J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
39 (1992), 379-383.

On maximal versions of the Large Sieve, II

By P. D. T. A. ELLIOTT

In the present paper I obtain a doubly infinite maximal version of the Large Sieve inequality which, for the first time, approaches in strength the standard model.

THEOREM. *Let* $\varepsilon>0$. *Then*

$$\sum_{\substack{qr\leq Q\\(q,r)=1}}\frac{q}{\phi(qr)}\sum_{\chi(\mathrm{mod}\ q)}^{*}\max_{0\leq v-u\leq y}\left|\sum_{u<n\leq v}a_n\chi(n)c_r(n)\right|^2\ll(y+Q^2(\log y)^{2+\varepsilon})\sum_{n=-\infty}^{\infty}|a_n|^2$$

where $*$ *denotes that* χ *runs through the primitive Dirichlet characters (mod* q*), and* $c_r(n)$ *is the Ramanujan sum*

$$\sum_{\substack{b=1\\(b,r)=1}}^{r}\exp(2\pi ibn/r).$$

The inequality is uniform in $Q\geq1$, $y\geq2$ *and square summable sequences of complex numbers* a_n.

COROLLARY.

$$\sum_{q\leq Q}\sum_{\chi(\mathrm{mod}\ q)}^{*}\max_{0\leq v-u\leq y}\left|\sum_{u<p\leq v}a_p\chi(p)\right|^2\ll\left(\frac{y}{\log y}+Q^{2+\varepsilon}\right)\sum_{p\geq2}|a_p|^2$$

where p *denotes a prime number.*

The theorem generalises Theorem 1 of [1], but I apply the earlier result during the proof of the present result. Apart from decreasing the value of the implied constant and removing the factor $(\log y)^{2+\varepsilon}$, the leading factor in the upper bound is best possible. In the earlier treatment I decomposed a certain Mellin integral into three pieces. Here I decompose its analogue into four.

It follows from Theorem 1 of [1] that the present theorem is certainly valid if u in the maximum is confined to the interval $(1, y]$. It will therefore suffice to establish the theorem under the assumption that $a_n=0$ for $n\leq y$. I denote by $\widehat{\sum}$ the multiple summation operator

$$\sum_{\substack{qr\le Q\\(q,r)=1}} \frac{q}{\phi(qr)} \sideset{}{^*}\sum_{\chi(\bmod q)} .$$

I note from Lemma 1 of [1] that $\widehat{\sum}|\chi(n)c_r(n)|^2 \ll Q^2$ uniformly in $Q\ge 1$ and integers n. I shall also apply the following inequality derived from the standard Large Sieve, and appearing as Lemma 2 in [1]:

LEMMA 1. *If* $T\ge 1$

$$\widehat{\sum}\int_{-T}^{T}\Big|\sum_{n\le x} a_n\chi(n)c_r(n)n^{i\tau}\Big|^2 d\tau \ll \sum_{n\le x}|a_n|^2(n+Q^2T).$$

The bulk of the remainder of the proof of the present theorem is contained in the next result.

LEMMA 2. *The inequality*

$$\widehat{\sum}\max_{\substack{0\le v-u\le y\\ w<u\le 2w}}\Big|\sum_{u<n\le v} a_n\chi(n)c_r(n)\Big|^2 \ll (y+Q^2(\log y)^2)\sum_{w<n\le 3w}|a_n|^2$$

holds uniformly for $w\ge y\ge 2$.

PROOF OF LEMMA 2. I temporarily denote the sum $\sum|a_n|^2$, $w<n\le 3w$, by $|\mathrm{a}|^2$. This is a notation slightly at odds with the notation of [1]. Let $\sigma=(\log w)^{-1}$. For real positive α not an integer

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\alpha^s}{s}ds=\begin{cases}1 & \text{if } \alpha>1,\\ 0 & \text{if } \alpha<1,\end{cases}$$

the integration being taken over the vertical line $\mathrm{Re}(s)=\sigma$ in the complex s-plane. In terms of the kernel $K(s)=K(u,v,s)=s^{-1}(v^s-u^s)$ there is a representation

$$\sum_{u<n\le v} a_n\chi(n)c_r(n)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\sum_{w<n\le 3w} a_n\chi(n)c_r(n)n^{-s}K(s)ds.$$

Note that since u belongs to the interval $(w, 2w]$, $v\le 3w$. By assuming that u, v are half odd integers, as we clearly may, we ensure that u/n, v/n are not 1 for any positive integer n. I break the integral into four pieces I_j, $j=1,2,3,4$, corresponding to the ranges $|\tau|\le 2wy^{-1}$, $2wy^{-1}<|\tau|\le 2wy^{-1}\log y$, $wy^{-1}\log y<|\tau|\le w$, $|\tau|>w$ of the variable $\tau=-i\mathrm{Im}(s)$. I shall treat these integrals differently.

