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Types of blocks with dihedral or quaternion defect groups

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The main purpose of this paper is to prove the following

THEOREM 1. *If two 2-blocks B, B' of finite groups G and G' have the same Brauer category with the same defect group D containing a cyclic subgroup of index 2, then they have the same type.*

The terminology used above is taken from [Br2]. The main step toward this theorem consists in defining a perfect isometry between B and B'. A perfect isometry may be viewed as a *correspondence with signs* between irreducible characters of B and B' which essentially preserves contribution matrices (see [B2]). The existence of such an isometry has numerous consequences on the block algebras $\mathcal{O}GB$, $\mathcal{O}G'B'$ over a complete valuation ring $\mathcal{O}$ with residual field $k=\mathcal{O}/J(\mathcal{O})$ of characteristic 2. It may also result from an equivalence of their derived categories over $\mathcal{O}$ (see [Br2] 3 and 6)[1]. Brauer-Olsson theorems ([B3], [O]) on generalized decomposition numbers of characters in blocks with dihedral and generalized quaternion defect provide enough information to define perfect isometries; this will be our main source and we won't need to use the methods of [E]. Part of the present paper shows how to restate in a compact way most of Brauer-Olsson results (see II and III below), mainly by use of the Broué-Puig $*$ construction [Br-P1].

The case where D is a generalized quaternion group and $G'=C_G(Z(D))$ deserves special attention: when B and B' are the principal blocks, the corresponding block algebras are *equal* (this is a consequence of Brauer-Suzuki theorem on groups with generalized quaternion Sylow). We show that in the general case the signs in the isotypie may be removed:

THEOREM 2. *If B is a 2-block of the finite group G with a generalized quaternion defect group D, if $H=C_G(Z(D))$ and b is the block of H inducing up to B, then there is an isotypie I: $\mathrm{CF}(G, B; \mathcal{O})\to \mathrm{CF}(H, b; \mathcal{O})$ which sends $\mathrm{Irr}(B)$ onto $\mathrm{Irr}(b)$.*

1) Using Erdmann's classification of the corresponding modular block algebras ([E]), M. Linckelmann shows that two blocks satisfying the hypothesis of theorem 1 are derived equivalent over k when D is dihedral [Li].

The paper is organized as follows. Part I is devoted to background results and notations on local block theory, mainly via the approach of [A-Br], in order to set the main definitions of [Br2]. In part II we recall the results on fusion of subpairs for this kind of defect group. In particular we show how the theory of essential groups leads to a quick determination of Brauer-Olsson cases of fusion.

In part III we show how Brauer-Olsson results on generalized decomposition numbers may be stated in terms of the $*$ construction. This provides a precise parametrization of non-rational characters in $\mathrm{Irr}(B)$ by non-rational characters in $\mathrm{Irr}(D)$. This is then used in IV to properly define perfect isometries and check Theorem 1.

Part V is devoted to the proof of Theorem 2. It consists mainly in imitating what has already been done for principal blocks (Brauer-Suzuki theorem [B1] VII, see also [D] 14): one studies restrictions to $H=C_G(Z(D))$ of integral combinations of characters in B which are zero on $G_{2'}$. In our case a truncated restriction provides an *isometry* on those central functions (Step 1), this is due to the strong condition of control satisfied by H in G. Moreover, we have *coherence* (Step 2): this isometry extends to all integral combinations of characters into a perfect isometry. The equality of all signs involved is checked by ad hoc computations mainly using sums of involutions in the group algebra.

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I. Notations and background.

Let l be a prime, G and G' two finite groups, $\mathcal{O}$ a complete valuation ring of characteristic zero containing primitive $|G|$ and $|G'|$-th roots of unity and having residual field k of characteristic l. Let K be the fraction field of $\mathcal{O}$, irreducible characters of G are considered as characters of the group algebra KG, so they are elements of $\mathrm{CF}(G;\mathcal{O})$ the set of central functions on G with values in $\mathcal{O}$. We denote similarly $\mathrm{CF}(G;K)=K\otimes\mathrm{CF}(G;\mathcal{O})$; we recall its standard inner product defined by $(f_1,f_2)_G=|G|^{-1}\sum_{g\in G}f_1(g)f_2(g^{-1})$.

It is well known that the elements of $\mathrm{Irr}(G)\cup\mathrm{Irr}(G')$ take values in a finite extension $K'\supset\boldsymbol{Q}$. We denote by Γ the Galois group of K' over its subfield generated by roots of unity of order prime to l. If $\sigma\in\Gamma$, $\chi\in\mathrm{Irr}(G)$ and $g\in G$, the formula $\sigma(\chi)(g)=\sigma(\chi(g))$ defines an action of Γ. The characters of G fixed by Γ are called "l-rational".

1. Conjugation. If $g \in G$, we denote by $\mathrm{int}(g)$ the interior automorphism defined by $\mathrm{int}(g)(g')=gg'g^{-1}$. If H is a subgroup of G and if g, g' are elements of G, we write $g=_H g'$ when $g=\mathrm{int}(h)(g')$ for some $h \in H$.

2. Blocks and characters. Any l-block B of G may be considered as a primitive idempotent in $Z(\mathcal{O}G)$, the center of the group algebra over $\mathcal{O}$; this induces an orthogonal projection $f \mapsto B.f$ in $\mathrm{CF}(G;\mathcal{O})$ by $B.f(g)=f(Bg)$ (where f has been extended to $\mathcal{O}G$ by linearity), the image is denoted by $\mathrm{CF}(G,B;\mathcal{O})$. The projection sends $\mathrm{CF}_{l'}(G;\mathcal{O})=\{f \in \mathrm{CF}(G;\mathcal{O});\ f(G \backslash G_{l'})=0\}$ into itself, so we denote by $\mathrm{CF}_{l'}(G,B;\mathcal{O})$ the set of elements of $\mathrm{CF}(G,B;\mathcal{O})$ which are zero outside l-regular elements. We use analogous notations $\mathrm{CF}(G,B;K)$, $\mathrm{CF}_{l'}(G;K)$, $\mathrm{CF}_{l'}(G,B;K)$ for corresponding vector spaces over K. Irreducible characters in B are supposed to be characters of KG-modules, their set is denoted by $\mathrm{Irr}(B):=\mathrm{Irr}(G) \cap \mathrm{CF}(G,B;\mathcal{O})$, with the same notation for Brauer characters $\mathrm{IBr}(B)=\mathrm{IBr}(G) \cap \mathrm{CF}_{l'}(G,B;K)$. One has $\mathrm{CF}_{l'}(G,B;\mathcal{O})=\mathcal{O}[\mathrm{IBr}(B)]$ and $\mathrm{CF}(G,B;K)=K[\mathrm{Irr}(B)]$.

3. Decomposition map. If $x \in G_l$, one has the decomposition map $d_G^x : \mathrm{CF}(G;\mathcal{O}) \to \mathrm{CF}_{l'}(C_G(x);\mathcal{O})$ defined by $d_G^x f(g)=f(xg)$; it is onto. If b is an l-block of $C_G(x)$, one writes $d_G^{(x,b)}f=b.d_G^x f$. We shall often abbreviate by omitting the subscript G when there is no ambiguity.

4. Subpairs. We freely use the setting of subpairs, "Brauer elements", fusion of subpairs, conjugation families, as taken from [A-Br]. If a maximal subpair (D,b_D) is given in G, subpairs included in it are just indexed by the corresponding subgroups of D: one writes them (X,b_X). If $\mathcal{S}$ is a system of representatives of Brauer elements in (D,b_D) mod. G-conjugation, the family of maps $(d^{(x,b_x)})_{(x,b_x)\in\mathcal{S}}$ provides an isometry

$$(d^{(x,b_x)})_{(x,b_x)\in\mathcal{S}} : \mathrm{CF}(G,b_1;\mathcal{O}) \longrightarrow \overset{\perp}{\bigoplus_{(x,b_x)\in\mathcal{S}}} \mathrm{CF}_{l'}(C_G(x),b_x;\mathcal{O}).$$

5. Isotypies. As in [Br2], the fact that two l-blocks B and B' in G and G' have the same Brauer category means the following: they have a common defect group D, there are maximal subpairs $(D,b_D)\supset(1,B)$ and $(D,b'_D)\supset(1,B')$ in G and G' such that for all pair X, Y of subgroups of D one has $\{\sigma \in \mathrm{Hom}(X,Y);\ \exists g \in G$ such that $g(X,b_X)g^{-1}\subset(Y,b_Y)$ and $\sigma=\mathrm{int}(g)_{|X}\}=\{\sigma \in \mathrm{Hom}(X,Y);\ \exists g' \in G'$ such that $g'(X,b_X)g'^{-1}\subset(Y,b_Y)$ and $\sigma=\mathrm{int}(g')_{|X}\}$. In particular the fusion of subpairs and Brauer elements included in (D,b_D) (see [A-Br]) are the same in G and G'.

If B, B' are l-blocks of G, G' with same Brauer category, let's consider a linear bijection $I: \mathrm{CF}(G, B; K) \to \mathrm{CF}(G', B'; K)$ such that

(i) $\forall \chi \in \mathrm{Irr}(B)$, $I(\chi)$ or $-I(\chi)$ is in $\mathrm{Irr}(B')$

(ii) $\forall u \in D\backslash\{1\}$ and $\forall \chi, \xi \in \mathrm{Irr}(B)$, one has $(d^{(u, b'_u)}(I(\chi)), d^{(u, b'_u)}(I(\xi)))_{C_{G'}(u)} = (d^{(u, b_u)}\chi, d^{(u, b_u)}\xi)_{C_G(u)}$.

Such a map is an isometry and defines a family of isometries $I_{l'}^{\langle u \rangle}$ from $\mathrm{CF}_{l'}(C_G(u), b_u; K)$ onto $\mathrm{CF}_{l'}(C_{G'}(u), b'_u; K)$ by $d^{(u, b'_u)} \circ I = I_{l'}^{\langle u \rangle} \circ d^{(u, b_u)}$, so I is fusion compatible in the sense of [Br2] 4.3.

