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# *Smooth normed spaces of (BD)-type*

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## 1. Introduction.

Further on, $X$ will be a normed linear space over the real number field $\boldsymbol{R}$. Consider the mapping

$$(\ ,\ )_T : X\times X\longrightarrow \boldsymbol{R}, \qquad (x, y)_T := \lim_{t\downarrow 0} (\|y+tx\|^2 - \|y\|^2)/2t$$

which is well-defined for all $x, y$ in $X$. This semi-inner-product on $X$ will be called semi-inner-product in the sense of Tapia or $T$-semi-inner-product, for short ([6], [1-2]). For the sake of completeness, we list some usual properties of $T$-semi-inner-product that will be used in the sequel:

(i) $(x, x)_T = \|x\|^2$ for all $x$ in $X$;
(ii) $(\alpha x, \beta y)_T = \alpha\beta(x, y)_T$ if $\alpha\beta \geqq 0$ and $x, y\in X$;
(iii) $(\alpha x+y, x)_T = \alpha\|x\|^2 + (y, x)_T$ for all $\alpha\in \boldsymbol{R}$ and $x, y\in X$;
(iv) $(-x, y)_T = (x, -y)_T$ for $x, y\in X$;
(v) $(x+y, z)_T \leqq \|x\|\|z\| + (y, z)_T$ for all $x, y, z\in X$;
(vi) $|(x, y)_T| \leqq \|x\|\|y\|$ if $x, y\in X$;
(vii) $(\ ,\ )_T$ is continuous subadditive in the first variable;
(viii) $X$ is smooth iff $(\ ,\ )_T$ is linear in the first variable or iff $(\ ,\ )_T$ is homogeneous in the second.

We also recall Tapia's theorem of representation for the continuous linear functional on smooth normed spaces ([6], [1]):

THEOREM 1. *Let $X$ be a Banach space. Then the following statements are equivalent:*

(i) *$X$ is reflexive and smooth;*
(ii) *for all $f\in X^*$ there exists an element $u_f\in X$ so that*

$$f(x) = (x, u_f)_T \qquad \textit{for all } x\in X \textit{ and } \|f\| = \|u_f\|.$$

For the other properties of $T$-semi-inner-product see [1-2] and [5-6].

## 2. Smooth normed spaces of $(D)$-type.

We start with the next definition:

DEFINITION 1. The $T$-semi-inner-product $(\ ,\ )_T$ is said to be continuous on $X$ if:

$$\lim_{t\to 0}(y, x+ty)_T=(y, x)_T \qquad \text{for all } x, y \text{ in } X. \tag{1}$$

PROPOSITION 1. *Let $X$ be a real normed space. Then $X$ is smooth if and only if the $T$-semi-inner-product is continuous.*

PROOF. "$(\Leftarrow)$". By the $T$-semi-inner-product properties we have:

$$(y, x)_T/\|x\| \leqq (\|x+ty\|-\|x\|)/t \leqq (y, x+ty)_T/\|x+ty\| \tag{2}$$

and

$$(y, x+sy)_T/\|x+sy\| \leqq (\|x+sy\|-\|x\|) \leqq (y, x)_T/\|x\| \tag{3}$$

for all $x, y$ in $X$, $x\neq 0$ and $t>0$, $s<0$ so that $x+ty$, $x+sy\neq 0$.

Since $(\ ,\ )_T$ is continuous, we derive:

$$\lim_{t\downarrow 0}(\|x+ty\|-\|x\|)/t=(y, x)_T/\|x\|$$

and

$$\lim_{s\uparrow 0}(\|x+sy\|-\|x\|)/s=(y, x)_T/\|x\|$$

for all $x, y\in X$, $x\neq 0$, i.e., the space $X$ is smooth.

"$(\Rightarrow)$". By relations (2) and (3) we have:

$$(y, x)_T\|x+ty\|/\|x\| \leqq (y, x+ty)_T \leqq (\|x+2ty\|-\|x+ty\|)\|x+ty\|/t \tag{4}$$

and

$$\|x+sy\|(\|x+2sy\|-\|x+sy\|)/s \leqq (y, x+sy)_T \leqq (y, x)_T\|x+sy\|/\|x\|, \tag{5}$$

for all $x, y\in X$, $x\neq 0$ and $t>0$, $s<0$.

Since $X$ is smooth, the inequalities (4) and (5) yield that $\lim_{t\to 0}(y, x+ty)_T =(y, x)_T$ and the proof is finished. We omit the details.

Now, let $X$ be a smooth normed space and $(\ ,\ )_T$ be the $T$-semi-inner-product of it. Then $(\ ,\ )_T$ will be called derivable on $X$ if the following limit exists:

$$(y, x)_T' := \lim_{t\to 0}[(y, x+ty)_T-(y, x)_T]/t$$

for all $x, y$ in $X$.

