty, \tau_0 < \infty) \leq C n^{1-(1/2)^d} \{G(0, x) + G(x, 0)\}$$

for $x \in \mathbb{Z}^d$.

A simple calculation shows

LEMMA 2.4. *For $x \neq 0$, let*

$$b(x) = P_x(\tau_0 < \infty) P_x(\tau_x = \infty) - P_x(\tau_0 < \infty, \tau_x = \infty).$$

Then

$$b(x) = P_x(\tau_0 < \tau_x < \infty) P_x(\tau_0 = \infty).$$

The last lemma in this section is

LEMMA 2.5. *If $p < 1$, then*

$$\sum_{l=1}^m p_{0x}^l(0, x) - p \sum_{l=1}^m p_x^l(0, x) = \sum_{l=1}^m p_x^l(0, x) \{P_0(m-l < \tau_0 < \infty) + P_0(\tau_x < \tau_0 \leq m-l)\}$$

for $x \neq 0$.

§3. The estimates of the covariances.

We first define some indicator random variables to show the existence of $\lim_{n \rightarrow \infty} \text{Cov}(R_n^{(k)}, R_n^{(l)})/n$ and $\lim_{n \rightarrow \infty} \text{Cov}(Q_n^{(k)}, Q_n^{(l)})/n$. The first step is to approximate $R_n^{(k)}$ by a partial sum of a stationary sequence.

Let $k \geq 2$. For $0 < j < l < n$,

$$Z_{j,l}^n(k) = \begin{cases} 1 & \text{if there are exactly } (k-2) \text{ indices } \alpha \text{ in} \\ & \{j+1, j+2, \dots, l-1\} \text{ such that } S_\alpha = S_j, \\ & S_l = S_j, \text{ and } S_{\alpha'} \neq S_j \text{ } \alpha' = l+1, \dots, n, \\ 0 & \text{otherwise,} \end{cases}$$

and for $0 < j < n$,

$$Z_{j,n}^n(k) = \begin{cases} 1 & \text{if there are exactly } (k-2) \text{ indices } \alpha \text{ in} \\ & \{j+1, j+2, \dots, n-1\} \text{ such that } S_\alpha = S_j, \\ & \text{and } S_n = S_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$R_n^{(k)} = \sum_{j=1}^{n-1} Z_j^n(k),$$

where $Z_j^n(k) = \sum_{l=j+1}^n Z_{j,l}^n(k)$. For $0 < j < n$, $j < l < \infty$, and $k \geq 2$, set

$$Z_{j,l}(k) = \begin{cases} 1 & \text{if there are exactly } (k-2) \text{ indices } \alpha \text{ in} \\ & \{j+1, j+2, \dots, l-1\} \text{ such that } S_\alpha = S_j, \\ & S_l = S_j, \text{ and } S_{\alpha'} \neq S_j \text{ } \alpha' = l+1, l+2, \dots, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Y_n^{(k)} = \sum_{j=1}^n \sum_{l=j+1}^n Z_{j,l}(k).$$

Then

$$Q_n^{(k)} - Y_n^{(k)} = U_n(k) - V_n(k),$$

where $U_n(k) = \sum_{j=1}^{n-1} \sum_{l=j+1}^n \{Z_{j,l}^n(k) - Z_{j,l}(k)\}$ and $V_n(k) = \sum_{j=1}^{n-1} \sum_{l=n+1}^{\infty} Z_{j,l}(k)$. For the sake of convenience, we put $W_{j,l}^n(k) = Z_{j,l}^n(k) - Z_{j,l}(k) \geq 0$ and we omit k if we don't confuse, *i.e.*

$$W_{j,l}^n = W_{j,l}^n(k), \quad Z_{j,l} = Z_{j,l}(k), \quad \dots, \text{ and etc..}$$

The following two lemmas assure that $R_n^{(k)}$ is approximately $Y_n^{(k)}$.

LEMMA 3.1. *If $p < 1$, then*

$$E|U_n(k)|^2 = \begin{cases} O(1) & \text{if } d \geq 5 \\ O\{(\log n)^2\} & \text{if } d = 4 \end{cases}$$

for each $k \geq 2$.

$$\begin{aligned} \text{PROOF.} \quad E|U_n(k)|^2 &= \sum_{j=1}^{n-1} \sum_{l=j+1}^n EW_{j,l}^n \\ &\quad + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{l=j+1}^n EW_{j,l}^n W_{j,i}^n \\ &\quad + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{l=j+1}^n EW_{j,l}^n W_{i,j}^n \\ &\quad + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{l=j+1}^n \sum_{h=i+1}^{j-1} EW_{j,l}^n W_{i,h}^n \\ &\quad + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{l=j+1}^n \sum_{h=j+1}^{l-1} EW_{j,l}^n W_{i,h}^n \\ &\quad + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{l=j+1}^n \sum_{h=l+1}^n EW_{j,l}^n W_{i,h}^n \\ &= \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI}. \end{aligned}$$

We will employ the lemmas in section 2 and Markov property to estimate each term. For $0 < j < l \leq n$,

$$\begin{aligned} EW_{j,l}^n &\leq p^{l-j}(0, 0)P_0(n-l < \tau_0 < \infty) \\ &\leq C(l-j)^{-(1/2)d}(n-l+1)^{1-(1/2)d}. \end{aligned}$$

Hence

$$I \leq \begin{cases} C & \text{if } d \geq 5 \\ C \log n & \text{if } d = 4. \end{cases}$$

For $0 < i < j < l < n$,

$$\begin{aligned} EW_{j,l}^n W_{i,l}^n &\leq p^{j-i}(0,0) p^{l-j}(0,0) P_0(n-l < \tau_0 < \infty) \\ &\leq C(j-i)^{-(1/2)d} (l-j)^{-(1/2)d} (n-l+1)^{1-(1/2)d} \end{aligned}$$

and

$$EW_{j,l}^n W_{i,j}^n = 0.$$

Therefore the second term has the same bound as the first term and the third term can be neglected. Since, for $0 < i < h < j < l \leq n$,

$$\begin{aligned} EW_{j,l}^n W_{i,h}^n &\leq \sum_{x \neq 0} p^{h-i}(0,0) p^{j-h}(0,x) p^{l-j}(x,x) P_0(n-l < \tau_0, \tau_x < \infty) \\ &\leq C(h-i)^{-(1/2)d} (l-j)^{-(1/2)d} (n-l+1)^{1-(1/2)d} \\ &\quad \times \sum_x p^{j-h}(0,x) \{G(0,x) + G(x,0)\} \\ &\leq C(h-i)^{-(1/2)d} (l-j)^{-(1/2)d} (n-l+1)^{1-(1/2)d} (j-h)^{1-(1/2)d}, \end{aligned}$$

we obtain that

$$IV \leq \begin{cases} C & \text{if } d \geq 5 \\ C(\log n)^2 & \text{if } d = 4. \end{cases}$$

The remainder terms can be estimated in the same manner.

$$V, VI \leq \begin{cases} C & \text{if } d \geq 5 \\ C(\log n)^2 & \text{if } d = 4 \end{cases}$$

Therefore we can derive the conclusion.

q.e.d.

