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# *Non-stationary Boussinesq equations*

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## 1. Introduction.

Let $\Omega$ be a bounded domain in $R^n$ with the boundary $\partial\Omega$ such that

$$\partial\Omega=\Gamma_1\cup\Gamma_2, \qquad \Gamma_1\cap\Gamma_2=\emptyset .$$

We consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}+(u\cdot\nabla)u=-\dfrac{1}{\rho}\nabla p+\nu\Delta u+\beta g\theta & \\ \operatorname{div} u=0, & x\in\Omega,\ t>0 , \\ \dfrac{\partial\theta}{\partial t}+(u\cdot\nabla)\theta=\chi\Delta\theta , & \end{cases} \tag{1}$$

$$\begin{cases} u(x,t)=0, \quad \theta(x,t)=\xi(x,t), & x\in\Gamma_1,\ t>0 , \\ u(x,t)=0, \quad \dfrac{\partial}{\partial n}\theta(x,t)=\eta(x,t), & x\in\Gamma_2,\ t>0 , \end{cases} \tag{2}$$

$$\begin{cases} u(x,0)=a_0(x), & \\ & x\in\Omega , \\ \theta(x,0)=\tau_0(x), & \end{cases} \tag{3}$$

where $u=(u_1,u_2,\cdots,u_n)$ is the fluid velocity, $p$ is the pressure, $\theta$ is the temperature, $u\cdot\nabla=\sum_{j=1}^{n}u_j\dfrac{\partial}{\partial x_j}$, $\dfrac{\partial\theta}{\partial n}$ denotes the outer normal derivative of $\theta$ at $x$ to $\partial\Omega$, $g(x,t)$ is the gravitational vector function, and $\rho$ (density), $\nu$ (kinematic viscosity), $\beta$ (coefficient of volume expansion), $\chi$ (thermal diffusivity) are positive constants. $\xi(x,t)$ (resp. $\eta(x,t)$) is a function defined on $\Gamma_1\times(0,T)$ (resp. $\Gamma_2\times(0,T)$). $a_0(x)$ (resp. $\tau_0(x)$) is a vector (resp. scalar) function defined on $\Omega$.

The system of equations (1) describes the motion of fluid of heat convection (Boussinesq approximation). For $\Omega=R^n$, Canon and DiBenedetto [3] showed the local unique existence of strong solution. Ôeda [14] studied the moving boundary case with Dirichlet boundary condition. See also Hisida [5].

For an unbouded domain $\Omega$, which is bounded by two parallel planes, the problem is called the Bénard problem. When the boundary condition for $\theta$ is given as constant on each plane, we can easily find a trivial solution. But in general domain, the existence of solution is not trivial. That is a reason why we begin to study this problem. In our previous papers [12, 13], we showed the existence of weak solution of the stationary problem.

In this paper, the existence of a weak solution of evolutional problem (1), (2), (3) is proved (Theorem 1). Uniqueness and some regularity property are also discussed (Theorem 2, 3). For the definition of weak solution, we use an extension of $\xi$ and solve the system of equations corresponding to it. If $\xi$ is sufficiently smooth, we can extend it to $\bar{\Omega}\times[0, T]$ satisfying certain smallness condition (Lemma 2). Using this result, the existence of a weak solution of (1), (2) satisfying (3) is proved for $2\leqq n\leqq 4$. The argument is based on the construction of approximate solutions by the Galerkin method and a passage to the limit using an a priori estimate on the fractional derivative in time of the approximate solutions and a compactness theorem. (c.f. J. Leray [8, 9], E. Hopf [6], J. L. Lions [10, 11], R. Temam [15]).

Let $\{u, \theta\}$ be a weak solution of (1), (2). If they satisfy the following condition:

$$u(x, 0)=u(x, T), \qquad \theta(x, 0)=\theta(x, T), \tag{4}$$

then we say they have reproductive property (Kaniel-Shinbrot [7]). Under some conditions, we can show the existence of solutions with reproductive property (Theorem 4).

In the Section 2, we define the weak solution of (1), (2), and state our results. They are proved in the Section 4 and the Section 5. The Section 3 contains some lemmas necessary to the proof of theorems.

## 2. Notations and results.

$\Omega$ stands for a bounded domain in $R^n$, and its boundary $\partial\Omega$ satisfies the following:

CONDITION (H). $\partial\Omega$ is of class $C^1$ and divided as follows: $\partial\Omega=\Gamma_1\cup\Gamma_2$, $\Gamma_1\cap\Gamma_2=\emptyset$, measure of $\Gamma_1\neq 0$, and the intersection $\bar{\Gamma}_1\cap\bar{\Gamma}_2$ is an $n-2$ dimensional $C^1$ manifold.

This condition is used in order to extend the function $\xi(x, t)$.

The functions considered in this paper are all real valued. $L^p(\Omega)$ and the Sobolev space $W_p^l(\Omega)$ are defined as usual. We also denote $H^l(\Omega)=W_2^l(\Omega)$. Whether the elements of space are scalar or vector functions is understood from the contexts unless stated explicitly.

