

On real James numbers

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Abstract. We determine James numbers of real Stiefel manifolds and real stunted projective spaces. As an application, we decide when the composite of the double covering map and the quotient map generates a free direct summand of the homotopy group of the stunted projective space.

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1. Introduction

The purpose of this note is to determine the real James numbers. Throughout the note n, l, k denote integers with $n \geq l \geq k \geq 1$ and $n \geq 2$. Let P_k denote the real projective space of dimension $k-1$, $P_{l,k} = P_l/P_{l-k}$ the stunted projective space, and $V_{l,k} = O(l)/O(l-k)$ the Stiefel manifold of orthonormal k -frames in \mathbf{R}^l . Note that $P_{k,k}$ is the union of P_k and a disjoint base point. Write $q: V_{l,k} \rightarrow V_{l,1} = S^{l-1}$ for the projection on to the last component, and $q: P_{l,k} \rightarrow P_{l,1} = S^{l-1}$ for the quotient map. There is a commutative square [6]:

$$\begin{array}{ccc} V_{n,k} & \xrightarrow{q} & V_{n,1} \\ \cup \uparrow & & \parallel \\ P_{n,k} & \xrightarrow{q} & P_{n,1} \end{array}$$

The unstable real James numbers $V\{n, k\}$ and $P\{n, k\}$ are non-negative integers which generate respectively the images of

$$q_* : \pi_{n-1}(V_{n,k}) \longrightarrow \pi_{n-1}(S^{n-1}) = \mathbf{Z},$$

$$q_* : \pi_{n-1}(P_{n,k}) \longrightarrow \pi_{n-1}(S^{n-1}) = \mathbf{Z}.$$

In the same way, replacing homotopy group $\pi_{n-1}(-)$ by stable homotopy group ${}^s\pi_{n-1}(-)$ we have the stable real James numbers $V^s\{n, k\}$ and $P^s\{n, k\}$. Let us denote the exponent of 2 in a positive integer k by $\nu_2(k)$; define

$\varphi(k)$ to be the number of integers s such that $0 < s < k$ and $s \equiv 0, 1, 2, 4 \pmod{8}$. Our results are

THEOREM (1.1). *We have $P^s\{n, k\} = V^s\{n, k\} = V\{n, k\}$ which is equal to 0, 1, or 2 according as $n \equiv 1 \pmod{2}$ and $k \geq 2$, $\nu_2(n) \geq \varphi(k)$, or $1 \leq \nu_2(n) < \varphi(k)$.*

THEOREM (1.2). *We have $V\{n, k\} = P\{n, k\}$ except for the following cases: (1) if $(n, k) = (4, 3), (8, 5), (8, 6), (8, 7), (16, 9)$, then $V\{n, k\} = 1$ and $P\{n, k\} = 2$; (2) if $n = k = 2m$ with $m = 1, 2, 4$, then $V\{n, k\} = 1$ and $P\{n, k\} = 0$; (3) if $n = k = 2m$ with $m \neq 1, 2, 4$, then $V\{n, k\} = 2$ and $P\{n, k\} = 0$.*

Let $p_n: S^{n-1} \rightarrow P_n$ be the canonical double covering map and $p_{n,k}: S^{n-1} \rightarrow P_{n,k}$ ($n > k$) the composition of p_n with the quotient map.

COROLLARY (1.3). *The rank of $\pi_{n-1}(P_{n,k})$, for $n > k$, is 0, 2, or 1 according as $n \equiv 1 \pmod{2}$ and $2 \leq k \leq n-2$, $n = 2k$ and $k \equiv 0 \pmod{2}$, or otherwise. The map $p_{n,k}$ generates a free direct summand of $\pi_{n-1}(P_{n,k})$ if and only if $n = k + 1 \geq 3$, $P\{n, k\} = 2$ or $(n, k) = (4, 2), (8, 4), (16, 8)$.*

Note that a part of (1.1) is not new. Indeed $V\{n, k\}$ was already known [1, 4, 5]. We shall calculate it again by using codegree [3, 8, 9]. We shall prove (1.1) in §2, and (1.2), (1.3) in §3.

2. $V\{n, k\}$

The symbol $a|b$ means that $b = ma$ for some integer m .

LEMMA (2.1). (1) $V\{n, n\} = V\{n, n-1\}$; $P^s\{n, 1\} = V^s\{n, 1\} = V\{n, 1\} = P\{n, 1\} = 1$; $P\{n, n\} = 0$.