DEFINITION 2. A smooth normed space $(X;\|\cdot\|)$ is called of $(D)$-type if its $T$-semi-inner-product is derivable on $X$.

*Examples* 1.1. Every inner-product space $(X;(\ ,\ ))$ is a smooth normed space of $(D)$-type.

Indeed, for every $x, y\in X$ we have:

$$(y, x)'=\lim_{t\to 0}[(y, x+ty)-(y, x)]/t=\|y\|^2 .$$

2. Let $(X;(\ ,\ ))$ be an inner-product space over real number field and $A:\mathcal{D}(A)\subset X\to X$ be an operator on linear subspace $\mathcal{D}(A)$ with the properties:

(a) $A(\alpha x)=\alpha A(x)$ for $\alpha\in\boldsymbol{R}$ and $x\in\mathcal{D}(A)$;

(aa) $(x, Ax)\geqq 0$ for $x\in\mathcal{D}(A)$ and $(x, Ax)=0$ implies $x=0$;

(aaa) $|(x, Ay)|^2\leqq(x, Ax)(y, Ay)$ for all $x, y\in\mathcal{D}(A)$;

(av) the Gâteaux differential $(VA)(x)\cdot y:=\lim_{t\to 0}[A(x+ty)-A(x)]/t$ exists for all $x, y\in\mathcal{D}(A)$;

then $(\mathcal{D}(A);\|\cdot\|_A)$ where $\|x\|_A:=(x, Ax)^{1/2}$ for $x\in\mathcal{D}(A)$ is a smooth normed linear space of $(D)$-type.

Indeed, a simple calculus gives:

$$(y, x)_{TA}:=\lim_{t\to 0}(\|x+ty\|_A^2-\|x\|_A^2)/2t=(y, Ax) \qquad \text{for } x\in\mathcal{D}(A),$$

$(\ ,\ )_{TA}$ is continuous on $X$ and:

$$(y, x)'_{TA}=(y, (VA)(x)\cdot y) \qquad \text{for all } x, y\in\mathcal{D}(A).$$

3. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set $\Omega$, a $\sigma$-algebra $\mathcal{A}$ of subsets of $\Omega$ and a countably additive and positive measure $\mu$ on $\mathcal{A}$ with values in $\boldsymbol{R}\cup\{\infty\}$. If $L_r^p(\Omega)$ is the real Banach space of $p$-integrable functions on $\Omega$ with $p>1$, then it is well-known that (see for example [7]):

$$\lim_{t\to 0}(\|x+ty\|_p-\|x\|_p)/t=\|x\|_p^{1-p}\int_\Omega|x(s)|^{p-1}(\operatorname{sgn} x(s))y(s)d\mu(s)$$

for all $x, y\in L_r^p(\Omega)$, $x\neq 0$.

Suppose $p\geqq 2$ and put $p=2k+2$, $k\geqq 0$. Then

$$(y, x)_{Tp}:=\lim_{t\to 0}(\|x+ty\|_p^2-\|x\|_p^2)/2t=\|x\|_p^{-2k}\int_\Omega[x(s)]^{2k+1}y(s)d\mu(s)$$

for all $x, y \in L_r^p(\Omega)$, $x \neq 0$ and $(y, 0)_{Tp} = 0$ if $y \in L_r^p(\Omega)$.

A simple calculus gives:

$$(y, x)'_{Tp} = \|x\|_p^{-2k}(2k+1)\int_\Omega x^{2k}(s)y^2(s)d\mu(s)$$

$$-2k\|x\|_p^{-2k-2}\left(\int_\Omega x^{2k+1}(s)y(s)d\mu(s)\right)^2$$

for all $x, y \in L_r^p(\Omega)$, $x \neq 0$ and $(y, 0)'_{Tp} = \|y\|_p^2$ if $y \in L_r^p(\Omega)$.

Consequently, the real Banach space $L_r^p(\Omega)$, $p \geqq 2$, is a smooth Banach space of $(D)$-type.

Now, we shall give some usual properties of $T$-semi-inner-product derivative on a smooth normed space of $(D)$-type.

PROPOSITION 2. *If $X$ is as above, then the following statements are valid:*

(i) $(y, y)'_T = \|y\|^2$ *for all* $y \in X$;

(ii) $(y, 0)'_T = \|y\|^2$ *for all* $y \in X$;

(iii) $(\alpha y, x)'_T = \alpha^2 (y, x)'_T$ *for all* $\alpha \in \boldsymbol{R}$ *and* $x, y \in X$;

(iv) $(y, \alpha x)'_T = (y, x)'_T$ *for all* $\alpha \in \boldsymbol{R}\setminus\{0\}$ *and* $x, y \in X$;

(v) $\|x\|^2 (y, x)'_T \geqq (y, x)_T^2$ *for all* $x, y \in X$.

