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Moduli and deformation for Fuchsian projective connections on a Riemann surface

Dedicated to Professor Tosihusa Kimura on the occasion of his sixtieth birthday

By Katsunori IWASAKI

Abstract. Moduli spaces of Fuchsian projective connections on a closed Riemann surface are constructed and their geometric structures as an analytic space or a complex manifold are studied. Moduli of projective representations of the fundamental group of a Riemann surface with punctures are also studied. The projective monodromy map of the moduli space of connections into that of representations is defined and its properties are studied in connection with the gauge equivalence. The moduli space of connections is studied more deeply by considering a Cousin problem associated with a family of holomorphic line bundles on the Riemann surface. After a suitable moduli theory is obtained, the monodromy preserving deformation for projective connections is studied. It is shown that there exists a closed 2-form Ω on the moduli space of connections such that an Ω-Lagrangian foliation describes the monodromy preserving deformation. Moreover, a Poisson structure of the moduli space is discussed and the sheaf of monodromy changing Hamiltonians is introduced.

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§ 0. Introduction.

0.1. History and general introduction.

The theory of monodromy preserving deformation of linear differential equations is not only classical but also modern. As Miwa-Jimbo-Ueno [MJU] said, like in many other branches of modern mathematics, B. Riemann was the first to introduce this fruitful idea. In 1857, he proposed the problem of constructing a system of functions with regular singular points admitting the prescribed monodromy property, and studying them as functions of the regular singular points when the monodromy is kept invariant. He foresaw that this problem would naturally lead to a generalization of the theory of Abelian functions and theta functions. Early in the twentieth century, R. Fuchs [F], L. Schlesinger [Sc] and R. Garnier [Gar] tackled the problem of the monodromy preserving deformation in a precise framework for the first time. They derived (systems of) nonlinear differential equations which describe the monodromy preserving deformation of second order or first order systems of linear differential equations on the Riemann sphere. The celebrated equations of P. Painlevé ([P], [Gam]) are important special examples of such nonlinear equations.

Almost at the same time, there was another interesting problem which seems completely independent of the monodromy preserving deformation: Find new transcendental special functions defined as solutions of nonlinear differential equations in a complex domain! What nonlinear equation are "good" for this purpose? It is well-known that many of classical special functions satisfy second order linear differential equations on the Riemann sphere. The above problem is merely an attempt to generalize this fact to the nonlinear case. However, the occurence of the so-called movable branch points makes the study of nonlinear differential equations much more complicated than that of linear ones. Painlevé considered that nonlinear equations without movable branch points are simple and hence "good". With this idea, Painlevé [P] and this student B. Gambier [Gam] tackled the problem of classifying second order nonlinear ordinary differential equations without movable branch points and discovered the six kinds of the Painlevé equations $(P_{\mathrm{I}})-(P_{\mathrm{VI}})$.

The fact that the two kinds is of completely different ideas mentioned above led to the same nonlinear differential equations should have implied richness and good characters of such nonlinear equations. It was, however, not the case. The importance of the monodromy preserving deformation and the resulting nonlinear deformation equations was not recognized sufficiently, and the works of Fuchs, Schlesinger and Garnier have been almost forgotten for a long period. Exceptionally K. Okamoto was aware of its importance and has started to study the deformation theory on an elliptic curve at the begining of 1970's ([Ok1-Ok7]).

In the latter half of 1970's, the discovery of an unexpected connection of the monodromy preserving deformation with mathematical physics provided mathematicians and mathematical physicists with a revival of interest and a new insight into this theory ([BMW][JMSM][JMMS][SMJ][WMTB]). It has been recognized that the nonlinear differential equations obtained by the monodromy preserving deformation can be applied effectively to the theory of particular solutions of various nonlinear partial differential equations in mathematical physics such as the K-dV, nonlinear Schrödinger and sine-Gordon equations, to the theory of correlation functions of the Ising model and so on. This discovery has begun to establish the position of solutions of those nonlinear equations as important special functions. The holonomic quantum field theory due to Sato-Miwa-Jimbo and Môri ([JMSM][SMJ]) also had an influence on the monodromy preserving deformation. Moreover, in 1981, Miwa-Jimbo-Ueno [MJU][JM1, 2] generalized the work of Schlesinger, established a general theory of monodromy preserving deformation for first order systems of ordinary differential equations with rational coefficients on the Riemann sphere and developed the theory of the so-called τ-functions. As for the deformation theory for second order single ordinary differential equations, Okamoto and H. Kimura made an extensive study of the Painlevé equations and their generalizations to several variables, the Garnier systems, concerning transformation groups of the solution spaces, Hamiltonian structures, foliations and the construction of the space of initial values ([Ok4, 5][KH2]). The differential Galois theory and the proof of the transcendency of the Painlevé equations due to H. Umemura [Um1-3] and K. Nishioka [Ni] are also an important subject.

So far the deformation theory has been developed chiefly for ordinary differential equations on the Riemann sphere $\boldsymbol{P}^1$ (genus 0). Although there is a series of works due to Okamoto [Ok1-3, 6] which are concerned with the deformation theory on an elliptic curve (genus 1), no one has considered it on a closed Riemann surface of an arbitrary genus from a general point of view. The aim of the present paper is to develop such a deformation theory on a Riemann surface of an arbitrary genus as a fairly abstract generalization of Okamoto's work.

To develop the deformation theory, Schlesinger [Sc] considered the first order Fuchsian systems of the form

$$(C)\qquad \frac{dY}{dx}=\sum_{j=1}^{n}\frac{C_j}{x-a_j}Y \qquad \text{on } \boldsymbol{P}^1,$$

where C_j $(j=1,\cdots,n)$ are m-by-m constant matrices. He derived the condition under which the monodromy of (C) is kept invariant when the location of the singular points a_j $(j=1,\cdots,n)$ is changed. His result is summarised into the following completely integrable system of nonlinear differential equations for the matrices C_j:

$$(S)\qquad \begin{cases} \dfrac{\partial C_i}{\partial a_j}=\dfrac{[C_i,C_j]}{a_j-a_i} & (i\neq j), \\[2ex] \dfrac{\partial C_i}{\partial a_i}=-\displaystyle\sum_{k(\neq i)}\dfrac{[C_k,C_i]}{a_k-a_i}. \end{cases}$$

Exaggeratingly stated, the linear differential equations (C) are special type of Fuchsian connections on the trivial vector bundle over the Riemann sphere. To develop the deformation theory on a Riemann surface M of an arbitrary genus, the first question to be considered is: For what class of linear differential equations on M the deformation theory works out satisfactorily? More precisely the question is: Should we consider what class of connections on what fiber bundle over M? Contrary to the fact that the Riemann sphere has a positive curvature, a Riemann surface M of genus $g\geqq 2$ has a negative curvature, whence the number of differential forms on M get larger. In higher genus case, this fact makes the above question much more difficult. After some try and errors, we decide to develop our deformation theory for a certain class of Fuchsian $PSL(2;\boldsymbol{C})$-connections on M. Reflecting the geometry of Riemann surfaces, this class can be an object of very beautiful deformation theory.