LEMMA 3.2. If $p < 1$, then

$$E|V_n(k)|^2 = \begin{cases} O(1) & \text{if } d \geq 5 \\ O\{(\log n)^2\} & \text{if } d = 4 \end{cases}$$

for each $k \geq 2$.

PROOF.
$$E|V_n(k)|^2 = \sum_{j=1}^{n-1} \sum_{l=n+1}^{\infty} EZ_{j,l}$$

$$\begin{aligned}
& + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{l=n+1}^{\infty} EZ_{j,l} Z_{i,l} \\
& + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{h=n+1}^{\infty} \sum_{l=h+1}^{\infty} EZ_{j,l} Z_{i,h} \\
& + 2 \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{l=n+1}^{\infty} \sum_{h=l+1}^{\infty} EZ_{j,l} Z_{i,h} \\
& = \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned}$$

The first two terms can be computed in the same way as Lemma 3.1., and we have

$$\text{I}, \text{II} \leq \begin{cases} C & \text{if } d \geq 5 \\ C \log n & \text{if } d = 4. \end{cases}$$

On the other hand, for $0 < i < j \leq n < h < l < \infty$,

$$\begin{aligned}
EZ_{j,l} Z_{i,h} & \leq \sum_x p^{j-i}(0, x) p^{h-j}(x, 0) p^{l-h}(0, x) \\
& \leq A(h-j)^{-(1/2)d} \sum_x p^{j-i}(0, x) p^{l-h}(0, x).
\end{aligned}$$

Hence

$$\begin{aligned}
\text{III} & \leq 2A \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} \sum_{h=n+1}^{\infty} (h-j)^{-(1/2)d} \sum_x p^{j-i}(0, x) G(0, x) \\
& \leq C \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} (n-j)^{1-(1/2)d} (j-i)^{1-(1/2)d} \\
& \leq \begin{cases} C & \text{if } d \geq 5 \\ C(\log n)^2 & \text{if } d = 4. \end{cases}
\end{aligned}$$

Similarly we have

$$\text{IV} \leq \begin{cases} C & \text{if } d \geq 5 \\ C(\log n)^2 & \text{if } d = 4. \end{cases}$$

Then we conclude the assertion.

q.e.d.

For $n \geq 1$ and $0 < j < n$, let

$$\begin{aligned}
Z_j^*(1) & = \begin{cases} 1 & \text{if } S_j \neq S_\alpha \quad j < \alpha \leq n, \\ 0 & \text{otherwise,} \end{cases} \\
Z_j(1) & = \begin{cases} 1 & \text{if } S_j \neq S_\alpha \quad j < \alpha < \infty, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

$$Z_n^n(1) = 1.$$

We can express $R_n^{(1)} = \sum_{j=1}^n Z_j^n(1)$. The estimate

$$E|R_n^{(1)} - Y_n^{(1)}|^2 = \begin{cases} O(1) & \text{if } d \geq 5 \\ O\{(\log n)^2\} & \text{if } d = 4 \end{cases}$$

was shown, where $Y_n^{(1)} = \sum_{j=1}^n Z_j(1)$. (c.f. [4] Theorem 1)

Summing up the above lemmas, we have

LEMMA 3.3. *If $p < 1$, then*

$$E|R_n^{(k)} - Y_n^{(k)}|^2 = \begin{cases} O(1) & \text{if } d \geq 5 \\ O\{(\log n)^2\} & \text{if } d = 4 \end{cases}$$

for each $k \geq 1$.

In the second step we try to approximate $Y_n^{(k)}$ by a sum of *i.i.d.* random variables which was an important method in [4].

Let $R^{(k)}(a, b)$ be the number of points with multiplicity k from time a up to time b , that is, $R^{(k)}(a, b) = \sum_{j=a}^{b-1} \sum_{l=j+1}^b Z_{j,l}^b(k)$ if $k \geq 2$, and $R^{(1)}(a, b) = \sum_{j=a}^b Z_j^b(1)$ if $k = 1$. In the case that $k = 1$, the bound

$$\text{Var}\left(\sum_{j=1}^N R^{(1)}((j-1)n+1, jn) - Y_{Nn}^{(1)}\right) = \begin{cases} O(N) & \text{if } d \geq 5 \\ O(N \log n \log Nn) & \text{if } d = 4 \end{cases}$$

can be obtained along the same line as Theorem 3 in [4]. In what follows, we extend the above estimate to a general $k \geq 2$. For $N, n \geq 1$ and $k \geq 2$, we define

$$L_n^N(k) = \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{l=r+1}^{jn} W_{r,l}^{jn}(k)$$

and

$$M_n^N(k) = \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{l=jn+1}^{\infty} Z_{r,l}(k).$$

Then

$$\sum_{j=1}^N R^{(k)}((j-1)n+1, jn) - Y_{Nn}^{(k)} = L_n^N(k) - M_n^N(k).$$

The bound of $L_n^N(k)$ is obtained analogously to the proof for $k = 1$ in [4]. That is

LEMMA 3.4. *If $p < 1$, then*

$$\text{Var} L_n^N(k) = \begin{cases} O(N) & \text{if } d \geq 5 \\ O(N \log n \log Nn) & \text{if } d = 4 \end{cases}$$

for each $k \geq 2$.

PROOF.

$$\begin{aligned} & \text{Var} L_n^N(k) \\ &= \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{s=(j-1)n+1}^{jn} \sum_{l=r+1}^{jn} \sum_{m=s+1}^{jn} \text{Cov}(W_{r,l}^{jn}, W_{s,m}^{jn}) \\ & \quad + 2 \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{s=(i-1)n+1}^{in} \sum_{l=r+1}^{jn} \sum_{m=s+1}^{in} \text{Cov}(W_{r,l}^{jn}, W_{s,m}^{in}) \\ &= \text{I} + \text{II}. \end{aligned}$$

Generally if $X \geq 0$, $Y \geq 0$, then $\text{Cov}(X, Y) \leq EXY$. Replacing each term in I with $EW_{r,l}^{jn} W_{s,m}^{jn}$, we can estimate I in the same fashion as Lemma 3.1., which yields

$$\text{I} \leq \begin{cases} CN & \text{if } d \geq 5 \\ CN(\log n)^2 & \text{if } d = 4. \end{cases}$$