If, for all u in $D\backslash\{1\}$, there exists an isometry $I^{\langle u \rangle}$ from $\mathrm{CF}(C_G(u), b_u; K)$ onto $\mathrm{CF}(C_{G'}(u), b'_u; K)$ such that

$(\mathrm{i})_u$ $\forall \chi \in \mathrm{Irr}(b_u)$, $I(\chi)$ or $-I(\chi)$ is in $\mathrm{Irr}(b'_u)$

$(\mathrm{ii})_u$ $I_{l'}^{\langle u \rangle} \circ d_{C_G(u)}^{(1, b_u)} = d_{C_{G'}(u)}^{(1, b'_u)} \circ I^{\langle u \rangle}$,

then I is a perfect isometry (in the sense of [Br2] 1.4). Moreover, if each $I^{\langle u \rangle}$ is a perfect isometry, then I is an isotypie from B to B'.

Proofs of the above are easy to derive from [Br2] 4.5 and 4.6.

6. Basic sets and contribution matrices. If (u, b_u) is a B-Brauer element, a basic set for b_u is any $\boldsymbol{Z}$-basis Φ_u of $\boldsymbol{Z}[\mathrm{IBr}(b_u)] \subset \mathrm{CF}_{l'}(C_G(u), b_u; \mathcal{O})$. The Cartan matrix $C(\Phi_u)$ for this basic set is such that $C(\Phi_u)^{-1} = ((\phi, \phi')_{C_G(u)})_{\phi, \phi' \in \Phi_u}$.

According to [B2] 5, if $\chi, \xi \in \mathrm{Irr}(B)$, one defines the "contribution of (u, b_u) to $(\chi, \xi)_G$" as $(d^{(u, b_u)}\chi, d^{(u, b_u)}\xi)_{C_G(u)}$. Let $\Delta_u = (n_{\chi\phi})_{\chi \in \mathrm{Irr}(B),\, \phi \in \Phi_u}$ be the generalized decomposition matrix with respect to Φ_u: $d^{(u, b_u)}\chi = \sum_{\phi \in \Phi_u} n_{\chi\phi}\phi$. Then $\Delta_u^t \Delta_u = C(\Phi_u)$ and the contributions are given by the matrix $\Delta_u C(\Phi_u)^{-1} \Delta_u^t$.

7. Broué-Puig $*$ construction. If (D, b_D) is a maximal subpair for G, a "(G, b_D)-stable" generalized character of D is any generalized character η of D such that $\eta(u) = \eta(v)$ each time there is $g \in G$ such that $(u, b_u) = g(v, b_v)g^{-1}$ with both u, v in D. Then, if $\chi \in \mathrm{Irr}(b_1)$, one defines the central function $\chi * \eta \in \mathrm{CF}(G, b_1; \mathcal{O})$ by $d^{(u, b_u)}(\chi * \eta) = \eta(u) d^{(u, b_u)}\chi$. The main result in [Br-P1] is that $\chi * \eta$ is a generalized character.

If $\sigma \in \Gamma$, then $\sigma(\chi * \eta) = \sigma(\chi) * \sigma(\eta)$.

II. Fusion.

Let us recall the notion of "essential" subpair:

DEFINITION. A subpair (U, b) in G is said to be essential if, and only if, b is of defect $Z(U)$ in $C_G(U)$ and $N = N_G(U, b)/UC_G(U)$ contains a proper subgroup M such that l divides $|M|$ and $\forall g\in N\backslash M$, $|M\cap M^g|$ is prime to l.

The main application to the fusion of subpairs is the following description of conjugation families (see [Br1] 2.9): if B is an l-block of G, a set of B-subpairs contained in a maximal one (D, b_D) is a conjugation family if, and only if, it contains (D, b_D) and a conjugate of each essential B-subpair (for a complete study, see [L]).

From now on we assume $l=2$, B is a 2-block of G, (D, b_D) is a maximal B-subpair and D contains a cyclic subgroup C of index 2. The inspection of essential subpairs is made easy by the following elementary fact applied to subgroups of D: if U has a cyclic subgroup of index $\leqq 2$, then $\mathrm{Aut}(U)$ is a 2-group except when U is kleinian (then Aut is S_3) or quaternion of order 8 (then Aut is S_4).

Let us denote by x a generator of C, $|x|$ its order and let y be an element of minimal order in $D\backslash C$. One denotes $z=x^{|x|/2}$, it is central. Then, either D is cyclic, generalized quaternion (then y is of order 4), or a semidirect product $C\rtimes\langle y\rangle$ with $yxy^{-1}=x$ (abelian), $yxy^{-1}=x^{-1}$ (dihedral), $yxy^{-1}=zx$ with $|x|\geqq 4$ (semidihedral) or $yxy^{-1}=zx^{-1}$ with $|x|\geqq 8$ (quasidihedral). Note that the Klein group is considered both as abelian and dihedral. One checks easily the following: D has kleinian subgroups U with $C_D(U)=U$ if, and only if, D is dihedral or quasidihedral, then they are D-conjugate to $\langle z, y\rangle$ or $\langle z, xy\rangle$; D has quaternion subgroups of order 8 if, and only if, D is generalized quaternion or quasidihedral, then they are D-conjugate to $\langle x^{|x|/4}, xy\rangle$ or $\langle x^{|x|/4}, y\rangle$.

One then finds a saturated system of the essential subpairs in (D, b_D) mod. G-conjugation as follows:

— if D is dihedral of order $\geqq 8$: one takes the pairs (U, b_U) such that $U=\langle z, xy\rangle$ or $\langle z, y\rangle$ (both kleinian) and $N_G(U, b_U)/C_G(U)\cong S_3$,

— if D is quaternion of order $\geqq 16$: one takes the pairs (U, b_U) such that $U=\langle x^{|x|/4}, xy\rangle$ or $\langle x^{|x|/4}, y\rangle$ (both quaternion of order 8) and $N_G(U, b_U)/UC_G(U)\cong S_3$,

— if D is quasidihedral of order $\geqq 16$: one takes the pairs (U, b_U) such that $U=\langle z, y\rangle$ (kleinian) or $\langle x^{|x|/4}, xy\rangle$ (quaternion of order 8) and $N_G(U, b_U)/UC_G(U)\cong S_3$,

— otherwise there is none.

The cases are labeled (aa) when the two subpairs listed above are essential or when $|N_G(D)/C_G(D)|=3$ (which implies that D is kleinian or quaternion of order 8).

We label (ab) (resp. (ba)) the cases when only the first (resp. the second) is essential. Note that they are different only when D is quasidihedral: otherwise, one may replace y by xy.

The remaining cases for fusion are labeled (bb) (this includes D abelian non kleinian and D semidihedral).

Note that if (U, b_U) is essential then $N_G(U, b_U)$ is transitive on the elements of U of given order[2]. One then obtains easily in each case a system of representatives $\mathcal{S}$ of the Brauer elements (u, b_u) mod. G-conjugation. For instance, if D is dihedral, a system of representatives is $\{(u, b_u)\}_{u\in\mathcal{S}'}$ where $\mathcal{S}'$ is a system of representatives of C mod. inversion, plus $\{(y, b_y)\}$ in case (ab), nothing in case (aa).

The following is straightforward:

PROPOSITION 0. (i) *B is nilpotent if and only if the fusion falls into case* (bb),

(ii) *if $u\in D\backslash\{1\}$ and (u, b_u) is not conjugate to (z, b_z), then b_u is nilpotent,*

(iii) *if D is dihedral of order $\geqq 8$, generalized quaternion or quasidihedral, b_z has a reduction mod. z denoted $\bar{b}_z$ with dihedral defect $D/\langle z\rangle$ in $C_G(z)/\langle z\rangle$. When D is quasidihedral and case* (aa) (*resp.* (ab)) *occurs for B, then case* (ab) (*resp.* (bb)) *occurs for b_z. When D is dihedral, case* (bb) *occurs for b_z. Otherwise (D generalized quaternion or D quasidihedral with case* (ba) *or* (bb)) *$C_G(z)$ is a B-control subgroup (see* [A-Br] 4.20), *so the fusion case for b_z is the one labeled the same for B.*

III. Brauer-Olsson's theorems and the $*$ construction.

We now return to our particular 2-blocks. We have seen above that except in the cases (aa), (ab) and (ba), B is nilpotent. Then the problem of perfect isometries and types is solved (see [Br2] 5B, in fact the blocks have the same source algebra thus are Morita equivalent by [P]). So we concentrate on the cases already studied by Brauer-Olsson where D is dihedral, generalized quaternion or quasidihedral. Then $D/[D, D]$ is kleinian. One denotes the four linear characters of D as follows: $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ with λ_1 the trivial character, $\mathrm{Ind}_C^D 1=\lambda_1+\lambda_2$, $\lambda_3(xy)=1$ in the dihedral case, -1 in the others. For technical reasons we denote $\eta_0=-2\lambda_1+\lambda_2+\lambda_3+\lambda_4$ when D is kleinian or quaternion of order 8, $\eta_0=-\lambda_2+\lambda_3+\lambda_4$ otherwise; then it is easy to check that η_0 is (G, b_D)-stable in all cases for fusion.

If $\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, it is of the form $\mu_\lambda=\mathrm{Ind}_C^D\lambda$ where $\lambda\in\mathrm{Irr}(C)$ and $|\lambda|\geqq 4$. Moreover $\mu_\lambda=\mu_{\lambda'}$ if, and only if, $\lambda=\lambda'$ or λ'^y.

2) This would prove at once that the saturated system of essential subpairs given above is *minimal*, hence a system of representatives mod. G-conjugation.

We gather next all the information we need on $\mathrm{Irr}(D)$. We denote by ε the character of C of order 2.