The result of this paper was announced in the symposium "Die gewöhnlichen Differentialgleichungen und spetial Functionen" held at Oberwolfach in April, 1989. It was announced also as a special lecture in the annual meeting of the Mathematical Society of Japan held in October, 1989. This paper is the final version of the author's preprint [Iw1].

0.2. An aspect of the monodromy preserving deformation.

We shall introduce an aspect of the theory of the monodromy preserving deformation by giving a review of a result due to K. Okamoto [Ok5], by which we shall make clear the issue of the problem involved for our general deformation theory. Okamoto [Ok5] considered the following second order Fuchsian differential equation (Q) on the Riemann sphere:

$$(Q) \qquad \frac{d^2z}{dx^2}=Q(x)z\,,$$

where $Q(x)$ is given by

$$\begin{aligned}Q(x)=&\frac{a_0}{x^2}+\frac{a_1}{(x-1)^2}+\frac{a_\infty}{x(x-1)}\\&+\sum_{j=1}^{N}\left\{\frac{b_j}{(x-t_j)^2}+\frac{t_j(t_j-1)h_j}{x(x-1)(x-t_j)}\right\}\\&+\sum_{k=1}^{N}\left\{\frac{3}{4(x-\lambda_k)^2}-\frac{\lambda_k(\lambda_k-1)\mu_k}{x(x-1)(x-\lambda_k)}\right\}.\end{aligned}$$

Hence the differential equation (Q) depends on the parameters

$$\begin{aligned}&c=(a_0, a_1, a_\infty\,;\, b_1, \cdots, b_N)\,,\\&t=(t_1, \cdots, t_N)\,, \qquad \lambda=(\lambda_1, \cdots, \lambda_N)\,,\\&\mu=(\mu_1, \cdots, \mu_N)\,, \qquad h=(h_1, \cdots, h_N)\,.\end{aligned}$$

When emphasis on this dependence is neccessary, (Q) is written as $(Q(c, t, \lambda, \mu, h))$. The difference of the characteristic exponents of (Q) at $x=\lambda_k$ is an integer $(=2)$. Hence Frobenius' method in the theory of Fuchsian differential equations implies that there exist rational function $H_j(c\,;\, t, \lambda, \mu)\in \boldsymbol{C}(c, t, \lambda, \mu)$ such that the singular points $x=\lambda_k$ $(k=1, \cdots, N)$ are logarithmic or non-logarithmic according to $h_j\neq H_j(c\,;\, t, \lambda, \mu)$ or not. Now we assume

$$(A) \qquad h_j=H_j(c\,;\, t, \lambda, \mu) \qquad (j=1, \cdots, N)\,,$$

namely, $x=\lambda_k$ $(k=1, \cdots, N)$ are non-logarithmic singular points. Under this condition (Q) depends on the parameters (c, t, λ, μ) : $(Q)=(Q(c, t, \lambda, \mu))$. We call $H_j(c\,;\, t, \lambda, \mu)$ the *Hamiltonians*. The reason for this naming will be clear soon (see Theorem 0.2).

The differential equation (Q) determines the linear monodromy representation (up to conjugacy)

$$(LM) \qquad \pi_1(\boldsymbol{P}^1\backslash\{0, 1, \infty, t_1, \cdots, t_N, \lambda_1, \cdots, \lambda_N\})\longrightarrow SL(2\,;\, \boldsymbol{C})\,.$$

By the assumption (A), it is easy to see that the circuit matrix C_k at $x=\lambda_k$ determined by (LM) belongs to the centor of $SL(2\,;\, \boldsymbol{C})$; in fact $C_k=-I$, where I is the unit matrix in $SL(2\,;\, \boldsymbol{C})$. Passing to the quotient $PSL(2\,;\, \boldsymbol{C})=SL(2\,;\, \boldsymbol{C})/\{\pm I\}$, we obtain from (LM) the projective monodromy representation (up to conjugacy)

$$(PM) \qquad \pi_1(\boldsymbol{P}^1\backslash\{0, 1, \infty, t_1, \cdots, t_N\}) \longrightarrow PSL(2\,; \boldsymbol{C})\,.$$

Thus the singular points $x=\lambda_k$ have no effect on the projective monodromy representation. For this reason, they are called *apparent* singular points. Let us consider the following problem.

Problem 0.1. Find a necessary and sufficient condition such that the deformation of the parameters $(c\,; t, \lambda, \mu)$ subjecting this condition does not change the conjugacy class of projective monodromy representation of (Q).

We call such a deformation of the differential equation $(Q(c\,; t, \lambda, \mu))$ the *monodromy preserving deformation.* Notice that the parameters $c=(a_0, a_1, a_\infty\,; b_1, \cdots, b_N)$ are the local monodromy data which are uniquely determined by the characteristic exponents of (Q) at the regular singular points. Hence c must be kept invariant under the monodromy preserving deformation. Hereafter we regard c as absolute constants. Under a certain generic condition on c, Okamoto [Ok5] proved the following theorem.

THEOREM 0.2 (Okamoto). *If $t=(t_1, \cdots, t_N)$ are viewed as deformation parameters, then the monodromy preserving deformation of* $(Q(c\,; t, \lambda, \mu))$ *is governed by the following completely integrable Hamiltonian system*

$$(G) \qquad \begin{cases} \dfrac{\partial \lambda_k}{\partial t_j}=\dfrac{\partial H_j}{\partial \mu_k}\,, \\[2mm] \dfrac{\partial \mu_k}{\partial t_j}=-\dfrac{\partial H_j}{\partial \lambda_k}\,. \end{cases} \qquad (j, k=1, \cdots, N)\,.$$

Okamoto called (G) the *Garnier system.* In fact, Garnier [Ga1] had already obtained essentially the same system of nonlinear differential equations. But the credit for the discovery of such an ingenious expression as (G) goes to Okamoto. In case $N=1$, (G) is known to be equivalent to the sixth Painlevé equation. Hence the Garnier system is an extension of the Painlevé equation to several variables. Essentially in the same method, Okamoto [Ok6] also obtained an analogous result for Fuchsian equations similar to (Q) on an elliptic curve, which can be explicitly represented in terms of the Weierstrass $\wp$-functions and ζ-functions instead of rational functions in the case of genus zero.