(2) $V^s\{n, k\} | V\{n, k\} | P\{n, k\}$; $V\{n, k\} | V\{n, l\}$ and $P\{n, k\} | P\{n, l\}$ if $n \geq l \geq k \geq 1$.

(3) $V\{2, 2\} = V\{4, 4\} = V\{8, 8\} = V\{16, 9\} = 1$.

(4) ([7; 4.2]) $P^s\{n, k\} = V^s\{n, k\}$.

(5) If $n \geq 2k$, then $P^s\{n, k\} = V^s\{n, k\} = V\{n, k\} = P\{n, k\}$.

(6) ([10; 23.4, 25.6], [5; 2.3]) If n is even or $k = 1$, then $V\{n, k\} = 1$ or 2. If n is odd and $k \geq 2$, then $V\{n, k\} = 0$.

PROOF. By definition, (1) and (2) are obvious. As is well-known, if $n = 2, 4, 8$, then $V\{n, n\} = 1$ (cf., [11; p. 200]). By [6; p. 4], we have $V\{16, 9\} = 1$. This proves (3). Since $P_{n,k}$ is $(n-k-1)$ -connected, it follows from suspension theorem that $P^s\{n, k\} = P\{n, k\}$ if $n \geq 2k$. Hence (5) follows from (2) and (4).

PROPOSITION (2.2). *The number $V^s\{n, k\}$ is 0, 1, or 2 according as $n \equiv 1 \pmod{2}$ and $k \geq 2$, $\nu_2(n) \geq \varphi(k)$, or $1 \leq \nu_2(n) < \varphi(k)$.*

PROOF. Let $L_k \rightarrow P_k$ be the canonical line bundle. Then L_k is of order $2^{\varphi(k)}$ in the J-group of P_k [2]. If a positive integer m satisfies $m+n \equiv 0 \pmod{2^{\varphi(k)}}$, then $P^s\{n, k\} = {}^s\text{cdg}(P_k^{mL}, m)$ by stable duality [6; (7.9)], where ${}^s\text{cdg}(-)$ is the stable codegree [3, 8, 9] which was denoted by $\text{cd}(mL_k)$ in [8], and P_k^{mL} is the Thom space of mL_k . Then the assertion follows from (2.1) (4) and [8; 3.5] (cf., [3]).

PROPOSITION (2.3). $V^s\{n, k\} = V\{n, k\}$.

To prove (2.3), we need

- LEMMA (2.4). (1) *If $k \geq 10$, then $2^{\varphi(k)} > 2k$. If $1 \leq k \leq 9$, then $2^{\varphi(k)} < 2k$.*
 (2) *Conditions $2k > n \geq k \geq 2$ and $n \equiv 0 \pmod{2^{\varphi(k)}}$ are satisfied if and only if (n, k) is (2, 2), (4, 3), (4, 4), (8, 5), (8, 6), (8, 7), (8, 8) or (16, 9).*
 (3) $V\{n, k\} = 1$ for every (n, k) in (2).

PROOF. Write $k-1 = 8x+y$ with $0 \leq y \leq 7$. Then $\varphi(k) = 4x+z$ such that z is 0 (if $y=0$), 1 (if $y=1$), 2 (if $y=2, 3$), and 3 (if $4 \leq y \leq 7$). If $x \geq 2$, that is, if $k \geq 17$, then $2^{\varphi(k)} \geq 2^{4x} > 16(x+1) \geq 2k$. Hence the following table completes the proof of (1).

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$2k$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$2^{\varphi(k)}$	1	2	4	4	8	8	8	8	16	32	64	64	128	128	128	128

If $2k > n \geq k \geq 2$ and $n \equiv 0 \pmod{2^{\varphi(k)}}$, then $k \leq 9$ by (1), hence (2) follows from the table. We have (3) by (2.1) (2) (3).

PROOF OF PROPOSITION (2.3). By (2.1) (5) (6) and (2.2), it suffices to consider the case: $2k > n \equiv 0 \pmod{2}$. If $1 \leq \nu_2(n) < \varphi(k)$ and $n < 2k$, then $V^s\{n, k\} = V\{n, k\} = 2$ by (2.1) (2) (6) and (2.2). If $2k > n \equiv 0 \pmod{2^{\varphi(k)}}$, then $V^s\{n, k\} = V\{n, k\} = 1$ by (2.2) and (2.4) (3).

PROOF OF THEOREM (1.1). This follows from (2.1) (4), (2.2) and (2.3).