Characters of a dihedral group of order≧8

	λ_1	λ_2	λ_3	λ_4	$\mathrm{Ind}_C^D \lambda$ $4 \leq \lvert\lambda\rvert < \lvert x\rvert$	$\mathrm{Ind}_C^D \lambda$ $\lvert\lambda\rvert = \lvert x\rvert$
1	1	1	1	1	2	2
z	1	1	1	1	2	-2
y	1	-1	-1	1	0	0
xy	1	-1	1	-1	0	0
Res_C	1	1	ε	ε	$\lambda+\lambda^y$	$\lambda+\lambda^y$

Characters of a generalized quaternion or quasidihedral group

	λ_1	λ_2	λ_3	λ_4	$\mathrm{Ind}_C^D \lambda$ $4 \leq \lvert\lambda\rvert < \lvert x\rvert/2$	$\mathrm{Ind}_C^D \lambda$ $\lvert\lambda\rvert = \lvert x\rvert/2$	$\mathrm{Ind}_C^D \lambda$ $\lvert\lambda\rvert = \lvert x\rvert$
1	1	1	1	1	2	2	2
z	1	1	1	1	2	2	-2
$x^{\lvert x\rvert/4}$ if $\lvert D\rvert \geq 16$	1	1	1	1	2	-2	0
y	1	-1	1	-1	0	0	0
xy	1	-1	-1	1	0	0	0
Res_C	1	1	ε	ε	$\lambda+\lambda^y$	$\lambda+\lambda^y$	$\lambda+\lambda^y$

To describe irreducible characters in B and their generalized decomposition numbers, we shall keep the same notations as in [B3] for the dihedral case and [O] 4.6 for the others. By [B3] (6C, 6H) and [O] 4.6, $\mathrm{Irr}(B)$ contains four characters $\chi_1, \chi_2, \chi_3, \chi_4$ of height zero; they satisfy $d^x\chi_i=\delta_i\phi_x$ with δ_i a sign and $\mathrm{IBr}(b_x)=\{\phi_x\}$; if $\mathrm{IBr}(b_{x^2})$ has just one element ϕ_{x^2} (that is $|D|\geqq 8$), then $d^{(x^2, b_{x^2})}\chi_i=\gamma_i\phi_{x^2}$ where γ_i is a sign. The numbering satisfies: $\gamma_1\delta_1=\gamma_2\delta_2=-\gamma_3\delta_3=-\gamma_4\delta_4$.

III.1. Dihedral defect.

Assume D is dihedral and case (aa) or (ab).

LEMMA 1. *Let χ be any irreducible character of height zero in B and*

η, η' any (G, b_D)-stable generalized characters of D. Then we have

$$(\chi*\eta, \chi*\eta')_G = \begin{cases} \frac{1}{2}(\eta, \eta')_C + \frac{1}{2}\eta(1)\eta'(1), & \text{in case (aa)} \\ \frac{1}{2}(\eta, \eta')_C + \frac{1}{2}(\eta, \eta')_{\langle y \rangle}, & \text{in case (ab)} \end{cases}$$

where $(\eta, \eta')_Y$ denotes the inner product of restrictions to $Y \subset D$.

PROOF. By the isometry in I.4 and the definition of the $*$ construction (I.7), $(\chi*\eta, \chi*\eta')_G$ is $\sum_{(u, b_u) \in S} \eta(u)\eta'(u^{-1})(d^{(u, b_u)}\chi, d^{(u, b_u)}\chi)_{C_G(u)}$. When $u \neq 1$ b_u is nilpotent (II.0.(i)) and the generalized decomposition number of χ is a *sign*, so $((d^{(u, b_u)}\chi, d^{(u, b_u)}\chi)_{C_G(u)} = |D(u)|^{-1}$ where $D(u)$ is a defect group of b_u. In case (aa) one may take $u \in C$, so $D(z) = D$ and $D(u) = C$ for others. In case (ab) one has in addition $D(y) = \langle z, y \rangle$. This determines all $(d^{(u, b_u)}\chi, d^{(u, b_u)}\chi)_{C_G(u)}$'s for $u \neq 1$, then $(d^1\chi, d^1\chi)_G = 1 - \sum_{u \neq 1}(d^{(u, b_u)}\chi, d^{(u, b_u)}\chi)_{C_G(u)}$ making $|D|^{-1} + 1/2$ in case (aa), resp. $|D|^{-1} + 1/4$ in case (ab). This implies the formulas of Lemma 1.

Let's consider the following (G, b_D)-stable generalized characters of D: if $\mu = \mathrm{Ind}_C^D \lambda$ with $4 \leqq |\lambda| \leqq |x|$, let

$$\eta_\mu = \begin{cases} \mu - \lambda_2, & \text{if } 4 \leqq |\lambda| < |x|, \\ \mu + \lambda_2 - 2.\lambda_1, & \text{if } |\lambda| = |x| \text{ in case (aa)}, \\ \mu - \lambda_3 + \lambda_4 - \lambda_1, & \text{if } |\lambda| = |x| \text{ in case (ab)}. \end{cases}$$

PROPOSITION 1. *There is a parametrization $\mu \mapsto \chi_\mu$ of the elements of* Irr(B) *of height $\neq 0$ by* Irr$(D) \backslash \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ *such that*

$$\delta_1\chi_1 * \eta_0 = -\delta_2\chi_2 - \delta_3\chi_3 - \delta_4\chi_4,$$

$$\delta_1\chi_1 * \eta_\mu = \chi_\mu - \delta_2\chi_2.$$

In case (ab), *one may assume moreover $\delta_1\delta_3 = \delta_2\delta_4 = -1$ and $\chi_1 * \lambda_3 = \chi_3$.*

PROOF. We compute $(\chi_1*\eta, \chi_1*\eta')_G$ using Lemma 1 for the described (D, b_D)-generalized characters η, η'.

	$\chi_1*\lambda_1$	$\chi_1*\eta_0$	$\chi_1*\eta_\mu$	$\chi_1*\lambda_3$ only in case (ab)
$\chi_1*\lambda_1$	1	0	0	0
$\chi_1*\eta_0$	0	3	1	1
$\chi_1*\eta_{\mu'}$	0	1	$1+\delta_{\mu,\mu'}$	0
$\chi_1*\lambda_3$ only in case (ab)	0	1	0	1

So $\delta_1\chi_1*\eta_0$ is a linear combination with signs of three distinct irreducible characters all different from χ_1. If they are not all of height zero, only one is and, since $\delta_1\chi_1*\eta_0$ is 2-rational, the other two must be the elements of F_1, the only class of cardinality 2 under the action of the Galois group Γ on $\mathrm{Irr}(B)$ (this forces $|D|\geqq 16$); in particular, the rational part of their generalized decomposition numbers on (x, b_x) is zero and, as $\eta_0(x)=-3$, this contradicts the fact that decomposition numbers of characters of height zero on (x, b_x) are signs (see [B3] 4C, 4E). So these three are χ_2, χ_3, χ_4 and the study of decomposition numbers at (x, b_x) shows that $\delta_1\chi_1*\eta_0=-\sum_{i=2}^4\delta_i\chi_i$. Assume now $|D|\geqq 8$. The $\delta_1\chi_1*\eta_\mu$'s, being of square norm 2, are each a linear combination with signs of two distinct characters. Moreover the inner products with $-\sum_{i=2}^4\delta_i\chi_i$ is 1, so $\delta_1\chi_1*\eta_\mu=\varepsilon_\mu\chi_\mu-\delta_{i_0}\chi_{i_0}$ for $i_0\in\{2, 3, 4\}$ and $\varepsilon_\mu\in\{\pm 1\}$. The mutual inner products are 1 and $(\delta_1\chi_1*\eta_\mu+\delta_{i_0}\chi_{i_0})(1)=\delta_1\chi_1(1)+\delta_{i_0}\chi_{i_0}(1)$ has a constant sign, so one may write $\delta_1\chi_1*\eta_\mu=\varepsilon\chi_\mu-\delta_{i_0}\chi_{i_0}$ with distinct χ_μ's in $\mathrm{Irr}(B)\backslash\{\chi_1, \chi_2, \chi_3, \chi_4\}$. This provides a bijection $\mu\mapsto\chi_\mu$ between $\mathrm{Irr}(D)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and the characters of height $\neq 0$ by [B3] Theorem 1. When $\mu=\mathrm{Ind}_C^D\lambda$ with $|\lambda|=4$, then χ_μ must be the only 2-rational character which is not of height zero and the results on the generalized decomposition numbers ([B3] 6C) at (x^2, b_{x^2}) readily imply $\varepsilon=1$ and $i_0=2$.

In case (ab), $\delta_1\delta_3=\delta_2\delta_4=-1$ and $d^1\chi_1$, $d^1\chi_2$ are independent with $d^1\chi_3=d^1\chi_1$ and $d^1\chi_4=d^1\chi_2$ by [B3] 6H. On the other hand $\chi_1*\lambda_3$ is orthogonal to χ_1, of norm 1, height zero and same d^1 as χ_3, so it is χ_3.

III.2. Quaternion defect.

Assume D is generalized quaternion and case (ab) or (aa). We keep the notations of [O]. In particular, we take a numbering of the characters of height zero $\chi_1, \chi_2, \chi_3, \chi_4$ satisfying [O] 4.6. We write $F_{n-2}=\{\chi_5\}$, $F_{n-1}=\{\chi_6\}$ (characters of height $n-2$ where $|x|=2^{n-1}$) when they exist. Also we shall use the signs $\varepsilon_1, \kappa, \rho$ defined in [O] 4.6.

LEMMA 2. *Let* χ *be any irreducible character of height zero in* B *and* η, η' *any* (G, b_D)*-stable generalized characters of* D. *Then we have*

$$(\chi*\eta, \chi*\eta')_G = \begin{cases} \frac{1}{2}(\eta, \eta')_C + \frac{1}{2}(\eta, \eta')_{\langle z\rangle}, & \textit{in case (aa)} \\ \frac{1}{2}(\eta, \eta')_C + \frac{1}{2}(\eta, \eta')_{\langle y\rangle}, & \textit{in case (ab)} \end{cases}$$

where $(\eta, \eta')_Y$ *denotes the inner product of restrictions to* $Y \subset D$.

