J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
39 (1992), 87-139.

On unirationality of threefolds which contain toric surfaces with ample normal bundles

By Madoka EBIHARA

Abstract. Let X be a nonsingular three-dimensional algebraic variety containing a nonsingular toric surface S with the normal bundle $N_{S/X}$ ample. First, we study the formal neighbourhood $(X, S)^\wedge$ of S in terms of semi-groups which we shall call scopes. Next, we take a nonsingular rational curve C on S and study the formal neighbourhood $(X, C)^\wedge$ of C. We shall prove that there exists a dominant morphism from the formal completion $(\boldsymbol{P}^3, line)^\wedge$ of $\boldsymbol{P}^3$ along a line to $(X, C)^\wedge$ by estimating the scope of the neighbourhood $(X, S)^\wedge$ of S. Then we can prove that X is unirational by using the fact that a connected closed subscheme of positive dimension of $\boldsymbol{P}^n$ is $G3$, which is proved by H. Hironaka and H. Matsumura.

1991 *Mathematics Subject Classification.* Primary 14J30; Secondary 14B20, 14M20, 14M25.

§ 0. Introduction.

In this paper, we shall prove:

MAIN THEOREM. *Let X be a nonsingular complete algebraic variety of dimension three defined over an algebraically closed field k. Assume that X contains a nonsingular projective toric surface S and that the normal bundle $N_{S/X}$ of S in X is ample. Then X is unirational.*

This theorem is considered to be a partial answer to the following question.

QUESTION. *Assume that a nonsingular complete algebraic variety X of dimension three contains a nonsingular rational surface S with $N_{S/X}$ ample. Then is X unirational?*

An algebraic variety X is called unirational if there exists a dominant rational map $\varphi: \boldsymbol{P}^n \to X$. As easily follows by definition, any unirational

This work is partially supported by the Fûjukai Foundation.

variety contains many unirational subvarieties. Then it is natural to ask conversely whether varieties which contain many unirational subvarieties are unirational or not. In fact, we have the following fact due to M. Noether in the two-dimensional case.

FACT (M. Noether). *Assume that a nonsingular projective surface S contains a nonsingular rational curve C with* $(C^2)_S>0$. *Then S is a rational surface.*

In this context, the above question is considered to be a starting point of an attempt to generalize M. Noether's theorem to higher-dimensional cases. But it is not suitable to ask whether threefolds we consider above are rational or not, which is an essentially different point from the two-dimensional case. For example, let X be a nonsingular cubic hypersurface in $\boldsymbol{P}^4$ and $S=X\cap H$, where H denotes a general hyperplane. Then X and S satisfy the assumption of the above question. As is well-known, X is unirational, but not rational (Cf. [CG] Theorem 13.12). This fact suggests that the notion of unirationality is more essential than that of rationality so far as we stand in the framework of our question.

Up to now, there are various kinds of results classifying varieties which contain a given variety as an ample divisor. L. Bădescu ([Ba1], [Ba2]) classified all the normal projective varieties of dimension three that contain $\boldsymbol{P}^2$ or a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ as an ample divisor. According to his classification, such threefolds turn out to be rational and consequently unirational. This result also gives a partial answer to our question by the following fact.

FACT (Cf. [Ha] Th. 4.2). *Let X be a complete algebraic variety and let Y an effective Cartier divisor on X. Then the following are equivalent.*

(1) $N_{Y/X}$ *is ample.*

(2) *There exists a birational morphism* $f: X\to X'$ *such that f is an isomorphism in a neighbourhood of Y and that* $f(Y)$ *is an ample effective divisor on* X'.

By using this fact, we can write our main result in the following alternative form.

MAIN THEOREM (Alternative Version). *Let X be a normal projective algebraic variety of dimension three. Assume that X contains a nonsingular projective toric surface S as an ample divisor and that X is nonsingular along S. Then X is unirational.*

Far from the method of L. Bădescu, we do not aim at precise and concrete classification of threefolds which contain toric surfaces as ample divisors. Practically speaking, it would be difficult to classify such threefolds for an arbitrarily given toric surface. Then how do we show the unirationality of such threefolds? One often constructs a dominant rational map from $\boldsymbol{P}^n$ to an explicitly given variety in order to show that it is unirational. But such a naive way based directly on the definition would be helpless for our question because the varieties whose unirationality we want to show would not be explicitly given.

In this paper, we use a method inspired by F. Campana [C] and Ma. Kato [K]. In a word, we observe the formal neighbourhood of a rational curve in a threefold. It is Theorem 2.1 due to H. Hironaka and H. Matsumura that plays an essential role in our theory (Cf. [HM]). This is a kind of continuation theorem which claims that any formal rational function defined along a closed connected subscheme of positive dimension of $\boldsymbol{P}^n$ extends to a rational function on $\boldsymbol{P}^n$. One can show immediately from this theorem that a variety X which contains a rational curve C with the formal completion $(X, C)^\wedge$ of X along C rationally dominated is unirational, where a neighbourhood $(X, C)^\wedge$ of a rational curve C is said to be rationally dominated if there exists a dominant morphism $\varphi: (\boldsymbol{P}^n, \mathit{line})^\wedge \to (X, C)^\wedge$ (Cf. Def. 2.4 and Prop. 2.5). It is easy to see that the normal bundle $N_{C/X}$ is a positive vector bundle if a neighbourhood (X, C) of a nonsingular rational curve C in X is rationally dominated. Thus we observe the formal neighbourhood of a suitable rational curve C in a given variety X with $N_{C/X}$ positive in order to show that X is unirational.

In this paper, we shall actually prove:

THEOREM 0. *Let S be any nonsingular projective toric surface. Then there exists a nonsingular rational curve C on S such that, for any regular formal neighbourhood (X, S) of S with $N_{S/X}$ being an ample line bundle, the neighbourhood $(X, C)^\wedge$ of C in X is rationally dominated.*

Our main theorem that we mentioned in the beginning immediately follows from Theorem 0. The proof of this theorem is divided into the following three steps in principle.

Step 1. For a given toric surface S and an ample line bundle N on S, we describe all the formal neighbourhoods (X, S) of S with $N_{S/X}$ isomorphic to N.

Step 2. We take a suitable nonsingular rational curve C on S with $(C^2)_S > 0$, which we shall call a reference curve.

Step 3. We prove that the formal neighbourhood $(X, C)^\wedge$ of the reference curve C in X is rationally dominated.

Let us explain our practical plan of the proof. In order to describe a regular formal neighbourhood of a given smooth variety, we cover it by affine open subsets and patch them together by giving the transition relations among their coordinates. Let us begin with neighbourhoods of $\boldsymbol{P}^1$. It can be covered by two sheets of affine open subsets. In order to describe it, we have only to give a transition relation between the coordinates of the two open sets. In § 2, we ask a preliminary question when a neighbourhood of $\boldsymbol{P}^1$ described in such a way is rationally dominated. After quite naive and elementary consideration, we prepare a lemma which provides a sufficient condition for a neighbourhood of $\boldsymbol{P}^1$ to be rationally dominated (Cf. Lemma 2.6). It is one of the most essential assertion in this paper, though it is quite easy to prove. Suppose a description of a neighbourhood of $\boldsymbol{P}^1$ by the transition relation is given. Then we plot the lattice points corresponding to the monomial terms which appear in the formal power series which determine the transition relation. These points form a subset of a Euclidean space, which contain infinitely many points in general. Lemma 2.6 claims that a neighbourhood of $\boldsymbol{P}^1$ is rationally dominated if the set of lattice points constructed in the above way satisfies a kind of boundedness condition. As Lemma 2.6 suggests, the rational dominatedness of a neighbourhood of $\boldsymbol{P}^1$ might essentially follow from some finiteness or boundedness conditions in general.

Now let S be a nonsingular toric surface, C a rational curve on S, and (X, S) a formal neighbourhood of the surface S. It is easy to write a description of the neighbourhood $(X, C)^{\wedge}$ of C in X explicitly by the transition relations if such a description of (X, S) and the defining equation of the curve C in S with respect to the coordinates on S are given. So the problem is reduced to the question how to describe neighbourhoods of toric surfaces.

In § 1. A, we recall a general theory in [SGA1] to construct regular neighbourhoods of a given smooth variety. Roughly speaking, its basic idea due to K. Kodaira and A. Grothendieck is a kind of methods of undetermined coefficients; one successively determine the n-th infinitesimal neighbourhoods containing a given $(n-1)$-th infinitesimal neighbourhood. Let $n \geqq 2$, for simplicity. Suppose some $(n-1)$-th infinitesimal neighbourhood (S_{n-1}, S) of a smooth variety S is given. Then the set of the isomorphism classes of the n-th infinitesimal neighbourhoods containing (S_{n-1}, S) is either empty or a $\mathrm{H}^1(S, \mathscr{F}_n)$-torsor, where $\mathscr{F}_n \cong \Theta_{S_{n-1}} \otimes \mathcal{O}_S \otimes S^n(N^{\vee}_{S/S_{n-1}})$, with $\Theta_{S_{n-1}}$ denoting the tangent bundle of S_{n-1}. The obstruction lies in $\mathrm{H}^2(S, \mathscr{F}_n)$.

We can interpret this general theory in terms of transition relations among the coordinates of open subsets. Suppose that a description of some $(n-1)$-th infinitesimal neighbourhood of S is given, that is, the transition functions are determined up to degree $n-1$ with respect to the coordinates which are transversal to the variety S. In order to construct one of the n-th infinitesimal neighbourhoods that are extensions of the given $(n-1)$-th neighbourhood, we have to add some terms of degree n to the transition functions. Giving such data is equivalent to giving a certain Čech 1-cochain of the sheaf $\mathscr{F}_n$. A technical difficulty in this construction lies in the fact that this cochain does not satisfy the cocycle condition in general. If we give transition functions up to degree $n-1$ to describe some $(n-1)$-th neighbourhood, they are patched together up to degree $n-1$, but they are not patched in degree n. We need to adjust such discrepancies by adding some terms of degree n of the transition functions. This is why the cochain of $\mathscr{F}_n$ that we mentioned above is not a cocycle in general. In many cases, such discrepancies actually appear successively. Thus we essentially need infinitely many terms in order to write down the transition functions, even if the cohomology group $\mathrm{H}^1(S, \mathscr{F}_n)$ vanishes for a sufficiently large n. It is a fatal problem of the method of undetermined coefficients which tries to analyze nonlinear phenomena into successive linear equations.

Now let S be again a nonsingular toric surface and N a line bundle on S. In § 3, we introduce the notion of the scope of a description of a neighbourhood (X, S) of S with $N_{S/X} \cong N$ in order to avoid such difficulty that one must handle infinitely many terms. It is a semigroup contained in the group $M \times \boldsymbol{Z}_{\geq 0}$, where M denotes the group of the characters of the toric surface S. Roughly speaking, it indicates what kinds of monomial terms possibly appear in the transition relations among the coordinates. It is generated as a semi-group by the monomial terms appearing in the transition functions divided by their top terms, and not generated by the monomial terms in the transition functions as they are. This is a key point of the definition of the scope.

By defining scopes in such a way, we can easily prove Theorem 3.8, which we shall call the fundamental theorem on scopes. It claims that we have only to consider Čech 1-cocycles of the sheaves $\Theta_S \otimes N^{-n}$ and N^{1-n} $(n \in \boldsymbol{N})$ in order to calculate the scope of a description of a neighbourhood (X, S) of S with $N_{S/X} \cong N$. It is the following two points that are essential in this statement.

(1) The sheaves $\Theta_S \otimes N^{-n}$ and N^{1-n} depend only on the surface S and

the line bundle N on S, while the sheaves $\mathscr{F}_n$ depend on the first infinitesimal neighbourhoods of S.

(2) We don't have to consider Čech cochains in general, but we have only to consider Čech cocycles. That is, we are released from the problem of successive discrepancies to write down the transition functions that we mentioned above. For example, we can prove as the corollary of Theorem 3.8 that a neighbourhood of a toric surface with the normal bundle N admits a description with a finitely generated scope, if $H^1(S, \mathscr{F}_n)=0$ for a sufficiently large n, especially if the line bundle N is ample.

The scope does not indicate precise transition relations, but it is enough to prove Theorem 0. Practically, we prove Theorem 0 by the induction on the Picard number $\rho(S)$ of S. So we need to estimate scopes in the following situation. Let $f: \tilde{S} \to S$ be an equivariant blowing-up along a point. Let $\tilde{N}$ and N be line bundles on $\tilde{S}$ and S respectively such that $\tilde{N} \cong f^*N \otimes \mathcal{O}(-cE)$, where E denotes the exceptional divisor of f and c a positive integer. In § 4, we compare the scopes of neighbourhoods of $\tilde{S}$ with the normal bundle $\tilde{N}$ with those of neighbourhoods of S with the normal bundle N. It is possible in principle because they are contained in the common lattice $M \times \boldsymbol{Z}_{\geq 0}$, where the group M of the characters is common to both surfaces S and $\tilde{S}$. Theorem 4.2 estimates the difference between the above two semi-groups, which enables us the induction. To prove Theorem 4.2, we use the Leray spectral sequence in terms of Čech cochains. In § 1. B, we prepare general discussions what kinds of correspondence between Čech cochains induce the edge sequence of the Leray spectral sequence.

In § 6, we finally prove Theorem 0. (Cf. Th. 6.1 and Cor. 6.2). For some technical reason, we classify nonsingular projective toric surfaces which are not isomorphic to $\boldsymbol{P}^2$ into three types in § 5. (Cf. Lemma 5.2). We take three types of reference curves, that is, suitable nonsingular rational curves with positive self-intersection numbers, on these three types of toric surfaces, respectively. The proof of Theorem 0 is done by estimating scopes of descriptions of neighbourhoods of toric surfaces of each kind and by applying Lemma 2.6 to the induced descriptions of neighbourhoods of the reference curves. These three arguments are reduced to the same kind of numerical statements on scopes, which we discuss in § 5.

When the surface S is not toric, the scope of a neighbourhood of S is not defined. Thus we restrict ourselves to the case where the surface S is toric. But the author believes that the question in the beginning would be affirmatively solved in general for the following reason.

The unirationality of threefolds follows from the rational dominatedness of neighbourhoods of rational curves on them. The rational dominatedness seems to follow from some finiteness conditions. In general, we need infinitely many parameters to describe neighbourhoods of curves with positive normal bundles. But, as Gieseker [G] pointed out, we need essentially finite parameters to describe neighbourhoods of surfaces with ample normal bundles, which is easily shown by Serre duality and Serre vanishing theorem. Thus the assumption of the question in the beginning would imply that the neighbourhood of the reference curve is finite in some sense. Such finiteness seems to be deeply related to the rational dominatedness of neighbourhoods of the reference curves.

Acknowledgement.

The author would like to express his thanks to Professor S. Iitaka and Professor Y. Kawamata for their advices and warm encouragement. He also expresses his thanks to Professor F. Campana and Professor Ma. Kato whose papers inspired him the central idea of this paper.

§ 1. Preliminaries.

In this section, we discuss general theories which are well-known and which we need to use later. We divide this section into two parts. In the first part, we discuss how to construct regular neighbourhoods of a given smooth variety with a given vector bundle as the normal bundle. In the second one, we discuss the Leray spectral sequence. Both theories are well-known, but we need to interpret them in terms of the Čech cohomology for later use. We briefly survey these theories and prepare some notation which we use later.

A. Construction of neighbourhoods.

Let S be a smooth variety and N a vector bundle on S. We discuss how to construct a regular neighbourhood (X, S) of S with $N_{S/X} \cong N$. There is a general theory in [SGA1]. (Cf. Exposé 3, Theorem 6.3). Though its statement is slightly different from ours, we can use the same argument.

First, we make some definitions.

DEFINITION 1.1. Let S and N be as above. In this paper, n-th infinitesimal neighbourhoods mean regular n-th infinitesimal neighbourhoods. That is, a pair (X, S) is said to be the n-th infinitesimal neighbourhood of S with the normal bundle N if the following are satisfied:

(1) X is a regular scheme and X contains S as the reduced subscheme;
(2) $N_{S/X} \cong N$;
(3) $I^{n+1}=0$, where I denotes the ideal defining S in X.

DEFINITION 1.2. (1) Let (X, S) and (X', S) be two first infinitesimal neighbourhoods of S with the normal bundle N. These neighbourhoods are said to be equivalent to each other if there exists an isomorphism $\varphi: X \to X'$ such that $\varphi \circ i = i'$, where i and i' denotes the inclusions $i: S \to X$ and $i': S \to X'$, respectively.

(2) Let $n \geqq 2$. Let (X_{n-1}, S) be one of the $(n-1)$-th infinitesimal neighbourhoods of S with the normal bundle N. Let (X_n, S) and (X'_n, S) be two n-th infinitesimal neighbourhoods of S containing (X_{n-1}, S). These two neighbourhoods are said to be equivalent to each other if there exists an isomorphism $\varphi: X_n \to X'_n$ such that $\varphi \circ i_n = i'_n$, where i_n and i'_n denote the inclusions $i_n: X_{n-1} \to X_n$ and $i'_n: X_{n-1} \to X'_n$, respectively.

The following is one of the most fundamental proposition, which we essentially use throughout this paper.

PROPOSITION 1.3. *Let S and N be as above.*

(1) *The set of the equivalence classes of the first infinitesimal neighbourhoods of S with the normal bundle N is an* $\mathrm{H}^1(S, \mathcal{G}_1)$*-torsor, where* $\mathcal{G}_1 \cong \Theta_S \otimes N^{\vee}$ *with* Θ_S *denoting the tangent bundle of S and* $N^{\vee}$ *the dual of N.*

(2) *Let $n \geqq 2$, and let (X_{n-1}, S) be one of the $(n-1)$-th infinitesimal neighbourhoods of S with the normal bundle N. Then the set of the equivalence classes of the n-th infinitesimal neighbourhoods which contain (X_{n-1}, S) is either empty or an* $\mathrm{H}^1(S, \mathcal{F}_n)$*-torsor, where* $\mathcal{F}_n \cong \Theta_{X_{n-1}}|_S \otimes S^n(N^{\vee})$ *with* $\Theta_{X_{n-1}}|_S$ *denoting the tangent bundle of* X_{n-1} *restricted to S and* $S^n(N^{\vee})$ *the n-th symmetric power of the dual of N. Moreover, the obstruction lies in* $\mathrm{H}^2(S, \mathcal{F}_n)$.

The proof is done by the same arguments as in [SGA1]. In order to fix the notation, we give here a rough sketch of the proof in the case where S is a nonsingular rational surface and where N is a line bundle, which is necessary to state the definition of scopes in § 3.

Since any nonsingular projective rational surface is covered by affine open subsets which are isomorphic to the affine plane $\boldsymbol{A}_k^2$ and since any regular formal neighbourhood of a smooth affine variety is trivial, we may start from the following situation. Let $\mathcal{U}=(U_i)_{i\in I}$ be an affine open cover, (X, S) a regular formal neighbourhood of S with $N_{S/X} \cong N$, and $\tilde{U}_i = X|_{U_i}$.

We assume that $U_i \cong \mathrm{Spec}\, k[t_i, u_i]$ and $\tilde{U}_i \cong \mathrm{Spf}\, k[t_i, u_i][[X_i]]$, where t_i, u_i and X_i denote the coordinates. On $\tilde{U}_i \cap \tilde{U}_j$ $(i, j \in I)$, the coordinates are related to each other by the following transition relation:

$$(t_i, u_i, X_i) = \Phi_{ij}(t_j, u_j, X_j),$$

where Φ_{ij} is a vector-valued formal power series in X_j with the coefficients in $\Gamma(U_i \cap U_j, \mathcal{O}_S)$. We put $\Phi_{ij} = (f_{ij}, g_{ij}, h_{ij})$, that is, $t_i = f_{ij}(t_j, u_j, X_j)$, $u_i = g_{ij}(t_j, u_j, X_j)$, and $X_i = h_{ij}(t_j, u_j, X_j)$. We expand f_{ij}, g_{ij}, and h_{ij} in the following way:

$$f_{ij} = f_{ij|0}(t_j, u_j) + f_{ij|1}(t_j, u_j, X_j) + \cdots + f_{ij|n}(t_j, u_j, X_j) + \cdots,$$

$$g_{ij} = g_{ij|0}(t_j, u_j) + g_{ij|1}(t_j, u_j, X_j) + \cdots + g_{ij|n}(t_j, u_j, X_j) + \cdots,$$

and

$$h_{ij} = h_{ij|1}(t_j, u_j, X_j) + \cdots + h_{ij|n}(t_j, u_j, X_j) + \cdots,$$

where $f_{ij|n}, g_{ij|n}, h_{ij|n} \in \Gamma(U_i \cap U_j, \mathcal{O}_S) \cdot X_j^n$. Note that $h_{ij|0} = 0$. We put $\Phi_{ij|n} = (f_{ij|n}, g_{ij|n}, h_{ij|n})$. We also use the following notation:

$$f_{ij}^{[n]} = f_{ij|0} + f_{ij|1} + \cdots + f_{ij|n},$$

$$g_{ij}^{[n]} = g_{ij|0} + g_{ij|1} + \cdots + g_{ij|n},$$

$$h_{ij}^{[n]} = h_{ij|1} + \cdots + h_{ij|n},$$

and

$$\Phi_{ij}^{[n]} = \Phi_{ij|0} + \Phi_{ij|1} + \cdots + \Phi_{ij|n}.$$

The collection $\{(f_{ij|0}, g_{ij|0})\}$ of the terms of degree zero is nothing but the transition functions that determine the surface S, which is already given. The collection $\{h_{ij|1}\}$ is nothing but the transition functions that determine the line bundle N, which is also given.

To construct (X, S), we have to give a collection $\{\Phi_{ij}\}$ satisfying the following:

(1) $\{(f_{ij|0}, g_{ij|0})\}$ determines S;
(2) $\{h_{ij|1}\}$ determines N;
(3) $\Phi_{ij}(\Phi_{jk}(t_k, u_k, X_k)) = \Phi_{ik}(t_k, u_k, X_k)$ for $i, j, k \in I$.