PROOF. We only prove the statement (v). The other sentences are obvious from the definition of $T$-semi-inner-product derivative.

(v). By the properties of $T$-semi-inner-product we have:

$$(y, x+ty)_T - (y, x)_T \geqq (y, x)_T(\|x+ty\| - \|x\|)/\|x\|$$

for all $x, y \in X$, $x \neq 0$ and $t \geqq 0$; which implies for $t > 0$:

$$[(y, x+ty)_T - (y, x)_T]/t \geqq (y, x)_T(\|x+ty\| - \|x\|)/(t\|x\|).$$

Passing at limit for $t \to 0$, $t > 0$, we derive:

$$(y, x)'_T \geqq (y, x)_T^2/\|x\|^2 \qquad \text{for all } x, y \in X,\ x \neq 0,$$

and the statement is proven.

Another result is embodied in the next proposition.

PROPOSITION 3. *Let $X$ be a smooth normed space of $(D)$-type and $x, y$ be two elements in $X$. Then the mapping:*

$$\varphi_{x,y} : \boldsymbol{R} \longrightarrow \boldsymbol{R}, \qquad \varphi_{x,y}(t) = \|x+ty\|^2,$$

*is derivable of two orders on* $\boldsymbol{R}$, *the second derivative is nonnegative on* $\boldsymbol{R}$ *and*:

$$\varphi'_{x,y}(t)=2(y,x+ty)_T, \qquad \varphi''_{x,y}(t)=2(y,x+ty)'_T$$

*for all* $t\in\boldsymbol{R}$.

The proof is obvious and we omit the details.

Further on, we shall give a characterization of inner-product spaces in the class of smooth normed linear spaces of $(D)$-type.

PROPOSITION 4. *Let* $X$ *be as above. Then the following statements are equivalent*:

(i) $X$ *is an inner-product space*;

(ii) *the mapping* $\psi_{x,y}: \boldsymbol{R}\to\boldsymbol{R}_+$, $\psi_{x,y}(t)=(y,tx)'_T$ *is continuous at* 0 *for all* $x,y$ *in* $X$,

(iii) *for every* $x,y\in X$ *there exists a sequence* $\alpha_n\in\boldsymbol{R}\setminus\{0\}$, $\alpha_n\to 0$ *so that* $\lim_{n\to\infty}(y,\alpha_n x)'_T=(y,0)'_T$;

(iv) *for every* $x,y\in X$ *we have*: $(y,x)'_T=\|y\|^2$.

PROOF. "(i)⇒(ii)⇒(iii)⇒(iv)". It's obvious.

"(iv)⇒(i)". By Taylor's formula for the mapping $\varphi_{x,y}$ $(x,y\in X)$ we have:

$$\|x+ty\|^2=\|x\|^2+2(y,x)_T t+\|y\|^2t^2 \qquad \text{for all } t\in\boldsymbol{R},$$

which implies the parallelogram identity:

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2) \qquad \text{for all } x,y\in X,$$

i.e., $X$ is an inner-product space.

## 3. Smooth normed spaces of $(BD)$-type.

Let $X$ be a smooth normed linear space of $(D)$-type. The $T$-semi-inner-product has a bounded derivative if there exists a real number $k\geqq 1$ so that:

$$(6) \qquad (y,x)'_T\leqq k^2\|y\|^2 \qquad \text{for all } x,y\in X.$$

The least number $k$ such that (6) is valid will be called the boundedness modulus of the derivative $(\ ,\ )'_T$ and we shall denote this number with $k_0$.

DEFINITION 3. A smooth normed space of $(D)$-type is called of $(BD)$-

type if its $T$-semi-inner-product has a bounded derivative.

*Examples* 2.1. Every inner-product space is a smooth normed space of $(BD)$-type.

2. Let $(X; (\ ,\ ))$ be an inner-product space and $A: \mathcal{D}(A) \subset X \to X$ be an operator satisfying conditions (a)-(av) from Example 1.2. Suppose, in addition, that $A$ is $M$-lischitzian ($M \geqq 1$), i.e.,

$$\|Ax - Ay\| \leqq M\|x-y\| \qquad \text{for all } x, y \in \mathcal{D}(A). \tag{aM}$$

Then $(\mathcal{D}(A); \|\cdot\|_A)$ is a smooth normed space of $(BD)$-type.

Indeed, from (aM) we derive:

$$\|(\nabla A)(x)\cdot y\| \leqq M\|y\| \qquad \text{for all } x, y \in \mathcal{D}(A),$$

which implies that:

$$(y, x)'_{TA} \leqq M\|y\|^2 \qquad \text{for all } x, y \in \mathcal{D}(A),$$

and the assertion is proven.