PROOF. The idea of the proof is the same as for Lemma 1. To determine $(\chi*\eta, \chi*\eta')_G$, one must moreover compute the contribution $(d^{(z,b_z)}\chi, d^{(z,b_z)}\chi)_{C_G(z)}$. It can be determined from [O] 4.6 by the formula recalled in I.6: $(d^{(z,b_z)}\chi, d^{(z,b_z)}\chi)_{C_G(z)}$ equals $|D|^{-1}+1/4$ in case (aa), resp. $|D|^{-1}+1/8$ in case (ab).

Let's consider the following (G, b_D)-stable generalized characters of D: If $\mu = \mathrm{Ind}_C^D \lambda$ with $4 \leqq |\lambda| \leqq |x|$, let

$$\eta_\mu = \begin{cases} \mu - \lambda_2, & \text{if } 4 \leqq |\lambda| < |x|/2, \\ \mu + \lambda_2 - 2\lambda_1, & \text{if } |\lambda| = |x|/2 \text{ in case (aa)}, \\ \mu + \lambda_3 - \lambda_4 - \lambda_1, & \text{if } |\lambda| = |x|/2 \text{ in case (ab)}, \\ \mu, & \text{if } |\lambda| = |x|. \end{cases}$$

We set $\mathcal{F} = \{\mu;\ \mu = \mathrm{Ind}_C^D \lambda \text{ with } 4 \leqq |\lambda| < |x|\}$ and $\mathcal{F}' = \{\mu;\ \mu = \mathrm{Ind}_C^D \lambda \text{ with } |\lambda| = |x|\}$.

PROPOSITION 2. *There is a parametrization* $\mu \mapsto \chi_\mu$ *of characters of height* 1 *by* $\mathrm{Irr}(D) \setminus \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ *such that (notations of* [O] 4.6):

$$\delta_1\chi_1*\eta_0 = -\delta_2\chi_2 - \delta_3\chi_3 - \delta_4\chi_4,$$

$$\delta_1\chi_1*\eta_\mu = \chi_\mu - \delta_2\chi_2, \quad \textit{if } \mu \in \mathcal{F},$$

$$\delta_1\chi_1*\eta_\mu = \begin{cases} \chi_\mu + \varepsilon_1\kappa\chi_5 + \varepsilon_1\rho\chi_6 & \textit{in case (aa)} \\ \chi_\mu - \varepsilon_1\kappa\chi_5 & \textit{in case (ab)} \end{cases} \quad \textit{if } \mu \in \mathcal{F}'.$$

In case (ab), *one may assume* $\delta_1\delta_4 = \delta_2\delta_3 = -1$ *and* $\chi_1*\lambda_4 = \chi_4$.

PROOF. As in Proposition 1, we compute $(\delta_1\chi_1*\eta, \delta_1\chi_1*\eta')_G$ by use of Lemma 2. The results are in $\{0, 1, 2, 3\}$ and are precisely the inner products of the expressions given in the statement.

The proof of the equalities of the proposition then goes as in Proposition 1, making use of [O] 4.6 to recognize classes of characters under the action of the Galois group Γ and to check the decomposition numbers at x, x^2 and z; it should be noted here that in the table of [O] 4.6 for the decomposition numbers at z, when $l(b_z)=3$, the rows corresponding to the characters of height zero should be multiplied by -1 (otherwise the first column would not be orthogonal to $a_0^{(n-2)}$).

III.3. Quasidihedral defect.

Assume D is quasidihedral and case (aa), (ab) or (ba). Keeping as in III.2 the numbering of characters of height zero and the notations of [O] 4.6, we write $F_{n-2}=\{\chi_5\}$ (character of height $n-2$ where $|x|=2^{n-1}$).

As in the dihedral and quaternion cases, one proves the following:

LEMMA 3. *Let χ be any irreducible character of height zero in B and η, η' any (G, b_D)-stable generalized characters of D. Then we have*

$$(\chi*\eta, \chi*\eta')_G=\begin{cases} \dfrac{1}{2}(\eta, \eta')_C+\dfrac{3}{8}\eta(1)\eta'(1)+\dfrac{1}{8}\eta(z)\eta'(z), & \textit{in case } \text{(aa)} \\ \dfrac{1}{2}(\eta, \eta')_C+\dfrac{1}{4}\eta(1)\eta'(1)+\dfrac{1}{4}\eta(xy)\eta'(xy), & \textit{in case } \text{(ab)} \\ \dfrac{1}{2}(\eta, \eta')_C+\dfrac{1}{2}(\eta, \eta')_{\langle z,y\rangle}, & \textit{in case } \text{(ba)} \end{cases}$$

where $(\eta, \eta')_Y$ denotes the inner product of restrictions to $Y\subset D$.

Consider the following (G, b_D)-stable generalized characters of D: If $\mu=\mathrm{Ind}_C^D\lambda$ with $4\leqq|\lambda|\leqq|x|$, let

$$\eta_\mu=\begin{cases} \mu-\lambda_2, & \text{if } 4\leqq|\lambda|<|x|/2 \text{ or if } |\lambda|=|x|/2 \text{ in case (ab)}, \\ \mu+\lambda_3-\lambda_4-\lambda_1, & \text{if } |\lambda|=|x|/2 \text{ in cases (aa) and (ba)}, \\ \mu+\lambda_4-\lambda_3-\lambda_1, & \text{if } |\lambda|=|x| \text{ in case (ab)}, \\ \mu+\lambda_2-\lambda_3-\lambda_1, & \text{if } |\lambda|=|x| \text{ in case (aa)}, \\ \mu, & \text{if } |\lambda|=|x| \text{ in case (ba)}. \end{cases}$$

Let $\mathscr{F}=\{\mu;\ \mu$ is in one of the above first three cases$\}$ and $\mathscr{F}'$ be the remaining cases, that is $\mathscr{F}'=\{\mu;\ \mu=\mathrm{Ind}_C^D\lambda$ with $|\lambda|=|x|$ in case (aa) or (ba)$\}$

PROPOSITION 3. *There is a parametrization of characters of height* 1 *by* $\mathrm{Irr}(D)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ *such that* (*notations of* [O] 4.6):

$$\delta_1\chi_1 * \eta_0 = -\delta_2\chi_2 - \delta_3\chi_3 - \delta_4\chi_4,$$

$$\delta_1\chi_1 * \eta_\mu = \chi_\mu - \delta_2\chi_2 \quad \text{if } \mu \in \mathscr{F},$$

$$\delta_1\chi_1 * \eta_\mu = \begin{cases} \chi_\mu + \delta_4\chi_4 - \varepsilon_1\kappa\chi_5 & \text{in case (aa)}, \\ \chi_\mu - \varepsilon_1\kappa\chi_5 & \text{in case (ba)}, \end{cases} \quad \text{if } \mu \in \mathscr{F}'.$$

In case (ab) (*resp.* (ba)), *one may assume* $\delta_1\delta_4 = \delta_2\delta_3 = -1$ *and* $\chi_1 * \lambda_3 = \chi_4$ (*resp.* $\chi_1 * \lambda_4 = \chi_4$).

PROOFS. The inner products $(\delta_1\chi_1 * \eta_i, \delta_1\chi_1 * \eta_j)_G$ for $i, j \in \{\mu\} \cup \{0\}$ can be computed using Lemma 3. The outcomes are in $\{-1, 0, 1, 2, 3\}$ and coincide with the inner products of the expressions given in the statement.

The proof of the equalities then goes as in Proposition 1, making use of [O] 4.6 to recognize classes of characters under the action of the Galois group and to check the decomposition numbers at x, x^2 and z.

III.4. The case when z is central ($|D| \geqq 8$).

The following proposition is used in IV.2 and V. We assume one of the cases described in 1, 2, 3 occurs and we keep the same notation. We consider b_z and determine the characters with z in their kernel.

PROPOSITION 4. *If* $z \in Z(G)$ *then* $\{\chi \mid \chi \in \mathrm{Irr}(b_z),\ \chi(z) = \chi(1)\} = \{\chi_i, \chi_\mu \mid i = 1, 2, 3, 4,\ \mu = \mathrm{Ind}_C^D \lambda,\ 4 \leqq |\lambda| \leqq |x|/2\}$ *with notations of* III. 1, 2, 3.

PROOF. The block $\bar{b}_z$ of $C_G(z)/\langle z\rangle$, being of dihedral defect (II.0.(iii)), has four characters of height zero. If one makes them into characters of $C_G(z)$ with z in their kernel, they are in b_z with same degrees, so they remain of height zero. Since there are four of them, they are all the characters of height zero in b_z.

Now let θ the endomorphism of $\mathrm{CF}(G; \mathcal{O})$ defined by $\theta(f)(h) = f(zh)$. If $\chi \in \mathrm{Irr}(b_z)$ then $\theta(\chi) \in \{\pm\chi\}$. So $\theta(f) = f$ if and only if $(f, \chi)_G = 0$ for each χ such that $\theta(\chi) = -\chi$. This implies that the components of $\chi_1 * \eta_\mu$ satisfy $\theta(\chi) = \chi$ when $\mu = \mathrm{Ind}_C^D \lambda$ with $4 \leqq |\lambda| \leqq |x|/2$ and $\theta(\chi) = -\chi$ when $|\lambda| = |x|$. This finishes the proof by Propositions 1, 2, 3 above.

IV. Proof of Theorem 1.

We now prove Theorem 1. Let B, B' be as in the hypotheses of the theorem, with characters χ, χ' respectively. We keep the notations of the preceding section except that we put a prime $'$ on each character or sign for B'. We choose for $B, B', \bar{b}_z, \bar{b}'_z$ a parametrization of irreducible characters satisfying Propositions 1, 2, 3.

IV.1. Fusion compatible isometries.