Let us write $n = (2a+1)2^{b+4c}$, where a, b, c are integers and $0 \leq b \leq 3$; let us define $\rho(n) = 2^b + 8c$. As is easily shown, $\nu_2(n) \geq \varphi(k)$ if and only if $\rho(n) \geq k$. Hence we have the following by Theorem (1.1).

THEOREM (2.5) (Eckmann, Adams). *The fibration $q: V_{n,k} \rightarrow V_{n,1}$ has a cross section if and only if $\rho(n) \geq k$.*

3. $P\{n, k\}$

Let $\iota_k \in \pi_k(S^k)$ be the class of the identity map of S^k . Then the following is well-known.

LEMMA (3.1). *The homotopy class of $p_{n,1}$ is $2\iota_{n-1}$ or 0 according as n is even or odd; $\pi_{n-1}(P_n) = \mathbf{Z}\{\varepsilon \cdot p_n\}$, where ε is 1 or $1/2$ according as $n \geq 3$ or $n=2$.*

LEMMA (3.2). *If n is even, then $P\{n, n-1\} = 2$ for $n \geq 4$ and $P\{n, k\} = 1$ or 2 for $n > k$. If n is odd and $k \geq 2$, then $P\{n, k\} = 0$.*

PROOF. If n is even, then $P\{n, n-1\}$ is 1 or 2 according as $n=2$ or $n \geq 4$ by (2.1) (1) and (3.1), hence $P\{n, k\} \equiv 2$ provided $n > k$ by (2.1) (2). The second assertion follows from (2.1) (2) (6).

LEMMA (3.3). *If $n=2, 4, 8$, then $\pi_{2n-1}(P_{2n, n+1}) = \mathbf{Z}\{p_{2n, n+1}\} \oplus \text{Tor}$.*

PROOF. Let $n=2, 4, 8$. The assertion is obvious by (3.1) when $n=2$. Let $\omega_n: S^{2n-1} \rightarrow S^n$ be the Hopf map. We denote by Tor the torsion subgroup of any group. Then $\pi_{2n-1}(S^n) = \mathbf{Z}\{\omega_n\} \oplus \text{Tor}$. Let \mathcal{TOR} be the class of torsion groups. By mod \mathcal{TOR} Hurewicz theorem, we have

$$(3.4) \quad \pi_*(P_{2n-1, n}) \text{ and } \pi_*(P_{2n-1, n-2}) \text{ are torsion groups for } n=4, 8.$$

It follows from the homotopy exact sequence of the pair $(P_{2n, n+1}, P_{2n-1, n})$ that the rank of $\pi_{2n-1}(P_{2n, n+1})$ is 1 and $p_{2n, n+1}$ is of infinite order for $n=4, 8$. To complete the proof, it suffices to prove

$$(3.5) \quad \pi_{2n-1}(P_{2n, n}) = \mathbf{Z}\{p_{2n, n}\} \oplus \mathbf{Z} \oplus \text{Tor} \quad \text{for } n=2, 4, 8.$$

We shall prove (3.5). Since the manifold P_n is parallelizable and the Whitney sum of the tangent bundle of P_n with a trivial line bundle is nL_n , we have $P_{2n, n} = P_n^{nL} = S^n \vee (S^n \wedge P_n) = S^n \vee P_{2n-1, n-2} \vee S^{2n-1}$ up to homotopy. Hence $\pi_{2n-1}(P_{2n, n}) \cong \pi_{2n-1}(S^n) \oplus \pi_{2n-1}(P_{2n-1, n-2}) \oplus \pi_{2n-1}(S^{2n-1})$ by [11; (1.5) in p. 492, (7.12) in p. 368], where the isomorphism is induced by inclusion maps, and the rank of $\pi_{2n-1}(P_{2n, n})$ is 2 by (3.4). We can write $p_{2n, n} = i_{1*}(a_n \omega_n + t) + i_{2*}(b) + i_{3*}(2\iota_{2n-1})$ by (3.1), where $a_n \in \mathbf{Z}$, $t \in \text{Tor}(\pi_{2n-1}(S^n))$, $b \in \pi_{2n-1}(P_{2n-1, n-2})$, and i_k is a respective inclusion map. As is well-known, $\omega_n = f \circ p_{2n, n}$ for some map $f: P_{2n, n} \rightarrow S^n$. Write $f = x\iota_n \vee y \vee (z\omega_n + t')$ with

$x, z \in \mathbf{Z}$, $t' \in \text{Tor}(\pi_{2n-1}(S^n))$ and $y: P_{2n-1, n-2} \rightarrow S^n$, where equality means that $f \circ i_1 = x \iota_n$, $f \circ i_2 = y$, and $f \circ i_3 = z \omega_n + t'$. Then $\omega_n = f \circ p_{2n, n} \equiv (a_n x^2 + 2z) \omega_n \pmod{\text{Tor}}$, hence a_n is odd, therefore (3.5) follows. This completes the proof of (3.3).