We will estimate the second term and change the order to summing on s and m . For $0 < a < b \leq c < d$ and $k \geq 2$, we introduce

$$W_{a,b}^{c,d}(k) = Z_{a,b}^d(k) - Z_{a,b}^c(k).$$

Since, for $0 < (i-1)n < s < m \leq in \leq (j-1)n < r < l \leq jn$, $W_{s,m}^{in}(k) = W_{s,m}^{in,r}(k) + W_{s,m}^r(k)$ and $W_{r,l}^{jn}(k)$ and $W_{s,m}^{in,r}(k)$ are independent, it holds

$$\begin{aligned} \text{Cov}(W_{r,l}^{jn}, W_{s,m}^{in}) &= \text{Cov}(W_{r,l}^{jn}, W_{s,m}^r) \\ &\leq EW_{r,l}^{jn} W_{s,m}^r \\ &= EW_{r,l}^{jn} W_{s,m}^{r,l} + EW_{r,l}^{jn} W_{s,m}^l. \end{aligned}$$

The first term can be computed in the following way.

$$\begin{aligned} EW_{r,l}^{jn} W_{s,m}^{r,l} &\leq \sum_{h=1}^{l-r} \sum_x p^{m-s}(0, 0) p^{r-m}(0, x) p^h(x, 0) p^{l-r-h}(0, x) P_x(jn-l < \tau_z < \infty) \\ &\leq C(m-s)^{-(1/2)d} (r-m)^{-(1/2)d} (jn-l+1)^{1-(1/2)d} \\ &\quad \times \sum_{h=1}^{l-r} \sum_x p^{l-r-h}(0, x) p^h(x, 0) \\ &\leq C(m-s)^{-(1/2)d} (r-m)^{-(1/2)d} (l-r)^{1-(1/2)d} (jn-l+1)^{1-(1/2)d} \end{aligned}$$

since $\sum_x p^a(0, x)p^b(x, 0) = p^{a+b}(0, 0)$ for $a, b \geq 1$. Then

$$\begin{aligned} & \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{m=(i-1)n+1}^{in} \sum_{l=r+1}^{jn} \sum_{s=(i-1)n+1}^{m-1} EW_{r,l}^{jn} W_{s,m}^l \\ & \leq C \sum_{j=1}^N \sum_{m=1}^{(j-1)n} \sum_{r=(j-1)n+1}^{jn} \sum_{l=r+1}^{jn} (r-m)^{-(1/2)d} (l-r)^{1-(1/2)d} (jn-l+1)^{1-(1/2)d} \\ & \leq \begin{cases} CN & \text{if } d \geq 5 \\ CN(\log n)^2 & \text{if } d = 4. \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} EW_{r,l}^{jn} W_{s,m}^l & \leq \sum_{x \neq 0} p^{m-s}(0, 0) p^{r-m}(0, x) p^{l-r}(x, x) P_x(\tau_0 < \infty, jn-l < \tau_x < \infty) \\ & \leq C(m-s)^{-(1/2)d} (l-r)^{-(1/2)d} (jn-l+1)^{1-(1/2)d} \\ & \quad \times \sum_{x \neq 0} p^{r-m}(0, x) \{G(0, x) + G(x, 0)\} \\ & \leq C(m-s)^{-(1/2)d} (r-m)^{1-(1/2)d} (l-r)^{-(1/2)d} (jn-l+1)^{1-(1/2)d}. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{m=(i-1)n+1}^{in} \sum_{l=r+1}^{jn} \sum_{s=(i-1)n+1}^{m-1} EW_{r,l}^{jn} W_{s,m}^l \\ & \leq C \sum_{j=1}^N \sum_{m=1}^{(j-1)n} \sum_{r=(j-1)n+1}^{jn} \sum_{l=r+1}^{jn} (r-m)^{1-(1/2)d} (l-r)^{-(1/2)d} (jn-l+1)^{1-(1/2)d} \\ & \leq C \sum_{j=1}^N \sum_{m=1}^{(j-1)n} \sum_{r=(j-1)n+1}^{jn} (r-m)^{1-(1/2)d} (jn-r+1)^{1-(1/2)d} \\ & \leq \begin{cases} CN & \text{if } d \geq 5 \\ CN \log n \log Nn & \text{if } d = 4. \end{cases} \end{aligned}$$

Accordingly we obtain the conclusion.

q.e.d.

LEMMA 3.5. If $p < 1$, then

$$\text{Var} M_n^N(k) = \begin{cases} O(N) & \text{if } d \geq 5 \\ O(N \log n \log Nn) & \text{if } d = 4 \end{cases}$$

for each $k \geq 2$.

PROOF.

$$\begin{aligned} & \text{Var} M_n^N(k) \\ & = \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{l=jn+1}^{\infty} \sum_{m=jn+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{r,m}) \end{aligned}$$

$$\begin{aligned}
& +2 \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{s=(j-1)n+1}^{r-1} \sum_{l=jn+1}^{\infty} \sum_{m=jn+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,m}) \\
& +2 \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{s=(i-1)n+1}^{in} \sum_{l=jn+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,r}) \\
& +2 \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{s=(i-1)n+1}^{in} \sum_{l=jn+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,l}) \\
& +2 \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{s=(i-1)n+1}^{in} \sum_{l=jn+1}^{\infty} \sum_{m=l+1}^{\infty} \text{Cov}(Z_{r,l}, Z_{s,m}) \\
& +2 \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{s=(i-1)n+1}^{in} \sum_{l=jn+1}^{\infty} \sum_{m=r+1}^{l-1} \text{Cov}(Z_{r,l}, Z_{s,m}) \\
& +2 \sum_{j=1}^N \sum_{i=1}^{j-1} \sum_{r=(j-1)n+1}^{jn} \sum_{s=(i-1)n+1}^{in} \sum_{l=jn+1}^{\infty} \sum_{m=in+1}^{r-1} \text{Cov}(Z_{r,l}, Z_{s,m}) \\
& = \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI} + \text{VII}.
\end{aligned}$$

The first four terms can be estimated in the same manner as above lemmas and each term has the bound $O(N)$ if $d \geq 5$ and $O(N \log n \log Nn)$ if $d=4$.