Let

$$I: \mathrm{CF}(G, B;\, K) \longrightarrow \mathrm{CF}(G', B';\, K)$$

defined by $I(\delta_i\chi_i)=\delta'_i\chi'_i$ for $i=1, 2, 3, 4$, $I(\varepsilon_1\kappa\chi_5)=\varepsilon'_1\kappa'\chi'_5$, $I(\varepsilon_1\rho\chi_6)=\varepsilon'_1\rho'\chi'_6$ when they exist and $I(\chi_\mu)=\chi'_\mu$ for each $\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$.

Let us show that I is a fusion compatible isometry. We must check the equalities of the inner products (ii) given in I.5, or equivalently that the matrix of mutual inner products of the $d^{(u,b_u)}$'s of $\delta_1\chi_1$, $\delta_2\chi_2$, $\delta_3\chi_3$, $\delta_4\chi_4$, $(\chi_\mu)_{\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}}$, $\varepsilon_1\kappa\chi_5$, $\varepsilon_1\rho\chi_6$ is the same in G and G'. Let Δ_u be the decomposition matrix of $\delta_1\chi_1$, $\delta_2\chi_2$, $\delta_3\chi_3$, $\delta_4\chi_4$, $(\chi_\mu)_{\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}}$, $\varepsilon_1\kappa\chi_5$, $\varepsilon_1\rho\chi_6$ at (u, b_u) with respect to a basic set. Then, the Cartan matrix of this basic set is still equal to $\Delta_u^t\Delta_u$ since changing the rows of Δ_u by signs does not affect $\Delta_u^t\Delta_u$. So, the matrix we seek is $\Delta_u(\Delta_u^t\Delta_u)^{-1}\Delta_u^t$. It is clear from [B3] 6C, 6H and [O] 4.6, 4.8 that the generalized decomposition matrices of $\delta_1\chi_1$, $\delta_2\chi_2$, $\delta_3\chi_3$, $\delta_4\chi_4$, $\varepsilon_1\kappa\chi_5$, $\varepsilon_1\rho\chi_6$ and $\delta'_1\chi'_1$, $\delta'_2\chi'_2$, $\delta'_3\chi'_3$, $\delta'_4\chi'_4$, $\varepsilon'_1\kappa'\chi'_5$, $\varepsilon'_1\rho'\chi'_6$ with respect to a suitable basic set are the same up to a sign (this sign is $\varepsilon_m\varepsilon'_m$ when $u\in C$ is of order 2^m, otherwise it is 1). It is also the case at the row μ by the formulas of Propositions 1, 2, 3. This gives $\Delta_u(\Delta_u^t\Delta_u)^{-1}\Delta_u^t=\Delta'_u(\Delta'^t_u\Delta'_u)^{-1}\Delta'^t_u$ as required.

IV.2. Isotypies.

Let's show how to extend those isometries into isotypies. As said in I.5 we have to extend each $I_{2'}^{\langle u\rangle}: \mathrm{CF}_{2'}(C_G(u), b_u;\, K)\to\mathrm{CF}_{2'}(C_{G'}(u), b'_u;\, K)$. If b_u is nilpotent, then $\mathrm{CF}_{2'}(C_G(u), b_u;\, \mathcal{O})=\mathcal{O}\phi_u$ where $\{\phi_u\}=\mathrm{IBr}(b_u)$ (see [Br-P]). One has $d^{(u,b_u)}\chi_i=\pm\phi_u$ for $i=1, 2, 3, 4$, so $I_{2'}^{\langle u\rangle}(\phi_u)=\varepsilon_u\phi'_u$ with $\varepsilon_u\in\{\pm1\}$. Choose any χ (resp. χ') of height zero in $\mathrm{Irr}(\mathrm{b}_u)$ (resp. $\mathrm{Irr}(b'_u)$), then $\phi_u=d^1\chi$. Thus $I^{\langle u\rangle}$ defined by $I^{\langle u\rangle}(\chi*\eta)=\varepsilon_u\chi'*\eta$ extends $I_{2'}^{\langle u\rangle}$ and it is a fusion compatible perfect isometry (see [Br2] 5B) satisfying $(\mathrm{i})_u$ and $(\mathrm{ii})_u$ of I.5.

If b_u is not nilpotent with $u\neq1$, then $u=z$ and D is generalized quaternion with fusion (aa) or (ab), or quasidihedral with fusion (aa) or (ba), b_z has defect D and $|D|\geqq8$ (Proposition 0). Let $(\tilde{\chi}_i)_{i=1,\dots,6}$, $(\tilde{\chi}_\mu)_\mu$ be the elements of $\mathrm{Irr}(b_z)$ in a numbering satisfying Proposition 2 or 3, with associated signs $(\tilde{\delta}_i)_{i=1,2,3,4}$, $\tilde{\varepsilon}_1, \tilde{\kappa}, \tilde{\rho}$. When reducing mod. $\langle z\rangle$ one gets a block $\bar{b}_z$ of $C_G(z)/\langle z\rangle$ with dihedral defect $D/\langle z\rangle$. The characters $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3, \tilde{\chi}_4$ have z in their kernel and one denotes by $\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3, \bar{\chi}_4$ the corresponding characters in $\bar{b}_z$ (Proposition 4). Then Proposition 1 is satisfied since the numbering of the characters of height zero is determined by the products of generalized decomposition numbers at x and x^2 and those are preserved. Their associated signs are $\bar{\delta}_i=\tilde{\delta}_i$.

Now let's take the following basic sets for $\bar{b}_z$ in $C_G(z)/\langle z\rangle$:

$$\phi_1=d^1(\bar{\delta}_1\bar{\chi}_1+\bar{\delta}_2\bar{\chi}_2),\quad \phi_2=-d^1(\bar{\delta}_2\bar{\chi}_2),\quad \text{when } |\mathrm{IBr}(\bar{b}_z)|=2,$$

$$\phi_1=d^1(\bar{\delta}_1\bar{\chi}_1+\bar{\delta}_2\bar{\chi}_2),\quad \phi_2=-d^1(\bar{\delta}_4\bar{\chi}_4),\quad \phi_3=-d^1(\bar{\delta}_1\bar{\chi}_1),\quad \text{when } |\mathrm{IBr}(\bar{b}_z)|=3 \text{ and } |D|=8,$$

$$\phi_1=d^1(\bar{\delta}_1\bar{\chi}_1+\bar{\delta}_2\bar{\chi}_2),\quad \phi_2=-d^1(\bar{\delta}_3\bar{\chi}_3),\quad \phi_3=-d^1(\bar{\delta}_1\bar{\chi}_1),\quad \text{when } |\mathrm{IBr}(\bar{b}_z)|=3 \text{ and } |D|\geqq 16.$$

The relations of III.1 for $C_G(z)/\langle z\rangle$ show that we have defined basic sets and allow to compute the decomposition and Cartan matrices. One finds the Cartan matrix $\begin{pmatrix} |D|/8+1 & 2 \\ 2 & 4 \end{pmatrix}$ or $\begin{pmatrix} |D|/8+1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ for $|\mathrm{IBr}(\bar{b}_z)|=2$ or 3 respectively. So they are twice the above when $(\phi_i)_i$ is considered as basic set of b_z in $\mathrm{CF}(C_G(z), b_z; \mathcal{O})$. Thus we obtain the Cartan matrices used in [O] p.227 and 229. Olsson proved that if the Cartan matrix is as above, then the generalized decomposition matrix at z of $\chi_1, \chi_2, \chi_3, \chi_4, (\chi_\mu)_\mu, \chi_5, \chi_6$ is one of the following:

$$\begin{pmatrix} \delta_1\varepsilon_1 & 0 & 0 & -\delta_4\varepsilon_1 & \varepsilon_1 \cdots \varepsilon_1 & -\varepsilon_1 \cdots -\varepsilon_1 & \kappa \\ \delta_1\varepsilon_1 & -\delta_2\varepsilon_1 & \delta_3\varepsilon_1 & -\delta_4\varepsilon_1 & 0 \cdots 0 & 0 \cdots 0 & 2\kappa \end{pmatrix}^t,$$

$$\begin{pmatrix} 0 & \delta_2\varepsilon_1 & 0 & -\delta_4\varepsilon_1 & \varepsilon_1 \cdots \varepsilon_1 & -\varepsilon_1 \cdots -\varepsilon_1 & \kappa & 0 \\ 0 & 0 & -\delta_3\varepsilon_1 & \delta_4\varepsilon_1 & 0 \cdots 0 & 0 \cdots 0 & -\kappa & \rho \\ -\delta_1\varepsilon_1 & \delta_2\varepsilon_1 & 0 & 0 & 0 \cdots 0 & 0 \cdots 0 & \kappa & \rho \end{pmatrix}^t,$$

$$\begin{pmatrix} 0 & \delta_2\varepsilon_1 & 0 & -\delta_4\varepsilon_1 & \varepsilon_1 \cdots \varepsilon_1 & -\varepsilon_1 \cdots -\varepsilon_1 & \kappa & 0 \\ \delta_1\varepsilon_1 & -\delta_2\varepsilon_1 & 0 & 0 & 0 \cdots 0 & 0 \cdots 0 & -\kappa & -\rho \\ 0 & 0 & \delta_3\varepsilon_1 & -\delta_4\varepsilon_1 & 0 \cdots 0 & 0 \cdots 0 & \kappa & -\rho \end{pmatrix}^t,$$

with ε_1 written $|D|/8-1$ times and $-\varepsilon_1$ written $|D|/8$ times.

When $|\mathrm{IBr}(b_z)|=|\mathrm{IBr}(b'_z)|=2$, this implies that $d^z(\delta_1\chi_1)=\varepsilon_1(\phi_1+\phi_2)$, $d^z(\delta_2\chi_2)=-\varepsilon_1\phi_2$. So, the isometry I described in 1 above thus satisfies $I_{2'}^{\langle z\rangle}(\varepsilon_1\phi_i)=\varepsilon'_1\phi'_i$ for $i=1, 2$. If the generalized decomposition matrix of $\mathrm{Irr}(B)$ and $\mathrm{Irr}(B')$ correspond to the same of the above last two cases, then similarly $I_{2'}^{\langle z\rangle}(\varepsilon_1\phi_i)=\varepsilon'_1\phi'_i$ for $i=1, 2, 3$. Otherwise $I_{2'}^{\langle z\rangle}(\varepsilon_1\phi_1)=-\varepsilon'_1\phi'_1$, $I_{2'}^{\langle z\rangle}(\varepsilon_1\phi_2)=\varepsilon'_1\phi'_3$, $I_{2'}^{\langle z\rangle}(\varepsilon_1\phi_3)=\varepsilon'_1\phi'_2$.