An integration by parts shows that for $\alpha>0$, $\alpha\neq 1$,

$$\frac{1}{2\pi i}\int_{|\tau|>w}\frac{\alpha^s}{s}ds\ll\frac{\alpha^\sigma}{w|\log\alpha|}.$$

Hence

$$I_4\ll w^{-1}\sum_{w<n\leq 3w}|a_n||\chi(n)c_r(n)|\left(\min\left\{\left|\log\frac{u}{n}\right|,\ \left|\log\frac{v}{n}\right|\right\}\right)^{-1}.$$

An application of the Cauchy-Schwarz inequality gives

$$\hat{\sum}\max_{u,v}|I_4|^2\ll w^{-2}\max_{w<u\leq 2w}\sum_{m\leq 3w}\left|\log\frac{u}{m}\right|^{-2}\hat{\sum}\sum_{w<n\leq 3w}|a_n|^2|\chi(n)c_r(n)|^2.$$

For a typical u in the interval $(w, 2w]$, $|\log u/m|$ is bounded below away from zero for $m\leq 2u/3$, $m>3u/2$, and the corresponding terms in the sum over the integers $m\leq 3w$ contribute $\ll w$. For the integers $m=u+k-1/2$, $1\leq k\leq(u+1)/2$, where k is an integer, we have $|\log u/m|\gg k/u\gg k/w$, and a corresponding contribution of

$$\ll w^2\sum_{k\geq 1}k^{-2}\ll w^2.$$

We may likewise treat the terms with $m=u-k+1/2$, $1\leq k\leq(u/3)+1/2$. Bearing in mind our earlier remark concerning the average size of Ramanujan sums, we see that

$$\hat{\sum}\max_{u,v}|I_4|^2\ll Q^2|\mathrm{a}|^2.$$

The integral I_1 is dealt with by applying the Cauchy-Schwarz inequality to the integral representation:

$$\hat{\sum}\max_{u,v}|I_1|^2\ll\max_{u,v}\int_{|\tau|\leq 2wy^{-1}}|K(s)|^2d\tau\cdot\int_{|\tau|\leq 2wy^{-1}}\hat{\sum}\left|\sum_{w<n\leq 3w}a_n\chi(n)c_r(n)\right|^2d\tau.$$

Since

$$|K(s)|=\left|\int_u^v t^{s-1}dt\right|\leq\int_u^v t^{\sigma-1}dt\ll yw^{-1}$$

uniformly for $w<u\leq 2w$, $w<v\leq 3w$, $\sigma=(\log w)^{-1}$, the integral involving the square of the kernel is $\ll yw^{-1}$. The second integral over τ is by Lemma 1 $\ll(w+Q^2wy^{-1})|\mathrm{a}|^2$. Hence

$$\hat{\sum}\max_{u,v}|I_1|^2\ll(y+Q^2)|\mathrm{a}|^2.$$

To estimate I_2 I apply the Cauchy-Schwarz inequality in a third way:

$$|I_2|^2 \leq \int |K(s)|^2 |s|^{1/2} d\tau \cdot \int |s|^{-1/2} \left| \sum_{w<n\leq 3w} a_n \chi(n) c_r(n) \right|^2 d\tau,$$

where the integrals are taken over the range $J: 2wy^{-1} < |\tau| \leq 2wy^{-1} \log y$. Since $K(s) \ll |s|^{-1}$ uniformly on $\mathrm{Re}(s) = (\log w)^{-1}$, $w < u \leq 3w$, $w < v \leq 3w$, the first of these two integrals is $\ll (yw^{-1})^{-1/2}$. I cover the range J by pairs of intervals $z < |\tau| \leq 2z$, where z runs through the powers of 2 in the interval $[wy^{-1}, 2wy^{-1} \log y]$. The second integral in the majorant for $|I_2|^2$ then breaks into $O(\log\log y)$ smaller integrals, typically

$$\ll z^{-1/2} \int_{z<|\tau|\leq 2z} \left| \sum_{w<n\leq 3w} a_n \chi(n) c_r(n) \right|^2 d\tau.$$