We will estimate the remainder terms. For $0 < s < r < l < m < \infty$,

$$\begin{aligned}
\text{Cov}(Z_{r,l}, Z_{s,m}) & \leq E Z_{r,l} Z_{s,m} \\
& \leq \sum_x p^{r-s}(0, x) p^{l-r}(x, x) p^{m-l}(x, 0) \\
& \leq 2A(l-r)^{-(1/2)d} (r-s+m-l)^{-(1/2)d}.
\end{aligned}$$

Then

$$\begin{aligned}
\text{V} & \leq C \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{s=1}^{(j-1)n} \sum_{l=jn+1}^{\infty} (l-r)^{-(1/2)d} (r-s)^{1-(1/2)d} \\
& \leq C \sum_{j=1}^N \sum_{s=1}^{(j-1)n} \sum_{r=(j-1)n+1}^{jn} (jn-r+1)^{1-(1/2)d} (r-s)^{1-(1/2)d} \\
& \leq \begin{cases} CN & \text{if } d \geq 5 \\ CN \log n \log Nn & \text{if } d = 4. \end{cases}
\end{aligned}$$

For $0 < s < r < m < l < \infty$,

$$\begin{aligned}
\text{Cov}(Z_{r,l}, Z_{s,m}) & \leq E Z_{r,l} Z_{s,m} \\
& \leq \sum_x p^{r-s}(0, x) p^{m-r}(x, 0) p^{l-m}(0, x).
\end{aligned}$$

Therefore we obtain the following bound by changing the order of summing on m and l in the sixth term.

$$\begin{aligned}
\text{VI} &\leq \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{s=1}^{(j-1)n} \sum_x p^{r-s}(0, x) G(x, 0) G(0, x) \\
&\leq \begin{cases} C \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{s=1}^{(j-1)n} (r-s)^{-(1/2)d} & \text{if } d \geq 5 \\ C \sum_{j=1}^N \sum_{r=(j-1)n+1}^{jn} \sum_{s=1}^{(j-1)n} (r-s)^{-2} \log(r-s) & \text{if } d=4 \end{cases} \\
&\leq \begin{cases} CN & \text{if } d \geq 5 \\ CN \log n \log Nn & \text{if } d=4. \end{cases}
\end{aligned}$$

The last term is more complicated. To estimate its bound, we introduce new notations:

$$\begin{aligned}
q^h(x) &= P_x(x \text{ is the lattice point entered exactly } (k-2) \text{ times} \\
&\quad \text{in the first } (h-1) \text{ steps and } S_h=x) \\
q_y^h(x) &= P_x(x \text{ is the lattice point entered exactly } (k-2) \text{ times} \\
&\quad \text{in the first } (h-1) \text{ steps, } S_h=x, \text{ and } \tau_y \geq h).
\end{aligned}$$

For $0 < s < m < r < l < \infty$,

$$\begin{aligned}
EZ_{r,l} Z_{s,m} &= \sum_{x \neq 0} q^{m-s}(0) p_0^{r-m}(0, x) q_0^{l-r}(x) P_x(\tau_0, \tau_x = \infty) \\
&\leq q^{m-s}(0) q^{l-r}(0) \sum_{x \neq 0} p_0^{r-m}(0, x) P_x(\tau_0, \tau_x = \infty) \\
EZ_{r,l} Z_{s,m} &= q^{m-s}(0) P_0(\tau_0 = \infty) q^{l-r}(0) P_0(\tau_0 = \infty) \\
&= q^{m-s}(0) q^{l-r}(0) \sum_{x \neq 0} p_0^{r-m}(0, x) P_x(\tau_0 = \infty) P_x(\tau_x = \infty)
\end{aligned}$$

since $P_x(\tau_x = \infty) = P_0(\tau_0 = \infty) = \sum_{x \neq 0} p_0^h(0, x) P_x(\tau_0 = \infty)$ for each $h \geq 1$ and $q^{l-r}(x) = q^{l-r}(0)$. By lemma 2.4.,

$$\begin{aligned}
\text{Cov}(Z_{r,l}, Z_{s,m}) &\leq q^{m-s}(0) q^{l-r}(0) \sum_{x \neq 0} p_0^{r-m}(0, x) b(x) \\
&\leq p^{m-s}(0, 0) p^{l-r}(0, 0) \sum_x p^{r-m}(0, x) G(0, x) G(x, 0) \\
&\leq \begin{cases} C(m-s)^{-(1/2)d} (l-r)^{-(1/2)d} (r-m)^{-(1/2)d} & \text{if } d \geq 5 \\ C(m-s)^{-2} (l-r)^{-2} (r-m)^{-2} \log(r-m) & \text{if } d=0. \end{cases}
\end{aligned}$$

Therefore if $d \geq 5$,

$$\begin{aligned}
\text{VII} &\leq C \sum_{j=1}^N \sum_{s=1}^{(j-1)n} \sum_{r=(j-1)n+1}^{jn} (jn-r+1)^{1-(1/2)d} (r-s)^{-(1/2)d} \\
&\leq CN
\end{aligned}$$

and if $d=4$,

$$\begin{aligned} \text{VII} &\leq C \sum_{j=1}^N \sum_{s=1}^{(j-1)n} \sum_{r=(j-1)n+1}^{jn} (jn-r+1)^{-1} (r-s)^{-2} \log(r-s) \\ &\leq CN \log n \log Nn. \end{aligned}$$

Consequently we can conclude the assertion.

q.e.d.

Hence the next lemma follows immediately.

LEMMA 3.6. *If $p > 1$, then it holds that for each $k \geq 1$,*

$$\text{Var} \left(\sum_{j=1}^N R^{(k)}((j-1)n+1, jn) - Y_{Nn}^{(k)} \right) = \begin{cases} O(N) & \text{if } d \geq 5 \\ O(N \log n \log Nn) & \text{if } d = 4. \end{cases}$$

Combining this with Lemma 3.3.,

$$\text{Var} \left(\sum_{j=1}^N R^{(k)}((j-1)n+1, jn) - R_{Nn}^{(k)} \right) = \begin{cases} O(N) & \text{if } d \geq 5 \\ O(N \log n \log Nn) & \text{if } d = 4 \end{cases}$$

is derived by Schwarz's inequality. From this estimate one can easily obtain the following lemma along the same line as Lemma 4.1. in [5].

LEMMA 3.7. *If $p < 1$ and $d \geq 4$, then for each $k \geq 1$,*

$$\text{Var} R_n^{(k)} = O(n).$$

Then we can state the theorem in this section.

THEOREM 3.8. *If $p < 1$ and $d \geq 5$, then there exist*

$$\lim_{n \rightarrow \infty} \frac{\text{Cov}(R_n^{(k)}, R_n^{(l)})}{n} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\text{Cov}(Q_n^{(k)}, Q_n^{(l)})}{n}$$

for each $k, l \geq 1$.