PROOF OF THEOREM (1.2). As is easily shown, $\nu_2(n) \geq \varphi(n)$ if and only if $n=2, 4, 8$. Then the assertion for $n=k$ follows from (1.1) and (2.1) (1). If $V\{n, k\}$ is 0 or 2, then $P\{n, k\} = V\{n, k\}$ by (1.1), (2.1) (2) and (3.2). Suppose that $V\{n, k\}=1$ and $n > k$. Then $n \equiv 0 \pmod{2^{\varphi(k)}}$ by (1.1). If $k \geq 10$ or $k \leq 9$ and $n \geq 2k$, then $V\{n, k\} = P\{n, k\}$ by (2.1) (5) and (2.4) (1). If $k \leq 9$ and $n < 2k$, then (n, k) is (4, 3), (8, 5), (8, 6), (8, 7), or (16, 9) by (2.4) (2), and $P\{n, k\}=2$ except for $(n, k)=(8, 6)$ by (3.1), (3.2) and (3.3). We then have $P\{8, 6\}=2$ by (2.1) (2).

PROOF OF COROLLARY (1.3). The assertions are obvious when $k=1$ or $k=n-1$, by (3.1). Suppose $2 \leq k \leq n-2$. If n is odd, then $\pi_*(P_{n, k}) \in \mathcal{I}\mathcal{O}\mathcal{R}$ for k even by mod $\mathcal{I}\mathcal{O}\mathcal{R}$ Hurewicz theorem, and $i_*: \pi_*(S^{n-k}) \rightarrow \pi_*(P_{n, k})$ is a $\mathcal{I}\mathcal{O}\mathcal{R}$ -isomorphism for k odd by mod $\mathcal{I}\mathcal{O}\mathcal{R}$ Whitehead theorem. Let $j: P_{n-1, k-1} \rightarrow P_{n, k}$ be the inclusion map and $f: S^{n-1} \rightarrow P_{n, k}$ a map with $q_*(f) = P\{n, k\} \iota_{n-1}$. If n is even, then, by mod $\mathcal{I}\mathcal{O}\mathcal{R}$ Whitehead theorem, $q_*: \pi_*(P_{n, k}) \rightarrow \pi_*(S^{n-1})$ and $(f \vee j)_*: \pi_*(S^{n-1} \vee P_{n-1, k-1}) \rightarrow \pi_*(P_{n, k})$ are $\mathcal{I}\mathcal{O}\mathcal{R}$ -isomorphisms when k is odd and even respectively. Then the assertions can be proved easily by using (1.1), (1.2) and (3.5).

References

- [1] Adams, J.F., Vector fields on spheres, *Ann. of Math.* **75** (1962), 603-632.
- [2] Adams, J.F., On the groups $J(X)$ -II, *Topology* **3** (1965), 137-171.
- [3] Crabb, M.C. and K. Knapp, James numbers and the codegree of vector bundles I, preprint.
- [4] Eckmann, B., Stetige Lösungen linearer Gleichung systeme, *Comment. Math. Helv.* **15** (1942/3), 318-339.
- [5] James, I.M., Cross-sections of Stiefel manifolds, *Proc. London Math. Soc.* **8** (1958), 536-547.
- [6] James, I.M., *The Topology of Stiefel Manifolds*, Cambridge Univ. Press, Cambridge, 1976.
- [7] Ōshima, H., On stable James numbers of stunted complex or quaternionic projective spaces, *Osaka J. Math.* **16** (1979), 479-504.
- [8] Ōshima, H., Remarks on the j -codegree of vector bundles, *Japan. J. Math.* **16** (1990), 97-117.
- [9] Ōshima, H. and K. Takahara, Cohomotopy of Lie groups, *Osaka J. Math.* **28** (1991), 213-221.
- [10] Steenrod, N.E., *The Topology of Fibre Bundles*, Princeton Univ. Press, Princeton, 1951.

- [11] Whitehead, G. W., Elements of Homotopy Theory, Springer, New-York, 1978.

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