3. The real Banach spaces $L^p_r(\Omega)$ for $p \geqq 2$ are smooth normed linear spaces of $(BD)$-type.

Indeed, by Hölder's inequality for integrals, we have:

$$\int_\Omega x^{2k}(s) y^2(s) d\mu(s) \leqq \left(\int_\Omega x^{2k+2}(s) d\mu(s)\right)^{2k/(2k+2)} \left(\int_\Omega y^{2k+2}(s) d\mu(s)\right)^{2/(2k+2)}$$

and

$$\left(\int_\Omega x^{2k+1}(s) y(s) d\mu(s)\right)^2$$
$$\leqq \left(\int_\Omega x^{2k+2}(s) d\mu(s)\right)^{(4k+2)/(2k+2)} \left(\int_\Omega y^{2k+2}(s) d\mu(s)\right)^{2/(2k+2)}$$

where $p = 2k+2$, $k \geqq 0$.

Then we obtain the evaluation:

$$(y, x)'_{Tp} \leqq (4k+1)\|y\|_p^2 \qquad \text{for all } x, y \in L^p_r(\Omega),\ x \neq 0,$$

and the statement is proven.

The following result gives a characterization of inner-product spaces in the class of smooth normed linear spaces of $(BD)$-type.

PROPOSITION 5. *Let $X$ be as above. Then the following statements are equivalent*:

(i) *$X$ is an inner-product space*;

(ii) *we have $k_0 = 1$.*

PROOF. "(i)⇒(ii)". It's obvious.

"(ii)⇒(i)". By Taylor's formula for $\varphi_{x,y}$ $(x, y\in X)$ we obtain:

$$\|x+y\|^2 \leqq \|x\|^2+2(y, x)_T+\|y\|^2 \qquad \text{for all } x, y\in X,$$

which implies

$$\|x+y\|^2 \leqq \|x\|^2+2(x, y)_T+\|y\|^2 \qquad \text{for all } x, y\in X.$$

Since $X$ is smooth, we have:

$$\|x+ty\|^2 \leqq \|x\|^2+2(x, y)_T t+\|y\|^2 t^2,$$

for all $x, y\in X$ and $t\in \boldsymbol{R}$. If we assume that $t>0$, we derive:

$$(\|x+ty\|^2-\|x\|^2)/2t \leqq (x, y)_T+t\|y\|^2/2$$

hence:

$$(y, x)_T \leqq (x, y)_T \qquad \text{for all } x, y\in X$$

and by symmetry $(y, x)_T \geqq (x, y)_T$ for all $x, y\in X$, i.e., $X$ is an inner-product space, see [6].

Further on, we shall introduce two concepts of $\varepsilon$-orthogonality and we shall establish a result of $\varepsilon$-decomposition for smooth normed spaces of $(BD)$-type.

DEFINITION 4. Let $X$ be as above and $k_0$ be the boundedness modulus of $T$-semi-inner-product derivative. If $\varepsilon\in[0, 1)$, then the element $x\in X$ is said to be $\varepsilon-k_0$-orthogonal over $y\in X$ if

$$|(y, x)_T| \leqq \varepsilon k_0\|x\|\,\|y\|, \tag{7}$$

and we denote $x(1/\varepsilon)_{k_0}y$.

REMARK 1. If $X$ is an inner-product space, then in (7) we can put $k_0=1$. We denote $x(1/\varepsilon)y$.

If in the previous definition we choose $\varepsilon=0$, we recapture the usual orthogonality in the $T$-semi-inner-product sense or the usual orthogonality in prehilbertian spaces, respectively.

We also give now the following generalization of Birkhoff's orthogonality which works in general normed spaces (see also [3]).

DEFINITION 5. Let $X$ be a normed linear space, $\varepsilon\in[0, 1)$ and $x, y\in X$. The element $x$ is said to be $\varepsilon$-Birkhoff orthogonal over $y$ and we denote $x(1/\varepsilon)_B y$ if:

$$\|x+ty\| \geqq (1-\varepsilon)\|x\| \qquad \text{for all } t\in \boldsymbol{R}.$$

The following proposition establishes a connection between $\varepsilon-k_0$-orthogonality and $\varepsilon$-Birkhoff orthogonality in smooth normed linear spaces of $(BD)$-type.

PROPOSITION 6. *Let $X$, $k_0$ be as above and $x, y\in X$, $\varepsilon\in[0,1)$. Then the following statements are valid*:

(i) $x(1/\varepsilon)_B y$ *implies* $x(1/\delta(\varepsilon))_k y$ *with* $\delta(\varepsilon):=[\varepsilon(2-\varepsilon)]^{1/2}$;

(ii) $x(1/\eta(\varepsilon))_B y$ *implies* $x(1/\varepsilon)_{k_0} y$ *with* $\eta(\varepsilon):=1-(1-\varepsilon^2)^{1/2}$.