We define the inner product and the norm of $L^2(\Omega)$ as follows:

$$(u, v)=\int_{\Omega}\sum_{j=1}^{n}u_j(x)v_j(x)dx, \qquad \|u\|=\sqrt{(u, u)},$$

for vector $L^2(\Omega)$-functions $u=(u_1, \cdots, u_n)$, $v=(v_1, \cdots, v_n)$,

$$(\theta, \tau)=\int_{\Omega}\theta(x)\tau(x)dx, \qquad \|\theta\|=\sqrt{(\theta, \theta)}, \quad \text{for scalar } L^2(\Omega)\text{-functions } \theta, \tau.$$

Now we define the solenoidal function spaces:

$D_\sigma=\{$vector function $\varphi\in C^\infty(\Omega)|$ supp $\varphi\subset\Omega$, div $\varphi=0$ in $\Omega\}$,

$H=$completion of $D_\sigma$ under the $L^2(\Omega)$-norm,

$V=$completion of $D_\sigma$ under the $H^1(\Omega)$-norm.

It is well known that $V=H_0^1(\Omega)\cap H$, where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the $H^1(\Omega)$ norm ([15]). We define another basic function spaces as follows:

$D_0=\{$scalar function $\varphi\in C^\infty(\bar{\Omega})|$ $\varphi\equiv 0$ in a neighborhood of $\Gamma_1\}$,

$W=$completion of $D_0$ under the $H^1(\Omega)$-norm.

Let $\tilde{V}$ be the completion of $D_\sigma$ under the norm $\|u\|_{L^n(\Omega)}+\|u\|_V$, and $\tilde{W}$ the completion of $D_0$ under the norm $\|\theta\|_{L^n(\Omega)}+\|\theta\|_W$. We denote the dual space of $\tilde{V}$, $\tilde{W}$ by $\tilde{V}'$, $\tilde{W}'$. For $2\leqq n\leqq 4$, $\tilde{V}=V$ and $\tilde{W}=W$ because of Sobolev's imbedding theorem (e.g. Adams [1]).

Assume $\{u, p, \theta\}$ be a classical solution of (1), (2), (3). Let us take the $L^2$ inner product of $v\in D_\sigma$ (resp. $\tau\in D_0$) and the first equation of (1) (resp. the third equation of (1)). Then, using the integration by parts, we obtain:

$$\begin{aligned}&\frac{d}{dt}(u, v)+B(u, u, v)=-\nu(\nabla u, \nabla v)+(\beta g\theta, v),\\&\frac{d}{dt}(\theta, \tau)+b(u, \theta, \tau)=-\chi(\nabla\theta, \nabla\tau)+\chi(\eta, \tau)_{\Gamma_2},\end{aligned}\tag{5}$$

where

$$B(u, v, w)=\int_{\Omega}\sum_{i,j=1}^{n}u_j(x)\frac{\partial v_i}{\partial x_j}w_i(x)dx, \quad b(u, \theta, \tau)=\int_{\Omega}\sum_{j=1}^{n}u_j(x)\frac{\partial\theta(x)}{\partial x_j}\tau(x)dx,$$

$$(\eta, \tau)_{\Gamma_2}=\int_{\Gamma_2}\eta(x')\tau(x')d\sigma.$$

By continuity, the equation (5) holds for any $v\in\tilde{V}$, $\tau\in\tilde{W}$.

Now we define the weak solution of (1), (2).

DEFINITION 1. A pair of functions $\{u, \theta\}$ is called a weak solution of (1), (2) if $u \in L^2(0, T: V)$, $\theta - \theta_0 \in L^2(0, T: W)$, for some function $\theta_0(x, t)$ in $L^2(0, T: H^1(\Omega))$ such that $\theta_0(x, t) = \xi(x, t)$, on $\Gamma_1 \times (0, T)$, and $\{u, \theta\}$ satisfy (5) for any $v \in \tilde{V}$, $\tau \in \tilde{W}$, where the derivative with respect to $t$ is in the distribution sense $\mathcal{D}'(0, T)$.

If we suppose merely $u \in L^2(0, T: V)$ and $\theta - \theta_0 \in L^2(0, T: W)$, the condition (3) doesn't necessarily make sense but we have:

LEMMA 1. *Suppose*

$$g \in L^\infty(\Omega \times (0, T)), \quad \theta_0 \in L^2(0, T: H^1(\Omega)), \quad \eta \in L^2(\Gamma_2 \times (0, T))$$

$$u \in L^2(0, T: V), \quad \theta - \theta_0 \in L^2(0, T: W)$$

*and* $\{u, \theta\}$ *satisfy* (5) *for auy* $v \in \tilde{V}, \tau \in \tilde{W}$. *Then* $u$ *and* $\theta$ *are equal to an absolutely continuous function from* $[0, T]$ *into* $\tilde{V}'$ *and* $\tilde{W}'$, *respectively.*

Therefore conditions $u(x, 0) = a_0(x), \theta(x, 0) = \tau_0(x)$ make sense. In the next section, we shall prove Lemma 1 in a similar way to Temam [15].

As for the boundary condition, we can extend $\xi(x, t)$ defined on $\bar{\Gamma}_1 \times [0, T]$, onto $\bar{\Omega} \times [0, T]$ satisfying certain smallness condition.

LEMMA 2. *Suppose* $\Omega$ *satisfy Condition* (H) *and* $\xi(x, t) \in C^1(\bar{\Gamma}_1 \times [0, T])$. *Then for every* $\varepsilon > 0$ *and* $p > 1$, *there exists a function* $\theta_0(x, t)$ *such that*

$$\theta_0 \in C^1(\bar{\Omega} \times [0, T]), \quad \theta_0 = \xi \text{ on } \bar{\Gamma}_1 \times [0, T], \quad \sup_{0 \le t \le T} \|\theta_0(t)\|_{L^p(\Omega)} < \varepsilon .$$

For the proof, see [4], Lemma 6.38. See also Morimoto [12].

