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On multiplicators of hermitian forms of type D_n

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Introduction.

Let k be a field, D be an associative division algebra of finite dimension over k, V be a finite dimensional right D-vector space, Φ be a non-degenerate (skew-)hermitian form over V with values in D. In [T2] we have determined the group $M(\Phi)$ of multiplicators of similitudes of the form Φ over a local or global fields for all forms of type A and C and for many forms of type D. The aim of this note is to give another, more exact determination of the group $M(\Phi)$ in the case, where Φ is a form of type D, and also a correction to [T2]. As application, we give here a new proof of the weak approximation property in adjoint almost simple algebraic groups of type D_n over global fields and also application to Scharlau Norm Principle.

Throughout this paper, unless otherwise stated, we will denote by
k — a global field of char. $\neq 2$,
k_v — the completion of k at a valuation v of k,
D — a quaternion division algebra with centre k and standard involution J,
$D_v = D \otimes k_v$,
S — the finite set of all valuations v of k such that D_v is nontrivial,
$s = \mathrm{Card}(S)$,
Φ — a non-degenerate skew-hermitian form with respect to J with values in D and of rank n,
$\Phi_v = \Phi \otimes k_v$,
$d = (-1)^n d'$, where d' is the discriminant of the form Φ,
$k' = k(\sqrt{d})$,
$U(\Phi)$ (resp. $SU(\Phi)$, $GU(\Phi)$) denotes the unitary (resp. special unitary, similitude) algebraic k-group of the form Φ i.e. for any field K containing k, $U(\Phi)(K) = U(\Phi \otimes K, D \otimes K)$, etc.
If f is a quadratic form over k, then we denote by $O(f)$ (resp. $SO(f)$, $GO(f)$) the orthogonal (resp. special orthogonal, similitude) algebraic k-group of the form f.

Denote by $M = M(\Phi)$ the group of all multiplicators of similitudes of the form Φ. It is well-known and easy to prove that we have $GU(\Phi) =$

$U(\Phi)\cdot G_m$, where $U(\Phi)\cap G_m=(\pm 1)$. For a similitude $g\in GU(\Phi)$, let $m(g)$ be the multiplicator of g and denote by m the corresponding map $GU(\Phi)\to G_m$. Then M is the image of $GU(\Phi)(k)$ via m. We define the special similitude k-group

$$GU^{+}(\Phi)=SU(\Phi)\cdot G_m$$

which is the almost direct product since $SU(\Phi)\cap G_m=(\pm 1)$, and the group of special multiplicators of the form

$$M^{+}=M^{+}(\Phi)=m(GU^{+}(\Phi)(k))\,.$$

For any valuation v of k let $M_v=M(\Phi_v)$, $M_v^{+}=M^{+}(\Phi_v)$ and let

$$M'=M'(\Phi)=\bigcap_v (M_v\cap k^{\times})\,.$$

For $v\notin S$, D_v is a matrix algebra and as it is well-known (cf. e.g. [Sa]), there is a quadratic form f_v of rank $2n$, determined up to isomorphism over k_v by Φ_v such that $M(\Phi_v)=M(f_v)$.

In the case v is a real valuation, $v\notin S$, denote by $j(f_v)$ the inertia index of the form f_v in the sense of Silvester (cf. e.g. [D3]). In this paper all forms are supposed to be nondegenerate.

1. The special group of multiplicators.

In this section we determine the groups M^{+} and M_v^{+}. We have

PROPOSITION 1. $M^{+}=\bigcap_v (M_v^{+}\cap k^{\times})$.

PROOF. We consider the following commutative diagram with exact rows

$$\begin{array}{ccccc}
GU^{+}(\Phi)(k) & \xrightarrow{\ m\ } & G_m(k) & \xrightarrow{\ \delta\ } & H^1(k, SU(\Phi)) \\
\downarrow & & \downarrow & & \downarrow \alpha \\
\prod_v GU^{+}(\Phi)(k_v) & \xrightarrow{\ m'\ } & \prod_v G_m(k_v) & \xrightarrow{\ \delta'\ } & \prod_v H^1(k_v, SU(\Phi))
\end{array}$$

Now let $x\in\bigcap_v (M_v^{+}\cap k^{\times})$. Then $0=\delta'(x)=\alpha(\delta(x))$, hence $\delta(x)=0$, since α is injective (cf. [K]). Hence $x\in \mathrm{Im}\,(m)$. Q.E.D.