0.3. The problems involved for the deformation on a Riemann surface of an arbitrary genus.

In the present paper we want to develop a deformation theory on a closed Riemann surface of an arbitrary genus. Okamoto [Ok5, 6] obtained his result by direct and explicit calculation. In an arbitrary genus case, however, things are not so simple. We can not avoid speaking rather abstract language. We must recognize what are essential in the relevant problem and make up them into an abstract theory. More precisely, let M be a Riemann surface of genus $g \geqq 0$; of course our particular interest lies in the higher genus case $g \geqq 2$. We need to ask for what class of Fuchsian differential equations on M the theory of the monodromy preserving deformation works out beautifully. To make clear the issue involved, we must settle the following problems.

Problem 0.3. (i) What are the second order linear Fuchsian differential equations on the Riemann surface M? Which equations among them correspond to (Q) on the Riemann sphere? How many apparent singular points do we need in order to develop a suitable deformation theory?

(ii) In case $g \geqq 2$, Fuchsian equations on M admit no such explicit representation as in the cases $g=0$ and 1. Thus, to begin with, we must takcle the problem of constructing the *moduli space* of the relevant Fuchsian equations on M and studying its geometric structure as an analytic space or a complex manifold. In fact, this moduli problem with turn out to be much harder and more central than the problem of the monodromy preserving deformation itself. Many results of the deformation problem will naturally follow from those of the moduli problem.

(iii) To develop the deformation theory on a Riemann surface of an arbitrary genus, we must provide an intrinsic meaning with Okamoto's work [Ok5, 6] by using, for example, the Riemann-Roch formula, the Serre duality theorem, the Kodaira-Spencer theory and other geometric tools.

(iv) In addition to the monodromy preserving deformation, consider also the *monodromy changing deformation* (especially from the point of view of Poisson geometry).

0.4. SL-operators (Fuchsian projective connections).

Before describing our main results of the present paper, we shall briefly explain the object of our deformation theory. Let M be a closed Riemann surface of genus $g \geqq 0$. In Section 1, we shall define a class of second order Fuchsian differential operators on M and call them SL-operators (Definition 1.2). In geometric terminology, they are a kind of Fuchsian $PSL(2; \mathbf{C})$-connections on M. Moreover, we shall make various definitions related to SL-operators. Of particular importance among them are classification of singular points of SL-operators (*generic* singular points and *apparent* singular points) and the *multiplicity* at an apparent singular point (see §§1.2). Roughly speaking, an SL-operator is a differential operator $L:\mathcal{M}(\xi)\rightarrow\mathcal{M}(\xi\otimes\kappa^{\otimes 2})$, where $\mathcal{M}$ is the sheaf of germs of meromorphic functions on M, κ the canonical line bundle over M, ξ any fixed holomorphic line bundle over M with the first Chern class $c_1(\xi)=1-g\in\boldsymbol{Z}\cong H^2(M;\boldsymbol{Z})$, (see Remark 1.4). An SL-operator L is said to be of *ground state* if all of its apparent singular points are of multiplicity one; otherwise L is said to be of *excited state*. SL-operators of ground state are the main object of our research, though those of excited state will be considered as well in the first half of the present paper.

We shall see that the set of all SL-operators can be identified with the $\Gamma(M;\mathcal{M}(\kappa^{\otimes 2}))$-affine space Q of all meromorphic projective connections on M. By fixing a projective structure on M subordinate to its complex structure, we can further identify it with the linear space $\Gamma(M;\mathcal{M}(\kappa^{\otimes 2}))$ of all meromorphic quadratic differentials on M. Hence we use the letter Q as well as L to denote an SL-operator, where Q is the initial letter of quadratic differential. Only Fuchsian SL-operators will be considered in the present paper. Hereafter we mean by an SL-operator a *Fuchsian SL*-operator. Given an SL-operator Q, let

$$m:=\text{the number of generic singular points of } Q,$$

$$n:=\text{the number of apparent singular points of } Q.$$

We shall see that it is natural to assume

$$\begin{aligned} n&=m+3g-3 \\ &=\text{the moduli number of Riemann surfaces} \\ &\quad\ \text{of genus } g \text{ with } m \text{ punctures} \end{aligned} \tag{$\sharp$}$$

when we consider the deformation theory for SL-operators of ground state (see §§5.3).

0.5. Moduli and deformation for SL-operators.

Owing to the length of the present paper, it is better to provide the reader with a general picture of the paper by giving its outline in some detail. Although we shall be concerned with SL-operators of excited state as well as those of ground state in the first half of the paper, we confine ourselves to stating only those results which are concerned with SL-operators of ground state for the sake of simplicity. Moreover the statement of the outline below will not always be arranged in the order in which our theory is developed in the main body of the paper (§ 1-§ 9) when such a rearrangement seems to be helpful to the reader's understanding.

Let M be a closed Riemann surface of genus $g \geqq 0$, m a natural number such that $n := m+3g-3$ is positive. For a fixed $\theta = (\theta_1, \cdots, \theta_m) \in (\boldsymbol{C} \backslash \boldsymbol{Z})^m$, let

$$E(m\,;\,\theta) := \left\{ \begin{array}{l} SL\text{-operators with } m+n \textit{ ordered} \text{ regular singular} \\ \text{points such that, for } j=1, \cdots, m, \text{ the } j^{th} \\ \text{singular point has the characteristic exponents} \\ \dfrac{1}{2}(1 \pm \theta_j) \text{ and the last } n \text{ singular points are} \\ \textit{apparent} \text{ and } \textit{of ground state.} \end{array} \right\}.$$

Moreover we put

$$B(l) := \{(p_1, \cdots, p_l) \in M^l\,;\, i \neq j \text{ implies } p_i \neq p_j\}.$$

$B(l)$ is naturally an l dimensional complex manifold as an open submanifold of M^l. We have a natural map

$$\begin{array}{ccc} \pi : E(m\,;\,\theta) & \longrightarrow & B(m+n) \\ \cup\!\!\!\!\!\!\cup & & \cup\!\!\!\!\!\!\cup \\ Q & \longmapsto & \boldsymbol{r} = (p_1, \cdots, p_m, q_1, \cdots, q_n), \end{array}$$

where $\boldsymbol{r}$ is the ordered singular points of Q. By using the Riemann-Roch formula, the Kodaira-Spencer theory and elements of local analytic geometry, together with some basic facts concerning Fuchsian differential equations (Fuchs' criterion for regular singularity and Frobenius' method, etc.), we can show the following theorem.