Our results are the following theorems.

THEOREM 1. *Let* $n$ *be an integer* $2 \leqq n \leqq 4$, *and* $\Omega$ *a bounded domain in* $R^n$ *with* $C^1$ *boundary satisfying Condition* (H). *If* $g$ *is in* $L^\infty(\Omega \times (0, T))$, $\xi \in C^1(\bar{\Gamma}_1 \times [0, T])$, $\eta \in L^2(\Gamma_2 \times (0, T))$, $a_0 \in H$, $\tau_0 \in L^2(\Omega)$, *then there exists a weak solution* $\{u, \theta\}$ *of* (1), (2) *satisfying the initial condition* (3). *Furthermore*

$$u \in L^\infty(0, T: H), \qquad \theta \in L^\infty(0, T: L^2(\Omega)) .$$

THEOREM 2. *Let* $n = 2$. *The weak solution* $\{u, \theta\}$ *of* (1), (2) *satisfying the initial condition* (3) *is unique. Moreover,* $u$ *and* $\theta$ *are almost everywhere equal to a function continuous from* $[0, T]$ *to* $H$ *and* $L^2(\Omega)$, *respectively.*

THEOREM 3. *Let* $n \geqq 3$. *The weak solution* $\{u, \theta\}$ *of* (1), (2) *satisfying the initial condition* (3) *is unique if*

$$u \in L^2(0, T: V) \cap L^\infty(0, T: H) \quad and \quad \theta \in L^2(0, T: H^1(\Omega)) \cap L^\infty(0, T: L^2(\Omega))$$

*and*

$$u \in L^s(0, T: L^r(\Omega)) \quad and \quad \theta \in L^s(0, T: L^r(\Omega)),$$

*hold for some* $r > n$, $s = 2r/(r-n)$.

Let $g_\infty = \|g\|_{L^\infty(\Omega \times (0, T))}$, and $c_1, c_2$ be constants in the well-known Poincaré inequality:

$$\|u\| \leqq c_1 \|\nabla u\| \quad \text{for} \quad \forall u \in V, \qquad \|\theta\| \leqq c_2 \|\nabla \theta\| \quad \text{for} \quad \forall \theta \in W.$$

THEOREM 4. *Let* $2 \leqq n \leqq 4$, *and* $\Omega$ *be a bounded domain in* $R^n$ *with* $C^1$ *boundary satisfying Condition* (H). *Let* $g$ *be in* $L^\infty(\Omega \times (0, T))$, $\xi \in C^1(\bar{\Gamma}_1 \times [0, T])$ *and* $\eta \in L^2(\Gamma_2 \times (0, T))$. *If* $\beta g_\infty c_1 c_2 < \sqrt{\nu \chi}$, *then there exists a weak solution of* (1), (2) *having reproductive property* (4). *Furthermore*

$$u \in L^\infty(0, T: H), \qquad \theta \in L^\infty(0, T: L^2(\Omega)).$$

## 3. Preliminary lemmas.

In this section, we prepare some lemmas.

LEMMA 3. *Let* $n \geqq 3$. *There exist positive constants* $c_B$ *and* $c_b$ *such that*

$$|B(u, v, w)| \leqq c_B \|\nabla u\| \|\nabla v\| \|w\|_n \qquad for \quad u \in V, v \in H^1, w \in L^n,$$

$$|b(u, \theta, \tau)| \leqq c_b \|\nabla u\| \|\nabla \theta\| \|\tau\|_n \qquad for \quad u \in V, \theta \in H^1, \tau \in L^n.$$

LEMMA 4. *Let* $n$ *be* $2 \leqq n \leqq 4$. *There exist positive constants* $c_B$ *and* $c_b$ *such that*

$$|B(u, v, w)| \leqq c_B \|\nabla u\| \|\nabla v\| \|\nabla w\|, \qquad for \quad u, v, w \in V,$$

$$|b(u, \theta, \tau)| \leqq c_b \|\nabla u\| \|\nabla \theta\| \|\nabla \tau\|, \qquad for \quad u \in V, \theta, \tau \in W.$$

Lemma 3, 4 are proved by Hölder's inequality and Poincaré's inequality. See also e.g. Temam [15].

Now, we prove Lemma 1 for $n \geqq 3$. For $n = 2$, the proof is similar and is omitted.

The space $\tilde{V}$ and $\tilde{W}$ are contained and dense in $H$ and in $L^2(\Omega)$, respectively, and

$$\tilde{V} \subset H \subset \tilde{V}', \qquad \tilde{W} \subset L^2(\Omega) \subset \tilde{W}',$$

hold where each space is dense in the following one and the injections are continuous. Furthermore the injections $\tilde{V} \to H$, $\tilde{W} \to L^2(\Omega)$ are compact.

Since $(\nabla u, \nabla v)$ $(u \in V)$ is continuous linear functional on $\tilde{V}$, it defines a continuous linear mapping $A$ from $V$ into $\tilde{V}'$ as follows:

$${}_{\tilde{V}'}\langle Au, v\rangle_{\tilde{V}} = (\nabla u, \nabla v), \qquad u \in V, v \in \tilde{V},$$

and we have the estimate:

$$\|Au\|_{\tilde{V}'} \leqq \|\nabla u\| . \tag{6}$$

According to Lemma 3, we can define a continuous (nonlinear) mapping $B$ from $V$ to $\tilde{V}'$ as follows:

$${}_{\tilde{V}'}\langle B(u), v\rangle_{\tilde{V}} = B(u, u, v), \qquad u \in V, v \in \tilde{V} .$$

The inequality $\|B(u)\|_{\tilde{V}'} \leqq c_B \|\nabla u\|^2$ holds.