PROOF. We shall start with Taylor's formula:

$$\|x+ty\|^2=\|x\|^2+2(y,x)_T t+(y,x+\xi_t y)'_T t^2 \qquad \text{for } t\in\boldsymbol{R},$$

where $\xi_t$ is between 0 and $t$.

(i). If $x(1/\varepsilon)_B y$, then:

$$(1-\varepsilon)^2\|x\|^2 \leqq \|x+ty\|^2 \qquad \text{for all } t\in\boldsymbol{R},$$

which implies:

$$(\varepsilon^2-2\varepsilon)\|x\|^2 \leqq 2(y,x)_T t+k_0^2\|x\|^2 t^2 \qquad \text{for all } t\in\boldsymbol{R},$$

from where we get:

$$(y,x)_T^2 \leqq k_0^2\varepsilon(2-\varepsilon)\|x\|^2\|y\|^2,$$

i.e., $x(1/\delta(\varepsilon))_{k_0} y$ with $\delta(\varepsilon)$ is as above.

(ii). It follows from (i) substituting $\varepsilon$ by $\eta(\varepsilon)\in[0,1)$.

REMARK 2. In the case of inner-product spaces we have:

(i) $x(1/\varepsilon)_B y$ iff $x(1/\delta(\varepsilon))y$;

(ii) $x(1/\eta(\varepsilon))_B y$ iff $x(1/\varepsilon)y$;

where $\delta(\varepsilon)$ and $\eta(\varepsilon)$ are as above.

The proof is obvious and we omit the details.

Now, let $X$ be a normed linear space and $A$ be its nonempty subset. By $A^{(1/\varepsilon)B}$ we shall denote the set:

$$\{y\in X \mid y(1/\varepsilon)_B x \quad \text{for all } x\in A\},$$

where $\varepsilon$ is a given real number in $[0,1)$. This set will be called the $\varepsilon$-Birkhoff orthogonal complement of $A$. It is easy to see that $0\in A^{(1/\varepsilon)B}$ and $A\cap A^{(1/\varepsilon)B}\subseteqq\{0\}$ for all $\varepsilon\in[0,1)$.

The following lemma ([3]) is a variant of $F$. Riesz result (see for example [8, p. 84]):

LEMMA 1. *Let $X$ be a normed space and $E$ be its closed linear subspace. Suppose $E\neq X$. Then for every $\varepsilon\in(0,1)$ the $\varepsilon$-Birkhoff orthogonal complement of $E$ is nonzero.*

PROOF. Let $\bar{y}\in X\backslash E$. Since $E$ is closed, we have $d(\bar{y},E)=d>0$. Then there exists $y_\varepsilon\in E$ so that: $0\leqq\|\bar{y}-y_\varepsilon\|\leqq d/(1-\varepsilon)$. Putting $x_\varepsilon:=\bar{y}-y_\varepsilon$, we have $x_\varepsilon\neq 0$ and for every $y\in E$ and $\lambda\in\boldsymbol{R}$:

$$\|x_\varepsilon+\lambda y\|=\|\bar{y}-y_\varepsilon+\lambda y\|=\|\bar{y}-(y_\varepsilon-\lambda y)\|\geqq d\geqq(1-\varepsilon)\|x_\varepsilon\|,$$

which means that $x_\varepsilon\in E^{(1/\varepsilon)B}$ and the lemma is proven.

The following decomposition theorem holds.

THEOREM 2. *Let $X$ be a normed linear space and $E$ be its closed linear subspace. Then for any $\varepsilon\in(0,1)$ the following decomposition*

$$X=E+E^{(1/\varepsilon)B}, \tag{8}$$

*is valid.*

PROOF. Suppose $E\neq X$ and $x\in X$.

If $x\in E$, then $x=x+0$ with $x\in E$ and $0\in E^{(1/\varepsilon)B}$.

If $x\notin E$, then there exists $y_\varepsilon\in E$ such that $0<d=d(x,E)=\|x-y_\varepsilon\|\leqq d/(1-\varepsilon)$. Since $x_\varepsilon:=x-y_\varepsilon\in E^{(1/\varepsilon)B}$ (see the proof of the above lemma) we obtain $x=y_\varepsilon+x_\varepsilon$ and the relation (8) is valid.

In the sequel, we shall apply these results in the particular case of smooth normed spaces of $(BD)$-type.

Let $X$ be as above and $A$ be a nonempty subset of $X$. Then by $A^{(1/\varepsilon)k_0}$ we shall denote the set:

$$\left\{y\in X\,\middle|\; y(1/\varepsilon)_{k_0}x \quad \text{for all } x\in A\right\},\quad \varepsilon\in[0,1),$$

which will be called the $\varepsilon-k_0$-orthogonal complement of $A$ in $X$.