THEOREM A (see Theorem 2.4). *$E(m\,;\,\theta)$ carries a natural structure of an analytic space of pure dimension $m+2n$ such that π is a holomorphic surjection.*

Let $p : M^m \times M^n \supset B(m+n) \to B(m) \subset M^m$ be the projection into the first m factors. We have the following commutative diagram.

$$(D1) \qquad \begin{array}{ccc} & \overset{\pi}{\longleftarrow} & E(m\,;\,\theta) \\ B(m+n) & & \downarrow \varpi \\ & \underset{p}{\longrightarrow} & B(m) \end{array}$$

where π, p and ϖ are surjective. Given any $\boldsymbol{p} \in B(m)$, let

$$E(\boldsymbol{p}\,;\theta):=\{Q\in E(m\,;\theta)\,;\,\varpi(Q)=\boldsymbol{p}\}\,.$$

THEOREM B (see Theorem 2.5). *For any* $\boldsymbol{p}\in B(m)$, $E(\boldsymbol{p}\,;\theta)$ *is an analytic subspace of* $E(m\,;\theta)$ *of pure dimension* $2n$.

The fact that the dimension of $E(\boldsymbol{p}\,;\theta)$ is even will turn out to be very important in connection with the symplectic geometry.

Next we proceed to the reducibility and irreducibility of SL-operators. We give a rough definition of them.

DEFINITION (See Definition 3.1 for the rigorous definition). An SL-operator L is said to be *reducible* if there exists first order differential operator K such that $L=K\cdot K^*$, where K is locally written as $K=-(d/dx)+P$ and K^* is the formal adjoint of K. For a set $\mathcal{S}$ of SL-operators, we put

$$\mathcal{S}_{\text{red}}:=\{Q\in\mathcal{S}:Q \text{ is reducible}\}\,,$$

$$\mathcal{S}_{\text{irr}}:=\{Q\in\mathcal{S}:Q \text{ is irreducible}\}\,.$$

Reducible SL-operators cause some troubles in considering the moduli theory of SL-operators. The following theorem asserts that the reducible SL-operators form only a thin set.

THEOREM C (see Theorems 3.6–3.8). (i) $E(m\,;\theta)_{\text{red}}$ *is an analytic subspace of* $E(m\,;\theta)$ *whose condimension is at least* $n-1$. *For any* $\boldsymbol{p}\in B(m)$, $E(\boldsymbol{p}\,;\theta)_{\text{red}}$ *is an analytic subspace of* $E(\boldsymbol{p}\,;\theta)$ *whose codimension is at least* $n-1$.

(ii) $E(m\,;\theta)_{\text{red}}$ *is empty for* θ *in a Zariski open subset.*

Next we turn to the consideration of the space of projective representations of the fundamental group of the Riemann surface M with m punctures. Given $\boldsymbol{p}=(p_1,\cdots,p_m)\in B(m)$, let $|\boldsymbol{p}|$ be the unordering of $\boldsymbol{p}$ i.e. $|\boldsymbol{p}|=\{p_1,\cdots,p_m\}$. We denote the Lie group $PSL(2\,;\boldsymbol{C})$ by G. Let us consider the space

$$\hat{R}(\boldsymbol{p}):=\text{Hom}\,(\pi_1(M\backslash|\boldsymbol{p}|)\,;G)\,,$$

topologized with the compact-open topology, where $\pi_1(M\backslash|\boldsymbol{p}|)$ is regarded as a discrete group. The space $\hat{R}(\boldsymbol{p})$ can be embedded into the product space G^{m+2g} as a complex submanifold. Although such an embedding is not unique, the induced complex manifold structure on $\hat{R}(\boldsymbol{p})$ is unique. A representation $\rho\in\hat{R}(\boldsymbol{p})$ is said to be *irreducible* if the image group $\text{Im}\,\rho\subset G=\text{Aut}\,(\boldsymbol{P}^1)$ has no fixed point in $\boldsymbol{P}^1$. For any subset $\mathcal{R}$ of $\hat{R}(\boldsymbol{p})$, let $\mathcal{R}_{\text{irr}}$ be the subset of $\mathcal{R}$ consisting of all irreducible representations of $\mathcal{R}$. The inner automorphism group $\mathrm{Ad}(G)$ acts on $\hat{R}(\boldsymbol{p})$ by conjugacy. $\hat{R}(\boldsymbol{p})_{\mathrm{irr}}$ is invariant under this action. By Schur's lemma, $\mathrm{Ad}(G)$ acts on $\hat{R}(\boldsymbol{p})_{\mathrm{irr}}$ freely. We put

$$R(\boldsymbol{p})_{\mathrm{irr}} := \hat{R}(\boldsymbol{p})_{\mathrm{irr}}/\mathrm{Ad}(G)\,.$$

We see that $R(\boldsymbol{p})_{\mathrm{irr}}$ carries a natural complex manifold structure of dimension $3m+6g-6$ such that the canonical projection $\hat{R}(\boldsymbol{p})_{\mathrm{irr}} \to R(\boldsymbol{p})_{\mathrm{irr}}$ is a holomorphic principal $\mathrm{Ad}(G)$-bundle. Now we put

$$\hat{R}(\boldsymbol{p}\,;\theta)_{\mathrm{irr}} := \left\{ \rho \in \hat{R}(\boldsymbol{p})_{\mathrm{irr}}\,;\ \begin{array}{l} \text{the circuit matrix } (\mathrm{mod}\ \pm I) \text{ at } p_j \\ \text{induced by } \rho \text{ has eigenvalues} \\ \exp(\pm\pi\sqrt{-1}\theta_j)\ (\mathrm{mod}\ \pm 1),\ j=1,\cdots,m \end{array} \right\}.$$