In a similar way to the definition of $A$, a continuous linear mapping $Q$ from $H^1(\Omega)$ to $\tilde{W}'$ is defined as follows:

$${}_{\tilde{W}'}\langle Q\theta, \tau\rangle_{\tilde{W}} = (\nabla\theta, \nabla\tau), \qquad \theta \in H^1(\Omega), \tau \in \tilde{W},$$

and we have the estimate:

$$\|Q\theta\|_{\tilde{W}'} \leqq \|\nabla\theta\| . \tag{7}$$

Using Lemma 3, we can define a bilinear bounded operator $b$ from $V \times H^1$ to $\tilde{W}'$ as follows:

$${}_{\tilde{W}'}\langle b(u, \theta), \tau\rangle_{\tilde{W}} = b(u, \theta, \tau), \qquad u \in V, \theta \in H^1, \tau \in \tilde{W} .$$

The estimate $\|b(u, \theta)\|_{\tilde{W}'} \leqq c_b \|\nabla u\| \|\nabla\theta\|$ holds.

Since

$$|(\eta, \tau)_{\Gamma_2}| \leqq \|\eta\|_{\Gamma_2} \|\tau\|_{\Gamma_2} \leqq c_3 \|\eta\|_{\Gamma_2} \|\nabla\tau\|, \qquad \tau \in \tilde{W},$$

we can find $\Xi \in \tilde{W}'$ satisfying ${}_{\tilde{W}'}\langle \Xi, \tau\rangle_{\tilde{W}} = (\eta, \tau)_{\Gamma_2}$.

Using these operators, the equation (5) is transformed as follows:

$$\frac{d}{dt}(u, v) = {}_{\tilde{V}'}\langle -\nu Au - Bu + \beta g\theta, v\rangle_{\tilde{V}}, \qquad v \in \tilde{V}, \tag{8}$$

$$\frac{d}{dt}(\theta, \tau) = {}_{\tilde{W}'}\langle -\chi Q\theta - b(u, \theta) + \chi\Xi, \tau\rangle_{\tilde{W}}, \qquad \tau \in \tilde{W} . \tag{9}$$

If $u\in L^2(0, T: V)$ and $\theta-\theta_0\in L^2(0, T: W)$, then we see

$$-\nu Au-Bu+\beta g\theta\in L^1(0, T: \tilde{V}'), \quad -\chi Q\theta-b(u,\theta)+\chi\Xi\in L^1(0, T: \tilde{W}') .$$

By the same argument of Temam (Lemma 1.1, Chap. 3 [15]), $u$ and $\theta$ are a. e. equal to an absolutely continuous function from $[0, T]$ into $\tilde{V}'$ and $\tilde{W}'$, respectively. Therefore $u|_{t=0}$, $\theta|_{t=0}$ make sense.

## 4. Proof of the existence and the uniqueness.

We construct a weak solution of (1), (2) for $2\leqq n\leqq 4$ by the Galerkin method. Since $V$ is separable and $D_\sigma$ is dense in $V$, there exists a sequence $\{u_j\}$ of elements of $D_\sigma$, which is a basis of $V$ and orthonormal in $H$. Similarly, there exists a sequence $\{\theta_j\}$ of elements of $D_0$, which is a basis of $W$ and orthonormal in $L^2(\Omega)$ (scalar). For each $m$, we consider an approximate solution $\{u^{(m)}(t), \theta^{(m)}(t)+\theta_0\}$ of (5) as follows:

$$u^{(m)}(t)=\sum_{j=1}^{m} f_j^{(m)}(t)u_j, \qquad \theta^{(m)}(t)=\sum_{j=1}^{m} g_j^{(m)}(t)\theta_j , \tag{10}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}u^{(m)}, u_k\right)+B(u^{(m)}, u^{(m)}, u_k)\\ &=-\nu(\nabla u^{(m)}, \nabla u_k)+(\beta g\theta^{(m)}, u_k)+(\beta g\theta_0, u_k), \qquad 1\leqq k\leqq m ,\end{aligned} \tag{11}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\theta^{(m)}, \theta_k\right)+b(u^{(m)}, \theta^{(m)}, \theta_k)\\ &=-\chi(\nabla\theta^{(m)}, \nabla\theta_k)-b(u^{(m)}, \theta_0, \theta_k)-\left(\frac{\partial}{\partial t}\theta_0, \theta_k\right)\\ &\quad-\chi(\nabla\theta_0, \nabla\theta_k)+\chi(\eta, \theta_k)_{\Gamma_2}, \qquad 1\leqq k\leqq m ,\end{aligned} \tag{12}$$

$$u^{(m)}(0)=u_{m0}\equiv\sum_{j=1}^{m}(a_0, u_j)u_j, \qquad \theta^{(m)}(0)=\theta_{m0}\equiv\sum_{j=1}^{m}(\tau_0-\theta_0(\cdot, 0), \theta_j)\theta_j .$$

Substituting (10) into (11), (12) we obtain a system of nonlinear differential equations for $f_j^{(m)}(t)$, $g_j^{(m)}(t)$, $1\leqq j\leqq m$, with the initial conditions

$$f_j^{(m)}(0)=(a_0, u_j), \qquad g_j^{(m)}(0)=(\tau_0-\theta_0(\cdot, 0), \theta_j) .$$

This initial value problem has a maximal solution defined on some interval $[0, t_m]$. The a priori estimate which we are going to prove shows we can take $t_m=T$.