PROPOSITION 2. *If Φ is a nondegenerate quadratic form of even rank $2m$ over k or a skew-hermitian form over a split quaternion algebra D then $M(\Phi)=M^{+}(\Phi)$.*

PROOF. Assume that Φ is a quadratic form. We have $GO^{+}(\Phi)=\{g\in GO(\Phi):\det(g)=m(g)^{m}\}$, $GO^{+}(\Phi)(k)\cdot O(\Phi)(k)=GO(\Phi)(k)$, hence

$$\begin{aligned} M^{+} &= GO^{+}(\Phi)(k)\cdot O(\Phi)(k)/O(\Phi)(k) \\ &= GO(\Phi)(k)/O(\Phi)(k) \\ &= M. \end{aligned}$$

The case of skew-hermitian form is considered in a similar way. Q.E.D.

The following proposition has been proved in [T2] and [Sch1].

PROPOSITION 3. *If* $v\in S$, *then* $M_v=k_v^{\times}$. *Moreover, if* $v\in S$ *and* α *is a skew-quaternion in* D_v, *the equation* $x^J\alpha x=\lambda\alpha$ *has a solution in* D_v *with* $\mathrm{Nrd}(x)=\lambda$ (*resp.* $-\lambda$) *if and only if we have* $(\lambda,\alpha^2)_v=1$ (*resp.* -1), *where* $(\ ,\ )_v$ *denotes the Hilbert symbol in* k_v.

From [T2] we obtain the following description of the set M'

PROPOSITION 4. $M'=\{x\in k^{\times}$: x *is a norm from* k'_v *for all* $v\notin S$ *s.t.* $x>0$ *according to all real valuations* $v\notin S$ *with* $j_v(\Phi)\neq n\}$.

We have a description of the set M_v^{+} as follows.

PROPOSITION 5. *Let* $v\in S$ *and* $\lambda\in k_v^{\times}=M_v$. *Then* $\lambda\in M_v^{+}$ *iff* $(\lambda,d)_v=1$.

PROOF. From the realization of the group $GU^{+}(\Phi)$ as a matrix group over an algebraic closure $\bar{k}$ of k we see that

$$GU^{+}(\Phi)(k)=\{x\in GU(\Phi)(k):\mathrm{Nrd}(x)=(m(x))^{n}\},$$

where Nrd denotes the reduced norm map from $M_n(D)$ to k. (Note that for every $x\in GU(\Phi)(k)$ we have $\mathrm{Nrd}(x)=\pm m(x)^{n}$.) Now let $v\in S$ and let $\Phi=X_1^J\alpha_1X_1+\cdots+X_n^J\alpha_nX_n$ be a diagonalization of Φ, where α_i are skew-quaternions in D. From Prop. 3 it follows that there are $x_1,\cdots,x_n$ in D_v s. t. the following system holds

$$\begin{cases} x_1^J\alpha_1x_1=\lambda\alpha_1 \\ \vdots \\ x_n^J\alpha_nx_n=\lambda\alpha_n \end{cases}$$

Denote $X=\mathrm{diag}(x_1,\cdots,x_n)$. It is clear that $X\in GU(\Phi)(k_v)$ and $m(X)=\lambda$. Since $v\in S$, $U(\Phi)(k_v)=SU(\Phi)(k_v)$ (cf. [D1] or [K]) hence

$$\begin{aligned}\lambda\in M_v^+ &\Longleftrightarrow X\in GU^+(\Phi)(k_v)\\ &\Longleftrightarrow \mathrm{Nrd}\,(X)=m(X)^n\\ &\Longleftrightarrow \prod_{1\leq i\leq n}(\lambda,\alpha_i^2)_v=1\\ &\Longleftrightarrow (\lambda,\prod_{1\leq i\leq n}\alpha_i^2)_v=1\\ &\Longleftrightarrow (\lambda,d)_v=1\,.\end{aligned}$$

Q. E. D.

Finally we can describe the groups M^+ and M as follows.

PROPOSITION 6. *$M^+=\{x\in N_{k'/k}(k')$: $x>0$ according to real valuations $v\notin S$ such that $j_v(\Phi)\neq n\}$.*

PROOF. It follows immediately from Propositions 1, 2, 4, 5 and the Hasse Norm Theorem. Q. E. D.

PROPOSITION 7. *M is either equal to M^+ or contains the latter as a subgroup of index 2.*

PROOF. It follows from the fact that $GU^+(\Phi)$ is either equal to the group $GU(\Phi)$ or is of index 2 in the latter. Q. E. D.

As consequences we have

COROLLARY 1. *If $d\in k_v^{\times 2}$ for all $v\in S$ then $M'=M=M^+$.*

COROLLARY 2. *If the index $[M':M^+]>4$, then $M'=M$.*

The following is just another formulation of a result proved in [T1].