To do this, we successively construct the n-th infinitesimal neighbourhoods. We introduce another notation. For $f, g \in \Gamma(U_i \cap U_j, \mathcal{O}_S)[[X_j]]$, we write $f \equiv_n g$ if $f^{[n]} = g^{[n]}$. For $\Phi = (f_1, f_2, f_3)$, $\Psi = (g_1, g_2, g_3) \in \Gamma(U_i \cap U_j, \mathcal{O}_S)^{\oplus 3}[[X_j]]$, we write $\Phi \equiv_n \Psi$ if $f_i \equiv_n g_i$ for $i = 1, 2, 3$. To construct the first infinitesimal neighbourhood, we have to give a collection $\{(f_{ij|1}, g_{ij|1})\}$ and determine $\{\Phi_{ij}^{[1]}\}$ satisfying the following condition $(*)_1$:

$$(*)_1 \qquad \Phi_{ij}^{[1]}(\Phi_{jk}^{[1]}(t_k, u_k, X_k)) \equiv_1 \Phi_{ik}^{[1]}(t_k, u_k, X_k) \qquad \text{for} \quad i, j, k \in I.$$

To the pair $(f_{ij|1}, g_{ij|1})$, we attach an element $\lambda_{ij|1} \in \Gamma(U_i \cap U_j, \mathcal{G}_1)$ in the following way:

$$\begin{aligned}\lambda_{ij|1} &= \left(\frac{\partial}{\partial t_i}\right)^0 \otimes f_{ij|1}(f_{ji|0}, g_{ji|0}, h_{ji|1}) \qquad \text{mod } X_i^2 \\ &\quad + \left(\frac{\partial}{\partial u_i}\right)^0 \otimes g_{ij|1}(f_{ji|0}, g_{ji|0}, h_{ji|1}) \qquad \text{mod } X_i^2 \\ &= \left(\frac{\partial}{\partial t_i}\right)^0 \otimes \tilde{f}_{ij|1}(t_i, u_i)X_i \qquad \text{mod } X_i^2 \\ &\quad + \left(\frac{\partial}{\partial u_i}\right)^0 \otimes \tilde{g}_{ij|1}(t_i, u_i)X_i \qquad \text{mod } X_i^2,\end{aligned}$$

where $(\partial/\partial u_i)^0$ and $(\partial/\partial u_i)^0$ denote the local basis of the sheaf Θ_S on U_i and $X_i \bmod X_i^2$ is the local basis of the sheaf $N^\vee$ on U_i. Since $\tilde{f}_{ij|1}$ and $\tilde{g}_{ij|1}$ belong to $\Gamma(U_i \cap U_j, \mathcal{O}_S)$, we can consider $\lambda_{ij|1}$ to be an element of $\Gamma(U_i \cap U_j, \mathcal{G}_1)$. Thus we often identify a collection $\{(f_{ij|1}, g_{ij|1})\}_{i,j\in I}$ with $\lambda_1 = (\lambda_{ij|1})_{i,j\in I} \in C^1(\mathcal{U}, \mathcal{G}_1)$.

CLAIM 1.4. A collection $\{(f_{ij|1}, g_{ij|1})\}$ determines $\{\Phi_{ij}^{[1]}\}$ satisfying the condition $(*)_1$ if and only if the corresponding Čech cochain λ_1 satisfies the cocycle condition, i.e., $\lambda_1 \in Z^1(\mathcal{U}, \mathcal{G}_1)$.

PROOF. If we put

$$J_{ij}^0 = \begin{pmatrix} \dfrac{\partial f_{ij|0}}{\partial t_j} & \dfrac{\partial g_{ij|0}}{\partial t_j} & 0 \\ \dfrac{\partial f_{ij|0}}{\partial u_j} & \dfrac{\partial g_{ij|0}}{\partial u_j} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have

$$\begin{pmatrix} \dfrac{\partial}{\partial t_j} \\ \dfrac{\partial}{\partial u_j} \\ 0 \end{pmatrix} = J_{ij}^0 \cdot \begin{pmatrix} \dfrac{\partial}{\partial t_i} \\ \dfrac{\partial}{\partial u_i} \\ 0 \end{pmatrix}.$$

Since

$$
\begin{aligned}
&\Phi_{ik}-\Phi_{ij}(\Phi_{jk})\\
&\equiv_1 \Phi_{ik|0}+\Phi_{ik|1}-\Phi_{ij|0}(\Phi_{jk|0}+\Phi_{jk|1})-\Phi_{ij|1}(\Phi_{jk})\\
&\equiv_1 (f_{ik|1}, g_{ik|1}, h_{ik|1})-(f_{jk|1}, g_{jk|1}, 0)\cdot J^0_{ij}(\Phi_{jk|0})\\
&\quad -(f_{ij|1}(\Phi_{jk|0}), g_{ij|1}(\Phi_{jk|0}), h_{ik|1}),
\end{aligned}
$$

we have $\Phi_{ik}\equiv_1 \Phi_{ij}(\Phi_{jk})$ if and only if

$$
(f_{ik|1}, g_{ik|1})\cdot\begin{pmatrix}\dfrac{\partial}{\partial t_i}\\ \dfrac{\partial}{\partial u_i}\end{pmatrix}
$$

$$
=(f_{ij|1}(\Phi_{jk|0}), g_{ij|1}(\Phi_{jk|0}))\cdot\begin{pmatrix}\dfrac{\partial}{\partial t_i}\\ \dfrac{\partial}{\partial u_i}\end{pmatrix}+(f_{jk|1}, g_{jk|1})\cdot\begin{pmatrix}\dfrac{\partial}{\partial t_j}\\ \dfrac{\partial}{\partial u_j}\end{pmatrix},
$$

which proves the claim.

CLAIM 1.5. Let $\lambda, \lambda'\in Z^1(\mathcal{U}, \mathcal{G}_1)$. The first infinitesimal neighbourhoods determined by λ and λ' are equivalent to each other if and only if $\lambda'-\lambda\in B^1(\mathcal{U}, \mathcal{G}_1)$.

PROOF. Assume that there exists an element $\psi=(\psi_i)_{i\in I}$ of $C^0(\mathcal{U}, \mathcal{G}_1)$ such that $\lambda'-\lambda=d\psi$. We put $\psi_i=(p_i, q_i)\begin{pmatrix}\partial/\partial t_i\\ \partial/\partial u_i\end{pmatrix}$, where $p_i, q_i\in\Gamma(U_i, \mathcal{O}_S)X_i$. Let $\{\Phi_{ij}\}$ be a collection of transition functions determined by λ. If we replace the coordinates (t_i, u_i, X_i) on each $\tilde{U}_i$ by $(t_i', u_i', X_i)=(t_i+p_i, u_i+q_i, X_i)$, then the transition function Φ_{ij} on $\tilde{U}_i\cap\tilde{U}_j$ changes to a certain function Φ_{ij}'. After easy calculation, we have

$$
\Phi'_{ij|0}(t_j', u_j', X_j)=\Phi_{ij|0}(t_j', u_j', X_j)
$$

and

$$
\Phi'_{ij|1}(t_j', u_j', X_j)=\Phi_{ij|1}(t_j', u_j', X_j)+(p_i,\ q_i,\ 0)-(p_j,\ q_j,\ 0)J^0_{ij}.
$$

Then Claim 1.5 immediately follows.

Suppose that a description of the $(n-1)$-th infinitesimal neighbourhood is given ($n\geqq 2$), that is, a collection $\{\Phi_{ij}^{[n-1]}\}$ of the transition functions is determined up to degree $n-1$ with respect to the coordinates X_j's. We have $\Phi_{ij}^{[n-1]}(\Phi_{jk}^{[n-1]})\equiv_{n-1}\Phi_{ik}^{[n-1]}$. We put

$$\Psi_{ijk|n}=(\Phi_{ij}^{[n-1]}(\Phi_{jk}^{[n-1]})-\Phi_{ik}^{[n-1]})_{[n]}=(p_{ijk|n},\ q_{ijk|n},\ r_{ijk|n}),$$

where $\Psi_{[n]}$ denotes the terms of degree n in the coordinate X_k for $\Psi\in\Gamma(U_i\cap U_j\cap U_k,\ \mathcal{O}_S)[[X_k]]$. To the function $\Psi_{ijk|n}$, we attach an element $\mu_{ijk|n}\in\Gamma(U_i\cap U_j\cap U_k,\ \mathcal{F}_n)$ in the following way:

$$\begin{aligned}\mu_{ijk|n}=&\frac{\partial}{\partial t_i}\otimes p_{ijk|n}(f_{ki|0},\ g_{ki|0},\ h_{ki|1}) \mod X_i^{n+1}\\ &+\frac{\partial}{\partial u_i}\otimes q_{ijk|n}(f_{ki|0},\ g_{ki|0},\ h_{ki|1}) \mod X_i^{n+1}\\ &+\frac{\partial}{\partial X_i}\otimes r_{ijk|n}(f_{ki|0},\ g_{ki|0},\ h_{ki|1}) \mod X_i^{n+1}\\ =&\frac{\partial}{\partial t_i}\otimes \tilde{p}_{ijk|n}(t_i,\ u_i)X_i^n \mod X_i^{n+1}\\ &+\frac{\partial}{\partial u_i}\otimes \tilde{q}_{ijk|n}(t_i,\ u_i)X_i^n \mod X_i^{n+1}\\ &+\frac{\partial}{\partial X_i}\otimes \tilde{r}_{ijk|n}(t_i,\ u_i)X_i^n \mod X_i^{n+1},\end{aligned}$$

where $\partial/\partial t_i$, $\partial/\partial u_i$, and $\partial/\partial X_i$ denotes in this time the local basis of the sheaf $\Theta_X|_S$ on U_i and $X_i^n \bmod X_i^{n+1}$ is the local basis of the sheaf N^{-n} on U_i. Thus we often identify a collection $\{\Psi_{ijk|n}\}_{i,j,k\in I}$ with $\mu_n=(\mu_{ijk|n})\in C^2(\mathcal{U},\ \mathcal{F}_n)$.

CLAIM 1.6. $\mu_n\in Z^2(\mathcal{U},\ \mathcal{F}_n)$.

PROOF. If we put

$$J_{ij}=\begin{pmatrix}\dfrac{\partial f_{ij|0}}{\partial t_j} & \dfrac{\partial g_{ij|0}}{\partial t_j} & 0\\ \dfrac{\partial f_{ij|0}}{\partial u_j} & \dfrac{\partial g_{ij|0}}{\partial u_j} & 0\\ \dfrac{\partial f_{ij|1}}{\partial X_j} & \dfrac{\partial g_{ij|1}}{\partial X_j} & \dfrac{\partial h_{ij|1}}{\partial X_j}\end{pmatrix},$$

we have

$$\begin{pmatrix} \dfrac{\partial}{\partial t_j} \\ \dfrac{\partial}{\partial u_j} \\ \dfrac{\partial}{\partial X_j} \end{pmatrix} = J_{ij} \cdot \begin{pmatrix} \dfrac{\partial}{\partial t_i} \\ \dfrac{\partial}{\partial u_i} \\ \dfrac{\partial}{\partial X_i} \end{pmatrix}.$$

Since

$$\begin{aligned} \Psi_{ijl} &\equiv_n \Phi_{ij}(\Phi_{jl}) - \Phi_{il} \\ &\equiv_n \Phi_{ij}(\Phi_{jk}(\Phi_{kl}) - \Psi_{jkl}) - \Phi_{il} \\ &\equiv_n \Phi_{ij}(\Phi_{jk}(\Phi_{kl})) - \Psi_{jkl} \cdot J_{ij}(\Phi_{jl}) - \Phi_{il} \end{aligned}$$

and

$$\begin{aligned} \Psi_{ikl} &\equiv_n \Phi_{ik}(\Phi_{kl}) - \Phi_{il} \\ &\equiv_n \Phi_{ij}(\Phi_{jk}(\Phi_{kl})) - \Psi_{ijk}(\Phi_{kl}) - \Phi_{jl} \qquad \text{for} \quad i, j, k, l \in I, \end{aligned}$$

we have $\Psi_{ijl} - \Psi_{ikl} = \Psi_{ijk}(\Phi_{kl}) - \Psi_{jkl} \cdot J_{ij}(\Phi_{jl})$, which is nothing but the cocycle condition.

To construct the n-th infinitesimal neighbourhood, we have to add a collection $\{\Phi_{ij|n}\}$ of the terms of degree n to $\{\Phi_{ij}^{[n-1]}\}$ which is already determined and determine $\{\Phi_{ij}^{[n]}\}$ satisfying the following condition $(*)_n$:

$$(*)_n \qquad \Phi_{ij}^{[n]}(\Phi_{jk}^{[n]}(t_k, u_k, X_k)) \equiv_n \Phi_{ik}^{[n]}(t_k, u_k, X_k) \qquad \text{for} \quad i, j, k \in I.$$

To the function $\Phi_{ij|n} = (f_{ij|n}, g_{ij|n}, h_{ij|n})$, we attach an element $\lambda_{ij|n} \in \Gamma(U_i \cap U_j, \mathscr{F}_n)$ in the following way:

$$\begin{aligned} \lambda_{ij|n} = {} & \frac{\partial}{\partial t_i} \otimes f_{ij|n}(f_{ji|0}, g_{ji|0}, h_{ji|1}) \\ & + \frac{\partial}{\partial u_i} \otimes g_{ij|n}(f_{ji|0}, g_{ji|0}, h_{ji|1}) \\ & + \frac{\partial}{\partial X_i} \otimes h_{ij|n}(f_{ji|0}, g_{ji|0}, h_{ji|1}) \mod (X_i)^{n+1}. \end{aligned}$$

Thus we often identify a collection $\{\Phi_{ij|n}\}_{i,j \in I}$ with $\lambda_n = (\lambda_{ij|n})_{i,j \in I} \in C^1(\mathcal{U}, \mathscr{F}_n)$. In this time, λ_n does not satisfy the cocycle condition in general.

CLAIM 1.7. In order to satisfy $(*)_n$, λ_n must satisfy $d\lambda_n = -\mu_n$. In particular, if $\mu_n \notin B^2(\mathcal{U}, \mathscr{F}_n)$, there does not exist such a cochain λ_n.

PROOF. Since $\Phi_{ij}^{[n]} = \Phi_{ij}^{[n-1]} + \Phi_{ij|n}$, we have

$$\Phi_{ij}^{[n]}(\Phi_{jk}^{[n]})-\Phi_{ik}^{[n]}$$
$$\equiv_n \Phi_{ij}^{[n-1]}(\Phi_{jk}^{[n-1]}+\Phi_{jk|n})+\Phi_{ij|n}(\Phi_{jk}^{[n]})-\Phi_{ik}^{[n-1]}-\Phi_{ik|n}$$
$$\equiv_n \Psi_{ijk|n}+\Phi_{ij|n}(\Phi_{jk}^{[n-1]})-\Phi_{ik|n}+\Phi_{jk|n}\cdot J_{ij}(\Phi_{jk}^{[n-1]}) .$$

CLAIM 1.8. Assume that λ_n and λ_n' determine two n-th infinitesimal neighbourhoods. Then these neighbourhoods are equivalent to each other if and only if $\lambda_n'-\lambda_n\in \mathrm{B}^1(\mathcal{U}, \mathcal{F}_n)$.

PROOF. We can easily prove it by similar arguments to Claim 1.5. That is, we take an element $\psi=(\psi_i)_{i\in I}$ of $\mathrm{C}^0(\mathcal{U}, \mathcal{F}_n)$ such that $\lambda_n'-\lambda_n=d\psi$, put $\psi_i=(\partial/\partial t_i)\otimes p_i+(\partial/\partial y_i)\otimes q_i+(\partial/\partial X_i)\otimes r_i \mod (X_i)^{n+1}$, and replace the coordinates (t_i, u_i, X_i) on U_i by $(t_i', u_i', X_i')=(t_i+p_i, u_i+q_i, X_i+r_i)$.

Thus Proposition 1.3 is proved in the special case. These arguments also provide a method how to write down the transition relations.

B. The Leray spectral sequence.

Let $f: X\rightarrow Y$ be a morphism of algebraic varieties and $\mathcal{F}$ a sheaf on X. As is well-known, there exists the Leray spectral sequence with $E_2^{p,q}=\mathrm{H}^p(Y, R^qf_*\mathcal{F})\Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F})$. The beginning of this spectral sequence leads to the exact sequence

$$(L)\quad 0\longrightarrow \mathrm{H}^1(Y, f_*\mathcal{F})\xrightarrow{\alpha}\mathrm{H}^1(X, \mathcal{F})\xrightarrow{\beta}\mathrm{H}^0(Y, R^1f_*\mathcal{F})\xrightarrow{\gamma}\mathrm{H}^2(Y, f_*\mathcal{F}) .$$

We shall state an interpretation of this sequence (L) in terms of Čech cocycles without proof. Let $\mathcal{U}=(U_i)_{i\in I}$ be an affine open cover of Y and $\mathcal{V}=(V_j)_{j\in J}$ an affine open cover of X. Assume that, for each $j\in J$, there exists an element $i(j)\in I$ such that $f(V_j)\subset U_{i(j)}$. The Leray spectral sequence is induced by the double complex $K^{\cdot\cdot}$ with

$$K^{p,q}=\mathrm{C}^p(\mathcal{U}, f_*C^q(\mathcal{V}, \mathcal{F}))=\prod_{i_0<\cdots<i_p}\prod_{j_0<\cdots<j_q}\Gamma(i_0, \cdots, i_p; j_0, \cdots, j_q) ,$$

where

$$\Gamma(i_0, \cdots, i_p; j_0, \cdots, j_q)=\Gamma(f^{-1}(U_{i_0}\cap\cdots\cap U_{i_p})\cap V_{j_0}\cap\cdots\cap V_{j_q}, \mathcal{F}) .$$

First, we interpret the morphism $\alpha: \mathrm{H}^1(Y, f_*\mathcal{F})\rightarrow \mathrm{H}^1(X, \mathcal{F})$ in the sequence (L) in terms of Čech cocycles. Let $\varphi=(\varphi_{i_0i_1})\in \mathrm{Z}^1(\mathcal{U}, f_*\mathcal{F})$ with $\varphi_{i_0i_1}\in\Gamma(f^{-1}(U_{i_0i_1}), \mathcal{F})$. Then we have $\varphi_{i_0i_1}-\varphi_{i_0i_2}+\varphi_{i_1i_2}=0$ for $i_0, i_1, i_2\in I$. If we put $x_{i_0i_1; j}=\varphi_{i_0i_1}|_{f^{-1}(U_{i_0i_1})\cap V_j}$, we have $(x_{i_0i_1; j})\in K^{1,0}$, $d'(x_{i_0i_1; j_0})=0$, and $d''(x_{i_0i_1; j_0})=0$. If we put $x_{i_0; j_0}=x_{i(j_0)i_0; j_0}\in\Gamma(i(j_0)i_0; j_0)=\Gamma(i_0; j_0)$, then we have $(x_{i_0;\, j_0}) \in K^{0,0}$ and $d'(x_{i_0;\, j_0}) = (x_{i_0 i_1;\, j_0})$, that is, $x_{i_0;\, j_0} - x_{i_0;\, j_0} = x_{i_0 i_1;\, j_0}$. Letting $y_{i_0;\, j_0 j_1} = x_{i_0;\, j_0} - x_{i_0;\, j_1} \in \Gamma(i_0\, ;\, j_0 j_1)$, we have $d'(y_{i_0;\, j_0 j_1}) = 0$ and $d''(y_{i_0;\, j_0 j_1}) = 0$. In particular, a collection $(y_{i_0;\, j_0 j_1})_{i_0 \in I}$ is patched to an element $z_{j_0 j_1} \in \Gamma(V_{j_0 j_1}, \mathcal{F})$. It is easy to see that $z = (z_{j_0 j_1}) \in \mathrm{Z}^1(\mathcal{V}, \mathcal{F})$.

PROPOSITION 1.9. *The above correspondence from* $\varphi = (\varphi_{i_0 i_1})$ *to* $z = (z_{j_0 j_1})$ *induces the injection* $\alpha : \mathrm{H}^1(Y, f_* \mathcal{F}) \to \mathrm{H}^1(X, \mathcal{F})$ *in the sequence* (L).

Next, we construct a correspondence between Čech cocycles with induces the second morphism β in the sequence (L). Let $\psi = (\psi_{j_0 j_1}) \in \mathrm{Z}^1(\mathcal{V}, \mathcal{F})$ with $\psi_{j_0 j_1} \in \Gamma(V_{j_0 j_1}, \mathcal{F})$. If we put $x_{i;\, j_0 j_1} = \psi_{j_0 j_1}|_{f^{-1}(U_i) \cap V_{j_0 j_1}}$, we have $x = (x_{i_0;\, j_0 j_1}) \in K^{0,1}$, $d'x = 0$, and $d''x = 0$. For a fixed element $i_0 \in I$, a collection $(x_{i_0;\, j_0 j_1})_{j_0, j_1 \in J}$ determine an element $y_{i_0} \in \Gamma(U_{i_0}, R^1 f_* \mathcal{F})$. A collection $(y_{i_0})_{i_0 \in I}$ is patched to an element $z \in \mathrm{H}^0(Y, R^1 f_* \mathcal{F})$.

PROPOSITION 1.10. *The above correspondence from* ψ *to* z *induces the morphism* $\beta : \mathrm{H}^1(X, \mathcal{F}) \to \mathrm{H}^0(Y, R^1 f_* \mathcal{F})$ *in the sequence* (L).

The third map $\gamma : \mathrm{H}^0(Y, R^1 f_* \mathcal{F}) \to \mathrm{H}^2(Y, f_* \mathcal{F})$ is interpreted in the following way. An element of $\mathrm{H}^0(Y, R^1 f_* \mathcal{F})$ is represented by an element $x \in K^{0,1}$ such that $d''x = 0$ and that there exists an element $y \in K^{1,0}$ which satisfies $d'x = d''y$. Then $d'y \in K^{2,0}$ represents an element $w \in \mathrm{H}^2(Y, f_* \mathcal{F})$.

PROPOSITION 1.11. *The above correspondence from* x *to* w *induces the morphism* $\gamma : \mathrm{H}^0(Y, R^1 f_* \mathcal{F}) \to \mathrm{H}^2(Y, f_* \mathcal{F})$ *in the sequence* (L).

Finally, assume that an element $x \in K^{0,1}$ satisfying $d''x = 0$ represents an element $[x]$ of $\mathrm{Ker}(\gamma)$. Then there exist $y, z \in K^{1,0}$ such that $d''y = d'x$, $d'y = d'z$ and $d''z = 0$. Then we can construct an element $\varphi \in \mathrm{Z}^1(\mathcal{V}, \mathcal{F})$ such that $\beta([\varphi]) = [x]$, where $[\varphi] \in \mathrm{H}^1(X, \mathcal{F})$ is represented by φ. It is done in the following way. If we put $u = y - z = (u_{i_0 i_1;\, j_0}) \in K^{1,0}$, then we have $d'u = 0$ and $d''u = d'x$. We get an element $v = (v_{i_0;\, j_0}) \in K^{0,0}$ with $d'v = u$ by putting $v_{i_0;\, j_0} = u_{i(j_0) i_0;\, j_0}$. Moreover, if we put $w = x + d''v = (w_{i_0;\, j_0 j_1}) \in K^{0,1}$, then we have $d'w = 0$ and $d''w = 0$. In particular, a collection $(w_{i_0;\, j_0 j_1})_{i_0 \in I}$ is patched to an element $\varphi_{j_0 j_1} \in \Gamma(V_{j_0 j_1}, \mathcal{F})$. The element $\varphi = (\varphi_{j_0 j_1}) \in \mathrm{C}^1(\mathcal{V}, \mathcal{F})$ satisfies the required property.