Multiplying (11) by $f_k^{(m)}(t)$, (12) by $g_k^{(m)}(t)$ respectively, and summing with respect to $k$, we have:

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|^2+\nu\|\nabla u^{(m)}(t)\|^2=(\beta g\theta^{(m)}, u^{(m)})+(\beta g\theta_0, u^{(m)}), \tag{13}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\theta^{(m)}(t)\|^2+\chi\|\nabla\theta^{(m)}(t)\|^2\\&\quad=b(u^{(m)}, \theta^{(m)}, \theta_0)-\left(\frac{\partial}{\partial t}\theta_0, \theta^{(m)}\right)-\chi(\nabla\theta_0, \nabla\theta^{(m)})+\chi(\eta, \theta^{(m)})_{\Gamma_2},\end{aligned} \tag{14}$$

where we used $B(u^{(m)}, u^{(m)}, u^{(m)})=0, b(u^{(m)}, \theta^{(m)}, \theta^{(m)})=0$. Using Schwarz' and Poincaré's inequality for the right hand side of (13), we have:

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|^2+\frac{\nu}{2}\|\nabla u^{(m)}(t)\|^2\leqq\frac{(\beta g_\infty c_1)^2}{\nu}(\|\theta^{(m)}(t)\|^2+\|\theta_0(t)\|^2). \tag{15}$$

Similarly, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\theta^{(m)}\|^2+\frac{\chi}{8}\|\nabla\theta^{(m)}\|^2\\&\leqq\frac{c}{\chi}\|\theta_0\|_4^2\|\nabla u^{(m)}\|^2+\frac{c_2^2}{\chi}\left\|\frac{\partial}{\partial t}\theta_0\right\|^2+\chi\|\nabla\theta_0\|^2+2\chi c_3^2\|\eta\|_{\Gamma_2}^2,\end{aligned} \tag{16}$$

where $c, c_3$ are constants depending only on $\Omega$. Integrate (15) (16) with respect to $t$, and we obtain:

$$\begin{aligned}&\|u^{(m)}(t)\|^2+\nu\int_0^t\|\nabla u^{(m)}(s)\|^2ds\\&\leqq\|u_{m0}\|^2+\frac{2(\beta g_\infty c_1)^2}{\nu}\int_0^t(\|\theta^{(m)}(s)\|^2+\|\theta_0(s)\|^2)ds, \qquad 0\leqq t\leqq T,\end{aligned} \tag{17}$$

$$\begin{aligned}&\|\theta^{(m)}(t)\|^2+\frac{\chi}{4}\int_0^t\|\nabla\theta^{(m)}(s)\|^2ds\\&\leqq\|\theta_{m0}\|^2+\frac{2c}{\chi}\int_0^t\|\theta_0(s)\|_4^2\|\nabla u^{(m)}(s)\|^2ds\\&\quad+\frac{2c_2^2}{\chi}\int_0^t\left\|\frac{\partial}{\partial s}\theta_0\right\|^2ds+2\chi\int_0^t\{\|\nabla\theta_0(s)\|^2+2c_3\|\eta(s)\|_{\Gamma_2}^2\}ds,\\&\qquad\qquad 0\leqq t\leqq T.\end{aligned} \tag{18}$$

Substitute $c_2^2\|\nabla\theta^{(m)}(s)\|^2$ for $\|\theta^{(m)}(s)\|^2$ in (17). Using (18), we transform (17) as follows:

$$\|u^{(m)}(t)\|^2+\nu\int_0^t\|\nabla u^{(m)}(s)\|^2ds$$

$$\leqq\|u_{m0}\|^2+\frac{2(\beta g_\infty c_1)^2}{\nu}\int_0^t\|\theta_0(s)\|^2ds+\frac{8(\beta g_\infty c_1c_2)^2}{\nu\chi}\|\theta_{m0}\|^2$$

$$+\frac{8(\beta g_\infty c_1c_2)^2}{\nu\chi}\int_0^t\left\{\frac{2c}{\chi}\|\theta_0(s)\|_4^2\|\nabla u^{(m)}(s)\|^2+\frac{2c_2^2}{\chi}\left\|\frac{\partial}{\partial s}\theta_0\right\|^2\right.$$

$$\left.+2\chi\|\nabla\theta_0\|^2+4c_3\chi\|\eta\|_{\Gamma_2}^2\right\}ds\,.$$

We can choose $\theta_0$ for which the inequality

$$\nu'\equiv\nu-\frac{16c(\beta g_\infty c_1c_2)^2}{\nu\chi^2}\sup_{0\leq t\leq T}\|\theta_0(t)\|_4^2>0$$

holds (Lemma 2). Therefore

$$\|u^{(m)}(t)\|^2+\nu'\int_0^t\|\nabla u^{(m)}(s)\|^2ds$$

$$\leqq\|u_{m0}\|^2+\frac{2(\beta g_\infty c_1)^2}{\nu}\int_0^t\|\theta_0(s)\|ds \tag{19}$$

$$+\frac{8(\beta g_\infty c_1c_2)^2}{\nu\chi}\left\{\|\theta_{m0}\|+\frac{2c_2^2}{\chi}\int_0^t\left\|\frac{\partial}{\partial s}\theta_0\right\|^2ds+2\chi\int_0^t(\|\nabla\theta_0\|^2+2c_3^2\|\eta\|_{\Gamma_2}^2)ds\right\}.$$