PROOF. For a positive integer K , and real vector $\xi = (\xi_j)_{j=1}^K$, let

$$R_n^\xi = \sum_{k=1}^K \xi_k R_n^{(k)}, \quad Y_n^\xi = \sum_{k=1}^K \xi_k Y_n^{(k)},$$

and

$$R^\xi(a, b) = \sum_{k=1}^K \xi_k R^{(k)}(a, b).$$

Define $q_m = \left\lfloor \frac{m}{n} \right\rfloor$ and $r_m = m - q_m n$ for $m > 2n$. From the independence of $\{R^{(k)}((j-1)n+1, jn)\}$, Lemma 3.3., and Lemma 3.6.,

$$\begin{aligned}
& |(q_m \text{Var} R_n^\xi)^{1/2} - (\text{Var} R_{q_m n}^\xi)^{1/2}| \\
& \leq \left[\text{Var} \left(\sum_{j=1}^{q_m} R^\xi((j-1)n+1, jn) - R_{q_m n}^\xi \right) \right]^{1/2} \\
& = O(q_m^{1/2}).
\end{aligned}$$

On the other hand, by Lemma 3.3. and Lemma 3.7.,

$$\begin{aligned}
|(\text{Var} R_{q_m n}^\xi)^{1/2} - (\text{Var} R_m^\xi)^{1/2}| & \leq |(\text{Var} Y_{q_m n}^\xi)^{1/2} - (\text{Var} Y_m^\xi)^{1/2}| + C \\
& \leq (\text{Var} Y_{r_m}^\xi)^{1/2} + C \\
& \leq (\text{Var} R_{r_m}^\xi)^{1/2} + C \\
& = O(r_m)^{1/2}.
\end{aligned}$$

Combining these estimates, we have

$$|(q_m \text{Var} R_n^\xi)^{1/2} - (\text{Var} R_m^\xi)^{1/2}| = O(q_m^{1/2} + r_m^{1/2}).$$

Then we obtain

$$\begin{aligned}
(\text{Var} R_m^\xi)^{1/2} & \leq (q_m \text{Var} R_n^\xi)^{1/2} + O(q_m^{1/2} + r_m^{1/2}) \\
& \leq \left(\frac{m}{n} \text{Var} R_n^\xi \right)^{1/2} + O(m^{1/2} n^{-1/2} + n^{1/2}).
\end{aligned}$$

Therefore

$$\left(\frac{\text{Var} R_m^\xi}{m} \right)^{1/2} \leq \left(\frac{\text{Var} R_n^\xi}{n} \right)^{1/2} + O(n^{-1/2} + n^{1/2} m^{-1/2}).$$

Taking $\limsup_{m \rightarrow \infty}$ at first and then $\liminf_{n \rightarrow \infty}$,

$$\limsup_{m \rightarrow \infty} \left(\frac{\text{Var} R_m^\xi}{m} \right)^{1/2} \leq \liminf_{n \rightarrow \infty} \left(\frac{\text{Var} R_n^\xi}{n} \right)^{1/2}.$$

Consequently $\lim_{n \rightarrow \infty} \text{Var} R_n^\xi / n$ exists.

Let $\sigma_n^{k,l} = \text{Cov}(R_n^{(k)}, R_n^{(l)})/n$ for each $k, l \geq 1$ and Σ_n be the $K \times K$ matrix whose (k, l) -component is $\sigma_n^{k,l}$. Then the above result implies that

$$\lim_{n \rightarrow \infty} \langle \Sigma_n \xi, \xi \rangle \text{ exists for each } \xi \in R^K,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on R^K . By employing the equality: $4\langle \Sigma_n \xi, \eta \rangle = \langle \Sigma_n(\xi + \eta), \xi + \eta \rangle - \langle \Sigma_n(\xi - \eta), \xi - \eta \rangle$, we can conclude

$$\lim_{n \rightarrow \infty} \langle \Sigma_n \xi, \eta \rangle \text{ exists for each } \xi, \eta \in R^K.$$

Hence $\sigma_n^{k,l}$ converges to some constant $\sigma^{k,l}$.

Since $Q_n^{(k)} = R_n^{(k)} - R_n^{(k+1)}$, we can prove that there is some constant $\pi^{k,l}$ such that $\lim_{n \rightarrow \infty} \text{Cov}(Q_n^{(k)}, Q_n^{(l)})/n = \pi^{k,l}$ in the same fashion. q.e.d.

Let

$$\Sigma = (\sigma^{k,l})_{1 \leq k, l \leq K}$$

and

$$\Pi = (\pi^{k,l})_{1 \leq k, l \leq K}.$$

The strict positive definiteness of these matrices is unknown.

§4. The central limit theorem.

Fix a positive integer K . We define two random vectors:

$$\begin{aligned}\Phi_n &= (R_n^{(1)}, R_n^{(2)}, \dots, R_n^{(K)}) \\ \Psi_n &= (Q_n^{(1)}, Q_n^{(2)}, \dots, Q_n^{(K)}).\end{aligned}$$

To show the central limit theorem for Φ_n and Ψ_n , we have only to check the Lindeberg condition for triangular arrays.

THEOREM 4.1. *If $p < 1$ and $d \geq 5$, then $(\Phi_n - E\Phi_n)/\sqrt{n}$ converges to the normal distribution with mean 0 and variance Σ in law and the distribution of $(\Psi_n - E\Psi_n)/\sqrt{n}$ coincides asymptotically with the normal distribution with mean 0 and variance Π .*

PROOF. It is sufficient to show the central limit theorem for R_n^ξ for arbitrarily fixed $\xi \in R^K$. By Lemma 3.3., $(R_n^\xi - Y_n^\xi)/\sqrt{n}$ converges in mean to zero. Thus we should show the central limit theorem for Y_n^ξ .

Let $m = [n^{1/3}]$, $A_j^\xi = R^\xi((j-1)m+1, jm)$ $j=1, 2, \dots, m^2$, and

$$I_m^\xi = \sum_{j=1}^{m^2} A_j^\xi - Y_m^\xi.$$

Since, by putting $n=m$ and $N=m^2$ in Lemma 3.6.,

$$\begin{aligned}P(|I_m^\xi - EI_m^\xi| > \varepsilon n^{1/2}) &\leq \varepsilon^{-2} n^{-1} \text{Var} I_m^\xi \\ &= O(n^{-1} m^2) \\ &= O(n^{-1/3}),\end{aligned}$$

$(I_m^\xi - EI_m^\xi)/\sqrt{n} \rightarrow 0$ in probability. $\{A_j^\xi\}$ is the sequence of independent identically distributed random variables with the same distribution as R_m^ξ . Hence

$$\begin{aligned} \text{Var}\left(\sum_{j=1}^{m^2} \frac{A_j^\xi}{\sqrt{n}}\right) &= \frac{m^2}{n} \text{Var} R_m^\xi \\ &= \langle \Sigma \xi, \xi \rangle + o(1) \end{aligned}$$