$\hat{R}(\boldsymbol{p}\,;\theta)_{\mathrm{irr}}$ is also invariant under the action of $\mathrm{Ad}(G)$. We put

$$\hat{R}(\boldsymbol{p}\,;\theta)_{\mathrm{irr}} := \hat{R}(\boldsymbol{p}\,;\theta)_{\mathrm{irr}}/\mathrm{Ad}(G)\,.$$

THEOREM D (see Theorem 5.2). *$R(\boldsymbol{p}\,;\theta)_{\mathrm{irr}}$ is a $2n$-dimensional complex submanifold of $R(\boldsymbol{p})_{\mathrm{irr}}$.*

We put

$$R(m)_{\mathrm{irr}} := \coprod_{\boldsymbol{p}\in B(m)} R(\boldsymbol{p})_{\mathrm{irr}}\,,$$

$$R(m\,;\theta)_{\mathrm{irr}} := \coprod_{\boldsymbol{p}\in B(m)} R(\boldsymbol{p}\,;\theta)_{\mathrm{irr}}\,.$$

$R(m)_{irr}$ carries a structure of a local system over $B(m)$ (i.e. a covariant functor of the fundamental groupoid of $B(m)$ into the category of complex manifolds) whose characteristic homomorphism at a point $\boldsymbol{p}^\circ \in B(m)$ is given by

$$\begin{array}{ccc} Br(m) := \pi_1(B(m), \boldsymbol{p}^\circ) & \longrightarrow & \mathrm{Aut}(R(\boldsymbol{p}^\circ)_{\mathrm{irr}}) \\ \Cup & & \Cup \\ l & \longmapsto & [\rho \mapsto \rho\cdot l_*]\,, \end{array}$$

where $Br(m) \to \mathrm{Aut}(\pi_1(M\backslash|\boldsymbol{p}^\circ|))$, $l \mapsto l_*$ denotes the natural action of the braid group $Br(m)$ on the fundamental group $\pi_1(M\backslash|\boldsymbol{p}^\circ|)$. We see that

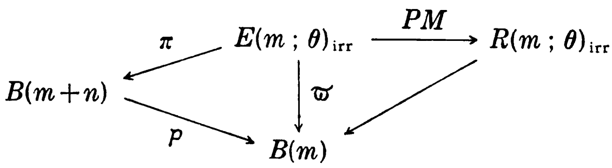
THEOREM E (see Theorem 5.4). *$R(m\,;\theta)_{\mathrm{irr}}$ is a local subsystem of $R(m)_{\mathrm{irr}}$ over $B(m)$. In particular, $R(m\,;\theta)_{\mathrm{irr}}$ is a complex manifold of dimension $m+2n$.*

Now we can define the *projective monodromy map PM* by

$$
\begin{array}{cccc}
PM: & E(m\,;\theta)_{\mathrm{irr}} & \longrightarrow & R(m\,;\theta)_{\mathrm{irr}} \\
 & \Downarrow & & \Downarrow \\
 & Q & \longmapsto & \begin{pmatrix}\text{the conjugacy class of projective} \\ \text{monodromy representation of } Q\end{pmatrix}.
\end{array}
$$

One can check that PM is a holomorphic map. We have the following commutative diagram:

(D2)



We notice that π, p and ϖ are surjective.

The analytic space $E(m\,;\theta)_{\mathrm{irr}}$ may contain singularities. It is difficult to investigate the behaviour of the projective monodromy map PM on the singularity set of $E(m\,;\theta)_{\mathrm{irr}}$. Thus we have to remove from $E(m\,;\theta)_{\mathrm{irr}}$ an analytic subset which contains all of the singularities of $E(m\,;\theta)_{\mathrm{irr}}$. A clever way to do this is to consider a family of holomorphic line bundles $\{\xi(\boldsymbol{r})\,;\boldsymbol{r}\in B(m+n)\}$ over M defined by

$$\xi(\boldsymbol{r}):=\kappa^{\otimes 2}\otimes[p_1+\cdots+p_m-(q_1+\cdots+q_n)]$$

for $\boldsymbol{r}=(p_1,\cdots,p_m,q_1,\cdots,q_n)\in B(m)$, where $[D]$ denotes the line bundle associated with a divisor D. The assumption ($\sharp$) $n=m+3g-3$ implies $c_1(\xi(\boldsymbol{r}))=g-1$. Hence the Riemann-Roch formula yields the following equality:

$$(FA)\qquad \dim H^0(M\,;\mathcal{O}(\xi(\boldsymbol{r})))=\dim H^1(M\,;\mathcal{O}(\xi(\boldsymbol{r})))$$

for $\boldsymbol{r}\in B(m+n)$. This is one of major advantages of the assumption ($\sharp$). We call (FA) the *Fredholm alternative*. The reason for this naming will be clear soon. Let

$$A(m):=\{\boldsymbol{r}\in B(m+n)\,;\dim H^0(M\,;\mathcal{O}(\xi(\boldsymbol{r})))>0\},$$

$$X(m):=B(m+n)\setminus A(m).$$

By Grauert's theorem [Gra, §7, Satz 3], $A(m)$ is an analytic subset of $B(m+n)$. In case $g=0$ or 1, $A(m)$ can be explicitly written down (see Example 6.2). Furthermore we have the following lemma.

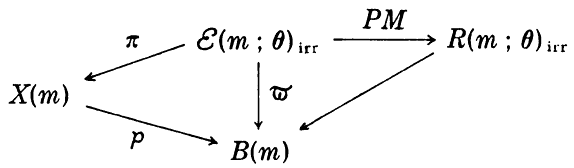
LEMMA (see Corollary 4.14). *$X(m)$ is a nonempty Zariski open subset of $B(m+n)$ such that the projection $p=p|X(m):X(m)\to B(m)$ is surjective.*

Now we put

$$\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}:=\pi^{-1}(X(m))\subset E(m\,;\,\theta)_{\mathrm{irr}}$$

and consider $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$ instead of $E(m\,;\,\theta)_{\mathrm{irr}}$. Then we obtain from $(D2)$ the following diagram:

$(D3)$



We note that π, p and ϖ are surjective. On the space $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$ the projective monodromy map PM has the following nice property.

THEOREM F (see Theorem 5.9). *$PM:\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}\rightarrow R(m\,;\,\theta)_{\mathrm{irr}}$ is a locally biholomorphic map. In particular, $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$ is a complex manifold.*

To establish this theorem, we need the concept of a kind of *gauge equivalence.* In the category of holomorphic integrable connections on a holomorphic vector bundle, it is a standard fact that the following property (GM) holds:

(GM) Two connections are gauge-equivalent if and only if they give rise to the same monodromy representation class.