The right hand side of (19) is bounded by a constant independent of $m=1,2,3,\cdots$, and $t\in[0,T]$. The sequence $\{u^{(m)}(t)\}_m$ is therefore a bounded sequence in $L^2(0,T\colon V)$ and in $L^\infty(0,T\colon H)$. Similarly, we see $\{\theta^{(m)}(t)\}_m$ is a bounded sequence in $L^2(0,T\colon W)$ and in $L^\infty(0,T\colon L^2(\Omega))$. Thereby we can choose subsequences of $\{u^{(m)}\}$, $\{\theta^{(m)}\}$, which we denote by the same symbols, such that

$$u^{(m)}\longrightarrow u \text{ weakly in } L^2(0,T\colon V),\text{ weakly* in } L^\infty(0,T\colon H),$$

$$\theta^{(m)}\longrightarrow \tilde\theta \text{ weakly in } L^2(0,T\colon W),\text{ weakly* in } L^\infty(0,T\colon L^2(\Omega))\,.$$

By the similar argument used in the proof of Theorem 3.1, Chap. 3 [15], we can show the sequence $\{|\tau|^\gamma\hat u^{(m)}(\tau)\}_m$ is bounded in $L^2(\boldsymbol{R}\colon H)$, and $\{|\tau|^\gamma\hat\theta^{(m)}(\tau)\}_m$ is bounded in $L^2(\boldsymbol{R}\colon L^2(\Omega))$, for some $0<\gamma<1/4$, where $\hat u^{(m)}$ (resp. $\hat\theta^{(m)}$) denotes the Fourier transform of the extension $\tilde u^{(m)}$ (resp. $\tilde\theta^{(m)}$) of $u^{(m)}$ (resp. $\theta^{(m)}$), that is,

$$\tilde u^{(m)}(t)=\begin{cases}u^{(m)}(t), & 0\leqq t\leqq T,\\ 0, & \text{otherwise},\end{cases}\qquad \tilde\theta^{(m)}(t)=\begin{cases}\theta^{(m)}(t), & 0\leqq t\leqq T,\\ 0, & \text{otherwise}.\end{cases}$$

Therefore, by Theorem 2.2, Chap. 3 ([15]), we can select subsequences which we denote by the same symbol such that

$$u^{(m)} \longrightarrow u \text{ strongly in } L^2(0, T: H),$$

$$\theta^{(m)} \longrightarrow \tilde{\theta} \text{ strongly in } L^2(0, T: L^2(\Omega)),$$

and $u, \theta(=\tilde{\theta}+\theta_0)$ satisfy (5) for any $v=u_j, \tau=\theta_j$. By continuity argument, (5) holds for any $v\in V, \tau\in W$; furthermore the initial conditions (3) are satisfied. Theorem 1 is proved.

PROOF OF THEOREM 2. Let $n=2$. First we prove the regularity property. From (8), (9), we have

$$\frac{du}{dt}=-\nu Au-B(u)+\beta g\theta, \qquad \frac{d\theta}{dt}=-\chi Q\theta-b(u, \theta)+\chi\Xi .$$

Since $u\in L^2(0, T: V)$ and $\theta-\theta_0\in L^2(0, T: W)$, (6), (7) show that

$$-\nu Au+\beta g\theta\in L^2(0, T: V'), \qquad -\chi Q\theta+\chi\Xi\in L^2(0, T: W') .$$

By Hölder's inequality and interpolation lemma ([2]), we obtain:

$$|\langle B(u), v\rangle|=|B(u, u, v)|\leqq\|u\|_4^2\|\nabla v\|\leqq c\|u\|\,\|\nabla u\|\,\|\nabla v\| .$$

Therefore, we have:

$$\|B(u)\|_{V'}\leqq c\|u\|\,\|\nabla u\|$$

and $B(u)\in L^2(0, T: V')$ holds. We calculate similarly, and obtain the estimate:

$$\|b(u, \theta)\|_{W'}\leqq\|u\|_4\|\theta\|_4\leqq c(\|u\|\,\|\nabla u\|\,\|\theta\|\,\|\nabla\theta\|)^{1/2} .$$

We can conclude $b(u, \theta)\in L^2(0, T: W')$. Thereby

$$\frac{du}{dt}\in L^2(0, T: V'), \qquad \frac{d\theta}{dt}\in L^2(0, T: W') .$$

Using Lemma 1.2 in Chap. 3 Section 1 of Temam [15] we have:

$$u\in C([0, T]: H) \quad\text{and}\quad \theta\in C([0, T]: L^2(\Omega)), \tag{20}$$

$$\frac{d}{dt}\|u\|^2=2\langle u', u\rangle, \qquad \frac{d}{dt}\|\theta\|^2=2\langle\theta', \theta\rangle, \tag{21}$$

where $'=(d/dt)$, and $\langle\ ,\ \rangle$ denotes the duality between $V'$ and $V$, or between $W'$ and $W$.