PROPOSITION 8. *If $[M:M^+]=2$, then the group $GU(\Phi)$ satisfies the (cohomological) Hasse principle and $[M':M]=2^{s-2}$.*

2. Examples and applications.

2.1. We give here some examples illustrating the connection between the groups M', M and M^+.

Example 1. Let $n=1$. Then it is easy to see that in this case $[M:M^+]=2$, hence by Prop. 8, $[M':M]=2^{s-2}$. Moreover, we can choose here explicitly a complete set of representatives of M' modulo M (cf. [T1], Example 1).

Example 2. Let $n=2$ and let $\Phi=X^J\alpha X+Y^J\beta Y$. Using similar arguments as in the proof of Prop. 3, we can choose the skew-hermitian form Φ such that $M=M^+$ or $M\neq M^+$, as we wish. In fact we can choose α, β such that the following system

$$\begin{cases} x^J\alpha x=\lambda\alpha \\ y^J\beta y=\lambda\beta \end{cases}$$

has a solution in D for some $\lambda\in k^\times$, such that $\mathrm{Nrd}\,(x)=-\mathrm{Nrd}\,(y)=\lambda$. Then $M\neq M^+$. Moreover, for any even number n and for the form

$$\Psi=X_1^J\alpha X_1+\cdots+X_{n-1}^J\alpha X_{n-1}+X_n^J\beta X_n$$

we have $M(\Psi)\neq M^+(\Psi)$, where α and β are chosen as above.

Example 3. Let Φ be arbitrary skew-hermitian form and n be a natural number. Denote by $n\Phi$ the orthogonal sum of n copies of Φ. Then we have

$$M(n\Phi)=\begin{cases} M'(n\Phi)=M^+(n\Phi), & \text{if } n \text{ is even,} \\ M(\Phi), & \text{if } n \text{ is odd and } M(\Phi)=M^+(\Phi)\,. \end{cases}$$

The first case is clear by Corollary 1 and the second follows from the facts that $M'(n\Phi)=M'(\Phi)$ (cf. Prop. 3), $M^+(n\Phi)=M^+(\Phi)$ (cf. Prop. 6), $M(n\Phi)\supseteqq M(\Phi)$ as it is easy to see and $[M'(n\Phi):M(n\Phi)]=[M'(n\Phi):M(\Phi)]$ $=2^{s-2}$ (by Prop. 8).

By the way, if $s>2$, then for any given skew-hermitian form we have the following equivalence

$$M'(\Phi)=M(\Phi)\Longleftrightarrow M'(\Phi)=M^+(\Phi)\,.$$

Indeed, if $M'=M^+$ then of course $M'=M$. Conversely, if $M'=M$, then $M=M^+$, since otherwise we might have $[M':M]=2^{s-2}>1$, which is impossible. Note that the above equivalence enables us to see (by using the condition $M'=M^+$) whether a given form Φ satisfies the Hasse principle for similarity in case $s>2$, which is the case if $s\leqq 2$.

2.2. Now we give an application to the problem of weak approximation in algebraic groups over global fields. The following (in fact more general) result is due to Harder and has been proved by various techniques (cf. [H1], [S], [KS]). We give here a proof using the above considerations.

PROPOSITION 9. *Let G be an adjoint almost simple k-group of type D_n over a global field k. Then G satisfies the weak approximation over k, i.e. for any finite set T of nonequivalent valuations of k, $G(k)$ is dense in the product $\prod_{v\in T} G(k_v)$ via the diagonal embedding.*

PROOF. Let $\tilde{G}$ be the special unitary k-group covering G. We may assume that char. $k\neq 2$, since otherwise, by virtue of the Cayley transformation in any characteristic (cf. [D2]), $\tilde{G}$ and thus also G satisfy the weak approximation over k. Now let $GU^{+}=\tilde{G}\cdot G_m$ be the almost direct product. By a theorem of Rosenlicht (cf. [R]), GU^{+} is birationally equivalent to the direct product $G\times G_m$. Hence we have only to show that the group GU^{+} has the weak approximation. First we need the following

LEMMA 1. *Let $1\rightarrow G_1\rightarrow G\xrightarrow{\pi} G_2\rightarrow 1$ be an exact sequence of reductive connected k-groups and π be a separable morphism. If G_1 has the weak approximation w.r. to a finite set T of nonequivalent valuations of k and $\pi(G(k))$ is dense in the product $\prod_{v\in T}\pi(G(k_v))$, then G has the weak approximation property w.r. to T.*