§ 2. Rationally dominated neighbourhoods of P^1.

Let us begin this section with discussing a principle how to show the unirationality of a given algebraic variety. It is the following theorem due to H. Hironaka and H. Matsumura that plays the most essential role in this paper.

THEOREM 2.1 (Hironaka-Matsumura. Cf. Th. (3.3) in [HM]). *Let Y be any connected closed subscheme of positive dimension of $\boldsymbol{P}^n$, $n \geqq 2$. Then Y is $G3$ in $\boldsymbol{P}^n$.*

For the definition of the term $G3$, we refer to Def. (2.9) in [HM]. We have the following corollary, which suffices for later arguments.

COROLLARY 2.2. *Let Y be a smooth irreducible subvariety of positive dimension of the projective space $P=\boldsymbol{P}^n$. Let $\hat{P}$ be the completion of P along Y, and let $c: \hat{P} \rightarrow P$ the natural morphism inducing the identity of Y. Let X be any algebraic variety. Then every morphism $\varphi: \hat{P} \rightarrow X$ is of the form $f \circ c$ with a rational map $f: P \rightarrow X$ which induces a morphism within a neighbourhood of Y. Moreover f is uniquely determined by φ.*

PROOF. We refer to Theorem V in [Hi]. Its statement is slightly weaker than this corollary, but the same arguments lead to the proof, because it is the fact that Y is $G3$ in P that is essentially used in the proof.

We now make some definitions.

DEFINITION 2.3. (1) Let Y be a smooth variety. In this paper, a formal neighbourhood of Y is always assumed to be regular. That is, we call a pair (X, Y) a formal neighbourhood of Y if X is a regular formal scheme with the reduced subscheme Y.

(2) Let (X_1, Y_1) and (X_2, Y_2) be formal neighbourhoods of smooth variety Y_1 and Y_2, respectively. A morphism $\varphi: (X_1, Y_1) \rightarrow (X_2, Y_2)$ of ringed spaces is said to be dominant if the induced morphism of the first infinitesimal neighbourhoods is dominant.

DEFINITION 2.4. Let (X, C) be a formal neighbourhood of a nonsingular rational curve C. The neighbourhood (X, C) is said to be rationally dominated if there exists a dominant morphism $\varphi: (\boldsymbol{P}^n, l)^{\wedge} \rightarrow (X, C)$, where l denotes a line in $\boldsymbol{P}^n$ and $(\boldsymbol{P}^n, l)^{\wedge}$ the formal completion of $\boldsymbol{P}^n$ along l.

As a corollary to Corollary 2.2, we have the following proposition, which gives an approach to the problem how to show the unirationality of a given variety.

PROPOSITION 2.5. *Let X be a nonsingular complete algebraic variety. Assume that there exists a nonsingular rational curve C such that $(X, C)^{\wedge}$ is rationally dominated. Then X is unirational.*

PROOF. It immediately follows from Corollary 2.2.

Let $C\cong \boldsymbol{P}^1$. It is easy to see that, if a formal neighbourhood (X, C) of C is rationally dominated, then the normal bundle $N_{C/X}$ of C in X is a positive vector bundle. We now consider the question when a formal neighbourhood (X, C) with $N_{C/X}$ positive is rationally dominated. From now on, we work on neighbourhoods of dimension three for simplicity. The following lemma gives a partial answer to the question. It is a key lemma in this paper, which we shall call the RD Lemma.

LEMMA 2.6 (RD LEMMA). *Let $(X, C)=\mathrm{Spf}(k[t_0][[X_0, Y_0]])\cup\mathrm{Spf}(k[t_1][[X_1, Y_1]])$ be a formal neighbourhood of a nonsingular rational curve C with the following transition relation of the coordinates:*

$$X_0=\sum_{\substack{i,j\geq 0\\ i+j\geq 1}} a_{\alpha ij}t_1^{-\alpha}X_1^iY_1^j,$$

$$Y_0=\sum_{\substack{i,j\geq 0\\ i+j\geq 1}} b_{\alpha ij}t_1^{-\alpha}X_1^iY_1^j,$$

$$t_0=t_1^{-1}+\sum_{\substack{i,j\geq 0\\ i+j\geq 1}} c_{\alpha ij}t_1^{-\alpha}X_1^iY_1^j.$$

Assume that there exists a positive integer r satisfying the following condition: If $a_{\alpha ij}\neq 0$, $b_{\alpha ij}\neq 0$ or $c_{\alpha ij}\neq 0$, then $\alpha\geqq(1/r)(i+j)$.

Then the neighbourhood (X, C) is rationally dominated.

PROOF. Let l be a line in $\boldsymbol{P}^3$. Then we have

$$(\boldsymbol{P}^3, l)^{\wedge}\cong\mathrm{Spf}(k[u_0][[Z_0, W_0]])\cup\mathrm{Spf}(k[u_1][[Z_1, W_1]])$$

with $u_0=u_1^{-1}$, $Z_0=u_1^{-1}Z_1$ and $W_0=u_1^{-1}W_1$. We can explicitly construct a dominant morphism $\varphi: (\boldsymbol{P}^3, l)^{\wedge}\to(X, C)$ by the following two homomorphisms ψ_0 and ψ_1 of rings:

$$\psi_0: k[t_0][[X_0, Y_0]]\longrightarrow k[u_0][[Z_0, W_0]],$$

$$X_0\longmapsto \sum a_{\alpha ij}u_0^{r\alpha-i-j}Z_0^iW_0^j,$$

$$Y_0\longmapsto \sum b_{\alpha ij}u_0^{r\alpha-i-j}Z_0^iW_0^j,$$

$$t_0\longmapsto u_0^r+\sum c_{\alpha ij}u_0^{r\alpha-i-j}Z_0^iW_0^j,$$

and

$$\phi_1 : k[t_1][[X_1,\ Y_1]] \longrightarrow k[u_1][[Z_1,\ W_1]],$$
$$X_1 \longmapsto Z_1,$$
$$Y_1 \longmapsto W_1,$$
$$t_1 \longmapsto u_1^r.$$

Lemma 2.6 provides a sufficient condition for a neighbourhood (X, C) to be rationally dominated, but not a necessary one. Note that the assumption of Lemma 2.6 depends not only on the isomorphism classes but also on their description by the transition functions.

We now consider a supplementary question how many rationally dominated neighbourhoods lie in the set of all the formal neighbourhoods of $\boldsymbol{P}^1$ with positive normal bundle, though it is not necessary for the proof of Main Theorem.

PROPOSITION 2.7. *Let (X, C) be a formal neighbourhood of $C \cong \boldsymbol{P}^1$ with $N_{C/X} \cong \mathcal{O}(p) \oplus \mathcal{O}(q)$. Then (X, C) admits the following description by the transition functions:*

$$(X,\ C) \cong \mathrm{Spf}(k[t_0][[X_0,\ Y_0]]) \cup \mathrm{Spf}(k[t_1][[X_1,\ Y_1]])$$

with

$$X_0 = t_1^{-p} X_1 + \sum_{\substack{i,j \geq 0 \\ i+j \geq 2}} a_{\alpha ij} t_1^{-\alpha} X_1^i Y_1^j,$$

$$Y_0 = t_1^{-q} Y_1 + \sum_{\substack{i,j \geq 0 \\ i+j \geq 2}} b_{\alpha ij} t_1^{-\alpha} X_1^i Y_1^j,$$

$$t_0 = t_1^{-1} + \sum_{\substack{i,j \geq 0 \\ i+j \geq 1}} c_{\alpha ij} t_1^{-\alpha} X_1^i Y_1^j$$

such that $a_{\alpha ij} \neq 0$ implies $p < \alpha < pi + qj$, $b_{\alpha ij} \neq 0$ implies $q < \alpha < pi + qj$, and that $c_{\alpha ij} \neq 0$ implies $2 < \alpha < pi + qj$.

PROOF. Let $\mathcal{U} = (U_i)_{i=0,1}$ be an affine open cover with $U_i = \mathrm{Spec}\, k[t_i]$ $(i = 0, 1)$, and let $N = \mathcal{O}(a) \oplus \mathcal{O}(b)$. We calculate the elements of $\mathrm{Z}^1(\mathcal{U}, \Theta_C \otimes N^{\check{}})$ modulo $\mathrm{B}^1(\mathcal{U}, \Theta_C \otimes N^{\check{}})$ and determine the first infinitesimal neighbourhoods. Let (X_1, C) be one of such neighbourhoods. Since $\mathrm{C}^2(\mathcal{U}, \mathcal{F}) = 0$ for any sheaf $\mathcal{F}$ on C, we have only to calculate the elements of $\mathrm{Z}^1(\mathcal{U}, \Theta_{X_1}|_C \otimes S^n(N^{\check{}}))$ modulo $\mathrm{B}^1(\mathcal{U}, \Theta_{X_1}|_C \otimes S^n(N^{\check{}}))$ for $n \geqq 2$. We can easily calculate the elements of $\mathrm{Z}^1(\Theta_C \otimes S^n(N^{\check{}}))$ modulo B^1 and those of $\mathrm{Z}^1(N \otimes S^n(N^{\check{}}))$ modulo B^1. We consider the exact sequences

$$0 \longrightarrow \Theta_C \otimes S^n(N^{\check{}}) \longrightarrow \Theta_{X_1}|_C \otimes S^n(N^{\check{}}) \longrightarrow N \otimes S^n(N^{\check{}}) \longrightarrow 0.$$

Elementary diagram chasing on the Čech complexes corresponding to the above exact sequence leads to the proof.

COROLLARY 2.8. *Let (X_n, C) be any n-th infinitesimal neighbourhood of $C\cong \boldsymbol{P}^1$ with positive normal bundle. Then there exists a rationally dominated formal neighbourhood $(\tilde{X}, C)$ of C which is an extension of (X_n, C).*

PROOF. It immediately follows from Lemma 2.6 and Proposition 2.7. In fact, if we put $a_{\alpha ij}=b_{\alpha ij}=c_{\alpha ij}=0$ for $i+j\geqq n+1$, then the assumption of Lemma 2.6 is satisfied for a sufficiently large r.

§ 3. Fundamental theorem on scopes.

In the first part of § 1, we state the way how to construct formal neighbourhoods of a given smooth variety. To do this, we need infinitely many times successive arguments on Čech cochains. Note that it is not sufficient to consider only Čech cocycles (cf. § 1). In this section, we restrict ourselves to considering formal neighbourhoods of nonsingular projective toric surfaces. We introduce certain semi-groups, which we shall call scopes, in order to describe these neighbourhoods in terms of Čech cocycles. Roughly speaking, the scope indicates how much twisted a description of a neighbourhood of a toric surface is.

We use the same notation as in § 1. Let S be a nonsingular projective toric surface, and let N a line bundle on S. To construct a formal neighbourhood (X, S) of S with $N_{S/X}\cong N$, we give collections $\{(f_{ij|1}, g_{ij|1})\}$ and $\{(f_{ij|n}, g_{ij|n}, h_{ij|n})\}$ $(n\geqq 2)$, or equivalently Čech cochains $\lambda_1\in Z^1(\mathcal{U}, \mathcal{G}_1)$ and $\lambda_n\in C^1(\mathcal{U}, \mathcal{F}_n)$ $(n\geqq 2)$ (cf. § 1). We also use the following notation throughout this paper. Since the surface S is toric, we may assume that the top terms $\{(f_{ij|0}, g_{ij|0})\}$ and $\{(h_{ij|1})\}$ of the transition functions are monomials. We put

$$f_{ij|0}(t_j, u_j)=t_j^{a(i,j)}u_j^{b(i,j)},$$

$$g_{ij|0}(t_j, u_j)=t_j^{c(i,j)}u_j^{d(i,j)},$$

$$h_{ij|1}(t_j, u_j)=t_j^{e(i,j)}u_j^{f(i,j)}X_j.$$

We also put

$$T(i, j)=\begin{pmatrix} a(i, j) & b(i, j) & 0 \\ c(i, j) & d(i, j) & 0 \\ e(i, j) & f(i, j) & 1 \end{pmatrix}\in M(3, 3; \boldsymbol{Z}).$$

Then it is easy to see the following.

CLAIM 3.1.
(1) $T(i, j)\in GL(3, \boldsymbol{Z})$ for any $i, j\in I$.
(2) $T(i, j)\cdot T(j, k)=T(i, k)$ for any $i, j, k\in I$.
(3) $f_{ij|0}^{\alpha(i)}g_{ij|0}^{\beta(i)}h_{ij|1}^{n}=t_{j}^{\alpha(j)}u_{j}^{\beta(j)}X_{j}^{n}$,
where $(\alpha(j), \beta(j), n)=(\alpha(i), \beta(i), n)T(i, j)$.

Finally, we put $\mathcal{F}_{n}\cong\Theta_{X}|_{S}\otimes N^{-n}$, $\mathcal{G}_{n}\cong\Theta_{S}\otimes N^{-n}$, and $\mathcal{H}_{n}\cong N\otimes N^{-n}$. Then we have the exact sequences

$$0\longrightarrow\mathcal{G}_{n}\stackrel{\iota_{n}}{\longrightarrow}\mathcal{F}_{n}\stackrel{\tau_{n}}{\longrightarrow}\mathcal{H}_{n}\longrightarrow 0\,.$$

We now define the scope of a description $\{\Phi_{ij}\}=\{(f_{ij}, g_{ij}, h_{ij})\}$ of a neighbourhood (X, S) of a toric surface S.

DEFINITION 3.2. Let $0\in I$. The scope of a description $\Phi=\{\Phi_{ij}\}_{i,j\in I}$ of a neighbourhood (X, S) with respect to the coordinates (t_{0}, u_{0}, X_{0}) is the semi-group contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ generated by the following elements:
(a) $(\alpha-a(i, j), \beta-b(i, j), n)T(j, 0)$ with $t_{j}^{\alpha}u_{j}^{\beta}X_{j}^{n}$ appearing in the function f_{ij},
(b) $(\alpha-c(i, j), \beta-d(i, j), n)T(j, 0)$ with $t_{j}^{\alpha}u_{j}^{\beta}X_{j}^{n}$ appearing in g_{ij}, and
(c) $(\alpha-e(i, j), \beta-f(i, j), n-1)T(j, 0)$ with $t_{j}^{\alpha}u_{j}^{\beta}X_{j}^{n}$ appearing in h_{ij},
where i and j run all over the index set I. We denote it by Scope$(\Phi\,; 0)$.

Intrinsically, the scope is a semi-group in $M\times\boldsymbol{Z}_{\geq 0}$, where M denotes the group of the characters of the toric surface S. Let $k, l\in I$. Then we clearly see

$$\begin{aligned}\mathrm{Scope}(\Phi\,;\,l)&=\mathrm{Scope}(\Phi\,;\,k)\cdot T(k, l)\\&=\{(\alpha, \beta, n)T(k, l)\,|\,(\alpha, \beta, n)\in\mathrm{Scope}(\Phi\,;\,k)\}\,.\end{aligned}$$

We also define the scopes of elements or subsets of $\mathrm{C}^{p}(\mathcal{U}, \mathcal{F}_{n})$, $\mathrm{C}^{p}(\mathcal{U}, \mathcal{G}_{n})$ and $\mathrm{C}^{p}(\mathcal{U}, \mathcal{H}_{n})$.

Let $\lambda=(\lambda_{i_{0}\cdots i_{p}})\in\mathrm{C}^{p}(\mathcal{U}, \mathcal{F}_{n})$ with $\lambda_{i_{0}\cdots i_{p}}\in\Gamma(U_{i_{0}\cdots i_{p}}, \mathcal{F}_{n})$. We write

$$\begin{aligned}\lambda_{i_{0}\cdots i_{p}}=&F_{i_{0}\cdots i_{p}}(t_{i_{0}}, u_{i_{0}})\frac{\partial}{\partial t_{i_{0}}}\otimes X_{i_{0}}^{n}\\&+G_{i_{0}\cdots i_{p}}(t_{i_{0}}, u_{i_{0}})\frac{\partial}{\partial u_{i_{0}}}\otimes X_{i_{0}}^{n}\\&+H_{i_{0}\cdots i_{p}}(t_{i_{0}}, u_{i_{0}})\frac{\partial}{\partial X_{i_{0}}}\otimes X_{i_{0}}^{n}\qquad\mathrm{mod}\ X_{i_{0}}^{n+1}\,,\end{aligned}$$

where $(\partial/\partial t_{i_0})\otimes X_{i_0}^n \bmod X_{i_0}^{n+1}$, $(\partial/\partial u_{i_0})\otimes X_{i_0}^n \bmod X_{i_0}^{n+1}$ and $(\partial/\partial X_{i_0})\otimes X_{i_0}^n \bmod X_{i_0}^{n+1}$ denote the local basis of $\mathcal{F}_n$ on $\mathcal{U}_{i_0}$.

DEFINITION 3.3. (1) Let $0\in I$. The scope of the above element $\lambda\in C^p(\mathcal{U}, \mathcal{F}_n)$ with respect to the coordinates (t_0, u_0, X_0) is the semigroup contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ generated by the following elements:

(a) $(\alpha-1, \beta, n)T(i_0, 0)$ with $t_{i_0}^{\alpha}u_{i_0}^{\beta}$ appearing in $F_{i_0\cdots i_p}$,

(b) $(\alpha, \beta-1, n)T(i_0, 0)$ with $t_{i_0}^{\alpha}u_{i_0}^{\beta}$ appearing in $G_{i_0\cdots i_p}$, and

(c) $(\alpha, \beta, n-1)T(i_0, 0)$ with $t_{i_0}^{\alpha}u_{i_0}^{\beta}$ appearing in $H_{i_0\cdots i_p}$,

where $i_0, \cdots, i_p$ run all over the set I. We denote it by $\mathrm{Scope}(\lambda; 0)$.

(2) Let $V\subset C^p(\mathcal{U}, \mathcal{F}_n)$. The scope $\mathrm{Scope}(V; 0)$ of the subset V of $C^p(\mathcal{U}, \mathcal{F}_n)$ with respect to the coordinates (t_0, u_0, X_0) is defined in the following way: $\mathrm{Scope}(V; 0)=\sum_{\lambda\in V}\mathrm{Scope}(\lambda; 0)$.

Let $\mu=(\mu_{i_0\cdots i_p})\in C^p(\mathcal{U}, \mathcal{G}_n)$ with $\mu_{i_0\cdots i_p}\in\Gamma(U_{i_0\cdots i_p}, \mathcal{G}_n)$. We write

$$\mu_{i_0\cdots i_p}=\tilde{F}_{i_0\cdots i_p}(t_{i_0}, u_{i_0})\left(\frac{\partial}{\partial t_{i_0}}\right)^0\otimes X_{i_0}^n + \tilde{G}_{i_0\cdots i_p}(t_{i_0}, u_{i_0})\left(\frac{\partial}{\partial u_{i_0}}\right)^0\otimes X_{i_0}^n \mod X_{i_0}^{n+1},$$

where $(\partial/\partial t_{i_0})^0$ and $(\partial/\partial u_{i_0})^0$ denote the local basis of the sheaf Θ_S on U_{i_0} with $\iota_n((\partial/\partial t_{i_0})^0\otimes X_{i_0}^n \bmod X_{i_0}^{n+1})=(\partial/\partial t_{i_0})\otimes X_{i_0}^n \bmod X_{i_0}^{n+1}$ and $\iota_n((\partial/\partial u_{i_0})^0\otimes X_{i_0}^n \bmod X_{i_0}^{n+1})=(\partial/\partial u_{i_0})\otimes X_{i_0}^n \bmod X_{i_0}^{n+1}$.

DEFINITION 3.4. (1) The scope of the above element $\mu\in C^p(\mathcal{U}, \mathcal{G}_n)$ with respect to the coordinates (t_0, u_0, X_0) is the semi-group contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ generated by the following elements:

(a) $(\alpha-1, \beta, n)T(i_0, 0)$ with $t_{i_0}^{\alpha}u_{i_0}^{\beta}$ appearing in $\tilde{F}_{i_0\cdots i_p}$, and

(b) $(\alpha, \beta-1, n)T(i_0, 0)$ with $t_{i_0}^{\alpha}u_{i_0}^{\beta}$ appearing in $\tilde{G}_{i_0\cdots i_p}$,

where $i_0, \cdots, i_p$ run all over the set I. We denote it by $\mathrm{Scope}(\mu; 0)$.

(2) Let $W\subset C^p(\mathcal{U}, \mathcal{G}_n)$. The scope $\mathrm{Scope}(W; 0)$ of W with respect to the coordinates (t_0, u_0, X_0) is defined in the following way: $\mathrm{Scope}(W; 0) =\sum_{\mu\in W}\mathrm{Scope}(\mu; 0)$.

Now let $\nu=(\nu_{i_0\cdots i_p})\in C^p(\mathcal{U}, \mathcal{H}_n)$ with $\nu_{i_0\cdots i_p}\in\Gamma(\mathcal{U}_{i_0\cdots i_p}, \mathcal{H}_n)$. We write

$$\nu_{i_0\cdots i_p}=\tilde{H}_{i_0\cdots i_p}(t_{i_0}, u_{i_0})\left[\frac{\partial}{\partial X_{i_0}}\right]\otimes X_{i_0}^n \mod X_{i_0}^{n+1},$$

where $[\partial/\partial X_{i_0}]\otimes X_{i_0}^n \bmod X_{i_0}^{n+1}$ denotes the local basis of $\mathcal{H}_n$ on U_{i_0} with

$$\left[\frac{\partial}{\partial X_{i_0}}\right]\otimes X_{i_0}^n \mod X_{i_0}^{n+1}=\tau_n\left(\frac{\partial}{\partial X_{i_0}}\otimes X_{i_0}^n \mod X_{i_0}^{n+1}\right).$$

DEFINITION 3.5. (1) The scope of the above element $\nu\in C^p(\mathcal{U}, \mathcal{H}_n)$ with respect to the coordinates (t_0, u_0, X_0) is the semi-group contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ generated by the elements $(\alpha, \beta, n-1)T(i_0, 0)$ with $t_{i_0}^{\alpha}u_{i_0}^{\beta}$ appearing in $\tilde{H}_{i_0\cdots i_p}$, where $i_0, \cdots, i_p$ run all over the set I. We denote it by Scope$(\nu\,;\,0)$.

(2) Let $Z\subset C^p(\mathcal{U}, \mathcal{H}_n)$. The scope Scope$(Z\,;\,0)$ of Z with respect to the coordinates (t_0, u_0, X_0) is defined in the following way: Scope$(Z\,;\,0)$ $=\sum_{\nu\in Z}$ Scope$(\nu\,;\,0)$.