LEMMA 2. *Let $X$ be a smooth normed linear space of $(BD)$-type, $E$ be its closed linear subspace and $\varepsilon\in(0,1)$. Assume $E\neq X$. Then the $\varepsilon-k_0$-orthogonal complement of $E$ is nonzero.*

PROOF. Let $\varepsilon\in(0,1)$ and $\eta(\varepsilon):=1-(1-\varepsilon^2)^{1/2}$. Then $\eta(\varepsilon)$ belongs to $(0,1)$. Applying Lemma 1 for $\eta(\varepsilon)$, then there exists an element $x_\varepsilon\neq 0$ and $x_\varepsilon\in E^{(1/\eta(\varepsilon))B}$. Since $E^{(1/\eta(\varepsilon))B}\subseteqq E^{(1/\varepsilon)k_0}$ (see Proposition 6) the lemma is thus proven.

Finally, we have:

THEOREM 3. *Let $X$ be a smooth normed space of (BD)-type, $E$ its closed linear subspace and $\varepsilon\in(0,1)$. Then the following decomposition holds*:

$$X=E+E^{(1/\varepsilon)k_0}.$$

The proof is obvious from Theorem 2 and Proposition 6 and we omit the details.

## 4. Tapia's subset of $X^*$.

Let $X$ be a smooth normed linear space over the real number field $\boldsymbol{R}$. The following subset of dual space $X^*$:

$$T(X):=\{f_y\in X^* \mid f_y(x)=(x,y)_T\ ;\ x,y\in X\}$$

will be called Tapia's subset of $X^*$ associated with $X$. We remark that, in general, $T(X)$ is not a linear subspace of $X^*$ and by Tapia's theorem of representation, a smooth Banach space $X$ is reflexive iff $T(X)=X^*$.

REMARK 3. If $(X;(\ ,\ ))$ is an inner-product space, then $T(X)$ is a linear subspace in $X^*$ which will be called Riesz's subspace of $X^*$ and will be denoted by $R(X)$. The mapping $\Delta: X\to X^*$ given by $\Delta(y):=f_y$ is a linear isometric operator to $X$ onto $R(X)$. Putting $(\ ,\ )^*: R(X)\times R(X)\to R$, $(f_y,f_x)^*:=(x,y)$, then $(\ ,\ )^*$ is an inner-product on $R(X)$ which generates the norm induced by dual space $X^*$ in $R(X)$ and by these considerations, $R(X)$ is isomorphic and isometric to $X$ as inner-product spaces.

The following proposition holds.

PROPOSITION 7. *Let $X$ be a smooth normed space of (BD)-type, $E$ be its closed linear subspace and $E\neq X$. Then for any $\varepsilon>0$, there exists a functional $f_\varepsilon\in T(X)$ so that*

$$\|f_\varepsilon\|\leqq 1 \quad \textit{and} \quad \|f_\varepsilon\|_E\leqq\varepsilon, \tag{9}$$

*where* $\|f\|_E:=\sup\{|f(x)|,\ \|x\|=1, x\in E\}$.

PROOF. If $\varepsilon\geqq 1$, the statement is clear.

Let assume that $\varepsilon\in(0,1)$. By Lemma 2 there exists a nonzero element $y_\varepsilon$ in $E^{(1/\varepsilon)k_0}$, i.e.,

$$|(x,y_\varepsilon)_T|\leqq\varepsilon k_0\|x\|\,\|y_\varepsilon\| \qquad \text{for all } x \text{ in } X.$$

Putting $x_\varepsilon := y_\varepsilon/(k_0\|y_\varepsilon\|)$ we have $\|x_\varepsilon\| = 1/k_0 \leqq 1$ and the functional $f_\varepsilon : X \to \boldsymbol{R}$, $f_\varepsilon(x) := (x, x_\varepsilon)_T$ satisfies the relation (9). The proof is finished.

The next theorem is important in the sequel.

THEOREM 4. *Let $X$ be a smooth normed space of (BD)-type and $f$ be a nonzero continuous linear functional on it. Then for any $\varepsilon > 0$ there exists a nonzero element $x_\varepsilon$ in $X$ so that*:

$$|f(x) - (x, x_\varepsilon)_T| \leqq \varepsilon \|x\| \qquad \textit{for all } x \in X. \tag{10}$$

PROOF. Since $f \neq 0$, the linear subspace $E := \mathrm{Ker}(f)$ is closed in $X$ and $E \neq X$.

Let $\varepsilon > 0$ and put $\delta(\varepsilon) := \varepsilon/(2\|f\|k_0) > 0$, where $k_0$ is the boundedness modulus of $(\ ,\ )'_T$.