PROOF. Let $x\in\prod_{v\in T} G((k_v))$. Since π is separable, it induces an open map with respect to the product topology $\pi_T:\prod_{v\in T} G(k_v)\rightarrow\prod_{v\in T} G_2(k_v)$ by the Implicit Function Theorem (cf. e.g. [H2]). We claim that the closure $\overline{G(k)}$ of $G(k)$ in the product topology is open in $\prod_{v\in T} G(k_v)$. We give here two arguments to prove this fact. First, since the underlying variety of a reductive algebraic group is unirational (by a theorem of Chevalley-Rosenlicht-Grothendieck, [Bo]), there is a surjective k-morphism of varieties $f: A\rightarrow G$, where A is an affine space over k. Again by the Implicit Function Theorem, f induces an open map $f_T:\prod_{v\in T} A(k_v)\rightarrow\prod_{v\in T} G(k_v)$ with respect to the product topology. In particular the image of f_T contains an open set in $\prod_{v\in T} G(k_v)$. But the weak approximation property holds in affine spaces, hence $A(k)$ is dense in $\prod_{v\in T} A(k_v)$ thus $\overline{G(k)}$, containing the image of $\overline{A(k)}$, also contains an open subset of $\prod_{v\in T} G(k_v)$ as required. (This argument is due to Platonov, cf. [P].)

Second, the similar statement for semisimple algebraic groups and for tori is well-known (cf. [H2] and [V] respectively). Now let $G=H\cdot S$, where H (resp. S) is semisimple k-group (resp. k-torus) and the product is almost direct. We have then a central k-isogeny $f: H\times S\rightarrow H\cdot S$. Let

$F=H\times S$. It is clear that one has only to show that $f(\prod_{v\in T} F(k_v))$ is open. If f is purely inseparable then the map f_v induced by f on $F(k_v)$ is bijective for any v, hence we are done. If f is separable, then the Implicit Function Theorem gives us the desired result.

From the above we see that $\pi_T(\overline{G(k)})$ is open, hence also a closed subgroup of $\prod_{v\in T} G_2(k_v)$. Therefore $\overline{\pi(G(k))}$ is contained in $\pi_T\overline{(G(k))}$. From the separability of π it follows that $\prod_{v\in T}\pi(G(k_v))$ is open, hence also closed in $\prod_{v\in T} G_2(k_v)$, and we conclude that $\pi_T\overline{(G(k))}\subseteqq\prod_{v\in T}\pi(G(k_v))$. Now from the assumption finally we have $\pi_T\overline{(G(k))}=\prod_{v\in T}(G(k_v))$. Therefore $\pi_T(x)\in\pi_T\overline{(G(k))}$, i.e. $x\in\overline{G(k)}\cdot\prod_{v\in T}G_1(k_v)=\overline{G(k)}\cdot\overline{G_1(k)}=\overline{G(k)}$. Q.E.D.

By this lemma we have only to prove the following

LEMMA 2. *With notation as above, let* $\tilde{G}=SU(\Phi)$. *Then* $M^+(\Phi)$ *is dense in the product* $\prod_{v\in T} M^+(\Phi_v)$.

PROOF. This follows immediately from the explicit description of the sets $M^+(\Phi_v)$, $M^+(\Phi)$, given above, by making use of the weak approximation in the group G_m. Q.E.D.

The proof of the Prop. 9 is therefore complete.

2.3. For a nondegenerate quadratic form f over a field F of char. $\neq 2$ the Scharlau Norm Principle holds for the group $M(f)$, hence by Prop. 2 also for the group $M^+(f)$, i.e. for any finite extension K of F we have

$$N_{K/F}(M^+(f\otimes K))\subseteqq M^+(f).$$

We show now that this is also true for skew-hermitian forms of type D over local and global fields.

PROPOSITION 10. *Let* Φ *be a nondegenerate skew-hermitian form of type* D *over a quaternion division algebra* D *over local or global field of char.* $\neq 2$. *Then the Scharlau Norm Principle holds for the group* $M^+(\Phi)$ *of special multiplicators of* Φ, *i.e. for any finite extension* K *of* k *we have* $N_{K/k}(M^+(\Phi\otimes K))\subseteqq M^+(\Phi)$.

PROOF. This follows from the description of the sets $M^+(\Phi)$ over local and global fields, from above, and also from the Scharlau Norm Principle for quadratic forms (cf. [L] or [Schl]). Q.E.D.