REMARK 3.6. (1) Since the surface S is toric, the algebraic torus $T\cong\boldsymbol{G}_m^2$ acts on S. The torus T also acts on the sheaf Θ_S. As is well-known, a pair $\{t_i(\partial/\partial t_i)^0,\ u_i(\partial/\partial u_i)^0\}$ $(i\in I)$ is a basis of the Lie algebra Lie(T) of the torus T. Suppose we are given certain action of T on the line bundle N, that is, N is an equivariant line bundle. Then the torus T acts on the sheaves $\mathcal{G}_n$ and $\mathcal{H}_n$. The vector space $C^p(\mathcal{U}, \mathcal{G}_n)$ and $C^p(\mathcal{U}, \mathcal{H}_n)$ $(n\geq 1,\ p\geq 0)$ are decomposed into the direct summands in the following way:

$$C^p(\mathcal{U}, \mathcal{G}_n)=\bigoplus_{m\in M}(C^p(\mathcal{U}, \mathcal{G}_n))_m$$

and

$$C^p(\mathcal{U}, \mathcal{H}_n)=\bigoplus_{m\in M}(C^p(\mathcal{U}, \mathcal{H}_n))_m,$$

where M denotes the group of the characters and $(C^p(\mathcal{U}, \mathcal{G}_n))_m$ (resp. $(C^p(\mathcal{U}, \mathcal{H}_n))_m$) the eigenspace of $C^p(\mathcal{U}, \mathcal{G}_n)$ (resp. $C^p(\mathcal{U}, \mathcal{H}_n)$) with respect to the character $m\in M$. Moreover, the Čech complexes $C^{\cdot}(\mathcal{U}, \mathcal{G}_n)$ and $C^{\cdot}(\mathcal{U}, \mathcal{H}_n)$ are decomposed as complexes, that is, the coboundary maps are compatible to the above decompositions.

These decompositions are closely related to the scope. Let $0\in I$, $\lambda\in C^p(\mathcal{U}, \mathcal{G}_n)$ and let $\varphi_0:\boldsymbol{Z}\times\boldsymbol{Z}\to M$ be an isomorphism of $\boldsymbol{Z}$-modules defined by $\varphi_0(\alpha, \beta)=\alpha[t_0]+\beta[u_0]$, where $[t_0]$ and $[u_0]$ denote the characters corresponding to the coordinates t_0 and u_0, respectively. By changing the action of T on N, if necessary, we may assume that the local basis $X_0^n \mod X_0^{n+1}$ of the sheaf N^{-n} on U_0 is invariant. Then it is easy to see that a bijection

$$\text{Scope}(\lambda\,;\,0)\cap(\boldsymbol{Z}\times\boldsymbol{Z}\times\{n\})\longrightarrow\{m\in M\mid \lambda_m\neq 0\}$$

is defined by $(\alpha, \beta, n)\mapsto\varphi_0(\alpha, \beta)$, where $\lambda=\sum\lambda_m$ with $\lambda_m\in(C^p(\mathcal{U}, \mathcal{G}_n))_m$.

Let $\mu\in C^p(\mathcal{U}, \mathcal{H}_n)$. Then a bijection

$$\text{Scope}(\mu\,;\,0)\cap(\boldsymbol{Z}\times\boldsymbol{Z}\times\{n-1\})\longrightarrow\{m\in M|\ \mu_m\neq 0\}$$

is defined by $(\alpha,\ \beta,\ n-1)\mapsto\varphi_0(\alpha,\ \beta)$.

(2) In particular, we see that the coboundary maps of the Čech complexes $C^{\cdot}(\mathcal{U},\ \mathcal{G}_n)$ and $C^{\cdot}(\mathcal{U},\ \mathcal{H}_n)$ preserve the scope, which we can directly prove by the following transition relations of local bases of Θ_S and N:

$$t_j\left(\frac{\partial}{\partial t_j}\right)^0=a(i,\ j)t_i\left(\frac{\partial}{\partial t_i}\right)^0+c(i,\ j)u_i\left(\frac{\partial}{\partial u_i}\right)^0,$$

$$u_j\left(\frac{\partial}{\partial u_j}\right)^0=b(i,\ j)t_i\left(\frac{\partial}{\partial t_i}\right)^0+d(i,\ j)u_i\left(\frac{\partial}{\partial u_i}\right)^0,$$

$$X_j\left[\frac{\partial}{\partial X_j}\right]=X_i\left[\frac{\partial}{\partial X_i}\right].$$

But, in general, the coboundary maps of the complex $C^p(\mathcal{U},\ \mathcal{F}_n)$ are not scope-preserving, because there exists no such decomposition of $C^{\cdot}(\mathcal{U},\mathcal{F}_n)$ corresponding to the definition of the scope as in (1), though we can estimate the behavior of the scope of cochains of $\mathcal{F}_n$ up to certain ambiguity. (Cf. Sublemma 3.16).

(3) To construct formal neighbourhoods, it is not sufficient to discuss $C^{\cdot}(\mathcal{U},\ \mathcal{F}_n)$ for a fixed integer n. The scope plays an essential role to link discussions on $C^{\cdot}(\mathcal{U},\ \mathcal{F}_n)$ for some integer n with those on $C^{\cdot}(\mathcal{U},\ \mathcal{F}_{n'})$ for another integer n'.

We make a definition before we state Theorem 3.8, which we shall call the fundamental theorem on scopes.

DEFINITION 3.7. Let $\mathcal{F}$ be any sheaf on S.

(1) A finite-dimensional vector subspace V of $Z^p(\mathcal{U},\ \mathcal{F})$ is said to be an H^p-slice of the sheaf $\mathcal{F}$ if V satisfies $\pi(V)=H^p(S,\ \mathcal{F})$, where π: $Z^p(\mathcal{U},\ \mathcal{F})\rightarrow H^p(S,\ \mathcal{F})$ denotes the canonical projection.

(2) Let V be an H^p-slice of $\mathcal{F}$. We call a basis $\{v_1,\cdots,\ v_k\}$ of V an H^p-basis of the sheaf $\mathcal{F}$.

THEOREM 3.8 (Fundamental theorem on scopes). *Let S and N be as before. Let V_n be an H^1-slice of the sheaf $\mathcal{G}_n$ $(n\geqq 1)$, and let W_n an H^1-slice of $\mathcal{H}_n$ $(n\geqq 2)$. For $0\in I$, we put*

$$\Omega=\sum_{n\geqq 1}\text{Scope}(V_n\,;\,0)+\sum_{n\geqq 2}\text{Scope}(W_n\,;\,0)\,.$$

Then any formal neighbourhood $(X,\ S)$ of the toric surface S with $N_{S/X}$ isomorphic to N admits a description $\Phi=\{\Phi_{ij}\}_{i,j\in I}$ by the transition func-

tions such that Scope(Φ ; 0)$\subset\Omega$.

COROLLARY 3.9. *Under the same situation as in Theorem* 3.8, *we further assume that the line bundle* N *is ample. Then any formal neighbourhood* (X, S) *of* S *with* $N_{S/X}\cong N$ *admits a description* Φ *such that* Scope(Φ ; 0) *is finitely generated and that*

$$\mathrm{Scope}(\Phi\,;\,0)\subset\sum_{n\geqq 1}\mathrm{Scope}(V_n\,;\,0)\,.$$

PROOF OF COROLLARY 3.9. It immediately follows from Serre duality, Serre vanishing theorem and Kodaira vanishing theorem. In fact, we can take $V_n=0$ for $n\gg 0$ and $W_n=0$ for $n\geqq 2$.

To prove Theorem 3.8, we prove the following four lemmas.

LEMMA 3.10. *Any first infinitesimal neighbourhood* (X_1, S) *of* S *with* $N_{S/X}\cong N$ *has a description* $\Phi^{[1]}=\{\Phi_{ij}^{[1]}\}_{i,j\in I}$ *such that* Scope($\Phi^{[1]}$; 0)$\subset$ Scope(V_1; 0) *for any* V_1.

The definition of the scope of $\mathrm{C}^{\cdot}(\mathcal{F}_n)$ depends on the description $\Phi^{[1]}$ of the first infinitesimal neighbourhood. From now on, we always assume that Scope($\Phi^{[1]}$; 0)$\subset$Scope(V_1; 0).

LEMMA 3.11. *Let* $n\geqq 2$. *Suppose that some* $(n-1)$-*th infinitesimal neighbourhood* (X_{n-1}, S) *of* S *is described by* $\Phi^{[n-1]}=\{\Phi_{ij}^{[n-1]}\}_{i,j\in I}$. *Let* $\mu_n=\{\mu_{ijk|n}\}_{i,j,k\in I}\in \mathrm{Z}^2(\mathcal{U}, \mathcal{F}_n)$ *corresponds to* $\Psi_n=\{\Psi_{ijk|n}\}$ *with* $\Psi_{ijk}=(\Phi_{ij}^{[n-1]}(\Phi_{jk}^{[n-1]})-\Phi_{ik}^{[n-1]})_{[n]}$. (*Cf.* § 1). *Then* Scope(μ_n; 0)$\subset$Scope($\Phi^{[n-1]}$; 0).

LEMMA 3.12. *Let* $n\geqq 2$. *For an element* $\varphi\in \mathrm{B}^2(\mathcal{U}, \mathcal{F}_n)$, *there exists* $\phi\in \mathrm{C}^1(\mathcal{U}, \mathcal{F}_n)$ *such that* $d\phi=\varphi$ *and that*

$$\mathrm{Scope}(\phi\,;\,0)\subset\mathrm{Scope}(\varphi\,;\,0)+\mathrm{Scope}(W_n\,;\,0)+\mathrm{Scope}(V_1\,;\,0)\,.$$

LEMMA 3.13. *Let* $n\geqq 2$. *There exists an* H^1-*slice* Y *of* $\mathcal{F}_n$ *such that*

$$\mathrm{Scope}(Y\,;\,0)\subset\mathrm{Scope}(V_n\,;\,0)+\mathrm{Scope}(W_n\,;\,0)+\mathrm{Scope}(V_1\,;\,0)\,.$$

PROOF OF LEMMA 3.10. It immediately follows from the definition of the scope, Claim 1.4 and Claim 1.5.

We make a preliminary definition before we prove Lemma 3.11.

DEFINITION 3.14. Let Ω be a semi-group contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geqq 0}$ such that, for any $n\in\boldsymbol{Z}_{\geqq 0}$, $\Omega\cap(\boldsymbol{Z}\times\boldsymbol{Z}\times\{n\})$ is a finite set. We define subsets

$UP(\Omega;\ t,\ u;\ X)$ and $P(\Omega;\ t,\ u;\ X)$ of $k[t,\ u,\ t^{-1},\ u^{-1}][[X]]$ in the following way:

$$UP(\Omega;\ t,\ u;\ X)=\left\{1+\sum_{\substack{(\alpha,\beta,\gamma)\in\Omega\\ \gamma\geq 1}} a_{\alpha\beta\gamma}t^{\alpha}u^{\beta}X^{\gamma}\mid a_{\alpha\beta\gamma}\in k\right\}$$

$$P(\Omega;\ t,\ u;\ X)=\left\{\sum_{(\alpha,\beta,\gamma)\in\Omega} a_{\alpha\beta\gamma}t^{\alpha}u^{\beta}X^{\gamma}\mid a_{\alpha\beta\gamma}\in k\right\}.$$

It is easy to see the following.

CLAIM 3.15. (1) $P(\Omega;\ t,\ u;\ X)$ is a subring of $k[t,\ u,\ t^{-1},\ u^{-1}][[X]]$,
(2) $UP(\Omega;\ t,\ u;\ X)$ is a group with respect to the natural multiplication.

PROOF OF LEMMA 3.11. We put $\Omega_i=\mathrm{Scope}(\Phi^{[n-1]};\ i)$ for $i\in I$. Note that $\Omega_j=\Omega_i T(i,\ j)$ for $i,\ j\in I$. Then $\Phi_{ij}^{[n-1]}=(f_{ij}^{[n-1]},\ g_{ij}^{[n-1]},\ h_{ij}^{[n-1]})$ is written in the following form:

$$f_{ij}^{[n-1]}(t_j,\ u_j,\ X_j)=t_j^{a(i,j)}u_j^{b(i,j)}\left(1+\sum_{(\alpha,\beta,\gamma)\in\Omega_j} f_{ij;\,\alpha\beta\gamma}t_j^{\alpha}u_j^{\beta}X_j^{\gamma}\right),$$

$$g_{ij}^{[n-1]}(t_j,\ u_j,\ X_j)=t_j^{c(i,j)}u_j^{d(i,j)}\left(1+\sum_{(\alpha,\beta,\gamma)\in\Omega_j} g_{ij;\,\alpha\beta\gamma}t_j^{\alpha}u_j^{\beta}X_j^{\gamma}\right),$$

$$h_{ij}^{[n-1]}(t_j,\ u_j,\ X_j)=t_j^{e(i,j)}u_j^{f(i,j)}X_j\left(1+\sum_{(\alpha,\beta,\gamma)\in\Omega_j} h_{ij;\,\alpha\beta\gamma}t_j^{\alpha}u_j^{\beta}X_j^{\gamma}\right)$$

with $f_{ij;\,\alpha\beta\gamma},\ g_{ij;\,\alpha\beta\gamma},\ h_{ij;\,\alpha\beta\gamma}\in k$. We calculate $\Phi_{ij}^{[n-1]}(\Phi_{jk}^{[n-1]})$. If we put $f_{jk}^{[n-1]}=t_k^{a(j,k)}u_k^{b(j,k)}F_{jk}$, $g_{jk}^{[n-1]}=t_k^{c(j,k)}u_k^{d(j,k)}G_{jk}$, and $h_{jk}^{[n-1]}=t_k^{e(j,k)}u_k^{f(j,k)}X_kH_{jk}$, then $F_{jk},\ G_{jk},\ H_{jk}\in UP(\Omega_k;\ t_k,\ u_k;\ X_k)$. We put $\Phi_{ij}^{[n-1]}(\Phi_{jk}^{[n-1]})=(f_{ijk},\ g_{ijk},\ h_{ijk})$. Noting that $T(i,\ j)T(j,\ k)=T(i,\ k)$ (cf. Claim 3.1), we see that

$$f_{ijk}=t_k^{a(i,k)}u_k^{b(i,k)}F_{jk}^{a(i,j)}G_{jk}^{b(i,j)}$$
$$\times\left\{1+\sum_{(\alpha,\beta,\gamma)\in\Omega_j} f_{ij;\,\alpha\beta\gamma}t_k^{\alpha(k)}u_k^{\beta(k)}X_k^{\gamma(k)}F_{jk}^{\alpha}G_{jk}^{\beta}H_{jk}^{\gamma}\right\},$$

$$g_{ijk}=t_k^{c(i,k)}u_k^{d(i,k)}F_{jk}^{c(i,j)}G_{jk}^{d(i,j)}$$
$$\times\left\{1+\sum_{(\alpha,\beta,\gamma)\in\Omega_j} g_{ij;\,\alpha\beta\gamma}t_k^{\alpha(k)}u_k^{\beta(k)}X_k^{\gamma(k)}F_{jk}^{\alpha}G_{jk}^{\beta}H_{jk}^{\gamma}\right\},$$

and that

$$h_{ijk}=t_k^{e(i,k)}u_k^{f(i,k)}X_kF_{jk}^{e(i,j)}G_{jk}^{f(i,j)}H_{jk}$$
$$\times\left\{1+\sum_{(\alpha,\beta,\gamma)\in\Omega_j} h_{ij;\,\alpha\beta\gamma}t_k^{\alpha(k)}u_k^{\beta(k)}X_k^{\gamma(k)}F_{jk}^{\alpha}G_{jk}^{\beta}H_{jk}^{\gamma}\right\},$$

where $(\alpha(k),\ \beta(k),\ \gamma(k))=(\alpha,\ \beta,\ \gamma)\cdot T(j,\ k)\in\Omega_k$. Thus $t_k^{-a(i,k)}u_k^{-b(i,k)}f_{ijk}$,

$t_k^{-c(i,k)}u_k^{-d(i,k)}g_{ijk}$ and $t_k^{-e(i,k)}u_k^{-f(i,k)}X_k^{-1}h_{ijk}$ belongs to $UP(\Omega_k;\ t_k,\ u_k;\ X_k)$, from which Lemma 3.11 follows.

Before we prove Lemma 3.12 and Lemma 3.13, we make a remark on the scopes. We consider exact sequences

$$0 \longrightarrow \mathcal{G}_n \xrightarrow{\iota_n} \mathcal{F}_n \xrightarrow{\tau_n} \mathcal{H}_n \longrightarrow 0\,.$$

We write $\iota=\iota_n$ and $\tau=\tau_n$ for simplicity. We also denote the morphism $\mathrm{C}^p(\mathcal{U},\ \mathcal{G}_n)\to\mathrm{C}^p(\mathcal{U},\ \mathcal{F}_n)$ (resp. $\mathrm{C}^p(\mathcal{U},\ \mathcal{F}_n)\to\mathrm{C}^p(\mathcal{U},\ \mathcal{H}_n)$) which is induced by $\iota:\ \mathcal{G}_n\to\mathcal{F}_n$ (resp. $\tau:\ \mathcal{F}_n\to\mathcal{H}_n$) by the same symbol ι (resp. τ). The coboundary maps of the complexes $\mathrm{C}^\cdot(\mathcal{U},\ \mathcal{G}_n)$ and $\mathrm{C}^\cdot(\mathcal{U},\ \mathcal{H}_n)$ are scope-preserving (Cf. Remark 3.6). The following sublemma enables us to estimate the scopes of elements appearing in diagram chasing on the above exact sequences.

SUBLEMMA 3.16. *Let* $n\geqq 2$, $p\geqq 0$ *and* $0\in I$.

(1) $\mathrm{Scope}(\iota(x);\ 0)\subset\mathrm{Scope}(x;\ 0)$ *for* $x\in\mathrm{C}^p(\mathcal{U},\ \mathcal{G}_n)$.

(2) *For an element* $y\in\mathrm{C}^p(\mathcal{U},\ \mathcal{H}_n)$, *there exists an element* $z\in\mathrm{C}^p(\mathcal{U},\ \mathcal{F}_n)$ *such that* $\tau(z)=y$ *and that*

$$\mathrm{Scope}(z;\ 0)\subset\mathrm{Scope}(y;\ 0)+\mathrm{Scope}(V_1;\ 0)\,.$$

(3) $\mathrm{Scope}(dw;\ 0)\subset\mathrm{Scope}(w;\ 0)+\mathrm{Scope}(V_1;\ 0)$ *for* $w\in\mathrm{C}^p(\mathcal{U},\ \mathcal{F}_n)$.

PROOF OF SUBLEMMA 3.16. We calculate the transition relation between the local bases of $\mathcal{F}_n$ on U_i and U_j. Let $\Omega_j=\mathrm{Scope}(V_1;\ j)$. Then $\Phi_{ij}^{[1]}=(f_{ij}^{[1]},\ g_{ij}^{[1]},\ h_{ij}^{[1]})$ is written in the following form:

$$f_{ij}^{[1]}=t_j^{a(i,j)}u_j^{b(i,j)}(1+f)\,,$$
$$g_{ij}^{[1]}=t_j^{c(i,j)}u_j^{d(i,j)}(1+g)\,,$$
$$h_{ij}^{[1]}=t_j^{e(i,j)}u_j^{f(i,j)}X_j\,,$$

where $f,\,g\in P(\Omega_j;\ t_j,\ u_j;\ X_j)\cap k[t_j,\ u_j,\ t_j^{-1},\ u_j^{-1}]X_j$. Then we easily see the following:

$$t_j\frac{\partial}{\partial t_j}=a(i,\ j)t_i\frac{\partial}{\partial t_i}+c(i,\ j)u_i\frac{\partial}{\partial u_i}\,,$$

$$u_j\frac{\partial}{\partial u_j}=b(i,\ j)t_i\frac{\partial}{\partial t_i}+d(i,\ j)u_i\frac{\partial}{\partial u_i}\,,$$

$$X_j\frac{\partial}{\partial X_j}=ft_i\frac{\partial}{\partial t_i}+gu_i\frac{\partial}{\partial u_i}+X_i\frac{\partial}{\partial X_i}\,.$$

Let Σ be a semi-group in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geqq 0}$ such that, for each $n\geqq 0$, $\Sigma\cap(\boldsymbol{Z}\times\boldsymbol{Z}\times\{n\})$ is a finite set, and let P, Q, $R\in P(\Sigma\,;\, t_j,\, u_j\,;\, X_j)$. Then we have the following relation:

$$\begin{aligned} & t_j\frac{\partial}{\partial t_j}\otimes P+u_j\frac{\partial}{\partial u_j}\otimes Q+X_j\frac{\partial}{\partial X_j}\otimes R \\ = & t_i\frac{\partial}{\partial t_i}\otimes(a(i,\ j)P+b(i,\ j)Q+fR) \\ & +u_i\frac{\partial}{\partial u_i}\otimes(c(i,\ j)P+d(i,\ j)Q+gR) \\ & +X_i\frac{\partial}{\partial X_i}\otimes R\,. \end{aligned}$$

Note that $a(i,\ j)P+b(i,\ j)Q+fR$ and $c(i,\ j)P+d(i,\ j)Q+fR$ belong to the ring $P(\Omega_j+\Sigma\,;\, t_j,\, u_j\,;\, X_j)$. The assertion (1), (2) and (3) immediately follow from the above relation. Thus Sublemma 3.16 is proved.

PROOF OF LEMMA 3.12. Let $\varphi\in \mathrm{B}^2(\mathcal{U},\ \mathcal{F}_n)$. We find an element $\psi\in \mathrm{C}^1(\mathcal{U},\ \mathcal{F}_n)$ such that $d\psi=\varphi$ and that $\mathrm{Scope}(\psi\,;\,0)\subset\mathrm{Scope}(\varphi\,;\,0)+\mathrm{Scope}(W_n\,;\,0)+\mathrm{Scope}(V_1\,;\,0)$ in the following way.