If $\delta(\varepsilon) \geqq 1$, then there exists an element $y_\varepsilon \in X \backslash E$ so that

$$|(y, y_\varepsilon)_T| \leqq \delta(\varepsilon)\|y\|\,\|y_\varepsilon\| \leqq \delta(\varepsilon) k_0 \|y\|\,\|y_\varepsilon\|\,. \tag{11}$$

If $0 < \delta(\varepsilon) < 1$, then by Lemma 2 there exists an element $y_\varepsilon \in X \backslash E$ so that (11) is valid too.

Let put $z_\varepsilon := y_\varepsilon/\|y_\varepsilon\|$. Then for all $x \in X$ we have:

$$y := f(x) z_\varepsilon - f(z_\varepsilon) x \in \mathrm{Ker}(f)\,,$$

and then:

$$|(f(x)z_\varepsilon - f(z_\varepsilon)x, z_\varepsilon)_T| \leqq k_0\delta(\varepsilon)\|f(x)z_\varepsilon - f(z_\varepsilon)x\| \leqq 2k_0\delta(\varepsilon)\|f\|\,\|x\| \leqq \varepsilon\|x\|,$$

for all $x \in X$.

On the other hand, we have:

$$(f(x)z_\varepsilon - f(z_\varepsilon)x, z_\varepsilon)_T = f(x) - (x, f(z_\varepsilon)z_\varepsilon)_T$$

for all $x \in X$ and putting $x_\varepsilon := f(z_\varepsilon)z_\varepsilon$ the relation (10) is obtained.

Now, we shall give the main result of this section.

THEOREM 5. *Let $X$ be a smooth normed space of (BD)-type. Then Tapia's subset $T(X)$ of $X^*$ is dense in $X^*$ endowed with the strong topology.*

PROOF. Let $f \in X^*$ and $\varepsilon > 0$. Then by Theorem 4 there exists an element $x_\varepsilon \in X$ so that:

$$|f(x) - f_\varepsilon(x)| \leqq \varepsilon\|x\| \qquad \text{for all } x \in X,$$

where $f_\varepsilon(x) := (x, x_\varepsilon)_T$, $x \in X$. Consequently, $\|f - f_\varepsilon\| \leqq \varepsilon$ and the assertion is proven.

REMARK 4. Let $[\ ,\ ]: X\times X\to K$ ($K=\boldsymbol{R}, C$) be a semi-inner-product on normed linear space $X$ (see for example [4]) which generates its norm. In paper [3] we introduced the concept of normed linear spaces of $(APP)$-type relative to $[\ ,\ ]$, i.e., a normed space so that for every nonzero continuous linear functional $f$ on it and for any $\varepsilon\in(0,1)$ there exists a nonzero element $y_\varepsilon$ in $X$ so that:

$$|[y, y_\varepsilon]|\leqq\varepsilon\|y\|\,\|y\| \qquad \text{for all } y\in\operatorname{Ker}(f).$$

We also proved that if $X$ is such a space, then the Lumer subset $L(X):=\{f_y\in X^*|\ f_y(x):=[x,y] \text{ for } x,y\in X\}$, of dual space $X^*$ associated to semi-inner-product $[\ ,\ ]$ is dense in $X^*$ endowed with the strong topology.

If $X$ is a smooth real normed space it is well-known that there exists a unique semi-inner-product which generates the norm and coincides with $T$-semi-inner-product (see [1] or [5]) and then $L(X)=T(X)$. We also remark that every smooth normed space of $(BD)$-type is a normed linear space of $(APP)$-type relative with $T$-semi-inner-product.

Now, we shall give a corollary of Theorem 5.

COROLLARY. *Let $X$ be a smooth Banach space of $(BD)$-type. Then the following statements are equivalent:*

( i ) *$X$ is reflexive;*

(ii) *$T(X)$ is closed in $X^*$;*

(iii) *$T(X)=X^*$.*

PROOF. The equivalence "(i)⇔(iii)" follows by Tapia's theorem of representation and the equivalence "(ii)⇔(iii)" is obvious by the above theorem.

The case of prehilbertian spaces is embodied in the next proposition.

PROPOSITION 8. *Let $X$ be an inner-product space. Then the following statements are equivalent:*

( i ) *$X$ is a Hilbert space;*

(ii) *$R(X)$ is closed in $X^*$;*

(iii) *$R(X)=X^*$.*

The proof follows by Remark 3 and Theorem 5 for inner-product spaces.

## 5. Applications to operator equations.

In this section we shall use Theorem 4 to establish some existence results for $\varepsilon$-solutions of the operator equation:

$$(A\,;y) \qquad Ax=y, \qquad x\in\mathcal{D}(A), \qquad y\in X,$$

where $A:\mathcal{D}(A)\subset X\to X$ is an operator defined on dense linear subspace $\mathcal{D}(A)$ of Hilbert space $X$ and having the properties (a)-(av) and (aM) from Example 2.2.