When $I_{2'}^{\langle z\rangle}(\varepsilon_1\phi_1)=\varepsilon'_1\phi'_1$, define $I^{\langle z\rangle}:\mathrm{CF}(C_G(z), b_z; K)\to\mathrm{CF}(C_{G'}(z), b'_z; K)$ by $I^{\langle z\rangle}(\tilde{\delta}_i\tilde{\chi}_i)=\varepsilon_1\varepsilon'_1\tilde{\delta}'_i\tilde{\chi}'_i$ for $i=1, 2, 3, 4$, $I^{\langle z\rangle}(\tilde{\chi}_\mu)=\varepsilon_1\varepsilon'_1\tilde{\chi}'_\mu$, $I^{\langle z\rangle}(\tilde{\varepsilon}_1\tilde{\kappa}\tilde{\chi}_5)=\varepsilon_1\varepsilon'_1\tilde{\varepsilon}'_1\tilde{\kappa}'\tilde{\chi}'_5$, $I^{\langle z\rangle}(\tilde{\varepsilon}_1\tilde{\rho}\tilde{\chi}_6)=\varepsilon_1\varepsilon'_1\tilde{\varepsilon}'_1\tilde{\rho}'\tilde{\chi}'_6$. Then IV.1 above tells us that $I^{\langle z\rangle}$ extends $I_{2'}^{\langle z\rangle}$ into a fusion compatible isometry. Then, by I.5, I is perfect and so is $I^{\langle z\rangle}$. Thus I is an isotypie and this completes the proof of Theorem 1.

When $I_{2'}^{\langle z\rangle}(\varepsilon_1\phi_1)=-\varepsilon'_1\phi'_1$, it suffices to compose the map $I^{\langle z\rangle}$ defined above with a perfect isometry $\sigma:\mathrm{CF}(C_G(z), b_z; \mathcal{O})\to\mathrm{CF}(C_G(z), b_z; \mathcal{O})$ such that $\sigma^2=\mathrm{Id}$, $\sigma(\tilde{\delta}_1\tilde{\chi}_1)=\tilde{\delta}_4\tilde{\chi}_4$ and $\sigma(\tilde{\delta}_2\tilde{\chi}_2)=\tilde{\delta}_3\tilde{\chi}_3$ if $|D|=8$, resp. $\sigma(\tilde{\delta}_1\tilde{\chi}_1)=\tilde{\delta}_3\tilde{\chi}_3$ and $\sigma(\tilde{\delta}_2\tilde{\chi}_2)=\tilde{\delta}_4\tilde{\chi}_4$ if $|D|\geqq 16$. This clearly implies $\sigma(\phi_2)=\phi_3$, $\sigma(\phi_3)=\phi_2$ and also $\sigma(\phi_1)=-\phi_1$ since $\phi_1=d^1(\tilde{\delta}_1\tilde{\chi}_1+\tilde{\delta}_2\tilde{\chi}_2)=-d^1(\tilde{\delta}_3\tilde{\chi}_3+\tilde{\delta}_4\tilde{\chi}_4)$. The existence of such an isometry is checked as in IV.1.

REMARK. In the generalized quaternion case (cf. Proposition 0 (iii), this includes the above case when $|\mathrm{IBr}(b_z)|=3$) another proof is as follows. We prove independently in V below that there is an isotypie between $\mathrm{CF}(G, B; \mathcal{O})$ and $\mathrm{CF}(C_G(z), b_z; \mathcal{O})$. So it remains to find an isotypie between $\mathrm{CF}(C_G(z), b_z; \mathcal{O})$ and $\mathrm{CF}(C_{G'}(z), b'_z; \mathcal{O})$. But in this case of a central z, it is easily checked that the isometry I of IV.1 is an isotypie: the Brauer elements (u, b_u) with $u\neq 1, z$ still give no trouble while on the other hand $I_{2'}^{\langle z\rangle}=I_{2'}^1$, which is extended by I.

V. The quaternion case.

From now on assume that B is a 2-block of G with generalized quaternion defect group D and set $(D, b_D)\supset(Z(D), b_z)\supset(\{1\}, B)$. We write $H=C_G(z)$.

Let us define a linear map:

$$\mathcal{R}:\mathrm{CF}(G, B; \mathcal{O})\longrightarrow\mathrm{CF}(H, b_z; \mathcal{O})$$

$$f\longmapsto b_z.\,\mathrm{Res}_H^G(f)\,.$$

Let $\mathrm{CF}_0(G)$ (resp. $\mathrm{CF}_0(H)$) denote the subspace of $\mathrm{CF}(G, B; \mathcal{O})$ (resp. $\mathrm{CF}(H, b_z; \mathcal{O})$) equal to $\ker d^1$. Then $\mathrm{CF}(G, B; \mathcal{O})=\mathrm{CF}_{2'}(G, B; \mathcal{O})\oplus^{\perp}\mathrm{CF}_0(G)$ and $\mathrm{CF}(H, b_z; \mathcal{O})=\mathrm{CF}_{2'}(H, b_z; \mathcal{O})\oplus^{\perp}\mathrm{CF}_0(H)$ (see I.4). Moreover, one has $\mathcal{R}(\mathrm{CF}_{2'}(G, B; \mathcal{O}))\subset\mathrm{CF}_{2'}(H, b_z; \mathcal{O})$ and $\mathcal{R}(\mathrm{CF}_0(G))\subset\mathrm{CF}_0(H)$ (see I.2).

We will prove the following additional properties.

Step 1. If $f\in \mathrm{CF}(G, B; \mathcal{O})$ and $f_0\in \mathrm{CF}_0(G)$, then $(f, f_0)_G=(\mathcal{R}(f), \mathcal{R}(f_0))_H$. Thus $\mathcal{R}$ induces an isometry from $\mathrm{CF}_0(G)\cap \boldsymbol{Z}[\mathrm{Irr}(B)]$ onto $\mathrm{CF}_0(H)\cap \boldsymbol{Z}[\mathrm{Irr}(b_z)]$ whose inverse map coincides with $B.\mathrm{Ind}_H^G$.

Step 2. (coherence) There exists an isotypie $\tilde{\mathcal{R}}: \mathrm{CF}(G, B; \mathcal{O})\rightarrow \mathrm{CF}(H, b_z; \mathcal{O})$ which coincides with $\mathcal{R}$ on $\mathrm{CF}_0(G)$.

Step 3. $\tilde{\mathcal{R}}(\mathrm{Irr}(B))=\mathrm{Irr}(b_z)$ or $-\mathrm{Irr}(b_z)$.

It is obvious that Theorem 2 follows from 2 and 3 above.

The following shows that the hypothesis on D and fusion of subpairs is a bit stronger than control by the subgroup H. We denote $D^{\#}=\{u\in D / z\in\langle u\rangle\}=D\backslash\{1\}$ and $H^{\#}=\{h\in H / h_2\in D^{\#}\}$.

LEMMA 4. *If* $u\in D^{\#}$, $f\in \mathrm{CF}(G, B; \mathcal{O})$ *and* $h\in H^{\#}$, *then*

(i) *the block* b_u *is the same for* G *and* H,

(ii) $d^{(u,b_u)}f=d^{(u,b_u)}\mathcal{R}f$ *over* $C_G(u)=C_H(u)$,

(iii) $f(h)=\mathcal{R}f(h)$.

PROOF. We first show that if b is a block of $C_G(u)=C_H(u)$ then the inclusion $(1, B)\subset(\langle u\rangle, b)$ in G is equivalent to the inclusion $(1, b_z)\subset(\langle u\rangle, b)$ in H. This proves (i) at once. Since $\langle z\rangle\subset\langle u\rangle$, we only need to show this for $u=z$. So let b be a block of H such that $(1, B)\subset(\langle z\rangle, b)$, then there exists $g\in G$ such that $g(\langle z\rangle, b)g^{-1}\subset(D, b_D)$. But z is the only involution in D, so $g\in H$ and $b=b_z$. Clearly, (i) implies (ii) which implies (iii).

We now check Step 1.

By I.4, one has $(f, f_0)_G=\sum_{(u,b_u)\in\mathcal{S}}(d^{(u,b_u)}f, d^{(u,b_u)}f_0)_{C_G(u)}$ for $\mathcal{S}$ a system of representatives of Brauer elements (u, b_u) in (D, b_D) mod. G-conjugacy. One has $d^{(1,B)}f_0=0$ and $d^{(1,b_z)}\mathcal{R}f_0=0$. Then $(f, f_0)_G=\sum_{(u,b_u)\in\mathcal{S}}(d^{(u,b_u)}f, d^{(u,b_u)}f_0)_{C_H(u)}=(\mathcal{R}f, \mathcal{R}f_0)_G$ since $\mathcal{S}$ is a system of representatives for subpairs of H in (D, b_D) by Lemma 4 (i) and Proposition 0 (iii).

Then $\mathcal{R}$ induces an isometry from $\mathrm{CF}_0(G)$ into $\mathrm{CF}_0(H)$.