Since $\tau(\varphi)\in \mathrm{B}^2(\mathcal{U},\ \mathcal{H}_n)$, there exists an element $a\in \mathrm{C}^1(\mathcal{U},\ \mathcal{H}_n)$ such that $da=\tau(\varphi)$ and that $\mathrm{Scope}(a\,;\,0)\subset\mathrm{Scope}(\tau(\varphi)\,;\,0)$ (Cf. Remark 3.6). We can take an element $b\in \mathrm{C}^1(\mathcal{U},\ \mathcal{F}_n)$ such that $\tau(b)=a$ and that $\mathrm{Scope}(b\,;\,0)\subset\mathrm{Scope}(a\,;\,0)+\mathrm{Scope}(V_1\,;\,0)\subset\mathrm{Scope}(\varphi\,;\,0)+\mathrm{Scope}(V_1\,;\,0)$. Let $c\in W_n\subset \mathrm{Z}^1(\mathcal{U},\mathcal{H}_n)$. Then there exists an element $e\in \mathrm{C}^1(\mathcal{U},\ \mathcal{F}_n)$ such that $\tau(e)=c$ and that $\mathrm{Scope}(e\,;\,0)\subset\mathrm{Scope}(W_n\,;\,0)+\mathrm{Scope}(V_1\,;\,0)$. Then there exists an element $f\in \mathrm{C}^2(\mathcal{U},\ \mathcal{G}_n)$ such that $\iota(f)=\varphi-db-de$ and that $\mathrm{Scope}(f;\,0)\subset\mathrm{Scope}(\varphi\,;\,0)+\mathrm{Scope}(W_n\,;\,0)+\mathrm{Scope}(V_1\,;\,0)$. By taking $c\in W_n$ suitably, we may assume $f\in \mathrm{B}^2(\mathcal{U},\ \mathcal{G}_n)$. Then there exists an element $g\in \mathrm{C}^1(\mathcal{U},\ \mathcal{G}_n)$ such that $dg=-f$ and that $\mathrm{Scope}(g\,;\,0)\subset\mathrm{Scope}(f;\,0)$ (cf. Remark 3.6). If we put $\psi=\iota(f)+b+e\in \mathrm{C}^1(\mathcal{U},\ \mathcal{F}_n)$, then ψ satisfies the required property. Thus Lemma 3.12 is proved.

PROOF OF LEMMA 3.13. We construct an H^1-slice Y of $\mathcal{F}_n$ in the following way. Let $a\in W_n$. Then there exists an element $b\in \mathrm{C}^1(\mathcal{F}_n)$ such that $\tau(b)=a$ and that $\mathrm{Scope}(b\,;\,0)\subset\mathrm{Scope}(W_n\,;\,0)+\mathrm{Scope}(V_1\,;\,0)$. There exists an element $c\in \mathrm{Z}^2(\mathcal{G}_n)$ such that $\iota(c)=db$ and that $\mathrm{Scope}(c\,;\,0)\subset\mathrm{Scope}(W_n\,;\,0)+\mathrm{Scope}(V_1\,;\,0)$. Whether the element c belongs to $\mathrm{B}^2(\mathcal{U},\ \mathcal{G}_n)$ or not depends on the choice of $a\in W_n$. We consider an element $a\in W_n$ such that the element c determined by a in the above way belongs to $\mathrm{B}^2(\mathcal{G}_n)$. Such elements make a vector subspace of W_n, which we denote by W_n'. Let $a_1, \cdots, a_l$ be a basis of W_n'. For $a_i \in W_n'$ $(i=1, \cdots, l)$, we take elements $b_i \in \mathrm{C}^1(\mathcal{F}_n)$ and $c_i \in \mathrm{B}^2(\mathcal{G}_n)$ in the above way. Then there exist $e_i \in \mathrm{C}^1(\mathcal{G}_n)$ such that $de_i = c_i$ and that $\mathrm{Scope}(e_i; 0) \subset \mathrm{Scope}(W_n; 0) + \mathrm{Scope}(V_1; 0)$. If we put $b_i' = b_i - \iota(e_i)$, then $b_i' \in \mathrm{Z}^1(\mathcal{F}_n)$ and $\mathrm{Scope}(b_i'; 0) \subset \mathrm{Scope}(W_n; 0) + \mathrm{Scope}(V_1; 0)$. Let $f_1, \cdots, f_m$ be a basis of V_n. Then $\iota(f_i) \in \mathrm{Z}^1(\mathcal{F}_n)$ and $\mathrm{Scope}(\iota(f_i); 0) \subset \mathrm{Scope}(V_n; 0) + \mathrm{Scope}(V_1; 0)$. Let Y be the vector subspace of $\mathrm{Z}^1(\mathcal{F}_n)$ generated by $b_1', \cdots, b_l'$ and $\iota(f_1), \cdots, \iota(f_m)$. Then Y satisfies the required property. Thus Lemma 3.13 is proved.

PROOF OF THEOREM 3.8. We construct neighbourhoods in such a way as in § 1. Then Theorem 3.8 immediately follows from Lemma 3.10, Lemma 3.11, Lemma 3.12 and Lemma 3.13.

§ 4. Further properties on scopes.

Let S be a nonsingular projective toric surface and N a line bundle on S. Roughly speaking, Theorem 3.8 tells us that we can choose a semi-group Ω such that any formal neighbourhood (X, S) of S with $N_{S/X} \cong N$ admits a description by the transition functions with its scope with respect to certain coordinates contained in Ω. If the line bundle N is ample, we have only to determine an H^1-slice of the sheaf $\mathcal{G}_n \cong \Theta_S \otimes N^{-n}$ for each $n>0$ in order to calculate such a semi-group Ω (Cf. Corollary 3.9). This section provides a preparation of § 6, in which we shall estimate the scope of an H^1-slice of $\mathcal{G}_n$ by the induction on the Picard number $\rho(S)$ of S. Let $f: \tilde{S} \to S$ be an equivariant blowing-up of S along a point, and we put $\tilde{N} = f^*N \otimes \mathcal{O}(-cE)$ with E denoting the exceptional curve of f and $c>0$. In this section we aim at proving Theorem 4.2, which enables us to compare an H^1-slice of $\tilde{\mathcal{G}}_n = \Theta_{\tilde{S}} \otimes \tilde{N}^{-n}$ with an H^1-slice of $\mathcal{G}_n$. To prove it, we need an interpretation of the Leray spectral sequence in terms of Čech cochains which is stated in § 1.B.

In order to fix notation which we shall use later, it is convenient to recall the correspondence between nonsingular toric surfaces and weighted circular graphs. (Cf. [O] for details).

Let S be a nonsingular projective toric surface on which an algebraic torus $T \cong \boldsymbol{G}_m^2$ acts. We denote the T-invariant prime divisors by $D_1, \cdots, D_s$ and we put $D = D_1 + \cdots + D_s$. As is well-known, each D_i is isomorphic to $\boldsymbol{P}^1$, D is a cycle of rational curves, and $S \backslash D$ is the unique open T-orbit. We denote by G_S the weighted dual graph of D. Then G_S turns out to be a circular graph with s vertices with weights $a_1, \cdots, a_s$, where $a_i = (D_i)^2$

$(i=1, \cdots, s)$.

We may assume that the weights $a_1, \cdots, a_s$ lie counter-clockwise such as in Figure A. Conventionally, we put $D_{ls+i}=D_i$ and $a_{ls+i}=a_i$ for $l \in \boldsymbol{Z}$. We also put $p_i=D_i \cap D_{i+1}$.

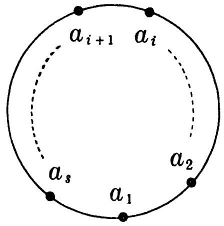


Figure A.

Conversely, the weighted dual graph G_S uniquely determines S up to isomorphism. Moreover, we can construct from G_S an affine open cover $S=\bigcup_{i=0}^{s-1} U_i$ of S with $U_i \cong \operatorname{Spec} k[t_i, u_i]$, and determine the transition functions between the coordinates (t_i, u_i) $(i=0, \cdots, s-1)$ in such a way that the following conditions are satisfied: $t_i=u_{i+1}^{-1}$ and $u_i=t_{i+1}u_{i+1}^{-a_{i+1}}$. The equation $t_i=0$ determines D_i on U_i, $u_i=0$ determines D_{i+1} on U_i, and $t_i=u_i=0$ determines p_i on U_i. From now on, we always take the coordinates (t_i, u_i) as above unless otherwise mentioned.

DEFINITION 4.1. We call the above affine open cover $\{U_i\}$ the canonical open cover of S determined by the weighted dual graph. We also call the coordinates (t_i, u_i) the canonical coordinates on U_i.

Let $B=\sum_{i=1}^{s} b_i D_i$ be an invariant divisor on S. In this paper, we describe the pair (S, B) by the double-weighted circular graph in Figure B.

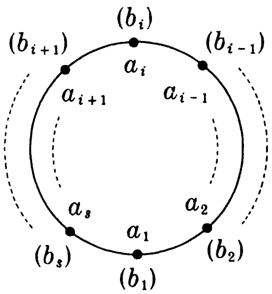


Figure B.

As is easily seen the divisor B is ample if and only if the following inequalities are satisfied for $i=1, \cdots, s: b_{i-1}+a_i b_i+b_{i+1}>0$.

Let $N=\mathcal{O}(B)$. From the double-weighted dual graph corresponding to B, we can recover the coordinates (t_i, u_i, X_i) $(i\in\{1, \cdots, s\})$ and the top terms $\{(f_{ij10}, g_{ij10})\}$ and $\{h_{ij11}\}$ of the transition functions describing a formal neighbourhood (X, S) of S with $N_{S/X}\cong N$. We can determine the matrices $T(i, j)\in GL(3, \boldsymbol{Z})$ $(i, j\in\{1, \cdots, s\})$ which are defined in §3 in such a way that the following conditions are satisfied:

$$(1) \qquad T(i, i+1)=\begin{pmatrix} 0 & -1 & 0 \\ 1 & -a_{i+1} & 0 \\ 0 & -d_{i+1} & 1 \end{pmatrix},$$

where $d_{i+1}=b_i+a_{i+1}b_{i+1}+b_{i+2}$;

(2) $T(i, j)T(j, k)=T(i, k)$ for $i, j, k\in\{1, \cdots, s\}$;

(3) $T(i, i)$ is the identity matrix.

It is easy to see that the normal bundle $N_{S/X}$ is isomorphic to N if we construct a formal neighbourhood (X, S) of S starting from the above matrices $T(i, j)$. We shall call these $T(i, j)$ the transition matrices with respect to the surface S and the line bundle N on S.

Next, we discuss what happens to weighted dual graphs when we blow up surfaces. Suppose that a nonsingular projective toric surface S and a line bundle $N=\mathcal{O}(B)$ are determined by the following double-weighted dual graph with $(D_i)^2=a_i$ and $B=\sum b_i D_i$:

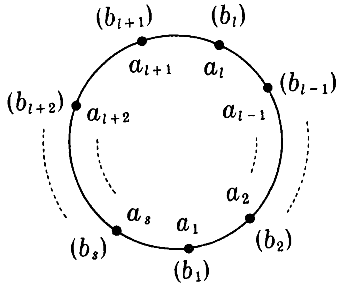


Figure C.

Let $f: \tilde{S}\to S$ be the equivariant blowing-up of S along $p_l=D_l\cap D_{l+1}$. We put $\tilde{N}=f^*N\otimes\mathcal{O}(-cE)$, where E denotes the exceptional divisor of f. Then $\tilde{S}$ and $\tilde{N}$ are determined by the double-weighted dual graph in

Figure D.

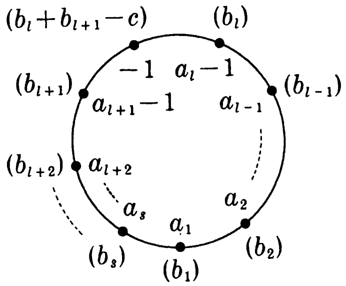


Figure D.

The surface S is covered by affine open subsets $U_i \cong \operatorname{Spec} k[t_i, u_i]$ ($i=1, \cdots, s$). Let $T(i, j)$ ($i, j \in \{1, \cdots, s\}$) be the transition matrices with respect to S and N. To get an affine open cover of $\tilde{S}$, we replace U_i by $\tilde{U}_i$ ($i \neq l$) and U_l by $\tilde{U}_{l-\varepsilon} \cup \tilde{U}_{l+\varepsilon}$ with $\tilde{U}_{l-\varepsilon} \cong \operatorname{Spec} k[t_{l-\varepsilon}, u_{l-\varepsilon}]$ and $\tilde{U}_{l+\varepsilon} \cong \operatorname{Spec} k[t_{l+\varepsilon}, u_{l+\varepsilon}]$, using a symbol ε. That is, $\tilde{S}$ is covered by open subsets $\tilde{U}_i$ ($i=1, \cdots, l-\varepsilon, l+\varepsilon, \cdots, s$). The transition matrices $\tilde{T}(i, j)$ with respect to $\tilde{S}$ and $\tilde{N}$ are calculated in the following way: $\tilde{T}(i, j)=T(i, j)$ if $i \neq l$ and $j \neq l$, $\tilde{T}(i, l+\varepsilon)=T(i, l)T(l, l+\varepsilon; c)$ and $\tilde{T}(i, l-\varepsilon)=T(i, l)T(l, l-\varepsilon; c)$, where

$$T(l, l+\varepsilon; c)=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ c & 0 & 1 \end{pmatrix}$$

and

$$T(l, l-\varepsilon; c)=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}.$$

We put $T(l+\varepsilon, l; c)=T(l, l+\varepsilon; c)^{-1}$ and $T(l-\varepsilon, l; c)=T(l, l-\varepsilon; c)^{-1}$. Note that the matrices $T(l, l+\varepsilon; c)$ and $T(l, l-\varepsilon; c)$ don't depend on l. If we put $p_{l+\varepsilon}=\{t_{l+\varepsilon}=u_{l+\varepsilon}=0\} \in U_{l+\varepsilon}$ and $p_{l-\varepsilon}=\{t_{l-\varepsilon}=u_{l-\varepsilon}=0\} \in U_{l-\varepsilon}$, we easily see $p_{l-\varepsilon}=D_l \cap E$ and $p_{l+\varepsilon}=D_{l+1} \cap E$.

THEOREM 4.2. *Let $f: \tilde{S} \to S$ be an equivariant blowing-up of S along $p_l \in S$ as above, and let N (resp. $\tilde{N}$) a line bundle on S (resp. $\tilde{S}$) with $\tilde{N}=f^*N \otimes \mathcal{O}(-cE)$, where E denotes the exceptional curve of f and $c>0$. Let $\{U_i\}_{i \in I}$ be the canonical open cover of S determined by the weighted dual graph. Let $\mathcal{G}_n=\Theta_S \otimes N^{-n}$ and $\tilde{\mathcal{G}}_n=\Theta_{\tilde{S}} \otimes \tilde{N}^{-n}$. Let $0 \in I \backslash \{l\}$. Since f*

is isomorphic on U_0, there canonically exists an element of $\tilde{I}$ indicating $f^{-1}(U_0)$, which we denote by the same symbol $0\in\tilde{I}$, where $\tilde{I}$ denotes the index set of the canonical open cover of $\tilde{S}$. Let V_n be any H^1-slice of $\mathcal{G}_n$. Then there exists an H^1-slice $\tilde{V}_n$ of $\tilde{\mathcal{G}}_n$ which satisfies the following condition:

$$\mathrm{Scope}(\tilde{V}_n\,;\,0)\subset\mathrm{Scope}(V_n\,;\,0)+\sum_{\substack{\alpha\leqq-1,\beta\leqq-1\\ \alpha+\beta+cn\geqq-1}}\boldsymbol{Z}_{\geqq0}(\alpha,\beta,n)T(l,0)\,.$$

Let us explain the outline of the proof of Theorem 4.2 before we go into details. We put $\mathcal{P}=\mathrm{Coker}(\Theta_{\tilde{S}}\to f^*\Theta_S)$, $\mathcal{P}_n=\mathcal{P}\otimes\tilde{N}^{-n}$ and $Q_n=f^*\Theta_S\otimes\tilde{N}^{-n}$. Then the exact sequence

$$0\longrightarrow\tilde{\mathcal{G}}_n\longrightarrow Q_n\longrightarrow\mathcal{P}_n\longrightarrow0$$

induces the exact sequence

$$\mathrm{H}^0(\tilde{S},\mathcal{P}_n)\longrightarrow\mathrm{H}^1(\tilde{S},\tilde{\mathcal{G}}_n)\longrightarrow\mathrm{H}^1(\tilde{S},Q_n)\longrightarrow\mathrm{H}^1(\tilde{S},\mathcal{P}_n)\,. \tag{4-1}$$

On the other hand, we consider the Leray spectral sequence on the sheaf Q_n and the morphism $f:\tilde{S}\to S$. Since $c>0$, we easily see $f_*Q_n\cong\mathcal{G}_n$ and $R^1f_*Q_n\cong\mathcal{G}_n\otimes R^1f_*\mathcal{O}(cnE)$. Thus we have the following exact sequence:

$$0\longrightarrow\mathrm{H}^1(S,\mathcal{G}_n)\longrightarrow\mathrm{H}^1(\tilde{S},Q_n)\longrightarrow\mathrm{H}^0(S,\mathcal{G}_n\otimes R^1f_*\mathcal{O}(cnE))\longrightarrow\mathrm{H}^1(S,\mathcal{G}_n). \tag{4-2}$$

By diagram chasing on the exact sequences (4-1) and (4-2), we construct an H^1-slice $\tilde{V}_n$ of the sheaf $\tilde{\mathcal{G}}_n$ starting from an H^1-slice V_n of $\mathcal{G}_n$. Since each sheaf appearing in these sequences admits the action of the torus T, we can introduce the notion of the scope on its Čech complexes.

Suppose $i\in I$. If $i\neq l$, then $f^*(\partial/\partial t_i)\otimes X_i^n \bmod X_i^{n+1}$ and $f^*(\partial/\partial u_i)\otimes X_i^n \bmod X_i^{n+1}$ are considered to be the local basis of the sheaf Q_n on $\tilde{U}_i$. On the open set $\tilde{U}_{l+\varepsilon}$ (resp. $\tilde{U}_{l-\varepsilon}$), $f^*(\partial/\partial t_l)\otimes X_{l+\varepsilon}^n \bmod X_{l+\varepsilon}^{n+1}$ and $f^*(\partial/\partial u_l)\otimes X_{l+\varepsilon}^n \bmod X_{l+\varepsilon}^{n+1}$ (resp. $f^*(\partial/\partial t_l)\otimes X_{l-\varepsilon}^n \bmod X_{l-\varepsilon}^{n+1}$ and $f^*(\partial/\partial u_l)\otimes X_{l-\varepsilon}^n \bmod X_{l-\varepsilon}^{n+1}$) are the local basis of Q_n. Let $\mathcal{U}=(U_i)_{i\in I}$ be the canonical open cover of S and let $\tilde{\mathcal{U}}=(\tilde{U}_i)_{i\in\tilde{I}}=((\tilde{U}_i)_{i\in I\setminus\{l\}},\tilde{U}_{l-\varepsilon},\tilde{U}_{l+\varepsilon})$ the canonical open cover of $\tilde{S}$. We can define the scopes of elements of $C^p(\tilde{\mathcal{U}},Q_n)$ and $C^p(\mathcal{U},f_*C^q(\tilde{\mathcal{U}},Q_n))$ as follows.

DEFINITION 4.3. (1) Let $\lambda\in\Gamma_{\mathrm{rat}}(\tilde{S},Q_n)$ be any rational section of Q_n and $0\in\tilde{I}$. We define the scope $\mathrm{Scope}(\lambda\,;\,0)$ of λ with respect to $0\in\tilde{I}$ in the following way.

(A) If $0\neq l-\varepsilon,\ l+\varepsilon$, we can write

$$\lambda = F(t_0, u_0) f^* \frac{\partial}{\partial t_0} \otimes X_0^n \quad \mathrm{mod}\ X_0^{n+1}$$

$$+ G(t_0, u_0) f^* \frac{\partial}{\partial u_0} \otimes X_0^n \quad \mathrm{mod}\ X_0^{n+1}$$

with F and G being Laurant power series. Then we define Scope$(\lambda; 0)$ to be the semi-group contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ generated by the following elements:

(A-1): $(\alpha-1, \beta, n)$ with $t_0^\alpha u_0^\beta$ appearing in $F(t_0, u_0)$,

and

(A-2): $(\alpha, \beta-1, n)$ with $t_0^\alpha u_0^\beta$ appearing in $G(t_0, u_0)$.

(B) If $0 = l+\varepsilon$, we write

$$\lambda = F(t_{l+\varepsilon}, u_{l+\varepsilon}) f^* \frac{\partial}{\partial t_l} \otimes X_{l+\varepsilon}^n \quad \mathrm{mod}\ X_{l+\varepsilon}^{n+1}$$

$$+ G(t_{l+\varepsilon}, u_{l+\varepsilon}) f^* \frac{\partial}{\partial u_l} \otimes X_{l+\varepsilon}^n \quad \mathrm{mod}\ X_{l+\varepsilon}^{n+1}.$$

We define Scope$(\lambda; 0)=$Scope$(\lambda; l+\varepsilon)$ to be the semi-group generated by the following elements:

(B-1): $(\alpha-1, \beta, n)$ with $t_{l+\varepsilon}^\alpha u_{l+\varepsilon}^\beta$ appearing in $F(t_{l+\varepsilon}, u_{l+\varepsilon})$,

and

(B-2): $(\alpha-1, \beta-1, n)$ with $t_{l+\varepsilon}^\alpha u_{l+\varepsilon}^\beta$ appearing in $G(t_{l+\varepsilon}, u_{l+\varepsilon})$.

(C) If $0 = l-\varepsilon$, we write

$$\lambda = F(t_{l-\varepsilon}, u_{l-\varepsilon}) f^* \frac{\partial}{\partial t_l} \otimes X_{l-\varepsilon}^n \quad \mathrm{mod}\ X_{l-\varepsilon}^{n+1}$$

$$+ G(t_{l-\varepsilon}, u_{l-\varepsilon}) f^* \frac{\partial}{\partial u_l} \otimes X_{l-\varepsilon}^n \quad \mathrm{mod}\ X_{l-\varepsilon}^{n+1}$$

and define Scope$(\lambda; l-\varepsilon)$ to be the semi-group generated by the following elements:

(C-1): $(\alpha-1, \beta-1, n)$ with $t_{l-\varepsilon}^\alpha u_{l-\varepsilon}^\beta$ appearing in $F(t_{l-\varepsilon}, u_{l-\varepsilon})$,

and

(C-2): $(\alpha, \beta-1, n)$ with $t_{l-\varepsilon}^\alpha u_{l-\varepsilon}^\beta$ appearing in $G(t_{l-\varepsilon}, u_{l-\varepsilon})$.

(2) Let $\mu = (\mu_{i_0 \cdots i_p}) \in \mathrm{C}^p(\mathcal{U}, Q_n)$ with $\mu_{i_0 \cdots i_p} \in \Gamma(\tilde{U}_{i_0 \cdots i_p}, Q_n)$. We define Scope$(\mu; 0)$ as follows:

$$\mathrm{Scope}(\mu; 0) = \sum_{i_0, \cdots, i_p} \mathrm{Scope}(\mu_{i_0 \cdots i_p}; 0).$$

(3) Let $\nu \in \mathrm{C}^p(\mathcal{U}, f_* C^q(\tilde{\mathcal{U}}, Q_n))$. We write $\nu = (\nu_{i_0 \cdots i_p; j_0 \cdots j_q})$ with $\nu_{i_0 \cdots i_p; j_0 \cdots j_q} \in \Gamma(f^{-1}(U_{i_0 \cdots i_p}) \cap \tilde{U}_{j_0 \cdots j_q}, Q_n)$. We define Scope$(\nu; 0)$ as follows:

$$\mathrm{Scope}\,(\nu\,;\,0)=\sum_{\substack{i_0,\cdots,i_p\\ j_0,\cdots,j_q}}\mathrm{Scope}\,(\nu_{i_0\cdots i_p;j_0\cdots j_q}\,;\,0)\,.$$

REMARK 4.4. (1) It is easy to see that the above definition of the scope is naturally induced by the decomposition of $\Gamma_{\mathrm{rat}}(\tilde{S}, \mathcal{Q}_n)$ into the eigenspaces with respect to the action of the torus T.