In the category of Fuchsian integrable connections, things are more complicated. If one considers only Fuchsian connections *with fixed singularities,* one can define the *meromorphic gauge equivalence* with the property (GM) in a similar manner as in the case of the ordinary holomorphic gaupe equivalence. To consider the deformation theory, however, we must consider Fuchsian connections whose singularities may be changed connection by connection. In such a situation it is difficult to define a precise gauge equivalence with the property (GM). In Section 4, we shall introduce a certain kind of "gauge equivalence" for SL-operators which enjoys the property (GM) as long as two SL-operators are sufficiently close. See Theorem 4.19, Theorem 5.6 and Theorem 5.8.

Recall that $R(m\,;\,\theta)_{\mathrm{irr}}$ has a structure of a local system over $B(m)$, (Theorem F), which determine a foliation on $R(m\,;\,\theta)_{\mathrm{irr}}$ such that each leaf is locally a horizontal section of $R(m\,;\,\theta)_{\mathrm{irr}}\rightarrow B(m)$. Since PM is locally biholomorphic, the above foliation induces a one on $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$. This foliation just describes the monodromy preserving deformation on $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$.

Let $\mathcal{E}(m;\theta)$ be the inverse image of $X(m)$ by the projection $\pi: E(m;\theta)\to B(m+n)$. We can investigate the structure of the space $\mathcal{E}(m;\theta)$ more deeply from a different point of view, i. e. by considering a Cousin problem associated with the line bundles $\xi(\boldsymbol{r})$ $(\boldsymbol{r}\in X(m))$. Here we would like to consider $\mathcal{E}(m;\theta)$ rather than $\mathcal{E}(m;\theta)_{\mathrm{irr}}$. First, we want to refer to an affine bundle structure of $\pi:\mathcal{E}(m;\theta)\to X(m)$. Introducing the concept of *accessory parameters* of an SL-operator, which play a role of fiber coordinates, we can show the following theorem.

THEOREM G (see Theorem 6.16). *The complex manifold $\mathcal{E}(m;\theta)$ has a natural structure such that $\pi:\mathcal{E}(m;\theta)\to X(m)$ is a holomorphic affine bundle of rank n.*

To establish the theorem, we need to solve a certain Cousin problem associated with the line bundles $\xi(\boldsymbol{r})$ $(\boldsymbol{r}\in X(m))$ which is stated as follows.

Problem (CP). Given a data on the location of singular points $\boldsymbol{r}\in X(m)$ and accessory parameters $\nu=(\nu_1,\cdots,\nu_n)\in \boldsymbol{C}^n$, construct the corresponding SL-operator. Moreover investigate the holomorphic dependence of the solution on the data $(\boldsymbol{r},\nu)$.

We abbreviate our Cousin problem as (CP). There is no room for explaining why the line bundles $\xi(\boldsymbol{r})$ are related to (CP). We confine ourselves to explaining why (FA) is called the Fredholm alternative. (FA) implies that the vanishing of $H^0(M;\mathcal{O}(\xi(\boldsymbol{r})))$ is equivalent to that of $H^1(M;\mathcal{O}(\xi(\boldsymbol{r})))$. On the other hand, the vanishing of $H^1(M;\mathcal{O}(\xi(\boldsymbol{r})))$ implies the solvability of (CP) and that of $H^0(M;\mathcal{O}(\xi(\boldsymbol{r})))$ implies the uniqueness of solution of (CP). Hence (FA) suggests that the solvability and the uniqueness of solution are equivalent in our Cousin problem. Thus, if one regards cohomology as an abstract version of the theory of integral equations, one can call (FA) the Fredholm alternative. From the very definition of $X(m)$, the following lemma is evident, but plays a key role in solving our Cousin problem.

LEMMA (see Lemma 6.4).
(i) $H^1(M;\mathcal{O}(\xi(\boldsymbol{r})))=0$ *and*
(ii) $H^0(M;\mathcal{O}(\xi(\boldsymbol{r})))=0$
(iii) *hold for all* $\boldsymbol{r}\in X(m)$.

The statement of the above lemma consists of one sentence. We remark that, as mentioned before, (i) guarantees the solvability of (CP), (ii) guarantees the uniqueness of solutions of (CP) and, by Kodaira-Spencer's theorem [KS, III, Theorem 7] (iii) guarantees the holomorphic dependence of the solution to (*CP*) on datum. Through the investigation of (*CP*), we can further establish the following theorems, which are the main results of the present paper.

THEOREM H (see Theorem 6.18). *There exists a canonically defined closed 2-form Ω on $\mathcal{E}(m;\theta)$ such that the restriction of Ω to each fiber $\mathcal{E}(\boldsymbol{p};\theta)$ of $\varpi:\mathcal{E}(m;\theta)\to B(m)$ determines a symplectic structure on $\mathcal{E}(\boldsymbol{p};\theta)$, $(\boldsymbol{p}\in B(m))$.*

THEOREM I (see Theorem 8.8 and 9.8). *The monodromy preserving deformation $\mathcal{E}(m;\theta)_{\mathrm{irr}}$ is characterized by the Ω-Lagrangian foliation on $\mathcal{E}(m;\theta)_{\mathrm{irr}}$ which is transverse to each fiber of $\varpi:\mathcal{E}(m;\theta)_{\mathrm{irr}}\to B(m)$.*

Moreover we state some results concerning a Poisson structure on the moduli space $\mathcal{E}(m;\theta)$, (see Section 9).

THEOREM J (see Proposition 9.1). *The moduli space $\mathcal{E}(m;\theta)$ admits a Poisson structure $\{\cdot,\cdot\}$ of constant rank $2n$ whose symplectic leaves consist of all fibers of $\varpi:\mathcal{E}(m;\theta)\to B(m)$.*

We can construct commuting vector fields $\mathcal{H}_i$ $(i=1,\cdots,m)$ on $\mathcal{E}(m;\theta)$ locally, (Theorem 9.6). Let t_i $(i=1,\cdots,m)$ be an "admissible" coordinate system of $B(m)$, then the $(1,1)$-tensor

$$D=\sum_{i=1}^{m}\mathcal{H}_i\otimes dt_i$$

makes sense globally on $\mathcal{E}(m;\theta)$, (Lemma 9.10). We put

$$\Omega^{(p)}=\mathcal{O}_{\mathcal{E}(m;\theta)}(\varpi^*\wedge^p T^*B(m)),$$

$$\mathcal{P}=\mathrm{Ker}\,[D:\Omega^{(0)}\to\Omega^{(1)}].$$

We call $\mathcal{P}$ *the sheaf of monodromy changing Hamiltonians,* and a (local) section of $\mathcal{P}$ a monodromy changing Hamiltonian. We expact that the Hamiltonian flow generated by a monodromy changing Hamiltonian describes a nice monodromy changing deformation in a certain sense.