The map $\mathcal{J}=B.\mathrm{Ind}_H^G: \mathrm{CF}(H, b_z; \mathcal{O})\rightarrow \mathrm{CF}(G, B; \mathcal{O})$ is adjoint to $\mathcal{R}$. Then $\mathcal{J}\circ\mathcal{R}$ fixes each element of $\mathrm{CF}_0(G)$ since $(\mathcal{J}\circ\mathcal{R}(f_0), f)_G=(\mathcal{R}(f_0), \mathcal{R}(f))_H=(f_0, f)_G$. On the other hand $\mathrm{rk}_{\mathcal{O}}\mathrm{CF}_0(G)=\sum_{(u,b_u)\in\mathcal{S}, u\neq 1}|\mathrm{IBr}(b_u)|=\mathrm{rk}_{\mathcal{O}}\mathrm{CF}_0(H)$. This implies that $\mathcal{R}(\mathrm{CF}_0(G))=\mathrm{CF}_0(H)$, $\mathcal{J}(\mathrm{CF}_0(H))=\mathrm{CF}_0(G)$, and $\mathcal{R}$, $\mathcal{J}$ give inverse isometries on those spaces. They also give rise to inverse isometries on generalized characters in CF_0's since $\mathcal{R}$ and $\mathcal{J}$ clearly preserve characters. This establishes Step 1.

We now turn to Step 2. We use the notations of III.2 for the elements of $\mathrm{Irr}(b_z)$ and associated signs: $(\chi_i)_{i=1,2,3,4,5,6}$, $(\chi_\mu)_{\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}}$ are the characters of b_z (in IV.2, they were denoted by $\tilde{\chi}$ to avoid confusion with $\mathrm{Irr}(B)$).

In case (aa), the following are generalized characters in $\mathrm{CF}_0(H)$: $\delta_1\chi_1+\delta_2\chi_2+\delta_3\chi_3+\delta_4\chi_4$, $\chi_\mu-\delta_1\chi_1-\delta_2\chi_2$ for $\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, $\varepsilon_1\kappa\chi_5-\delta_1\chi_1-\delta_4\chi_4$, $\varepsilon_1\rho\chi_6-\delta_1\chi_1-\delta_3\chi_3$. Concerning the last two this comes from the decomposition at z (see [O] 4.6) and Proposition 4. The others come from Proposition 2. In case (ab), one checks similarly that the following are in $\mathrm{CF}_0(H)$: $\delta_1\chi_1+\delta_4\chi_4$, $\delta_2\chi_2+\delta_3\chi_3$, $\chi_\mu-\delta_1\chi_1-\delta_2\chi_2$ for $\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, $\varepsilon_1\kappa\chi_5+\delta_1\chi_1-\delta_2\chi_2$. Let us consider the images by $\mathcal{J}$:

LEMMA 5. *In case* (aa) *there are* $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, (\psi_\mu)_{\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}}$ *in* $\mathrm{Irr}(B)\cup-\mathrm{Irr}(B)$ *and corresponding to distinct characters, such that*:

$$\mathcal{J}(\delta_1\chi_1+\delta_2\chi_2+\delta_3\chi_3+\delta_4\chi_4)=\psi_1+\psi_2+\psi_3+\psi_4,$$

$$\mathcal{J}(\chi_\mu-\delta_1\chi_1-\delta_2\chi_2)=\psi_\mu-\psi_1-\psi_2,$$

$$\mathcal{J}(\varepsilon_1\kappa\chi_5-\delta_1\chi_1-\delta_4\chi_4)=\psi_5-\psi_1-\psi_4,$$

$$\mathcal{J}(\varepsilon_1\rho\chi_6-\delta_1\chi_1-\delta_3\chi_3)=\psi_6-\psi_1-\psi_3.$$

LEMMA 6. *In case* (ab) *there are* $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, (\psi_\mu)_{\mu\in\mathrm{Irr}(D)\backslash\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}}$ *in* $\mathrm{Irr}(B)\cup-\mathrm{Irr}(B)$ *and corresponding to distinct characters, such that*:

$$\mathcal{J}(\delta_1\chi_1+\delta_4\chi_4)=\psi_1+\psi_4,$$

$$\mathcal{J}(\delta_2\chi_2+\delta_3\chi_3)=\psi_2+\psi_3,$$

$$\mathcal{J}(\chi_\mu-\delta_1\chi_1-\delta_2\chi_2)=\psi_\mu-\psi_1-\psi_2,$$

$$\mathcal{J}(\varepsilon_1\kappa\chi_5+\delta_1\chi_1-\delta_2\chi_2)=\psi_5+\psi_1-\psi_2.$$

PROOFS. The proofs are very similar to what was done in III: the inner products of the results are known since $\mathcal{J}$ is an isometry. Let's take case (aa). Then $\mathcal{J}(\delta_1\chi_1+\delta_2\chi_2+\delta_3\chi_3+\delta_4\chi_4)$ is a generalized character of square norm 4. Its value at 1 is 0, so it cannot be $\pm$ twice a character, hence $\mathcal{J}(\delta_1\chi_1+\delta_2\chi_2+\delta_3\chi_3+\delta_4\chi_4)$ is of the form announced in the lemma. The other images have square norm 3 and inner product -2 with $\psi_1+\psi_2+\psi_3+\psi_4$. So there exist ψ_μ, ψ_5, ψ_6 in $\pm\mathrm{Irr}(B)$ such that $\mathcal{J}(\chi_\mu-\delta_1\chi_1-\delta_2\chi_2)-\psi_\mu$, $\mathcal{J}(\varepsilon_1\kappa\chi_5-\delta_1\chi_1-\delta_4\chi_4)-\psi_5$, $\mathcal{J}(\varepsilon_1\rho\chi_6-\delta_1\chi_1-\delta_3\chi_3)-\psi_6$ are each a sum of two elements in $\{-\psi_1, -\psi_2, -\psi_3, -\psi_4\}$ with mutual inner products 1 or 2. Numbering ψ_1, ψ_2, ψ_3, ψ_4 such that the one common to $\mathcal{J}(\varepsilon_1\kappa\chi_5-\delta_1\chi_1-\delta_4\chi_4)-\psi_5$ and $\mathcal{J}(\varepsilon_1\rho\chi_6-\delta_1\chi_1-\delta_3\chi_3)-\psi_6$ is $-\psi_1$, then $-\psi_4$ in $\mathcal{J}(\varepsilon_1\kappa\chi_5-\delta_1\chi_1-\delta_4\chi_4)-\psi_5$ and $-\psi_3$ in $\mathcal{J}(\varepsilon_1\rho\chi_6-\delta_1\chi_1-\delta_3\chi_3)-\psi_6$, we obtain the desired result.

The case (ab) goes along the same line.

DEFINITION 7. Let $\tilde{\mathcal{R}}: \mathrm{CF}(G, B; K) \to \mathrm{CF}(H, b_z; K)$ defined by $\psi_i \mapsto \delta_i \chi_i$ for $i=1, 2, 3, 4$, $\psi_\mu \mapsto \chi_\mu$ for $\mu \in \mathrm{Irr}(D) \backslash \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. $\psi_5 \mapsto \varepsilon_1 \kappa \chi_5$, $\psi_6 \mapsto \varepsilon_1 \rho \chi_6$.

Then we have

LEMMA 8. *If $f \in \mathrm{CF}(G, B; K)$ and $h \in H^\sharp$ then $\tilde{\mathcal{R}}(f)(h) = \mathcal{R}(f)(h)$.*

PROOF. The generalized characters of $\mathrm{CF}_0(G)$ considered in Lemma 5 (resp. Lemma 6) form a free system of cardinality $|\mathrm{Irr}(B)|-3=\mathrm{rk}_{\mathcal{O}}\mathrm{CF}_0(G)$ (resp. $|\mathrm{Irr}(B)|-2=\mathrm{rk}_{\mathcal{O}}\mathrm{CF}_0(G)$). Moreover $\mathcal{I}\circ\tilde{\mathcal{R}}$ is the identity map on them, so $\tilde{\mathcal{R}}$ coincides with $\mathcal{R}$ on $\mathrm{CF}_0(G)$. Then $\tilde{\mathcal{R}}(f)-\mathcal{R}(f) \in \mathrm{CF}_0(H)^\perp = \mathrm{CF}_{2'}(H, b_z; \mathcal{O})$: if $f_0 \in \mathrm{CF}_0(G)$, $\mathcal{R}f_0 = \tilde{\mathcal{R}}f_0$ and $(\tilde{\mathcal{R}}f - \mathcal{R}f, \mathcal{R}f_0)_H = (\tilde{\mathcal{R}}f, \tilde{\mathcal{R}}f_0)_H - (\mathcal{R}f, \mathcal{R}f_0)_H = (f, f_0)_G - (f, f_0)_G = 0$ by Step 1 and the fact that $\tilde{\mathcal{R}}$ is an isometry. So $\tilde{\mathcal{R}}(f)(h) = \mathcal{R}(f)(h)$ if $h \in H \backslash H_{2'}$.

Let's check Step 2. $\tilde{\mathcal{R}}$ is well defined and provides a correspondence with signs which bijects $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b_z)$ since they have the same cardinality. Now let us verify the inner products of I.5.(ii). If $u \in D\backslash\{1\} = D^\sharp$, then $C_G(u) \subset H$. Thus we must check $(d^{(u,b_u)}\tilde{\mathcal{R}}(\psi_i), d^{(u,b_u)}\tilde{\mathcal{R}}(\psi_j))_{C_H(u)} = (d^{(u,b_u)}\psi_i, d^{(u,b_u)}\psi_j)_{C_H(u)}$. This clearly follows from the above Lemma and 4 (iii). So $\tilde{\mathcal{R}}$ satisfies (i) and (ii) of I.5. For all $u \in D^\sharp$, $\tilde{\mathcal{R}}_{l'}^{\langle u\rangle}$ is the identity map on $\mathrm{CF}_{l'}(C_H(u), b_u; K)$ by Lemmas 4 (ii) and 8. This implies that $\tilde{\mathcal{R}}$ is an isotypie: take $\tilde{\mathcal{R}}^{\langle u\rangle}$ to be the identity map on $\mathrm{CF}(C_H(u), b_u; K)$ ([Br2] 4.5, 4.6).

There remains Step 3. Let $\Sigma = (\sum_{g\in G} gzg^{-1})^2 \in Z(\mathcal{O}[G])$. The main tool to prove Step 3 consists in computing $f_0(\Sigma)$ for adequate f_0's in $\mathrm{CF}_0(G)$.