REMARK 5. Some examples of operators which verify the above conditions are the symmetric strictly positive operators which are densely defined on a real Hilbert space and satisfy condition:

$$\|Ax\|\leqq M\|x\|, \quad M\geqq 1 \qquad \text{for all } x\in\mathcal{D}(A)\,.$$

Now, let $\varepsilon>0$. The element $x_\varepsilon\in\mathcal{D}(A)$ is called an $\varepsilon$-solution for the equation $(A\,;y)$ if $\|Ax_\varepsilon-y\|\leqq\varepsilon$. It is known that (see Example 2.2) the mapping $\mathcal{D}(A)\ni x\overset{\|\cdot\|_A}{\longmapsto}(x,Ax)^{1/2}\in \boldsymbol{R}_+$ is a norm on $\mathcal{D}(A)$ and $(\mathcal{D}(A),\|\cdot\|_A)$ is a smooth normed space of $(BD)$-type. Then we can also introduce the following concept of approximative solutions.

DEFINITION 6. Let $\varepsilon>0$. The element $x_\varepsilon\in\mathcal{D}(A)$ is called an $A-\varepsilon$-solution for the equation $(A\,;y)$ if:

$$\sup_{\|x\|_A\leq 1}|(x,y-Ax_\varepsilon)|\leqq\varepsilon\,.$$

The next existence result for $A-\varepsilon$-solutions of the operatorial equation $(A\,;y)$ holds.

PROPOSITION 9. *Let $X$, $A$ be as above and $y$ be a nonzero element in $X$ satisfying the assumption*:

$$(12) \qquad |(x,y)|\leqq\mu(x,Ax)^{1/2} \qquad \textit{for all } x\in\mathcal{D}(A)\ (\mu>0)\,;$$

*then for every $\varepsilon>0$ the equation $(A;y)$ has an $A-\varepsilon$-solution.*

PROOF. Let $f_y:\mathcal{D}(A)\to\boldsymbol{R}$, $f_y(x):=(x,y)$. By condition (12) it follows that $f_y$ is continuous in $(\mathcal{D}(A),\|\cdot\|_A)$ and by Theorem 4 there exists an element $x_\varepsilon\in\mathcal{D}(A)\setminus\{0\}$ so that:

$$|f_y(x)-(x,x_\varepsilon)_{TA}|\leqq\varepsilon\|x\|_A \qquad \text{for all } x\in\mathcal{D}(A)\,,$$

which is equivalent to the existence of an $A-\varepsilon$-solution for the equation $(A; y)$.

COROLLARY. *Let $X$, $A$ be as above and, in addition, there exists a constant $\eta>0$ such that*:

$$\eta\|x\|^2 \leqq (x, Ax) \qquad \text{for all } x \in \mathcal{D}(A). \tag{13}$$

*Then for every $y \in X\backslash\{0\}$ and for any $\varepsilon>0$ the equation $(A; y)$ has an $A-$-$\varepsilon$-solution.*

The proof is obvious by Proposition 9 observing that Condition 13 implies Condition 12 for all $y \in X\backslash\{0\}$.

Finally, we have:

PROPOSITION 10. *Let $X$, $A$ be as in Proposition 9 and, in addition, there exists a constant $\gamma>0$ such that*:

$$(x, Ax) \leq \gamma\|x\|^2 \qquad \text{for all } x \in \mathcal{D}(A). \tag{14}$$

*If $y \in X\backslash\{0\}$ verifies the assumption (12), then for any $\varepsilon>0$ the equation $(A; y)$ has an $\varepsilon$-solution.*

PROOF. By condition (14) we have: $\|x\|_A \leqq \gamma^{1/2}\|x\|$ for all $x \in \mathcal{D}(A)$. Since the linear functional $f_y$ is continuous in $(\mathcal{D}(A); \|\cdot\|_A)$ then by Theorem 4, for any $\varepsilon \geqq 0$ there exists an element $x_\varepsilon \in \mathcal{D}(A)\backslash\{0\}$ such that:

$$|f_y(x)-(x, x_\varepsilon)_{TA}| \leqq (\varepsilon/\gamma^{1/2})\|x\|_A \qquad \text{for all } x \in \mathcal{D}(A),$$

from which results:

$$|(x, y-Ax_\varepsilon)| \leqq \varepsilon\|x\| \qquad \text{for all } x \in \mathcal{D}(A).$$

Since $\mathcal{D}(A)$ is dense in $X$, the proposition is proven.

REMARK 6. It would be interesting to find some concrete examples of nonlinear operators which satisfy the assumptions (a)-(av) and (aM) and to apply these results to partial differential equations.

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