On the other hand, noting $0 \leq R_m^{(k)} \leq m$, we easily see

$$\begin{aligned} \sum_{j=1}^{m^2} E\left[\left(\frac{A_j^\xi - EA_j^\xi}{\sqrt{n}}\right)^2; |A_j^\xi - EA_j^\xi| \geq \varepsilon n^{1/2}\right] &\leq \frac{m^4}{n} \left(\sum_{k=1}^K |\xi_k|\right)^2 P(|A_j^\xi - EA_j^\xi| \geq \varepsilon n^{1/2}) \\ &= O(m^6 n^{-2}) \\ &= O(n^{-1/3}). \end{aligned}$$

Then $(Y_m^{\xi_3} - EY_m^{\xi_3})/\sqrt{n}$ converges in law to a normal random variable with mean 0 and variance $\langle \Sigma \xi, \xi \rangle$ since the Lindeberg condition is satisfied. Lastly we should show the difference between Y_n^ξ and $Y_m^{\xi_3}$ is negligible.

$$\begin{aligned} P(|(Y_n^\xi - EY_n^\xi) - (Y_m^{\xi_3} - EY_m^{\xi_3})|) &\geq \varepsilon^{-2} n^{-1} \text{Var} Y_{n-m^3}^\xi \\ &= O\left(\frac{n - m^3}{n}\right) \\ &= O(n^{-1/3}) \end{aligned}$$

since $n - m^3 < 3n^{2/3}$. Thus, this term converges to zero in probability. q.e.d.

REMARK 4.2. *If the variance of the multiple point range has order n for $d=4$, we can show the assertion of this theorem holds by arguing in parallel.*

§5. The positivity of the variances.

At first we will extend a random walk path toward the negative time. Let

$$(\tilde{\Omega}, \tilde{\mathfrak{B}}, \tilde{P}) = \prod_{j=-\infty}^{+\infty} (\Omega_j, \mathfrak{B}_j, P_j),$$

where $\Omega_j = Z^d$, \mathfrak{B}_j consists of all subsets of Z^d , and $P_j = P \circ S_1^{-1}$. Then

$$\tilde{\mathcal{Q}} = \{\omega | \omega = (\omega_j), \omega_j \in \mathbf{Z}^d, -\infty < j < +\infty\}.$$

The shift transformation on $\tilde{\mathcal{Q}}$, denoted by φ , is defined by $(\varphi\omega)_j = \omega_{j+1}$. For $\omega = (\omega_j)$ and $n \geq 1$, let

$$S_0(\omega) = 0, \quad S_n(\omega) = \sum_{j=1}^n \omega_j, \quad \text{and} \quad S_{-n}(\omega) = \sum_{j=0}^{n-1} \omega_{-j}.$$

The part of the negative time of this process is independent of the part of the positive time. $\tilde{P}|_{\cup_{j>0} \mathfrak{B}_j}$ coincides with the probability measure of the original random walk, so we regard the original measure P as the extended one \tilde{P} from the beginning.

In addition to some indicator random variables in section 3, we will define others. For $0 < j \leq n$, let

$$\begin{aligned} \tilde{Z}_j(1) &= \begin{cases} 1 & \text{if } S_j = S_\alpha \text{ for } 0 < \alpha < j, \\ 0 & \text{otherwise,} \end{cases} \\ \hat{Z}_j(1) &= \begin{cases} 1 & \text{if } S_j = S_\alpha \text{ for } \alpha < j, \\ 0 & \text{otherwise,} \end{cases} \\ W_j^n(1) &= Z_j^n(1) - Z_j(1), \\ W_j(1) &= \tilde{Z}_j(1) - \hat{Z}_j(1). \end{aligned}$$

Then Lemma 3.1. asserts that if $p < 1$,

$$E \left| \sum_{j=1}^n W_j^n(1) \right|^2 = \begin{cases} O(1) & \text{if } d \geq 5 \\ O\{(\log n)^2\} & \text{if } d = 4. \end{cases}$$

A similar result is valid also for $W_j(1)$.

LEMMA 5.1. *If $p < 1$, then*

$$E \left| \sum_{j=1}^n W_j(1) \right|^2 = \begin{cases} O(1) & \text{if } d \geq 5 \\ O\{(\log n)^2\} & \text{if } d = 4. \end{cases}$$

PROOF. We will estimate the L.H.S. similarly in the case of $\sum_{j=1}^n W_j^n(1)$.

$$E \left| \sum_{j=1}^n W_j(1) \right|^2 = \sum_{i=1}^n E W_i + 2 \sum_{i=1}^n \sum_{j=i+1}^n E W_i W_j.$$

For $0 < i \leq n$,

$$\begin{aligned} EW_i &= \sum_x p_0^i(0, x) P_x(\tau_0 < \infty) \\ &\leq C i^{1-(1/2)^d}. \end{aligned}$$

and for $0 < i < j \leq n$,

$$\begin{aligned} EW_i W_j &= EW_0 W_{j-i} \\ &\leq \sum_{x \neq 0} p_0^{j-i}(0, x) P_x(i \leq \tau_0, \tau_x < \infty) \\ &\leq C^{1-(1/2)^d} (j-i)^{1-(1/2)^d}. \end{aligned}$$

Accordingly

$$E \left| \sum_{j=1}^n W_j(1) \right|^2 = \begin{cases} O(1) & \text{if } d \geq 5 \\ O\{(\log n)^2\} & \text{if } d = 4. \end{cases}$$

q.e.d.

Let $\hat{Y}_n^{(1)} = \sum_{j=1}^n \hat{Z}_j(1)$. Since we have $R_n^{(1)} = \sum_{j=1}^n \tilde{Z}_j(1)$, this lemma implies

$$Var R_n^{(1)} = Var \hat{Y}_n^{(1)} + o(n).$$

For a later purpose, we express the $Var R_n^{(1)}$ in a different form from that of Jain-Pruitt [4].

LEMMA 5.2. *If $p < 1$ and $d \geq 4$, then*

$$Var R_n^{(1)} = \sum_{j=1}^n \sum_{i=1}^{j-1} \left\{ \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_0(\tau_x = \infty) P_x(\tau_0 = \infty) - p^2 \right\} + o(n).$$

PROOF. By using the estimates of $Y_n^{(1)}$, $\hat{Y}_n^{(1)}$, $\sum W_j^n(1)$, and $\sum W_j(1)$, we obtain

$$\begin{aligned} Var R_n^{(1)} &= Cov(Y_n^{(1)}, \hat{Y}_n^{(1)}) + o(n) \\ &= \sum_{i=1}^n \sum_{j=1}^n Cov(Z_i, \hat{Z}_j) + o(n) \end{aligned}$$

holds. Here we have used $Var Y_n^{(1)}, Var \hat{Y}_n^{(1)} = O(n)$ and $E|\sum W_i^n(1)|^2, E|\sum W_j(1)|^2 = O\{(\log n)^2\}$. If $i \leq j$, $\hat{Z}_i = \tilde{Z}_i(1)$ and $Z_j = Z_j(1)$ are independent, therefore

$$Var R_n^{(1)} = \sum_{j=1}^n \sum_{i=1}^{j-1} Cov(Z_i, \hat{Z}_j) + o(n).$$

For $0 < i < j \leq n$,

$$\begin{aligned} EZ_i \hat{Z}_j &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_0(\tau_x = \infty) P_x(\tau_0 = \infty) \\ EZ_i E \hat{Z}_j &= p^2. \end{aligned}$$

Hence we can conclude the assertion.

q.e.d.