LEMMA 9. *If $\chi \in \mathrm{Irr}(B)$, then $\chi(\Sigma) = |G|^2(\chi(z)^2/\chi(1))$. If $f_0 \in \mathrm{CF}_0(G)$, then $f_0(\Sigma) = 0$.*

PROOF. The equality $\chi(\Sigma) = |G|^2(\chi(z)^2/\chi(1))$ is a consequence of Schur's Lemma: $\sum_{g\in G} gzg^{-1} \in Z(\mathcal{O}G)$, so it acts by a scalar on the representation space of χ. This scalar equals $(\chi(z)/\chi(1))|G|$. So Σ acts by its square and $\chi(\Sigma) = |G|^2(\chi(z)^2/\chi(1))$ as claimed.

It remains to check that $f_0(\Sigma)=0$. Let $T_G(z, H) = \{g \in G \mid gzg^{-1} \in H\}$ and $\Sigma' = (\sum_{g \in T_G(z,H)} gzg^{-1})^2 \in Z(\mathcal{O}[H])$. We first show that $f_0(\Sigma) = (G:H)\mathcal{R}f_0(\Sigma')$. Let $\mathcal{H} \subset H^\sharp$ be a system of representatives of the G-conjugacy classes of elements of G whose 2-parts are conjugate to some u in $D^\sharp$. Note that any h, h' in $H^\sharp$ which are G-conjugate are in fact H-conjugate as z is the only involution of $\langle h\rangle$ and $\langle h'\rangle$. So $\mathcal{H}$ is also a system of representatives of $H^\sharp$ mod. H-conjugacy. As $f_0 \in \mathrm{CF}_0(G)$ and $\mathcal{R}f_0 \in \mathrm{CF}_0(H)$, we have

$f_0(\Sigma)=\sum_{h\in\mathcal{H}} f_0(h)|\{(g, g')\in G\times G \,/\, gzg^{-1}g'zg'^{-1}={}_Gh\}|$ and

$\mathcal{R}f_0(\Sigma')=\sum_{h\in\mathcal{H}}\mathcal{R}f_0(h)|\{(g, g')\in T_G(z, H)\times T_G(z, H) \,/\, gzg^{-1}g'zg'^{-1}={}_Hh\}|.$

For $h\in\mathcal{H}$, $f_0(h)=\mathcal{R}f_0(h)$ by Lemma 4 (iii); moreover G (respectively H) acts by translation on the set $\{(g, g')\in G\times G \,/\, gzg^{-1}g'zg'^{-1}={}_Gh\}$ respectively $\{(g, g')\in T_G(z, H)\times T_G(z, H) \,/\, gzg^{-1}g'zg'^{-1}={}_Hh\})$ so its cardinality is $(G: C_G(h))|\{(g, g')\in G\times G \,/\, gzg^{-1}g'zg'^{-1}=h\}|$ (respectively $(H: C_H(h))|\{(g, g')\in T_G(z, H)\times T_G(z, H) \,/\, gzg^{-1}g'zg'^{-1}=h\}|$). But if $h=(gzg^{-1})(g'zg'^{-1})$ with $g, g'\in G$, these two involutions normalize $\langle h\rangle$ so they centralize z; thus g, g' are in fact in $T_G(z, H)$ and, recalling $C_G(h)=C_H(h)$, $|\{(g, g')\in G\times G \,/\, gzg^{-1}g'zg'^{-1}={}_Gh\}|=(G: H)|\{(g, g')\in T_G(z, H)\times T_G(z, H) \,/\, gzg^{-1}g'zg'^{-1}={}_Hh\}|$. So $f_0(\Sigma)=(G: H)\mathcal{R}f_0(\Sigma')$ as claimed.

Now it remains to show that $\mathcal{R}f_0(\Sigma')=0$. If $\chi\in\mathrm{Irr}(b_z)$, $\chi(gzg^{-1})=0$ when $g\notin H$ since such a gzg^{-1} cannot be H-conjugate to any u in $D^{\#}$. So $\chi(\Sigma')=\dfrac{\chi(\sum_{g\in T_G(z,H)} gzg^{-1})^2}{\chi(1)}=\dfrac{|H|^2\chi(z)^2}{\chi(1)}=|H|^2\chi(z^2)=|H|^2\chi(1)$. This implies $\mathcal{R}(f_0)(\Sigma')=|H|^2\mathcal{R}(f_0)(1)=0$.

Before we give the proof of Step 3, we need the following elementary argument:

LEMMA 10. *If $a, a', a'', b, b', b''\in K$ are such that $bb'b''\neq 0$ and $a+a'+a''=b+b'+b''=a^2/b+a'^2/b'+a''^2/b''=0$, then $a/b=a'/b'=a''/b''$.*

PROOF. $(ab'-a'b)^2=(b+b')(a^2b'+a'^2b)-(a+a')^2bb'=(-b'')(-a''^2bb'/b'')-a''^2bb'=0$, so $a/b=a'/b'$. Then apply symmetry.

Assume now that a sum of three characters with signs $\alpha\chi_i+\alpha'\chi_{i'}+\alpha''\chi_{i''}$ is in $\mathrm{CF}_0(H)$ and that $\mathcal{I}(\alpha\chi_i+\alpha'\chi_{i'}+\alpha''\chi_{i''})=\varepsilon\phi_i+\varepsilon'\phi_{i'}+\varepsilon''\phi_{i''}$ is one of the equalities in Lemma 5 or 6. Then $\alpha\chi_i(1)+\alpha'\chi_{i'}(1)+\alpha''\chi_{i''}(1)=0$ and $\varepsilon\phi_i(1)+\varepsilon'\phi_{i'}(1)+\varepsilon''\phi_{i''}(1)=0$. On the other hand, Lemma 9 tells us that $(\varepsilon\phi_i+\varepsilon'\phi_{i'}+\varepsilon''\phi_{i''})(\Sigma)=(\varepsilon\phi_i(z))^2/\varepsilon\phi_i(1)+(\varepsilon'\phi_{i'}(z))^2/\varepsilon'\phi_{i'}(1)+(\varepsilon''\phi_{i''}(z))^2/\varepsilon''\phi_{i''}(1)=0$. Then, by Lemma 8 and Proposition 4, $(\varepsilon\phi_i(z))^2/\varepsilon\phi_i(1)+(\varepsilon'\phi_{i'}(z))^2/\varepsilon'\phi_{i'}(1)+(\varepsilon''\phi_{i''}(z))^2/\varepsilon''\phi_{i''}(1)=(\varepsilon\chi_i(z))^2/\varepsilon\phi_i(1)+(\varepsilon'\chi_{i'}(z))^2/\varepsilon'\phi_{i'}(1)+(\varepsilon''\chi_{i''}(z))^2/\varepsilon''\phi_{i''}(1)=(\alpha\chi_i(1))^2/\varepsilon\phi_i(1)+(\alpha'\chi_{i'}(1))^2/\varepsilon'\phi_{i'}(1)+(\alpha''\chi_{i''}(1))^2/\varepsilon''\phi_{i''}(1)=0$. One then applies Lemma 10 to $\alpha\chi_i(1)$, $\alpha'\chi_{i'}(1)$, $\alpha''\chi_{i''}(1)$, and $\varepsilon\phi_i(1)$, $\varepsilon'\phi_{i'}(1)$, $\varepsilon''\phi_{i''}(1)$, it tells us that $\alpha\varepsilon\phi_i(1)$, $\alpha'\varepsilon'\phi_{i'}(1)$, $\alpha''\varepsilon''\phi_{i''}(1)$ have the same sign.

In case (aa), this provides the result we seek (Lemma 5).

In case (ab), the last two equations of Lemma 6 give the desired relation between $\delta_1\phi_1$, $\delta_2\phi_2$, $\varepsilon_1\kappa\phi_5$ and $(\phi_\mu)_{\mu\in\mathrm{Irr}(D)\setminus\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}}$. On the other hand

$\mathcal{I}(\chi_\mu-\delta_2\chi_2+\delta_4\chi_4)=\mathcal{I}(\chi_\mu-\delta_1\chi_1-\delta_2\chi_2)+\mathcal{I}(\delta_1\chi_1+\delta_4\chi_4)=\phi_\mu-\phi_2+\phi_4$ adds $\delta_4\phi_4$ to the relation. The sum of the second and the third relations of Lemma 6 adds $\delta_3\phi_3$. This completes the proof of Step 3.

REMARK. At this point, we can conclude that $\chi(z)=\chi(1)$ for any χ in $\mathrm{Irr}(B)$ of height zero, when B is the principal block, this essentially proves the theorem of Brauer-Suzuki. In the general case, we obtain:

There exists μ invertible in $\mathcal{O}$ (precisely $\mu=(H:D)\chi'(1)/(G:D)\chi(1)$ where $\chi\in\mathrm{Irr}(b_z)$ is of height zero and $\chi'\in\mathrm{Irr}(B)$ satisfies $\pm\tilde{\mathfrak{R}}(\chi')=\chi$) such that $\mu B\,\mathrm{Tr}_H^G(z)$ is an involution in $Z(\mathcal{O}GB)$. The following is an isomorphism (see [Br2] 1.5):

$$Z(\mathcal{O}Hb_2) \longrightarrow Z(\mathcal{O}GB)$$

$$b_z\,\mathrm{Tr}^H_{C_H(h)}(h) \longmapsto \begin{cases} \mu B\,\mathrm{Tr}^G_{C_G(h)}(h) & \text{if } h_2\in{}_HD^{\sharp} \\ \mu^2 B\,\mathrm{Tr}^G_H(z)\,\mathrm{Tr}^G_{C_G(h)}(hz) & \text{if } h_2\notin{}_HD^{\sharp}. \end{cases}$$

REMARK. In the other case of control, that is when D is quasidihedral and case (ba) occurs (see Proposition 0 (iii)), the same result can be obtained when the additional hypothesis is satisfied: *z and y are not G-conjugate.* The proof is similar using $D^{\sharp}=\{u\in D\,/\,z\in\langle u\rangle\}$.

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