(2) The natural map $\mathrm{C}^p(\tilde{\mathcal{U}}, \tilde{\mathcal{G}}_n)\to\mathrm{C}^p(\tilde{\mathcal{U}}, \mathcal{Q}_n)$ is scope-preserving.

(3) The differential maps of the double complex $\mathrm{C}^{\cdot}(\mathcal{U}, f_*\mathcal{C}^{\cdot}(\tilde{\mathcal{U}}, \mathcal{Q}_n))$ are scope-preserving.

For an element $\lambda\in\mathrm{C}^p(\tilde{\mathcal{U}}, \mathcal{P}_n)$, we define $\mathrm{Scope}\,(\lambda\,;\,0)$ in such a way that the natural map $\mathrm{C}^p(\tilde{\mathcal{U}}, \mathcal{Q}_n)\to\mathrm{C}^p(\tilde{\mathcal{U}}, \mathcal{P}_n)$ is scope-preserving. Let $\pi_\varepsilon : \Gamma(U_{l+\varepsilon}, \mathcal{Q}_n)\to\Gamma(U_{l+\varepsilon}, \mathcal{P}_n)$ and $\pi_{-\varepsilon} : \Gamma(U_{l-\varepsilon}, \mathcal{Q}_n)\to\Gamma(U_{l-\varepsilon}, \mathcal{P}_n)$ be the canonical projections. If we put

$$\pi_\varepsilon\Big(f^*\frac{\partial}{\partial u_l}\otimes X^n_{l+\varepsilon} \bmod X^{n+1}_{l+\varepsilon}\Big)=\xi_{l+\varepsilon,n}$$

and

$$\pi_{-\varepsilon}\Big(f^*\frac{\partial}{\partial t_l}\otimes X^n_{l-\varepsilon} \bmod X^{n+1}_{l-\varepsilon}\Big)=\xi_{l-\varepsilon,n}\,,$$

we have the isomorphisms

$$\Gamma(U_{l+\varepsilon}, \mathcal{P}_n)\cong k[u_{l+\varepsilon}]\cdot\xi_{l+\varepsilon,n}$$

and

$$\Gamma(U_{l-\varepsilon}, \mathcal{P}_n)\cong k[t_{l-\varepsilon}]\cdot\xi_{l-\varepsilon,n}\,.$$

The morphism π_ε and $\pi_{-\varepsilon}$ are determined by the following:

$$\pi_\varepsilon\Big(\lambda(t_{l+\varepsilon}, u_{l+\varepsilon})f^*\frac{\partial}{\partial t_l}\otimes X^n_{l+\varepsilon}+\mu(t_{l+\varepsilon}, u_{l+\varepsilon})f^*\frac{\partial}{\partial u_l}\otimes X^n_{l+\varepsilon} \bmod X^{n+1}_{l+\varepsilon}\Big)$$
$$=\{\mu(0, u_{l+\varepsilon})-u_{l+\varepsilon}\lambda(0, u_{l+\varepsilon})\}\xi_{l+\varepsilon,n}$$
$$\pi_{-\varepsilon}\Big(\lambda(t_{l-\varepsilon}, u_{l-\varepsilon})f^*\frac{\partial}{\partial t_l}\otimes X^n_{l-\varepsilon}+\mu(t_{l-\varepsilon}, u_{l-\varepsilon})f^*\frac{\partial}{\partial u_l}\otimes X^n_{l-\varepsilon} \bmod X^{n+1}_{l-\varepsilon}\Big)$$
$$=\{\lambda(t_{l-\varepsilon}, 0)-t_{l-\varepsilon}\mu(t_{l-\varepsilon}, 0)\}\xi_{l-\varepsilon,n}\,.$$

Note that the following transition relation are satisfied:

$$\xi_{l+\varepsilon,n}=-t^{1-cn}_{l-\varepsilon}\xi_{l-\varepsilon,n}\,.$$

DEFINITION 4.5. (1) For an rational section $\lambda\in\Gamma_{\mathrm{rat}}(E, \mathcal{P}_n)$, we define the scope $\mathrm{Scope}\,(\lambda\,;\,0)$ of λ with respect to the coordinates (t_0, u_0, X_0) in the following two ways which are equivalent to each other:

(A) If we write $\lambda=F(u_{l+\varepsilon})\xi_{l+\varepsilon,n}$, Scope$(\lambda;0)$ is the semi-group generated by the elements $(-1,\beta-1,n)\tilde{T}(l+\varepsilon,0)$ with $u_{l+\varepsilon}^{\beta}$ appearing in $F(u_{l+\varepsilon})$.

(B) If we write $\lambda=G(t_{l-\varepsilon})\xi_{l-\varepsilon,n}$, Scope$(\lambda;0)$ is the semi-group generated by the elements $(\alpha-1,-1,n)\tilde{T}(l-\varepsilon,0)$ with $t_{l-\varepsilon}^{\alpha}$ appearing in $G(t_{l-\varepsilon})$.

(2) We naturally induce the scope of an element $C^p(\tilde{\mathcal{U}},\mathcal{P}_n)$ by (1).

Then it is easy to see that the following exact sequence of complexes are scope-preserving:

$$0\longrightarrow C^{\cdot}(\tilde{\mathcal{U}},\tilde{\mathcal{G}}_n)\longrightarrow C^{\cdot}(\tilde{\mathcal{U}},\mathcal{Q}_n)\longrightarrow C^{\cdot}(\tilde{\mathcal{U}},\mathcal{P}_n)\longrightarrow 0\,.$$

PROOF OF THEOREM 4.2. Let A_n be a vector subspace of the space $C^0(\mathcal{U},f_*C^1(\tilde{\mathcal{U}},\mathcal{Q}_n))$ which represents $H^0(S,\mathcal{G}_n\otimes R^1f_*\mathcal{O}(cnE))$. By using the exact sequence (4-2), we see that there exists an H^1-slice B_n of $\mathcal{Q}_n$ such that Scope$(B_n;0)\subset$Scope$(V_n;0)+$Scope$(A_n;0)$. Note that each process in constructing the sequence (4-2) in terms of Čech cochains is scope-preserving. (Cf. §1, B and Remark 4.4). Let C_n be an H^0-slice of $\mathcal{P}_n$. Then there exists an H^1-slice $\tilde{V}_n$ of $\tilde{\mathcal{G}}_n$ such that Scope$(\tilde{V}_n;0)\subset$Scope$(B_n;0)+$Scope$(C_n;0)$. Thus we have reduced the proof to the calculation of Scope$(A_n;0)$ and Scope$(C_n;0)$.

We first calculate Scope$(A_n;0)$. We note that $H^0(S,R^1f_*\mathcal{Q}_n)\cong\Gamma(U_l,R^1f_*\mathcal{Q}_n)\cong H^1(U_{l+\varepsilon}\cup U_{l-\varepsilon},\mathcal{Q}_n)$. Let V be a vector subspace of $\Gamma(U_{l+\varepsilon}\cap U_{l-\varepsilon},\mathcal{Q}_n)$ generated by the elements $t_l^{\alpha}u_l^{\beta}f^*(\partial/\partial t_l)\otimes X_l^n \bmod X_l^{n+1}$ and $t_l^{\alpha}u_l^{\beta}f^*(\partial/\partial u_l)\otimes X_l^n \bmod X_l^{n+1}$ with $\alpha<0$, $\beta<0$ and $\alpha+\beta+cn\geqq 0$. Then we have $\pi(V)=H^1(U_{l+\varepsilon}\cup U_{l-\varepsilon},\mathcal{Q}_n)$, where $\pi:\Gamma(U_{l+\varepsilon}\cap U_{l-\varepsilon},\mathcal{Q}_n)\to H^1(U_{l+\varepsilon}\cup U_{l-\varepsilon},\mathcal{Q}_n)$ denotes the canonical projection. In fact, the above elements are considered to be elements of $\Gamma(U_{l+\varepsilon}\cap U_{l-\varepsilon},\mathcal{Q}_n)$ by the following equations:

$$\begin{aligned} t_l^{\alpha}u_l^{\beta}f^*\frac{\partial}{\partial t_l}\otimes X_l^n \bmod X_l^{n+1} &= t_{l+\varepsilon}^{\alpha+\beta+cn}u_{l+\varepsilon}^{\beta}f^*\frac{\partial}{\partial t_l}\otimes X_{l+\varepsilon}^n \bmod X_{l+\varepsilon}^{n+1} \\ &= t_{l-\varepsilon}^{\alpha}u_{l-\varepsilon}^{\alpha+\beta+cn}f^*\frac{\partial}{\partial t_l}\otimes X_{l-\varepsilon}^n \bmod X_{l-\varepsilon}^{n+1}\,, \end{aligned}$$

$$\begin{aligned} t_l^{\alpha}u_l^{\beta}f^*\frac{\partial}{\partial u_l}\otimes X_l^n \bmod X_l^{n+1} &= t_{l+\varepsilon}^{\alpha+\beta+cn}u_{l+\varepsilon}^{\beta}f^*\frac{\partial}{\partial u_l}\otimes X_{l+\varepsilon}^n \bmod X_{l+\varepsilon}^{n+1} \\ &= t_{l-\varepsilon}^{\alpha}u_{l-\varepsilon}^{\alpha+\beta+cn}f^*\frac{\partial}{\partial u_l}\otimes X_{l-\varepsilon}^n \bmod X_{l-\varepsilon}^{n+1}\,. \end{aligned}$$

We construct A_n from V in the following way. We define a map $l:\tilde{I}\to\{l+\varepsilon,l-\varepsilon\}$ of index sets as follows. If $i\neq l+\varepsilon,l-\varepsilon$, then the open subset

$U_i \cap U_l$ of S is contained in either $f(\tilde{U}_{l+\varepsilon})$ or $f(\tilde{U}_{l-\varepsilon})$. We choose $l(i) \in \{l+\varepsilon, l-\varepsilon\}$ in such a way that the following inclusion is satisfied: $U_i \cap U_l \subset f(\tilde{U}_{l(i)})$. We define $l(l+\varepsilon)=l+\varepsilon$ and $l(l-\varepsilon)=l-\varepsilon$.

From a given element $\psi=\psi_{l+\varepsilon, l-\varepsilon} \in \Gamma(U_{l+\varepsilon, l-\varepsilon}, Q_n)$, we construct $x=x(\psi) =(x_{i_0;j_0j_1}) \in C^0(\mathcal{U}, f_* C^1(\tilde{\mathcal{U}}, Q_n))$ with $x_{i_0;j_0j_1} \in \Gamma(f^{-1}(U_{i_0}) \cap \tilde{U}_{j_0j_1}, Q_n)$ in the following way. If $i_0 \neq l$, we define $x_{i_0;j_0j_1}=0$. We put $x_{l;j_0j_1}=\psi_{l(j_0)l(j_1)}$, where we use the following convention: $\psi_{l+\varepsilon, l+\varepsilon}=\psi_{l-\varepsilon, l-\varepsilon}=0$ and $\psi_{l-\varepsilon, l+\varepsilon}= -\psi_{l+\varepsilon, l-\varepsilon}$. Then $x(\psi)$ represents an element of $H^0(S, R^1 f_* Q_n)$. In fact, we easily see $x_{i_0;j_0j_1}-x_{i_0;j_0j_2}+x_{i_0;j_1j_2}=0$ for each i_0, j_0, j_1 and j_2. Moreover, we can find an element $y=(y_{i_0i_1;j_0}) \in C^1(\mathcal{U}, f_* C^0(\tilde{\mathcal{U}}, Q_n))$ with $y_{i_0i_1;j_0} \in \Gamma(f^{-1}(U_{i_0i_1}) \cap \tilde{U}_{j_0}, Q_n)$ such that $y_{i_0i_1;j_0}-y_{i_0i_1;j_1}=x_{i_0;j_0j_1}-x_{i_1;j_0j_1}$ for each i_0, i_1, j_0 and j_1 in the following way: We put $y_{i_0i_1;j_0}=0$ if $i_0 \neq l$ and $i_1 \neq l$, $y_{li_1;j_0}=\psi_{l(j_0)l(i_1)}$, and $y_{i_0l;j_0}=\psi_{l(i_0)l(j_0)}$.

Let $(\psi_1, \cdots, \psi_h)$ be a basis of the vector space V. If we take as A_n the vector space spanned by $x(\psi_1), \cdots, x(\psi_h)$, then A_n satisfies the required property. Since $\operatorname{Scope}(A_n; 0)=\operatorname{Scope}(V; 0)$, we obtain

$$\operatorname{Scope}(A_n; 0)=\sum_{\substack{\alpha, \beta \leqq -1 \\ \alpha+\beta+cn \geqq -1}} \boldsymbol{Z}_{\geqq 0}(\alpha, \beta, n) T(l, 0).$$

Next, we calculate $\operatorname{Scope}(C_n; 0)$. Since $\mathcal{P}_n \cong \iota_* \mathcal{O}_{P^1}(1-nc)$, where $\iota: E \to \tilde{S}$ denotes the natural inclusion, we have $H^0(\tilde{S}, \mathcal{P}_n)=0$ unless $n=c=1$. Suppose $n=c=1$. Then $\dim H^0(\tilde{S}, \mathcal{P}_1)=1$ and we can take as C_1 the set consisting of the elements $a\xi_{l+\varepsilon, 1}=-a\xi_{l-\varepsilon, 1}$ with $a \in k$. Thus we obtain

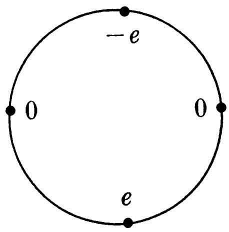
$$\begin{aligned}\operatorname{Scope}(C_1; 0) &= \boldsymbol{Z}_{\geqq 0}(-1, -1, 1) \tilde{T}(l+\varepsilon, 0) \\ &= \boldsymbol{Z}_{\geqq 0}(-1, -1, 1) \tilde{T}(l-\varepsilon, 0) \\ &= \boldsymbol{Z}_{\geqq 0}(-1, -1, 1) T(l, 0).\end{aligned}$$

Thus Theorem 4.2 is proved.

§ 5. Reduction to scopes.

In this section, we aim at proving at Proposition 5.6, which enables us to reduce the proof of Main Theorem to the estimation of scopes. In order to state Proposition 5.6, we fix the following notation concerning the Hirzebruch surface $\Sigma_e=\boldsymbol{P}_{P^1}(\mathcal{O} \oplus \mathcal{O}(-e))$.

The surface Σ_e is described by the following weighted dual graph:



Let $D=D_1+D_2+D_3+D_4$ be the corresponding invariant Cartier divisor with $(D_1)^2=e$, $(D_2)^2=0$, $(D_3)^2=-e$ and $(D_4)^2=0$, and let $p_0=D_4\cap D_1$, $p_1=D_1\cap D_2$, $p_2=D_2\cap D_3$ and $p_3=D_3\cap D_4$. Then Σ_e is covered by four sheets U_i ($i=0$, 1, 2, 3) of affine open subsets with $U_i=\operatorname{Spec} k[t_i, u_i]$. Then the following relations are satisfied: $t_0=u_1^{-1}$, $u_0=t_1u_1^{-e}$, $t_1=u_2^{-1}$, $u_1=t_2$, $t_2=u_3^{-1}$, $u_2=t_3u_3^{e}$, $t_3=u_0^{-1}$ and $u_3=t_0$.

DEFINITION 5.1. Let $f:\tilde{S}\to S$ be a proper birational morphism of nonsingular projective surfaces. We define the set Fund(f) of the fundamental points of f as follows:

$$\text{Fund}(f)=\{x\in S \mid f^{-1} \text{ is not defined at } x\}.$$

Using the above notation, we state the following lemma.

LEMMA 5.2. *Let S be a nonsingular projective toric surface. Assume that S is not isomorphic to $\boldsymbol{P}^2$. Then S is one of the following three types:*

(*Type* I): *There exists a proper birational morphism $f: S\to\Sigma_e$ which is a succession of equivariant blowing-ups such that $e\geqq 2$ and that* Fund(f) $\subset\{p_2, p_3\}$,

(*Type* II): *There exists a proper birational morphism $f: S\to\Sigma_0=\boldsymbol{P}^1\times\boldsymbol{P}^1$ which is a succession of equivariant blowing-ups such that* Fund(f) $\subset\{p_0, p_2\}$,

or

(*Type* III): *There exists a proper birational morphism $f: S\to\Sigma_1$ which is a succession of equivariant blowing-ups such that* Fund$(f)\subset$ $\{p_0, p_2, p_3\}$.

REMARK. The above three types of surfaces are not exclusive. For example, there exists a surface S of type II and type III at once. Precisely speaking, we consider the pair (S, f) of the surface S and the above morphism f when we say that S is of *type A* ($A=$I, II, III).

First we prove the following claim.

CLAIM 5.3. For any nonsingular projective toric surface S that is not isomorphic to $\boldsymbol{P}^2$, there exists a proper birational morphism $f: S \to \Sigma_e$ $(e \geqq 0)$ which is a succession of equivariant blowing-ups such that $\operatorname{Fund}(f) \subset \{p_0, p_2\}$.

PROOF OF CLAIM 5.3. We prove it by the induction on the Picard number $\rho(S)$ of S. If $\rho(S) \leqq 4$, then it is obvious. Suppose $\rho(S) \geqq 5$. Then, by the induction hypothesis, there exists a nonsingular toric surface S_1 and a sequence $S \xrightarrow{g} S_1 \xrightarrow{f} \Sigma_e$ such that f is a succession of equivariant blowing-ups with $\operatorname{Fund}(f) \subset \{p_0, p_2\}$ and that g is an equivariant blowing-up along a point $q \in S_1$. If $f(q) \in \{p_0, p_2\}$, then the composition $f \circ g$ satisfies the required property. Suppose $f(q) \in \{p_1, p_3\}$. Then there exists a nonsingular toric surface S_2 and a sequence $S \xrightarrow{f'} S_2 \xrightarrow{g'} \Sigma_e$ of morphisms such that g' is the equivariant blowing-up along $f(q)$ and that f' is a succession of equivariant blowing-ups with $g'(\operatorname{Fund}(f')) \subset \{p_0, p_2\}$. Replacing $S_2 \xrightarrow{g'} \Sigma_e$ by $S_2 \to \Sigma_{e+1}$ or $S_2 \to \Sigma_{e-1}$ by an elementary transformation, we get a morphism satisfying the required property.

PROOF OF LEMMA 5.2. We take such a morphism $f: S \to \Sigma_e$ as in Claim 5.3. If $e=0$, then the pair (S, f) is of type II. If $e=1$, then (S, f) is of type III. Suppose $e \geqq 2$. If $\operatorname{Fund}(f) \not\ni p_0$, then (S, f) is of type I. Suppose $\operatorname{Fund}(f) \ni p_0$. Then the morphism f is factorized into a succession $S \xrightarrow{g} S_1 \xrightarrow{h} \Sigma_e$ of morphisms, where h is a blowing-up along p_0. Replacing the morphism $h: S_1 \to \Sigma_e$ by its elementary transformation, we get a morphism $f_1: S \to \Sigma_{e-1}$ which is a succession of equivariant blowing-ups with $\operatorname{Fund}(f) \subset \{p_0, p_2, p_3\}$. If $e-1=1$, then (S, f_1) is of type III. If $e-1 \geqq 2$ and $\operatorname{Fund}(f_1) \not\ni p_0$, then (S, f_1) is of type I. If $e-1 \geqq 2$ and $\operatorname{Fund}(f_1) \ni p_0$, then f_1 factors via the blowing-up $h_1: S_2 \to \Sigma_{e-1}$ along p_0. Replacing h_1 by its elementary transformation, we get a morphism $f_2: S \to \Sigma_{e-2}$ which is a succession of equivariant blowing-ups with $\operatorname{Fund}(f_2) \subset \{p_0, p_2, p_3\}$. A succession of these arguments leads to the proof.

On toric surfaces of each type, we take the following nonsingular rational curve called reference curves, which we shall use to prove Main Theorem by applying RD Lemma (Lemma 2.6).

DEFINITION 5.4. (1) Let (S, f) be a toric surface of type I. Let D_1 denote the invariant curve on Σ_e with $(D_1)^2 = e$ as is stated before, that is, D_1 is determined by the equations $u_0=0$ on the open subset U_0 and $t_1=0$ on U_1. We call the strict transform C of D_1 with respect to f the reference curve of type I.

(2) Let (S, f) be a toric surface of type II. Let Γ be the diagonal curve on $\boldsymbol{P}^1\times\boldsymbol{P}^1$ defined by the equations $t_1=u_1$ on U_1 and $t_3=u_3$ on U_3. We call the strict transform C of Γ with respect to f the reference curve of type II.

(3) Let (S, f) be a toric surface of type III. Let D_1' be a displacement of the curve D_1 on Σ_1 defined by the equations $u_0=1$ on U_0 and $t_1=u_1$ on U_1. We call the strict transform C of D_1' with respect to f the reference curve of type III.

Moreover, we fix the following notation. Let (S, f) be a toric surface of *Type A* ($A=$I, II, III). Since Fund$(f)\not\ni p_1$, there exists an open subset U of S such that $f|_U: U\to U_1$ is an isomorphism. This open set U admits the natural coordinates which is induced by the coordinate (t_1, u_1) on U_1. We denote the open set U by U_1. We also denote the coordinates induced by (t_1, u_1) by the same symbols (t_1, u_1). Let (X, S) be a formal neighbourhood of S described by a collection $\Phi=\{\Phi_{ij}\}_{i,j\in I}$ of the transition functions. Since $X|_{U_1}\cong\mathrm{Spf}(k[t_1, u_1][[X_1]])$ for some coordinates X_1, we can define the scope Scope$(\Phi; 1)$ of the description Φ with respect to the coordinates (t_1, u_1, X_1).