THEOREM K (see Theorems 9.2 and 9.4). *$\mathcal{P}$ is a Poisson subalgebra of the structure sheaf $\mathcal{O}_{\mathcal{E}(m;\theta)}$ of the moduli space $\mathcal{E}(m;\theta)$. $\mathcal{P}$ admits the following resolution by $\mathcal{O}_{\mathcal{E}(m;\theta)}$-modules:*

$$0\longrightarrow\mathcal{P}\longrightarrow\Omega^{(0)}\stackrel{D}{\longrightarrow}\cdots\stackrel{D}{\longrightarrow}\Omega^{(m)}\longrightarrow 0.$$

Hamiltonian vector fields generated by a monodromy changing Hamiltonian commuts with the vector fields $\mathcal{H}_i$ $(i=1,\cdots,m)$.

The sheaf $\mathcal{P}$ and the monodromy changing deformation must be considered further in the future.

In the last section (Section 10), we shall give some open problems.

§ 1. *SL*-operators on a Riemann surface.

1.1. *GL*-operators and *SL*-operators.

We shall use the following notation:

M: a closed Riemann surface of genus $g \geqq 0$,

$\mathcal{O}$: the sheaf of germs of holomorphic functions over M,

$\mathcal{O}^*$: the sheaf of germs of nonzero holomorphic functions over M,

$\mathcal{M}$: the sheaf of germs of meromorphic functions over M,

$\xi \in H^1(M, \mathcal{O}^*)$: a holomorphic line bundle over M,

$\mathcal{U}=\{(U_j, x_j)\}$: a coordinate covering of M,

$(\xi_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$: a representative 1-cocycle of ξ with respect to the covering $\mathcal{U}$, and

$D_j = d/dx_j$: the differentiation with respect to x_j.

DEFINITION 1.1. A *GL-operator* on ξ for a coordinate covering $\mathcal{U}$ is, by definition, a collection $L=(L_j)$ of second order linear meromorphic differential operators

$$L_j = -D_j^2 + P_j D_j + Q_j\,, \qquad P_j, Q_j \in \mathcal{M}(U_j)\,, \tag{1.1}$$

such that, if $f_j = \xi_{jk} f_k$ in $U_j \cap U_k$, then the two differential equations $L_j f_j = 0$ and $L_k f_k = 0$ are equivalent as differential equations in $U_j \cap U_k$. Each L_j is called a *local expression* of L. Let L and L' be *GL*-operators on a lines bundle ξ for coordinate coverings $\mathcal{U}$ and $\mathcal{U}'$, respectively. L and L' are said to be *equivalent* if the union $L \cup L'$ is a *GL*-operator on ξ for the coordinate covering $\mathcal{U} \cup \mathcal{U}'$. A *GL-operator on* ξ is, by definition, an equivalence class of a *GL*-operator on ξ for a coordinate covering of M.

DEFINITION 1.2. An *SL-operator on* ξ is, by definition, a *GL*-operator on ξ which contains a representative *GL*-operator L on ξ for some coordinate covering $\mathcal{U}$ satisfying

$$P_j \equiv 0 \quad \text{in } U_j \text{ for every } j\,. \tag{1.2}$$

LEMMA 1.3. *Let* ξ *be a holomorphic line bundle over* M. *There exist*

SL-operators on ξ *if and only if the first Chern class* $c_1(\xi)$ *of* ξ *is* $1-g$. *Here* $H^2(M, \boldsymbol{Z})$ *is identified with* $\boldsymbol{Z}$.

PROOF. Given an SL-operator on ξ, let $L=(L_j)$ be a representative SL-operator for a coordinate covering $\mathcal{U}=\{(U_j, x_j)\}$ of M. Let κ be the canonical line bundle of M and let (ξ_{jk}), $(\kappa_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ be the representative 1-cocycles of ξ and κ, respectively. Notice that $\kappa_{jk}=dx_k/dx_j$ in $U_j \cap U_k$. As an easy calculation shows, (1.2) implies that $\eta_{jk}:=\xi_{jk}^2 \kappa_{jk}$ is constant in $U_j \cap U_k$. Hence $(\eta_{jk}) \in Z^1(\mathcal{U}, \boldsymbol{C})$ defines a holomorphic line bundle η with $c_1(\eta)=0$. Since $\xi^{\otimes 2}=\eta \otimes \kappa^{\otimes -1}$ and $c_1(\kappa)=2g-2$, we have $c_1(\kappa)=1-g$.

Conversely, for any holomorphic line bundle ξ with $c_1(\xi)=1-g$, there exists an SL-operator on ξ. To see this, we first consider a line bundle σ such that $\sigma^{\otimes 2}=\kappa^{\otimes -1}$. Since $c_1(\kappa^{\otimes -1})=2-2g$ is even and the Picard variety $Pic(M)$ of M is a torus of real dimension $2g$, there are 2^{2g} such line bundles. Fix any one of them. Let $(\sigma_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ be a representation 1-cocycle of σ. Let $\{f; x\}$ denote the Schwarzian derivative of f with respect to x:

$$\{f; x\}=\frac{3f''(x)^2-2f'(x)f'''(x)}{4f'(x)^2}.$$

Since $\sigma_{jk}^2=\kappa_{jk}^{-1}$ in $U_j \cap U_k$, an easy calculation shows that

$$\sigma_{jk}^{-1} D_j^2 \sigma_{jk}=\kappa_{jk}^2(D_k^2-\theta_{jk}) \tag{1.3}$$

holds as a differential operator in $U_j \cap U_k$, where

$$\theta_{jk}=\{x_j; x_k\}. \tag{1.4}$$

(θ_{jk}) defines a 1-cocycle with coefficients in $\kappa^{\otimes 2}$, i.e., $(\theta_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}(\kappa^{\otimes 2}))$. Consider an $H^0(M, \mathcal{M}(\kappa^{\otimes 2}))$-affine space

$$\mathcal{Q}=\{Q=(Q_j) \in C^0(\mathcal{U}, \mathcal{M}(\kappa^{\otimes 2})); \delta(Q_j)=(\theta_{jk})\}, \tag{1.5}$$

where δ is the coboundary homomorphism. For any element $Q=(Q_j) \in \mathcal{Q}$, if it exists, put

$$L_j=-D_j^2+Q_j \quad \text{in } U_j. \tag{1.6}$$

Then (1.3) implies the compatibility condition

$$L_j \sigma_{jk}=\sigma_{jk} \kappa_{jk}^2 L_k \quad \text{in } U_j \cap U_k. \tag{1.7}$$