Now we show the uniqueness. Let $\{u_1, \theta_1\}$, $\{u_2, \theta_2\}$ be two weak solutions of (1), (2) satisfying (3) and the conditions stated in Theorem 2. It is easy to verify $\theta_1-\theta_2\in W$. The pair of functions $\{u_i, \theta_i\}$ satisfies (5) for any $v\in V$ and $\tau\in W$. We subtract the corresponding equations. Putting $u=u_1-u_2$, $\theta=\theta_1-\theta_2$, we have

$$\frac{d}{dt}(u, v)+B(u, u_1, v)+B(u_2, u, v)+\nu(\nabla u, \nabla v)=\beta(g\theta, v),$$

$$\frac{d}{dt}(\theta, \tau)+b(u, \theta_1, \tau)+b(u_2, \theta, \tau)+\chi(\nabla\theta, \nabla\tau)=0.$$

Because of (21), we can take $v=u, \tau=\theta$, and we obtain:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|u\|^2+B(u, u_1, u)+\nu\|\nabla u\|^2=(\beta g\theta, u), \\ \dfrac{1}{2}\dfrac{d}{dt}\|\theta\|^2+b(u, \theta_1, \theta)+\chi\|\nabla\theta\|^2=0. \end{cases} \tag{22}$$

Following inequalities are easily obtained:

$$|B(u, u_1, u)|=|-B(u, u, u_1)|\leqq\frac{\nu}{2}\|\nabla u\|^2+C_\nu\|u\|^2\|u_1\|_4^4,$$

$$|b(u, \theta_1, \theta)|=|-b(u, \theta, \theta_1)|\leqq\chi\|\nabla\theta\|^2+\frac{\nu}{2}\|\nabla u\|^2+C_{\nu,\chi}\|u\|^2\|\theta_1\|_4^4,$$

where $C_\nu$, $C_{\nu,\chi}$ are constants depending only on $\nu$, $\chi$ and $\Omega$. Substituting these estimates into (22), we obtain:

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2+\|\theta\|^2)\leqq f(t)(\|u\|^2+\|\theta\|^2),$$

where $f(t)=C_\nu\|u_1\|_4^4+C_{\nu,\chi}\|\theta_1\|_4^4+\beta g_\infty$. Since $u_1$ is in $L^2(0, T: V)\cap L^\infty(0, T: H)$, we have:

$$\int_0^T\|u_1\|_4^4dt\leqq 2\|u_1\|_{L^\infty(0,T:H)}^2\|u_1\|_{L^2(0,T:V)}^2<+\infty.$$

We obtain similar result for the integral of $\|\theta_1\|_4^4$. Therefore $f(t)$ is integrable on $[0, T]$ and the following inequality holds:

$$\frac{d}{dt}\left[\exp\left(-2\int_0^t f(s)ds\right)\{\|u(t)\|^2+\|\theta(t)\|^2\}\right]\leqq 0.$$

Since $u(0)=0$, $\theta(0)=0$, we obtain the desired result.

PROOF OF THEOREM 3. Let $n\geqq 3$, and $\{u_i, \theta_i\}$, $i=1,2$, be two weak solutions satisfying the conditions stated in Theorem 3. Under the assumption

$$u\in L^s(0, T: L^r(\Omega)) \quad \text{and} \quad \theta\in L^s(0, T: L^r(\Omega)),$$

where $r>n$, $s=2r/(r-n)$, we can show

$$\frac{du}{dt}\in L^2(0, T: V'), \qquad \frac{d\theta}{dt}\in L^2(0, T: W')$$

(c.f. Lions [11], p. 84) and we obtain (22). Let $1/q+1/r=1/2$ and $\alpha=1-n/r$. Hölder's inequality and interpolation lemma (c.f. Brezis [2]) yield the following estimate:

$$\begin{aligned} |B(u, u_1, u)| = |B(u, u, u_1)| &\leqq \|u\|_q \|\nabla u\| \|u_1\|_r \\ &\leqq c\|u\|^{\alpha}\|\nabla u\|^{2-\alpha}\|u_1\|_r \\ &= \nu\|\nabla u\|^2 + C_\nu \|u\|^2 \|u_1\|_r^{2/\alpha} . \end{aligned} \tag{23}$$

Similarly, we have

$$|b(u, \theta_1, \theta)| \leqq \chi\|\nabla\theta\|^2 + C_\chi\|u\|^2\|\theta_1\|_r^{2/\alpha} .$$

Substituting these estimates into (22), we obtain:

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2+\|\theta\|^2) \leqq f(t)(\|u\|^2+\|\theta\|^2),$$

where $f(t)=C_\nu\|u_1\|_r^{2/\alpha}+C_\chi\|\theta_1\|_r^{2/\alpha}+\beta g_\infty$. Under our assumption, $f(t)$ is integrable and

$$\frac{d}{dt}\left[\exp\left\{-2\int_0^t f(t)dt\right\}(\|u(t)\|^2+\|\theta(t)\|^2)\right] \leqq 0 .$$

Since $u(0)=0$, $\theta(0)=0$, we have $u=u_1-u_2=0$, $\theta=\theta_1-\theta_2=0$, and Theorem 3 is proved.