Summing by the operator $\widehat{\sum}$ we see that after an application of Lemma 1

$$\begin{aligned} \widehat{\sum} \max_{u,v} |I_2|^2 &\ll (yw^{-1})^{1/2} \sum_z z^{-1/2} \sum_{w<n\leq 3w} |a_n|^2 (n + Q^2 z) \\ &\ll (yw^{-1})^{1/2} \sum_{w<n\leq 3w} |a_n|^2 (n(yw^{-1})^{1/2} + Q^2 (wy^{-1} \log y)^{1/2}) \\ &\ll (y + Q^2 (\log y)^{1/2}) |\mathrm{a}|^2. \end{aligned}$$

The integral I_3 is treated by yet another application of the Cauchy-Schwarz inequality:

$$|I_3|^2 \leq \int |K(s)|^2 |s| d\tau \int |s|^{-1} \left| \sum_{w<n\leq 3w} a_n \chi(n) c_r(n) \right|^2 d\tau,$$

where the integrals are defined over the range $wy^{-1} \log y < |\tau| \leq w$. With the bound $K(s) \ll |s|^{-1}$ the first integral is

$$\ll \int_{wy^{-1}}^{w} \tau^{-1} d\tau \ll \log y.$$

The range of the variable τ in the second integral is broken into $O(\log y)$ pieces covered by subranges $z < |\tau| \leq 2z$, and there is a corresponding estimate

$$\begin{aligned} \widehat{\sum} \max_{u,v} |I_3|^2 &\ll \log y \sum_z z^{-1} \sum_{w<n\leq 3w} |a_n|^2 (n + Q^2 z) \\ &\ll (y + Q^2 (\log y)^2) |\mathrm{a}|^2. \end{aligned}$$

This completes the proof of Lemma 2. Examination of the argument shows that for $Q \leq y^{1/2}(\log y)^{-2}$, the range $wy^{-1}(\log y)^{-1} < |\tau| \leq wy^{-1} \log y$ of the integral is responsible for the leading factor y in the upper bound.

PROOF OF THE THEOREM. We apply Lemma 2 with $w = 2^j y$, $j = 0, 1, 2, \cdots$ and add. Since

$$\sum_{j=0}^{\infty} \sum_{2^j y < n \leq 3(2^j y)} |a_n|^2 = \sum_{n>y} |a_n|^2 \sum_{n/(3y) \leq 2^j < n/y} 1 < 4 \sum_{n>y} |a_n|^2,$$

the theorem is established.

PROOF OF THE COROLLARY. Let $0 < \varepsilon < 1$. If $a_p = 0$ for $p > y^{\varepsilon/10}$, then the asserted bound follows directly from the theorem, using only $r = 1$.

If $a_p = 0$ for $p \leq y^{\varepsilon/10}$, then $c_r(p) = \mu(r)$ for $r \leq y^{\varepsilon/10}$. The inequality of the theorem asserts that

$$\sum_{q \leq Q} \frac{q}{\phi(q)} \sum_{\substack{r \leq y^{\varepsilon/10} \\ (r,q)=1}} \frac{|\mu(r)|}{\phi(r)} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\, q)} \max_{0 \leq v-u \leq y} \left| \sum_{u < p \leq v} a_p \chi(p) \right|^2$$
$$\ll (y + (Qy^{\varepsilon/10})^2 (\log y)^{2+\varepsilon}) \sum_{p \geq 2} |a_p|^2.$$

Since

$$\frac{q}{\phi(q)} \sum_{\substack{r \leq x \\ (r,q)=1}} \frac{|\mu(r)|}{\phi(r)} > \log x, \qquad x \geq 1,$$

(cf. [2], Lemma (3.1), p. 102), we obtain the asserted bound by considering the cases $y \leq Q^4$, $y > Q^4$ separately.

References

[1] Elliott, P.D.T.A., On maximal versions of the Large Sieve, J. Fac. Sci. Univ. Tokyo Sect. 1A, Math. **38** (1991), 141-164.
[2] Halberstam, H. and H.-E. Richert, Sieve Methods, Academic Press, London, New York, San Francisco, 1974.

(Received February 24, 1992)

Department of Mathematics
University of Colorado at Boulder
Campus Box 426
Boulder, Colorado 80309-0426
U. S. A.