For $x \neq 0$, let

$$\bar{b}(x) = P_0(\tau_x < \infty) P_0(\tau_0 = \infty) - P_0(\tau_x < \infty, \tau_0 = \infty).$$

From Lemma 2.4., we have

$$\bar{b}(x) = P_0(\tau_x < \tau_0 < \infty) P_0(\tau_x = \infty).$$

$b(x)$ and $\bar{b}(x)$ have the duality relation, that is

LEMMA 5.3.

$$b(x) P_0(\tau_x = \infty) = \bar{b}(x) P_x(\tau_0 = \infty) \quad \text{for } x \neq 0.$$

PROOF. By the proof of Lemma 5 in [4],

$$P_x(\tau_0 < \tau_x < \infty) = \frac{p P_0(\tau_x < \infty) P_x(\tau_0 < \infty)}{1 - P_0(\tau_x < \infty) P_x(\tau_0 < \infty)}$$

holds. Therefore $P_x(\tau_0 < \tau_x < \infty) = P_0(\tau_x < \tau_0 < \infty)$. Then

$$\begin{aligned} b(x) P_0(\tau_x = \infty) &= P_x(\tau_0 < \tau_x < \infty) P_x(\tau_0 = \infty) P_0(\tau_x = \infty) \\ &= P_0(\tau_x < \tau_0 < \infty) P_0(\tau_x = \infty) P_x(\tau_0 = \infty) \\ &= \bar{b}(x) P_x(\tau_0 = \infty). \end{aligned}$$

q.e.d.

The first theorem in this section is

THEOREM 5.4. *If $p < 1$ and $d \geq 4$, there is some positive constant $\theta_{(1)}^2$ such that $\text{Var} Q_n^{(1)} = \theta_{(1)}^2 n + o(n)$. In particular $\pi^{1,1} > 0$.*

PROOF. Let

$$\Gamma_n^{(1)} = \sum_{j=1}^n Z_j(1) \hat{Z}_j(1).$$

At first we will estimate $\text{Var} \Gamma_n^{(1)}$. For $i < j$,

$$\begin{aligned} \text{Cov}(Z_i \hat{Z}_i, Z_j \hat{Z}_j) &= E(Z_i \hat{Z}_i Z_j \hat{Z}_j) - (EZ_i \hat{Z}_i)(EZ_j \hat{Z}_j) \\ &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0, \tau_x = \infty) P_0(\tau_0, \tau_x = \infty) - p^4 \end{aligned}$$

$$\begin{aligned}
&= \sum_{x \neq 0} p_{0x}^{j-i} (0, x) b(x) \bar{b}(x) \\
&\quad + p \sum_{x \neq 0} p_{0x}^{j-i} (0, x) b(x) P_0(\tau_x = \infty) \\
&\quad + p \sum_{x \neq 0} p_{0x}^{j-i} (0, x) \bar{b}(x) P_x(\tau_0 = \infty) \\
&\quad + p^2 \left(\sum_{x \neq 0} p_{0x}^{j-i} (0, x) P_0(\tau_x = \infty) P_x(\tau_0 = \infty) - p^2 \right).
\end{aligned}$$

By Lemma 5.2. and Lemma 5.3.,

$$\begin{aligned}
Var \Gamma_n^{(1)} &= \sum_{j=1}^n Var Z_j \hat{Z}_j + 2 \sum_{j=1}^n \sum_{i=1}^{j-1} Cov(Z_i \hat{Z}_i, Z_j \hat{Z}_j) \\
&= p^2(1-p^2)n \\
&\quad + 2 \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i (0, x) b(x) \bar{b}(x) \\
&\quad + 4p \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i (0, x) b(x) P_0(\tau_x = \infty) \\
&\quad + 2p^2 Var R_n^{(1)} \\
&\quad + o(n).
\end{aligned}$$

Let $a_j = \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i (0, x) b(x) \bar{b}(x)$. a_j is nondecreasing and by Lemma 2.2. and Lemma 2.4.,

$$0 \leq a_j \leq \sum_{i=1}^{j-1} \sum_x p^i(0, x) G(x, 0) G(0, x) = O(1).$$

Then $\lim_{j \rightarrow \infty} a_j$ exists. Hence $\frac{1}{n} \sum_{j=1}^n a_j$ converges to the same value. The third term can be estimated similarly. Then there is some positive constant $\theta_{(1)}^2$ such that $Var \Gamma_n^{(1)} = \theta_{(1)}^2 n + o(n)$. Since

$$|Q_n^{(1)} - \Gamma_n^{(1)}| \leq \sum_{j=1}^n Z_j^n W_j + \sum_{j=1}^n W_j^n \hat{Z}_j,$$

we have

$$\begin{aligned}
(E|Q_n^{(1)} - \Gamma_n^{(1)}|^2)^{1/2} &\leq \left(E \left| \sum_{j=1}^n W_j \right|^2 \right)^{1/2} + \left(E \left| \sum_{j=1}^n W_j^n \right|^2 \right)^{1/2} \\
&= O(\log n) \\
&= o\{(Var \Gamma_n^{(1)})^{1/2}\}
\end{aligned}$$

by Lemma 3.1. and Lemma 5.1.. Thus, it holds $Var Q_n^{(1)} = \theta_{(1)}^2 n + o(n)$.

q.e.d.