DEFINITION 5.5. We define a semi-group Ω_{RD} contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ in the following way:

$$\Omega_{RD}=\{(\alpha, \beta, n)\in\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}\mid \alpha+\beta+n\leqq 0\}.$$

PROPOSITION 5.6. *Let S be a toric surface of type A ($A=$I, II or III) and let C the reference curve of type A on S. Let (X, S) be a formal neighbourhood of S such that $N_{S/X}\otimes_{\mathcal{O}_S}\mathcal{O}_C$ is ample on C. Assume that (X, S) is described by a collection Φ of the transition functions such that* Scope$(\Phi; 1)\subset\Omega_{RD}$. *Then the induced formal neighbourhood $(X, C)^\wedge$ of the curve C in X is rationally dominated. More precisely, $(X, C)^\wedge$ admits a description by the transition functions satisfying the assumption of Lemma* 2.6 *for $r=1$.*

PROOF. First, we assume that (S, f) is a toric surface of type I. Since Fund$(f)\not\ni p_0, p_1, f^{-1}(U_0)$ and $f^{-1}(U_1)$ are isomorphic to $\boldsymbol{A}^2$, which we denote by U_0 and U_1, respectively. Then we get

$$T(0,1)=\begin{pmatrix} 0 & -1 & 0 \\ 1 & -e & 0 \\ 0 & -a & 1 \end{pmatrix},$$

where $N_{C/X}\otimes_{\mathcal{O}_S}\mathcal{O}_C\cong\mathcal{O}_{P^1}(a)$ $(a>0)$. By the assumption, the transition relation between the coordinates (t_0, u_0, X_0) on $X|_{U_0}$ and (t_1, u_1, X_1) on $X|_{U_1}$ is written in the following way:

$$t_0=u_1^{-1}(1+\sum a_{\alpha\beta n}t_1^{\alpha}u_1^{\beta}X_1^{n}),$$

$$u_0=t_1u_1^{-e}(1+\sum b_{\alpha\beta n}t_1^{\alpha}u_1^{\beta}X_1^{n}),$$

$$X_0=u_1^{-a}X_1(1+\sum c_{\alpha\beta n}t_1^{\alpha}u_1^{\beta}X_1^{n}),$$

where $a_{\alpha\beta n}$, $b_{\alpha\beta n}$ or $c_{\alpha\beta n}\neq 0$ implies $\alpha+\beta+n\leqq 0$. Since the reference curve C is defined by the equations $u_0=0$ on U_0 and $t_1=0$ on U_1, we can consider the above equations to be a transition relation describing the neighbourhood $(X, C)^{\wedge}$ of C in X. We apply Lemma 2.6 to this description.

We now assume that (S, f) is a toric surface of type II. Since p_1, p_3 $\notin\mathrm{Fund}(f)$, $f^{-1}(U_1)$ and $f^{-1}(U_3)$ are isomorphic to $\boldsymbol{A}^2$, which we denote by U_1 and U_3, respectively. Then we get

$$T(3,1)=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -b & -a & 1 \end{pmatrix}$$

for some integer a and b. The transition relation between the coordinates (t_3, u_3, X_3) on $X|_{U_3}$ and (t_1, u_1, X_1) on $X|_{U_1}$ is written in the following way:

$$t_3=t_1^{-1}(1+\sum a_{\alpha\beta n}t_1^{\alpha}u_1^{\beta}X_1^{n}),$$

$$u_3=u_1^{-1}(1+\sum b_{\alpha\beta n}t_1^{\alpha}u_1^{\beta}X_1^{n}),$$

$$X_3=t_1^{-b}u_1^{-a}X_1(1+\sum c_{\alpha\beta n}t_1^{\alpha}u_1^{\beta}X_1^{n}),$$

where $a_{\alpha\beta n}$, $b_{\alpha\beta n}$ or $c_{\alpha\beta n}\neq 0$ implies $\alpha+\beta+n\leqq 0$. To obtain a transition relation of $(X, C)^{\wedge}$, we change the coordinates near the curve C in the following way: We put $T_3=u_3$, $X_3=X_3$ and $Y_3=u_3-t_3$ near $C\cap U_3$, and $T_1=t_1$, $X_1=X_1$ and $Y_1=u_1-t_1$ near $C\cap U_1$. Then the curve C is defined by the equation $X_3=Y_3=0$ and $X_1=Y_1=0$. The transition relation between the new coordinates (T_3, X_3, Y_3) and (T_1, X_1, Y_1) is easily calculated as follows:

$$T_3 = T_1^{-1}(1+T_1^{-1}Y_1)^{-1}\{1+\sum b_{\alpha\beta n}T_1^{\alpha+\beta}(1+T_1^{-1}Y_1)^{\beta}X_1^n\},$$

$$X_3 = T_1^{-a-b}(1+T_1^{-1}Y_1)^{-a}X_1\{1+\sum c_{\alpha\beta n}T_1^{\alpha+\beta}(1+T_1^{-1}Y_1)^{\beta}X_1^n\},$$

$$Y_3 = T_1^{-1}(1+T_1^{-1}Y_1)^{-1}\{1+\sum b_{\alpha\beta n}T_1^{\alpha+\beta}(1+T_1^{-1}Y_1)^{\beta}X_1^n\} - T_1^{-1}\{1+\sum a_{\alpha\beta n}T_1^{\alpha+\beta}(1+T_1^{-1}Y_1)^{\beta}X_1^n\}.$$

Since $N_{S/X}\otimes\mathcal{O}_C$ is ample, we see $a+b>0$. It is easy to see that, if a term $T_1^A X_1^n Y_1^m$ appears in the above transition functions, then $-A\geqq n+m$. We apply Lemma 2.6 to this description.

Finally, we assume that (S,f) is a toric surface of type III. As was stated before, the surface Σ_1 is covered by four affine open subsets U_0, U_1, U_2 and U_3. Corresponding to an equivariant blowing-up, we replace an affine open subset by two sheets of affine open sets. By replacing open covering in such a way as we stated before, we get an affine open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of S. The curve D_4 in Σ_1 is defined by the equations $t_0=0$ on U_0 and $u_3=0$ on U_3. There exists an element $\delta\in\Lambda$ such that $f(U_\delta)\subset U_0$ and that U_δ intersects with the strict transform of D_4 with respect to f. Then the transition matrix $T(\delta,1)$ is written in the following way. (Cf. §4). First, we formally put $T(\delta,1)=T(\delta,0)T(0,1)$. Then

$$T(0,1)=\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -a & 1 \end{pmatrix}$$

for some integer a, and $T(\delta,0)$ is a product of matrices of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}^{-1}=\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix}.$$

Hence $T(\delta,0)$ and $T(\delta,1)$ is written in the following form:

$$T(\delta,0)=\begin{pmatrix} 1 & -l & 0 \\ 0 & 1 & 0 \\ 0 & -p & 1 \end{pmatrix},$$

$$T(\delta,1)=\begin{pmatrix} -l & l-1 & 0 \\ 1 & -1 & 0 \\ -p & p-a & 1 \end{pmatrix}$$

for some $l, p \in \boldsymbol{Z}$. The transition relation between the coordinates $(t_\delta, u_\delta, X_\delta)$ on U_δ and (t_1, u_1, X_1) on U_1 is written in the following form:

$$t_\delta = t_1^{-l} u_1^{l-1}(1+\sum a_{\alpha\beta n} t_1^\alpha u_1^\beta X_1^n),$$

$$u_\delta = t_1 u_1^{-1}(1+\sum b_{\alpha\beta n} t_1^\alpha u_1^\beta X_1^n),$$

$$X_\delta = t_1^{-p} u_1^{p-a} X_1(1+\sum c_{\alpha\beta n} t_1^\alpha u_1^\beta X_1^n),$$

where $a_{\alpha\beta n}$, $b_{\alpha\beta n}$ or $c_{\alpha\beta n} \neq 0$ implies $\alpha+\beta+n \leqq 0$. In order to describe the formal neighbourhood $(X, C)^\wedge$ of C in X, we change coordinates near the curve C. We take coordinates $(\tilde{t}_\delta, \tilde{X}_\delta, Y_\delta)$ near $C \cap U_\delta$ as follows: $\tilde{t}_\delta = t_\delta u_\delta^l$, $\tilde{X}_\delta = u_\delta^p X_\delta$ and $Y_\delta = u_\delta - 1$. Note that we can take such coordinates around $C \cap U_\delta$, since $u_\delta \neq 0$ near $C \cap U_\delta$. We take coordinates $(\tilde{t}_1, \tilde{X}_1, Y_1)$ near $C \cap U_1$ as follows: $\tilde{t}_1 = u_1$, $\tilde{X}_1 = X_1$ and $Y_1 = t_1 - u_1$. Then the reference curve C is defined by the equations $\tilde{X}_\delta = Y_\delta = 0$ and $\tilde{X}_1 = Y_1 = 0$. The transition relation between the coordinates $(\tilde{t}_\delta, \tilde{X}_\delta, Y_\delta)$ and $(\tilde{t}_1, \tilde{X}_1, Y_1)$ is calculated as follows:

$$\tilde{t}_\delta = \tilde{t}_1^{-1}(1+\sum a_{\alpha\beta n}\varphi_{\alpha\beta n})(1+\sum b_{\alpha\beta n}\varphi_{\alpha\beta n})^l,$$

$$\tilde{X}_\delta = \tilde{t}_1^{-a}\tilde{X}_1(1+\sum b_{\alpha\beta n}\varphi_{\alpha\beta n})^p(1+\sum c_{\alpha\beta n}\varphi_{\alpha\beta n}),$$

$$Y_\delta = -1+(1+\tilde{t}_1^{-1}Y_1)(1+\sum b_{\alpha\beta n}\varphi_{\alpha\beta n}),$$

where $\varphi_{\alpha\beta n} = \tilde{t}_1^{\alpha+\beta}(1+\tilde{t}_1^{-1}Y_1)^\alpha \tilde{X}_1^n$. We apply Lemma 2.6 to this description, noting that $a>0$.

Thus Proposition 5.6 is proved.

As for $\boldsymbol{P}^2$, we fix the covering $\boldsymbol{P}^2 = U_0 \cup U_1 \cup U_2$ with $U_i \cong \operatorname{Spec} k[t_i, u_i]$ $(i=0,1,2)$ such that the following transition relations are satisfied: $t_0 = u_1^{-1}$ and $u_0 = t_1 u_1^{-1}$ on $U_0 \cap U_1$, and $t_1 = u_2^{-1}$ and $u_1 = t_2 u_2^{-1}$ on $U_1 \cap U_2$.

DEFINITION 5.7. We define a semi-group Ω'_{RD} contained in $\boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ in the following way:

$$\Omega'_{RD} = \left\{(\alpha, \beta, n) \in \boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0} \mid -\beta \geqq \frac{1}{2}(\alpha+n)\right\}.$$

PROPOSITION 5.8. *Let $S=\boldsymbol{P}^2$ and let C a nonsingular rational curve on S defined by the equations $u_0=0$ on U_0 and $t_1=0$ on U_1. Let (X, S) be a formal neighbourhood of S with $N_{S/X}$ ample. Assume that (X, S) is described by a collection Φ of the transition functions such that* Scope$(\Phi; 1)$ $\subset \Omega'_{RD}$. *Then the induced formal neighbourhood $(X, C)^\wedge$ of the curve C in X admits a description by the transition functions satisfying the assumption of Lemma 2.6 for $r=2$.*

PROOF. The transition relation between the coordinates (t_0, u_0, X_0) and (t_1, u_1, X_1) is written in the following way:

$$t_0 = u_1^{-1}(1 + \sum a_{\alpha\beta n} t_1^{\alpha} u_1^{\beta} X_1^{n}),$$

$$u_0 = t_1 u_1^{-1}(1 + \sum b_{\alpha\beta n} t_1^{\alpha} u_1^{\beta} X_1^{n}),$$

$$X_0 = u_1^{-a} X_1(1 + \sum c_{\alpha\beta n} t_1^{\alpha} u_1^{\beta} X_1^{n}),$$

with $a>0$, where $a_{\alpha\beta n}$, $b_{\alpha\beta n}$ or $c_{\alpha\beta n} \neq 0$ implies $-\beta \geqq (1/2)(\alpha+n)$. We can consider it to be a transition relation describing $(X, C)^{\wedge}$ as it is. We apply Lemma 2.6.

§ 6. The proof of Main Theorem.

In this section, we prove Main Theorem. We use the same notation as in § 5.

THEOREM 6.1. *Let S be a nonsingular projective toric surface and N an ample line bundle on S.*

(1) *If $S = \boldsymbol{P}^2$, then, for each $n>0$, there exists an H^1-slice V_n of $\mathcal{G}_n$ such that the following condition is satisfied:*

$$\mathrm{Scope}(V_n; 1) \subset \Omega'_{RD} = \left\{(\alpha, \beta, m) \mid -\beta \geqq \frac{1}{2}(\alpha+m)\right\}.$$

(2) *Assume that there exists a morphism $f: S \to \Sigma_e$ such that the pair (S, f) is of type A ($A=$ I, II or III). Then, for each $n>0$, there exists an H^1-slice V_n of $\mathcal{G}_n$ such that the following condition is satisfied:*

$$\mathrm{Scope}(V_n; 1) \subset \Omega_{RD} = \{(\alpha, \beta, m) \mid \alpha+\beta+m \leqq 0\}.$$

COROLLARY 6.2 (Theorem 0). *Let S be any nonsingular projective toric surface. Then there exists a nonsingular rational curve C on S such that, for any formal neighbourhood (X, S) of S with $N_{S/X}$ being an ample line bundle, the neighbourhood $(X, C)^{\wedge}$ of C on X is rationally dominated.*

COROLLARY 6.3 (Main Theorem). *Let X be a nonsingular complete algebraic variety of dimension three. Assume that X contains a nonsingular projective toric surface S and that the normal bundle $N_{S/X}$ of S in X is ample. Then X is unirational.*

PROOF OF COROLLARY 6.2 AND 6.3. Corollary 6.2 immediately follows from Theorem 6.1, Theorem 3.8, Proposition 5.6 and Proposition 5.8.

Corollary 6.3 follows from Corollary 6.2 and Proposition 2.5.

First, we prove Theorem 6.1 in the case where S is the projective space $\boldsymbol{P}^2$.

PROOF OF THEOREM 6.1. (1). As is easily seen, the cohomology group $\mathrm{H}^1(S, \Theta_{P^2}\otimes\mathcal{O}_{P^2}(-a))$ vanishes unless $a=3$. We also see

$$\dim \mathrm{H}^1(S, \Theta_{P^2}\otimes\mathcal{O}_{P^2}(-3))=1\,.$$

After elementary Čech cohomological calculation, we get the element $\phi=(\phi_{i_0 i_1})\in \mathrm{Z}^1(\mathcal{U}, \Theta_{P^2}\otimes\mathcal{O}_{P^2}(-3))$ with $\phi_{i_0 i_1}\in\Gamma(U_{i_0 i_1}, \Theta_{P^2}\otimes\mathcal{O}_{P^2}(-3))$ as an H^1-basis of the sheaf $\Theta_{P^2}\otimes\mathcal{O}_{P^2}(-3)$ as follows:

$$\phi_{01}=-t_0^{-1}\frac{\partial}{\partial u_0}\otimes\eta_0\,,$$

$$\phi_{02}=u_0^{-1}\frac{\partial}{\partial t_0}\otimes\eta_0\,,$$

$$\phi_{12}=t_0^{-1}\frac{\partial}{\partial u_0}\otimes\eta_0+u_0^{-1}\frac{\partial}{\partial t_0}\otimes\eta_0$$

$$=-t_1^{-1}\frac{\partial}{\partial u_1}\otimes\eta_1\,,$$

where η_i denotes the local basis of the sheaf $\mathcal{O}_{P^2}(-3)$ on U_i. Thus we can take an H^1-basis of $\mathcal{G}_n$ in the following way.

We easily see $\mathrm{H}^1(\boldsymbol{P}^2, \mathcal{G}_n)=0$ for all $n>0$, unless $N\cong\mathcal{O}(1)$ or $N\cong\mathcal{O}(3)$. Suppose $N\cong\mathcal{O}(1)$. Then we have the following element $\lambda=(\lambda_{i_0 i_1})\in \mathrm{Z}^1(\mathcal{U}, \mathcal{G}_3)$ with $\lambda_{i_0 i_1}\in\Gamma(U_{i_0 i_1}, \mathcal{G}_3)$ as an H^1-basis of $\mathcal{G}_3$:

$$\lambda_{01}=-t_0^{-1}\frac{\partial}{\partial u_0}\otimes X_0^3 \mod X_0^4\,,$$

$$\lambda_{02}=u_0^{-1}\frac{\partial}{\partial t_0}\otimes X_0^3 \mod X_0^4\,,$$

$$\lambda_{12}=-t_1^{-1}\frac{\partial}{\partial u_1}\otimes X_1^3 \mod X_1^4\,.$$

The scope $\mathrm{Scope}(\lambda;1)$ is generated by $(-1,-1,3)$.

If $N\cong\mathcal{O}(3)$, then we have $\mathrm{H}^1(\boldsymbol{P}^2, \mathcal{G}_n)=0$ unless $n=1$. We can take the following element $\mu=(\mu_{i_0 i_1})\in \mathrm{Z}^1(\mathcal{U}, \mathcal{G}_1)$ as an H^1-basis of $\mathcal{G}_1$:

$$\mu_{01} = -t_0^{-1}\frac{\partial}{\partial u_0}\otimes X_0 \mod X_0^2,$$

$$\mu_{02} = u_0^{-1}\frac{\partial}{\partial t_0}\otimes X_0 \mod X_0^2,$$

$$\mu_{12} = -t_1^{-1}\frac{\partial}{\partial u_1}\otimes X_1 \mod X_1^2.$$

The scope Scope$(\mu; 1)$ is generated by $(-1, -1, 1)$. Thus Theorem 6.1.(1) is proved.

REMARK 6.4. After more precise calculation of the transition functions, we can show that any formal neighbourhood $(X, \boldsymbol{P}^2)$ of $\boldsymbol{P}^2$ with the normal bundle N is isomorphic to the formal neighbourhood of the zero section of the bundle $N \to \boldsymbol{P}^2$. For example, the transition functions of the third infinitesimal neighbourhood $(X, \boldsymbol{P}^2)$ with $N_{P^2/X} \cong \mathcal{O}(1)$ is written in the following way: $t_0 \equiv u_1^{-1}$, $u_0 \equiv u_1^{-1}t_1 + cu_1^{-1}X_1^3$, $X_0 \equiv u_1^{-1}X_1$ on $X|_{U_{01}}$, $t_1 \equiv u_2^{-1}$, $u_1 \equiv t_2u_2^{-1} + cu_2^{-1}X_2^3$, $X_1 \equiv u_2^{-1}X_2$ on $X|_{U_{12}}$, and $t_2 \equiv u_0^{-1}$, $U_2 \equiv t_0u_0^{-1} + cu_0^{-1}X_0^3$, $X_2 \equiv u_0^{-1}X_0$ on $X|_{U_{02}}$ with $c \in k$, where the symbol $\equiv$ denotes the congruence modulo X_i^4 $(i = 0, 1, 2)$. But no fourth infinitesimal neighbourhood is actually an extension of the above third neighbourhood unless $c = 0$. We don't consider such obstructions so far as we discuss scopes.

Next, we prove Theorem 6.1 in the case where S is the Hirzebruch surface Σ_e. We make some preparations before we state the proof.

DEFINITION 6.5. Let $S = \Sigma_e$, $\pi: S \to C \cong \boldsymbol{P}^1$ the natural projection, F a fiber of π and s_o the section with $s_o^2 = -e$. For $a, b \in \boldsymbol{Z}$, we denote the invertible sheaf $\mathcal{O}(aF + bs_o)$ by the symbol $\mathcal{O}(a, b)$.

LEMMA 6.6. *The cohomology group* $\mathrm{H}^1(\Sigma_e, \mathcal{O}(p, q))$ *vanishes unless one of the following two conditions is satisfied*:

(a) $p \geqq e(q+1)$ *and* $q \leqq -2$

(b) $p \leqq eq - 2$ *and* $q \geqq 0$.

In the case (a), *we can take the following elements as an* H^1*-basis of* $\mathcal{O}(p, q)$:

$$\Psi^{\alpha,\beta} = (0, \varphi^{\alpha,\beta}, \varphi^{\alpha,\beta}, \varphi^{\alpha,\beta}, \varphi^{\alpha,\beta}, 0)$$

$$\in \Gamma(U_{01}, \mathcal{O}(p, q)) \times \Gamma(U_{02}, \mathcal{O}(p, q)) \times \Gamma(U_{03}, \mathcal{O}(p, q))$$

$$\times \Gamma(U_{12}, \mathcal{O}(p, q)) \times \Gamma(U_{13}, \mathcal{O}(p, q)) \times \Gamma(U_{23}, \mathcal{O}(p, q))$$

with $\alpha \geqq 0$, $\beta < 0$, $p - \alpha - e\beta \geqq 0$ *and* $q - \beta < 0$, *where* $\varphi^{\alpha,\beta} = t_0^\alpha u_0^\beta \eta_0$ *and* η_0 *de-*

notes the local basis of $\mathcal{O}(p, q)$ *on* U_0.

In the case (b), *we can take the following elements as an* H^1*-basis of* $\mathcal{O}(p, q)$:

$$\Phi^{\alpha,\beta}=(\varphi^{\alpha,\beta}, \varphi^{\alpha,\beta}, 0, 0, -\varphi^{\alpha,\beta}, -\varphi^{\alpha,\beta})$$

$$\in\Gamma(U_{01}, \mathcal{O}(p, q))\times\Gamma(U_{02}, \mathcal{O}(p, q))\times\Gamma(U_{03}, \mathcal{O}(p, q))$$

$$\times\Gamma(U_{12}, \mathcal{O}(p, q))\times\Gamma(U_{13}, \mathcal{O}(p, q))\times\Gamma(U_{23}, \mathcal{O}(p, q))$$

with $\alpha<0$, $\beta\geqq 0$, $p-\alpha-e\beta<0$ *and* $q-\beta\geqq 0$.

The proof is done by elementary calculation and we omit it.