## 5. The reproductive property.

Let us prove Theorem 4. We estimate the right hand side of the equations (13) and (14). Using Hölder's inequality, we have:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|^2 + \nu\|\nabla u^{(m)}\|^2 \\ &\leqq \beta g_\infty c_1 c_2 \|\nabla u^{(m)}\| \|\nabla\theta^{(m)}\| + \frac{\nu}{2}\|\nabla u^{(m)}\|^2 + f_1(t), \end{aligned} \tag{24}$$

where $f_1(t) \equiv \dfrac{(\beta g_\infty c_1)^2}{2\nu} \|\theta_0\|^2$.

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\theta^{(m)}\|^2+\chi\|\nabla\theta^{(m)}\|^2 \\ &\leqq \|u^{(m)}\|_4\|\nabla\theta^{(m)}\|\,\|\theta_0\|_4+\left\|\frac{\partial\theta_0}{\partial t}\right\|\|\theta^{(m)}\| \\ &\quad+\chi\|\nabla\theta_0\|\,\|\nabla\theta^{(m)}\|+c_3\chi\|\eta\|_{\Gamma_2}\|\nabla\theta^{(m)}\| \\ &\leqq \frac{\chi}{2}\|\nabla\theta^{(m)}\|^2+\frac{2c^2}{\chi}\|\theta_0\|_4^2\|\nabla u^{(m)}\|^2+f_2(t)\,, \end{aligned} \tag{25}$$

where $f_2(t) \equiv \dfrac{2c_2^2}{\chi}\left\|\dfrac{\partial\theta_0}{\partial t}\right\|^2+2\chi\|\nabla\theta_0\|^2+2\chi c_3^2\|\eta\|_{\Gamma_2}^2$.

From (24), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|^2+\frac{\nu}{2}\|\nabla u^{(m)}\|^2 \\ &\leqq \frac{\beta g_\infty c_1 c_2}{\sqrt{\nu\chi}}\left\{\frac{\nu}{2}\|\nabla u^{(m)}\|^2+\frac{\chi}{2}\|\nabla\theta^{(m)}\|^2\right\}+f_1(t)\,. \end{aligned} \tag{26}$$

Summing each side of (25) and (26), we obtain:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^{(m)}\|^2+\frac{\nu}{2}\left(1-\frac{\beta g_\infty c_1 c_2}{\sqrt{\nu\chi}}-\frac{2}{\nu}\frac{2c^2}{\chi}\|\theta_0\|_4^2\right)\|\nabla u^{(m)}\|^2 \\ &+\frac{1}{2}\frac{d}{dt}\|\theta^{(m)}\|^2+\frac{\chi}{2}\left(1-\frac{\beta g_\infty c_1 c_2}{\sqrt{\nu\chi}}\right)\|\nabla\theta^{(m)}\|^2 \\ &\leqq f_1(t)+f_2(t)\equiv f(t)\,. \end{aligned} \tag{27}$$

Assume $\beta g_\infty c_1 c_2 < \sqrt{\nu\chi}$. By virtue of Lemma 2, we can choose $\theta_0$ such that

$$\inf_t\left\{1-\frac{\beta g_\infty c_1 c_2}{\sqrt{\nu\chi}}-\frac{4c^2}{\nu\chi}\|\theta_0\|_4^2\right\} \geqq \frac{1}{2}\left(1-\frac{\beta g_\infty c_1 c_2}{\sqrt{\nu\chi}}\right)>0$$

holds. We put $\alpha=2\min\left\{\dfrac{\nu\delta}{4c_1^2}, \dfrac{\chi\delta}{2c_2^2}\right\}$, where $\delta=1-\dfrac{\beta g_\infty c_1 c_2}{\sqrt{\nu\chi}}$. From (27), we have:

$$\frac{d}{dt}\|U^{(m)}\|^2+\alpha\|U^{(m)}\|^2 \leqq 2f(t)\,,$$

where $U^{(m)}$ is the vector $(u^{(m)}, \theta^{(m)})$ and $\|U^{(m)}\|^2=\|u^{(m)}\|^2+\|\theta^{(m)}\|^2$. Integrating this estimate, we obtain:

$$e^{\alpha T}\|U^{(m)}(T)\|^2 \leqq \|U^{(m)}(0)\|^2 + 2\int_0^T e^{\alpha t} f(t)dt. \tag{28}$$

Let us define a mapping $\tau: R^{2m} \rightarrow R^{2m}$ as follows. $U^{(m)}(t)$ is identified with $R^{2m}$-valued function

$$F(t) = (f_1^{(m)}(t), \cdots, f_m^{(m)}(t),\ g_1^{(m)}(t), \cdots, g_m^{(m)}(t)).$$

Let $F_0$ be a constant vector $(f_{01}, \cdots, f_{0m}, g_{01}, \cdots, g_{0m})$ in $R^{2m}$, and $F(t)$ be a solution to the initial value problem (11), (12) with the initial value $F_0$. For $F_0$, we define $\tau(F_0) = F(T)$. Suppose $F_0 = F_0(\lambda)$ be a possible solution to the equation: $F_0 = \lambda\tau(F_0)$, $\lambda \in [0, 1]$.

From (28), we see $\|F_0\|^2$ is bounded by a constant independent of $\lambda$. Brouwer's fixed point theorem [4] shows us the existence of a fixed point of the mapping $\tau$, that is, the existence of a solution $\{u^{(m)}(t), \theta^{(m)}(t)\}$ of (11), (12) satisfying:

$$u^{(m)}(0) = u^{(m)}(T), \qquad \theta^{(m)}(0) = \theta^{(m)}(T).$$

The convergence to the weak solution with reproductive property is proved similarly to the non-stationary case, and is omitted.

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