We discuss here the validity of this principle in a more general situation. From now on we assume that k is a field of char. $\neq 2$, D is an associative division k-algebra of finite dimension over k, Φ is a non-degenerate hermitian form with respect to a k-linear involution J of D. Quite recently E. Bayer and H. W. Lenstra have proved a general theorem of Springer type about the injectivity of the map between the Witt groups of hermitian forms over D and over $D\otimes K$, induced by the inclusion of k into any finite extension K of odd degree (cf. [B], [B-L]). This theorem allows us to obtain the following

PROPOSITION 11. *The Scharlau Norm Principle holds*

a—*for the group $M(\Phi)$ of any hermitian form Φ and extension K as above,*

b—*for the group $M^+(\Phi)$ of any hermitian form Φ of type D and for finite extension K of k, which admits a normal closure L over k of odd degree $[L:K]$ such that L does not split D.*

PROOF. a. The proof (due to Scharlau) in the case of quadratic forms can be carried over to our case (cf. [L], [Sch1] for the proof). In fact, it is due to the following two points:

1. One can form a left $W(k)$—module structure on $W(D,J)$ (the Witt group of hermitian forms with respect to J over D) (cf. [Sch2]).

2. The validity of the theorem of Springer type mentioned above (cf. [B], [B-L]).

For details we refer to [L] and [Sch1].

b. First we assume that K is a normal extension of k. Denote by $G(K/k)$ the group of automorphisms of K over k. Let $x\in M^+(\Phi\otimes K)$. For any automorphism $s\in G(K/k)$ it is clear that $x^s\in M^+(\Phi\otimes K)$ hence $N_{K/k}(x)\in M^+(\Phi\otimes K)$. By part a., $N_{K/k}(x)\in M(\Phi)$. Let g (resp. h) be an element of $GU(\Phi)(k)$ (resp. $GU^+(\Phi)(K)$) s.t.

$$N_{K/k}(x)=m(g)=m(h)\,.$$

From this and from the fact that $U(\Phi)(K)=SU(\Phi)(K)$ by assumption ($K=L$ does not split D), we conclude that $g\in GU^+(\Phi)(k)$, i.e. $N_{K/k}(x)\in M^+(\Phi)$.

Assume now that L is a normal closure of K over k, s.t. L does not split D and $[L:K]=2m+1$, m is integer >0. Let $x\in M^+(\Phi\otimes K)$. Hence $x\in M^+(\Phi\otimes L)$ too and from above we have

$$N_{L/k}(x)=(N_{K/k}(x))^{2m+1}=(N_{K/k}(x))^{2m}\cdot N_{K/k}(x)\in M^+(\Phi)$$

hence $N_{K/k}(x)\in M^+(\Phi)$, since $k^{\times 2}\subseteq M^+(\Phi)$. Q. E. D.

2.4. Here we would like to make a correction to [T2]. Actually Lemma 1, stated there, is proved only for the two dimensional case, and we need only that case. Thus the Lemma should be stated as follows.

LEMMA 1′. *Let Φ be a nondegenerate two dimensional hermitian form with respect to the standard involution J of a quaternion division algebra D over a field k of char. $\neq 2$. Then Φ and ${}^t\Phi$ are equivalent, where Φ is considered as a matrix in $GL_2(D)$ in any fixed basis and ${}^t\Phi$ is the transpose of Φ.*

PROOF. Assume that in a given basis Φ has the following matrix

$$\begin{bmatrix} x & y \\ y^J & z \end{bmatrix}$$

where $x, z \in k$. It follows from a simple calculation that if $y \neq 0$, and $A := \mathrm{diag}(a, d)$, where d is any element from D with norm 1 and $a = ydy^{-J}$, we will have $\Phi = A^t\Phi^t A^J$. Since the case $y = 0$ is trivial, we are done. Q. E. D.

We note also a mistake in the proof of Landherr's Theorem (Lemma 4). The exact sequence stated on top of p. 797 should be as follows

$$1 \longrightarrow SU(\Phi) \longrightarrow U(\Phi) \longrightarrow T \longrightarrow 1$$

where T is a twisted form of G_m. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} T(k) & \longrightarrow & H^1(k, SU(\Phi)) & \xrightarrow{d} & H^1(k, U(\Phi)) & \longrightarrow & H^1(k, T) \\ \downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \\ \prod_v T(k_v) & \longrightarrow & \prod_v H^1(k_v, SU(\Phi)) & \longrightarrow & \prod_v H^1(k_v, U(\Phi)) & \longrightarrow & \prod_v H^1(k_v, T)\,. \end{array}$$

Since the Hasse principle and the weak approximation hold for T (cf. e. g. [S] or [V]), we have $\mathrm{Ker}(\alpha) \subseteq \mathrm{Im}(d)$. The rest of the proof goes through as stated in [T2], with T replacing G_m everywhere.

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