PROOF OF THEOREM 6.1 IN THE CASE WHERE $S=\Sigma_e$. We put $L=\mathrm{Ker}(\Theta_S\to\pi^*\Theta_C)$. Then we have the exact sequence

$$0\longrightarrow L\longrightarrow \Theta_S\longrightarrow \pi^*\Theta_C\longrightarrow 0\,.$$

It is easy to see that $L\cong\mathcal{O}(e, 2)$ and $\pi^*\Theta_C\cong\mathcal{O}(2, 0)$. In fact, $\partial/\partial u_0$, $\partial/\partial t_1$, $\partial/\partial u_2$ and $\partial/\partial t_3$ are the local basis of L on U_0, U_1, U_2 and U_3 respectively. Moreover, the sheaf $\pi^*\Theta_C$ admits $[\partial/\partial t_0]$, $[\partial/\partial u_1]$, $[\partial/\partial t_2]$ and $[\partial/\partial u_3]$ as the local bases on U_0, U_1, U_2 and U_3, respectively, where $[\partial/\partial t_i]$ and $[\partial/\partial u_i]$ denotes the images of $\partial/\partial t_i$ and $\partial/\partial u_i$, respectively ($i=0, 1, 2, 3$). We denote $L\otimes N^{-n}$ by L_n and $\pi^*\Theta_C\otimes N^{-n}$ by M_n. Then we have the following exact sequence

$$0\longrightarrow L_n\longrightarrow \mathcal{G}_n\longrightarrow M_n\longrightarrow 0\,.$$

Since L_n is a subsheaf of $\mathcal{G}_n$, we can naturally define the scope of an element of $\mathrm{C}^{\cdot}(\mathcal{U}, L_n)$ such that the natural map $\mathrm{C}^{\cdot}(\mathcal{U}, L_n)\to\mathrm{C}^{\cdot}(\mathcal{U}, \mathcal{G}_n)$ is scope-preserving. We can also define the scope of an element of the group $\mathrm{C}^{\cdot}(\mathcal{U}, M_n)$ such that the natural map $\mathrm{C}^{\cdot}(\mathcal{U}, \mathcal{G}_n)\to\mathrm{C}^{\cdot}(\mathcal{U}, M_n)$ is scope-preserving. Thus we have only to calculate the scopes of H^1-slices of L_n and M_n in order to calculate the scope of an H^1-slice of $\mathcal{G}_n$.

If we put $N\cong\mathcal{O}(a, b)$, then we obtain the isomorphisms $L_n\cong\mathcal{O}(e-na, 2-nb)$ and $M_n\cong\mathcal{O}(2-na, -nb)$. Since N is ample, the following inequalities are satisfied: $a>eb$, $b>0$. Since

$$T(0, 1)=\begin{pmatrix} 0 & -1 & 0 \\ 1 & -e & 0 \\ 0 & -a & 1 \end{pmatrix}$$

in this case, a vector (α, β, n) belongs to $\Omega_{RD}T(1, 0)$ if and only if $\alpha+(e-1)\beta+(a-1)n\geqq 0$.

First, we calculate the scope of an H^1-slice of L_n with respect to the coordinates (t_0, u_0, X_0). We put $p=e-na$ and $q=2-nb$. Since $a>eb$, we obtain the inequality $p-e(q+1)<0$. Thus the case (a) in Lemma 6.6 does not occur. Hence we have $nb\leqq 2$ if $\mathrm{H}^1(S, L_n)\neq 0$. Suppose $b=2$. Then $\mathrm{H}^1(S, L_n)=0$ if $n\geqq 2$. Noting that $\partial/\partial u_0 \otimes X_0^n \bmod X_0^{n+1}$ is the local basis of L_n on U_0, we can take an H^1-slice V_1 of L_1 such that

$$\mathrm{Scope}(V_1; 0)=\sum_{e-a<\alpha<0} \boldsymbol{Z}_{\geqq 0}(\alpha, -1, 1).$$

It is easy to see that $\mathrm{Scope}(V_1; 0)\subset \Omega_{RD}T(1, 0)$. Suppose $b=1$. Then $\mathrm{H}^1(S, L_n)=0$ if $n\geqq 3$. We can take an H^1-slice V_1 of L_1 such that $\mathrm{Scope}(V_1; 0)$ is generated by the vectors $(\alpha, -1, 1)$ with $e-a<\alpha<0$ and $(\alpha, 0, 1)$ with $-a<\alpha<0$. We can take an H^1-slice V_2 of L_2 such that $\mathrm{Scope}(V_2; 0)$ is generated by $(\alpha, -1, 2)$ with $e-2a<\alpha<0$. Both semi-groups $\mathrm{Scope}(V_1; 0)$ and $\mathrm{Scope}(V_2; 0)$ are contained in $\Omega_{RD}T(1, 0)$.

Next, we calculate the scope of an H^1-slice of M_n. We now put $p=2-na$ and $q=-nb$. Since $q<0$, the case (b) in Lemma 6.6 does not occur. We easily see that $\mathrm{H}^1(S, M_1)$ vanishes unless $a\leqq eb+2-e$ and $b\geqq 2$. Since $a>eb$, $\mathrm{H}^1(S, M_1)=0$ unless $e\leqq 1$. Noting that $[\partial/\partial t_0]\otimes X_0^n \bmod X_0^{n+1}$ is the local basis of M_n on U_0, we can take an H^1-slice W_1 of M_1 such that $\mathrm{Scope}(W_1; 0)$ is contained in the semi-group generated by the vectors $(\alpha-1, \beta, 1)$ with $\alpha\geqq 0$, $\beta<0$, $2-a-\alpha-e\beta\geqq 0$ and $-b-\beta<0$. Since $e\leqq 1$ and $a\geqq eb+1\geqq e+1$, $\mathrm{Scope}(W_1; 0)$ is contained in $\Omega_{RD}T(1, 0)$.

As for M_2, we see that $\mathrm{H}^1(S, M_2)=0$ unless $2a\leqq 2eb+2-e$. Since $a>eb$, $\mathrm{H}^1(S, M_2)=0$ unless $e=0$. We can take an H^1-slice W_2 of M_2 such that $\mathrm{Scope}(W_2; 0)$ is contained in the semi-group generated by the vectors $(\alpha-1, \beta, 1)$ with $\alpha\geqq 0$, $\beta<0$, $2-2a-\alpha\geqq 0$ and $-2b-\beta<0$. Then it is easy to see that $\mathrm{Scope}(W_2; 0)$ is contained in $\Omega_{RD}T(1, 0)$.

Suppose $n\geqq 3$. Then $\mathrm{H}^1(S, M_n)=0$ unless $a\leqq eb+(1/n)(2-e)$. Since $a\geqq eb+1$, $\mathrm{H}^1(S, M_n)=0$ for $n\geqq 3$.

Thus Theorem 6.1.(2) is partially proved in the case where $S=\Sigma_e$.

Before we prove Theorem 6.1.(2), we make the following definition.

PROPOSITION-DEFINITION 6.7. *Let $g: \tilde{S}\rightarrow S$ be a birational morphism between nonsingular projective surfaces and $\tilde{N}$ a line bundle on $\tilde{S}$. Then there uniquely exists a line bundle N on S such that $\tilde{N}=f^*N\otimes\mathcal{O}(D)$ where D is a divisor on $\tilde{S}$ with $\dim g(\mathrm{Supp}(D))=0$. We denote such a line bundle N by $g_{(*)}\tilde{N}$. If $\tilde{N}$ is ample, then $g_{(*)}\tilde{N}$ is also ample.*

PROOF OF THEOREM 6.1.(2). Suppose $\rho(S)\geqq 5$. Then the morphism $f: S\rightarrow\Sigma_e$ is written as the composition $S=S_m \xrightarrow{f_m} S_{m-1} \xrightarrow{f_{m-1}} S_{m-2}\rightarrow\cdots\rightarrow$

$S_1 \xrightarrow{f_1} S_0=\Sigma_e$ of equivariant blowing-ups. Let $N_j=(f_{j+1}\circ\cdots\circ f_m)_{(*)}N$ for $0\leqq j\leqq m-1$ and $N_m=N$. Let Fund $(f_j)=q_j\in S_{j-1}$ and $E_j=f_j^{-1}(q_j)\subset S_j$ $(1\leqq j\leqq m)$. Then, for each j with $1\leqq j\leqq m$, there exists a positive integer c_j such that $N_j\cong f_j^*N_{j-1}\otimes\mathcal{O}(-c_jE_j)$. We construct an affine open cover $\mathcal{U}^{(j)}=\{U_i\}_{i\in I(j)}$ of S_j in the following way. We take the open cover $\mathcal{U}^{(0)}=\{U_i\}_{i\in I(0)}$ of $S_0=\Sigma_e$ with $I(0)=\{0,1,2,3\}$ as before. Suppose the open cover $\mathcal{U}^{(j)}=\{U_i\}_{i\in I(j-1)}$ of S_{j-1} with $U_i=\operatorname{Spec} k[t_i,u_i]$ is determined. For $i\in I(j-1)$, we denote by p_i the point on S_{j-1} determined by the equation $t_i=u_i=0$. Using this notation, we can write $q_j=p_{s(j)}$ for some $s(j)\in I(j-1)$. Then we put

$$I(j)=(I(j-1)\setminus\{s(j)\})\cup\{s(j)+\varepsilon,\, s(j)-\varepsilon\},$$

with ε the symbol as is used in §4. For $i\in I(j-1)\setminus\{s(j)\}$, we denote $f_j^{-1}(U_i)$ the same symbol U_i and we use the same coordinates (t_i,u_i). We have $f_j^{-1}(U_{s(j)})=U_{s(j)+\varepsilon}\cup U_{s(j)-\varepsilon}$ with the coordinates satisfying the following: $t_{s(j)+\varepsilon}=t_{s(j)}$, $u_{s(j)+\varepsilon}=t_{s(j)}^{-1}u_{s(j)}$, $t_{s(j)-\varepsilon}=t_{s(j)}u_{s(j)}^{-1}$ and $u_{s(j)-\varepsilon}=u_{s(j)}$. By the induction on $\rho(S)$ and Theorem 4.2, it is enough to show

$$\{(\alpha,\beta,n)\mid \alpha\leqq -1,\beta\leqq -1,\alpha+\beta+c_mn\geqq -1\}\cdot T(s(m),1)\subset\Omega_{RD}.$$

Let $W=\{j\in\{1,2,\cdots,m\}\mid j=m$ or $f_{j+1}\circ\cdots\circ f_m(p_{s(m)})=p_{s(j)}\}\subset\{1,2,\cdots,m\}$. We put $W=\{j_1,j_2,\cdots,j_k\}$ with $j_1<j_2<\cdots<j_k$. Then, corresponding to W, we can construct a succession of equivariant blowing-ups $S'_k\xrightarrow{g_k}S'_{k-1}\xrightarrow{g_{k-1}}S'_{k-1}\to\cdots\to S'_1\xrightarrow{g_1}S_0=\Sigma_e$ with Fund $(g_\lambda)=p_{s(j_\lambda)}\in S'_{\lambda-1}$ for $1\leqq\lambda\leqq k$. There exists a birational morphism $h:S\to S'_k$ such that $g_1\circ\cdots\circ g_k\circ h=f$. Since the matrix $T(s(m),1)$ is determined by the data $s(j)$ and c_j with $j\in W$ and since $h_{(*)}N$ is ample, we may assume $W=\{1,2,\cdots,m\}$. That is, we may assume that f is a succession of equivariant blowing-ups along successive infinitely near points.

Thus we have $s(j+1)=s(j)+\varepsilon$ or $s(j+1)=s(j)-\varepsilon$ for $1\leqq j\leqq m-1$. We put $N_0=\mathcal{O}(a,b)$. Then we have $a>eb$ and $b>0$. We also put

$$T(s(j),1)=\begin{pmatrix} a_{11}(j) & a_{12}(j) & 0\\ a_{21}(j) & a_{22}(j) & 0\\ a_{31}(j) & a_{32}(j) & 1\end{pmatrix}.$$

We divide the proof into three cases.

Case I: $s(1)=0$. In this case, (S,f) is of type II or type III. Thus $e\leqq 1$. We put $r_1(j)=-a_{11}(j)-a_{12}(j)$, $r_2(j)=a_{21}(j)+a_{22}(j)$ and $r_3(j)=-a_{31}(j)-a_{32}(j)-1$ for $1\leqq j\leqq m$.

CLAIM 6.8. $r_i(j) \geqq 0$ for $i=1, 2$.

PROOF. We have

$$T(s(1), 1)=T(0, 1)=\begin{pmatrix} 0 & -1 & 0 \\ 1 & -e & 0 \\ 0 & -a & 1 \end{pmatrix}.$$

Thus $r_1(1)=1$, $r_2(1)=1-e \geqq 0$. If $s(j+1)=s(j)+\varepsilon$, we have

$$\begin{aligned} T(s(j+1), 1) &= T(s(j)+\varepsilon, s(j)\,;\, c_j)T(s(j), 1) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -c_j & 0 & 1 \end{pmatrix} T(s(j), 1). \end{aligned}$$

Thus we have $r_1(j+1)=r_1(j)$, $r_2(j+1)=r_1(j)+r_2(j)$ and $r_3(j+1)=-c_j r_1(j)+r_3(j)$.

If $s(j+1)=s(j)-\varepsilon$, then we have

$$\begin{aligned} T(s(j+1), 1) &= T(s(j)-\varepsilon, s(j)\,;\, c_j)T(s(j), 1) \\ &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -c_j & 1 \end{pmatrix} T(s(j), 1). \end{aligned}$$

Thus we have $r_1(j+1)=r_1(j)+r_2(j)$, $r_2(j+1)=r_2(j)$ and $r_3(j+1)=c_j r_2(j)+r_3(j)$. Thus Claim 6.8 is proved.

CLAIM 6.9. $r_3(j)-c_j r_1(j) \geqq 0$ for $1 \leqq j \leqq m$.

PROOF. First, we assume that $s(j+1)=s(j)+\varepsilon$ for all j. Since N is ample, the following inequalities are satisfied:

$$\begin{aligned} & a > c_1+c_2+\cdots+c_m\,, \\ & c_m > 0\,, \\ & c_{m-1} > c_m\,, \\ & \cdots \\ & c_1 > c_2\,, \\ & b > c_1\,, \end{aligned}$$

$$a>eb.$$

Then we have

$$T(s(j),1)=\begin{pmatrix} 1 & -(j-1) & 0 \\ 0 & 1 & 0 \\ 0 & -(c_1+\cdots+c_{j-1}) & 1 \end{pmatrix} T(0,1)$$

$$=\begin{pmatrix} -(j-1) & e(j-1)-1 & 0 \\ 1 & -e & 0 \\ -(c_1+\cdots+c_{j-1}) & e(c_1+\cdots+c_{j-1})-a & 1 \end{pmatrix}.$$

Thus

$$r_3(j)-c_j r_1(j)=a-c_j-1+(1-e)\{c_1+\cdots+c_{j-1}-(j-1)c_j\}\geqq 0.$$

Assume that $r_3(j)-c_j r_1(j)\geqq 0$ for some j. Moreover, we assume that the morphisms $f_j, f_{j+1}, \cdots, f_{j+t}$ $(t\geqq 1)$ are chosen such that the following are satisfied: $s(j+1)=s(j)-\varepsilon$, $s(j+\lambda)=s(j+\lambda-1)+\varepsilon$ for $2\leqq\lambda\leqq t$. Note that t may be equal to one. Since N is ample, the following inequalities are satisfied:

$$c_j>c_{j+1}+\cdots+c_{j+t},$$

$$c_{j+t}>0,$$

$$c_{j+t-1}>c_{j+t},$$

$$\cdots$$

$$c_{j+1}>c_{j+2}.$$

Then we have:

$$r_1(j+1)=r_1(j)+r_2(j),$$

$$r_2(j+1)=r_2(j),$$

$$r_3(j+1)=c_j r_2(j)+r_3(j);$$

$$r_1(j+t)=r_1(j+1),$$

$$r_2(j+t)=(t-1)r_1(j+1)+r_2(j+1),$$

$$r_3(j+t)=-(c_{j+1}+\cdots+c_{j+t-1})r_1(j+1)+r_3(j+1).$$

We obtain ·

$$\begin{aligned} & r_3(j+t)-c_{j+t}r_1(j+t) \\ &= r_3(j)-c_jr_1(j)+(c_j-c_{j+1}-\cdots-c_{j+t})r_1(j+1) \\ &\geqq 0 . \end{aligned}$$

Thus Claim 6.9 is proved.

Let $(\alpha, \beta, n)\in \boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ with $\alpha\leqq -1$, $\beta\leqq -1$ and $\alpha+\beta+c_mn\geqq -1$. Putting

$$(\alpha, \beta, n)T(s(m), 1)=(A, B, n),$$

we have:

$$\begin{aligned} -A-B-n &= r_1(m)\alpha-r_2(m)\beta+r_3(m)n \\ &= -(r_1(m)+r_2(m))\beta+r_1(m)(\alpha+\beta)+r_3(m)n \\ &\geqq r_2(m)+(r_3(m)-c_mr_1(m))n\geqq 0 . \end{aligned}$$

Case II: $s(1)=2$. In this case, we put $r_1(j)=a_{11}(j)+a_{12}(j)$, $r_2(j)=-a_{21}(j)-a_{22}(j)$ and $r_3(j)=-a_{31}(j)-a_{32}(j)-1$. We have

$$T(s(1), 1)=T(2, 1)=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -b & 0 & 1 \end{pmatrix}.$$

If $s(j+1)=s(j)+\varepsilon$, we have $r_1(j+1)=r_1(j)$, $r_2(j+1)=r_1(j)+r_2(j)$ and $r_3(j+1)=c_jr_1(j)+r_3(j)$. If $s(j+1)=s(j)-\varepsilon$, we have $r_1(j+1)=r_1(j)+r_2(j)$, $r_2(j+1)=r_2(j)$ and $r_3(j+1)=-c_jr_2(j)+r_3(j)$. It is easy to see that $r_i(j)\geqq 0$ for $i=1, 2$.

CLAIM 6.10. $r_3(j)-c_jr_2(j)\geqq 0$.

PROOF. First, we assume that $s(j+1)=s(j)-\varepsilon$ for all j. Since N is ample, the following inequalities are satisfied: $a>eb+c_1, c_1>c_2, \cdots, c_{m-1}>c_m$, $c_m>0$, and $b>c_1+c_2+\cdots+c_m$. Then we have $r_3(j)-c_jr_2(j)=b-1-(c_1+\cdots+c_j)\geqq 0$.

Assume that $r_3(j)-c_jr_2(j)\geqq 0$ for some j and that, for $t\geqq 1$, the following are satisfied: $s(j+1)=s(j)+\varepsilon$, $s(j+\lambda)=s(j+\lambda-1)-\varepsilon$ for $2\leqq\lambda\leqq t$. Since N is ample, the following inequalities are satisfied: $c_{j+1}>c_{j+2}, \cdots, c_{j+t-1}>c_{j+t}$, $c_{j+t}>0$, $c_j>c_{j+1}+\cdots+c_{j+t}$. Then we have

$$\begin{aligned} & r_3(j+t)-c_{j+t}r_2(j+t) \\ & =r_3(j)-c_jr_2(j)+(c_j-c_{j+1}-\cdots-c_{j+t})r_2(j+1) \\ & \geqq 0 . \end{aligned}$$

Thus Claim 6.10 is proved.

Let $(\alpha, \beta, n)\in \boldsymbol{Z}\times\boldsymbol{Z}\times\boldsymbol{Z}_{\geq 0}$ with $\alpha\leqq -1$, $\beta\leqq -1$ and $\alpha+\beta+c_m n\geqq -1$. Putting

$$(\alpha, \beta, n)T(s(m), 1)=(A, B, n),$$

we have:

$$\begin{aligned} -A-B-n &= -r_1(m)\alpha+r_2(m)\beta+r_3(m)n \\ &= -(r_1(m)+r_2(m))\alpha+r_2(m)(\alpha+\beta)+r_3(m)n \\ &\geqq r_1(m)+(r_3(m)-c_m r_2(m))n\geqq 0 . \end{aligned}$$

Case III: $s(1)=3$. In this case, (S,f) is of type I or type III. Thus $e\geqq 1$. We now put $r_1(j)=a_{11}(j)+a_{12}(j)$, $r_2(j)=-a_{21}(j)-a_{22}(j)$ and $r_3(j)=-a_{31}(j)-a_{32}(j)-1$ as in Case II. We have

$$T(s(1), 1)=T(3, 1)=\begin{pmatrix} -1 & e & 0 \\ 0 & -1 & 0 \\ -b & eb-a & 1 \end{pmatrix}.$$

By the same argument as before, we see that $r_j(j)\geqq 0$ for $i=1,2$. To prove Theorem 6.1.(2) in this case, it is sufficient to show that $r_3(j)-c_jr_2(j)\geqq 0$.

First, we assume that $s(j+1)=s(j)-\varepsilon$ for all j. Then the following inequalities are satisfied: $b>c_1$, $c_1>c_2,\cdots,c_{m-1}>c_m$, $c_m>0$, $a>eb+c_1+c_2+\cdots+c_m$. Then we have

$$\begin{aligned} r_3(j)-c_jr_2(j) &= r_3(1)-(c_1+\cdots+c_{j-1})r_2(j)-c_jr_2(j) \\ &= r_3(1)-(c_1+\cdots+c_j)r_2(1) \\ &= a-(e-1)b-1-(c_1+\cdots+c_j)\geqq 0 . \end{aligned}$$

Assume that $r_3(j)-c_jr_2(j)\geqq 0$ for some j and that, for $t\geqq 1$, the following are satisfied: $s(j+1)=s(j)+\varepsilon$, $s(j+\lambda)=s(j+\lambda-1)-\varepsilon$ for $2\leqq\lambda\leqq t$. Then we obtain $r_3(j+t)-c_{j+t}r_2(j+t)\geqq 0$ by the same argument as in Case II.

Thus Theorem 6.1.(2) is proved.

References

[B1] Bădescu, L., On ample divisors, Nagoya Math. J. **86** (1982), 155-171.

[B2] Bădescu, L., Hyperplane sections and deformations, in "Algebraic geometry, Bucharest 1982", Lecture Notes in Math. vol. 1056, Springer-Verlag, Berlin-New York, 1984, 1-33.

[C] Campana, F., Compactifications kähleriennes de voisinages ouverts de cycles géométriquement positifs, Ann. Sci. École Norm. Sup. (4) **23** (1990), 521-542.

[CG] Clemens, C. and P. Griffiths, The intermediate Jacobian of the cubic threefold, Ann. of Math. **95** (1972), 281-356.

[G] Gieseker, D., On two theorems of Griffiths about embeddings with ample normal bundle, Amer. J. Math. **99** (1977), 1137-1150.

[Ha] Hartshorne, R., Ample subvarieties of algebraic varieties, Lecture Notes in Math. vol. 156, Springer-Verlag, Berlin-New York, 1970.

[Hi] Hironaka, H., On some formal imbeddings, Illinois J. Math. **12** (1968), 587-602.

[HM] Hironaka, H. and H. Matsumura, Formal functions and formal embeddings, J. Math. Soc. Japan **20** (1968), 52-82.

[K] Kato, Ma., On compact complex 3-folds with lines, Japan. J. Math. **11** (1985), 1-58.

[O] Oda, T., Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties, Springer-Verlag, Berlin-New York, 1987.

[SGA1] Revêtemants Étale et Group Fondamentale, Lecture Notes in Math. vol. 224, Springer-Verlag, Berlin-New York, 1971, 58-86.

(Received June 14, 1991)

Department of Mathematics
Faculty of Science
Gakushuin University
1-5-1 Mejiro, Toshima-ku
Tokyo
171 Japan