The second theorem is

THEOREM 5.5. *If $p < 1$ and $d \geq 4$, there is a nonnegative constant $\sigma_{(2)}^2$ such that $\text{Var}R_n^{(2)} = \sigma_{(2)}^2 n + o(n)$. Moreover, if $p \leq \frac{1}{2}$ or $\frac{2}{3} \leq p < 1$, then $\sigma_{(2)}^2 > 0$. In particular $\sigma^{2,2} > 0$ under this condition.*

PROOF. Firstly we estimate $\text{Cov}(R_n^{(1)}, Q_n^{(1)})$. Since $\text{Cov}(Y_n^{(1)}, \Gamma_n^{(1)}) = O(n)$,

$$\text{Cov}(R_n^{(1)}, Q_n^{(1)}) = \text{Cov}(Y_n^{(1)}, \Gamma_n^{(1)}) + o(n)$$

by using the same argument of the above theorem. For $i < j$,

$$E(Z_j Z_i \hat{Z}_i) - (EZ_j)(EZ_i \hat{Z}_i) = p(EZ_j Z_i - EZ_j EZ_i)$$

and

$$\begin{aligned} EZ_j Z_i - EZ_j EZ_i &= \sum_{x \neq 0} p_{j-i}^{j-i}(0, x) \{P_x(\tau_0, \tau_x = \infty) - P_x(\tau_0 = \infty)P_x(\tau_x = \infty)\} \\ &= \sum_{x \neq 0} p_{j-i}^{j-i}(0, x) b(x). \end{aligned}$$

For $i < j$,

$$\begin{aligned} E(Z_i Z_j \hat{Z}_j) &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0, \tau_x = \infty) P_x(\tau_0 = \infty) \\ &= \sum_{x \neq 0} p_{0x}^{j-i}(0, x) b(x) P_0(\tau_x = \infty) \\ &\quad + \sum_{x \neq 0} p_{0x}^{j-i}(0, x) P_x(\tau_0 = \infty) P_0(\tau_x = \infty). \end{aligned}$$

Consequently

$$\begin{aligned} \sum_{j=1}^n \sum_{i=1}^{j-1} \text{Cov}(Z_i, Z_j \hat{Z}_j) &= \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_0(\tau_x = \infty) \\ &\quad + p \text{Var}R_n^{(1)} \\ &\quad + o(n) \end{aligned}$$

holds by Lemma 5.1..

On the other hand, Jain and Pruitt showed in [4] that

$$\text{Var}R_n^{(1)} = p(1-p)n + \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i(0, x) b(x) + o(n).$$

Then

$$\begin{aligned}
VarR_n^{(2)} &= VarR_n^{(1)} + VarQ_n^{(1)} - 2Cov(R_n^{(1)}, Q_n^{(1)}) \\
&= p(1-p)(1-3p+3p^2)n \\
&\quad + 2 \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i(0, x) b(x) \bar{b}(x) \\
&\quad + 2(1-p)(1-2p) \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_0^i(0, x) b(x) \\
&\quad + 2(2p-1) \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_0(\tau_x = \infty) \\
&\quad + o(n).
\end{aligned}$$

Hence there is some constant $\sigma_{(2)}^2$ such that $VarR_n^{(2)} = \sigma_{(2)}^2 n + o(n)$.

Next we will prove the positivity of $\sigma_{(2)}^2$.

$$\begin{aligned}
Cov(R_n^{(1)}, R_n^{(2)}) &= VarR_n^{(1)} - Cov(R_n^{(1)}, Q_n^{(1)}) \\
&= p(1-p)(1-2p)n \\
&\quad + (2-3p) \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_0^i(0, x) b(x) \\
&\quad - \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_0(\tau_x = \infty) \\
&\quad + o(n).
\end{aligned}$$

By Lemma 2.5. and $p_x^i(0, x) = p_0^i(0, x)$ for each $i \geq 0$, we see

$$\begin{aligned}
&\sum_{i=1}^{j-1} \sum_{x \neq 0} p_{0x}^i(0, x) b(x) P_0(\tau_x = \infty) \\
&= p \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) P_0(\tau_x = \infty) \\
&\quad + \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) P_0(\tau_x = \infty) P_0(j-i \leq \tau_0 < \infty) \\
&\quad + \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) P_0(\tau_x = \infty) P_0(\tau_x < \tau_0 < j-i) \\
&= \text{I} + \text{II} + \text{III}.
\end{aligned}$$

By Lemma 2.2. and Lemma 2.4., we have

$$\text{II} \leq C \sum_{i=1}^{j-1} (j-i)^{-1} i^{-2} \log i = O(j^{-1} \log j).$$

By estimating similarly the second term,

$$\text{III} = \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) P_0(\tau_x = \infty) P_0(\tau_x < \tau_0 < \infty) + O(j^{-1} \log j)$$

$$= \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) \bar{b}(x) + O(j^{-1} \log j),$$

where we have used the fact

$$\begin{aligned} P_0(\tau_x < \tau_0 < \infty) - P_0(\tau_x < \tau_0 < j-i) \\ &= P_0(\tau_x < \tau_0, j-i \leq \tau_0 < \infty) \\ &\leq C(j-i)^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} \text{Cov}(R_n^{(1)}, R_n^{(2)}) &= p(1-p)(1-2p)n \\ &\quad + 2(1-2p) \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) \\ &\quad + p \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) P_0(\tau_x < \infty) \\ &\quad - \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) \bar{b}(x) \\ &\quad + o(n) \\ &= p(1-p)(1-2p)n \\ &\quad + 2(1-2p) \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) \\ &\quad + \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) P_0(\tau_x < \infty, \tau_0 = \infty) \\ &\quad + o(n). \end{aligned}$$

Thus, if $p < \frac{1}{2}$, $\lim_{n \rightarrow \infty} \text{Cov}(R_n^{(1)}, R_n^{(2)})/n > 0$ and then we have $\sigma_{(2)}^2 > 0$. If $\frac{2}{3} \leq p < 1$, it holds that $\sigma_{(2)}^2 > 0$ by $\lim_{n \rightarrow \infty} \text{Cov}(R_n^{(1)}, R_n^{(2)})/n < 0$. If $p = \frac{1}{2}$, then

$$\text{Var} R_n^{(2)} = \frac{1}{16}n + 2 \sum_{j=1}^n \sum_{i=1}^{j-1} \sum_{x \neq 0} p_x^i(0, x) b(x) \bar{b}(x) + o(n).$$

Combining these estimates, we conclude that if $p \leq \frac{1}{2}$, $\frac{2}{3} \leq p < 1$, then $\sigma_{(2)}^2 > 0$. q.e.d.

REMARK 5.6. By the proof of Theorem 5.5., we can obtain that $\sigma^{1,2} > 0$ if $p < \frac{1}{2}$ and $\sigma^{1,2} < 0$ if $\frac{2}{3} \leq p < 1$.

By Remark 4.2., we can also derive the central limit theorems for $Q_n^{(1)}$ and $R_n^{(2)}$ in Z^d for $d \geq 4$.

COROLLARY 5.7. *If $p < 1$ and $d \geq 4$, then $(Q_n^{(1)} - p^2 n) / \sqrt{n}$ is asymptotically the normal with mean 0 and variance $\theta_{(1)}^2$ and the limit distribution is nondegenerate.*

COROLLARY 5.8. *If $p < 1$ and $d \geq 4$, then $(R_n^{(2)} - p(1-p)n) / \sqrt{n}$ converges to the normal with mean 0, variance $\sigma_{(2)}^2$ in the distribution sense. If $p \leq \frac{1}{2}$ or $\frac{2}{3} \leq p < 1$, the limit law is nondegenerate.*

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