Any line bundle ξ with $c_1(\xi)=1-g$ admits a decomposition $\xi=\eta \otimes \sigma$ with $\eta \in Pic(M)$. Let $(\eta_{jk}) \in Z^1(\mathcal{U}, \boldsymbol{C}^*)$ be a flat representative of η. Then we get the 1-cocycle (ξ_{jk}) defined by $\xi_{jk}=\eta_{jk}\sigma_{jk}$ as a representative of ξ. Now

(1.7) shows that the compatibility condition

$$L_j\xi_{jk}=\xi_{jk}\kappa_{jk}^2L_k \qquad \text{in } U_j\cap U_k \tag{1.8}$$

holds. Hence $L=(L_j)$ is an SL-operator on ξ for $\mathcal{U}$. The above argument shows that if Q is nonempty then an SL-operator exists. Let $\mathcal{U}$ be a covering of M such that any finite intersection of open sets in $\mathcal{U}$ is contractible. Then $H^1(\mathcal{U}, \mathcal{M}(\kappa^{\otimes 2}))=0$, i. e., $Z^1(\mathcal{U}, M(\kappa^{\otimes 2}))=\delta C^0(\mathcal{U}, \mathcal{M}(\kappa^{\otimes 2}))$. Hence Q is nonempty. ■

Hereafter we fix a line bundle ξ with $c_1(\xi)=1-g$. So an SL-operator on ξ is called simply an SL-operator.

REMARK 1.4. By (1.8), an SL-operator L can be regarded as a differential operator $L: \mathcal{M}(\xi)\to\mathcal{M}(\xi\otimes\kappa^{\otimes 2})$.

REMARK 1.5. The affine space Q defined by (1.5) is the space of meromorphic projective connections on M. The argument in the proof of Lemma 1.3 shows that there is a one-to-one correspondence between the set of SL-operators on ξ and that of meromorphic projective connections on M.

REMARK 1.6. The space Q (see (1.5)) parametrizing the SL-operators is an $H^0(M, \mathcal{M}(\kappa^{\otimes 2}))$-affine space. However, if a nice coordinate covering $\mathcal{U}=\{(U_j, x_j)\}$ is chosen, then Q is viewed as a linear space $H^0(M, \mathcal{M}(\kappa^{\otimes 2}))$ of meromorphic quadratic differentials on M. To see this, recall that any closed Riemann surface M admits projective structures subordinate to its complex structure, (see e. g., Gunning [Gu1, pp. 172–173]). In case $g\geqq 1$, such projective structures are not unique; in fact they depend on $\max\{1, 3g-3\}$ parameters. Fix one of them and let $\mathcal{U}=\{(U_j, x_j)\}$ be the corresponding projective coordinate covering. Then θ_{jk} defined by (1.4) are identically zero. Hence the space Q of SL-operators is identified with $H^0(M; \mathcal{M}(\kappa^{\otimes 2}))$ by the following correspondence

$$\begin{array}{ccc} \{SL\text{-operators}\} & = & H^0(M, \mathcal{M}(\kappa^{\otimes 2})) \\ \Cup & & \Cup \\ L=(-D_j^2+Q_j) & \longleftrightarrow & Q=(Q_j)\,. \end{array} \tag{1.9}$$

We fix a projective structure of M subordinate to its complex structure and keep the identification (1.9) throughout the paper. For this reason, hereafter, we shall often use the notation Q for an SL-operator in stead of L; Q is the initial letter of "qudratic differential".

1.2. Apparent singular points and their multiplicities.

In an obvious manner, various notions concerning local properties of meromorphic differential operators are extended to GL-operators on a Riemann surface M through their local expressions. In order to establish terminology, a brief explanation of them will be included here. For a GL-operator L, a point $p\in M$ is said to be a regular singular point of L if p is a regular singular point of its local expression. Characteristic exponents of L at p are those of its local expression. These definitions are independent of the choice of the local expression, whence well-defined. All GL- and SL-operators treated in the present paper are *Fuchsian*. So, hereafter, GL- and SL-operators mean Fuchsian GL- and SL-operators.

We introduce a total order $\succ$ into $\boldsymbol{C}$ by

$$\lambda\succ\mu \overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{either } \operatorname{Re}(\lambda)>\operatorname{Re}(\mu) \\ \text{or } \operatorname{Re}(\lambda)=\operatorname{Re}(\mu), \quad \operatorname{Im}(\lambda)\geqq\operatorname{Im}(\mu). \end{cases} \tag{1.10}$$

We put

$$\boldsymbol{C}_+ := \{\theta\in\boldsymbol{C}\,;\,\theta\succ 0\}\,. \tag{1.11}$$

If λ and μ are the characteristic exponents of L at a regular singular point p with $\lambda\succ\mu$, then $\theta:=\lambda-\mu\in\boldsymbol{C}_+$ is called the *difference* of the characteristic exponents of L at p.

REMARK 1.7. If L is an SL-operator, then $\lambda+\mu=1$. Hence we have

$$\lambda=\frac{1+\theta}{2}, \qquad \mu=\frac{1-\theta}{2}\,. \tag{1.12}$$

In particular, characteristic exponents are uniquely determined by its difference.

A regular singular point p is said to be *generic* if θ is not an integer. Otherwise it is said to be *nongeneric*. At a nongeneric regular singular point p the equation $Lf=0$ has linearly independent solutions of the form either

$$\text{(i)} \begin{cases} x^{\lambda}f(x) \\ x^{\mu}g(x)+x^{\lambda}h(x)\log x \end{cases} \quad\text{or}\quad \text{(ii)} \begin{cases} x^{\lambda}f(x) \\ x^{\mu}g(x)\,, \end{cases}$$

where x is a local coordinate of M at p with $x(p)=0$ and $f(x)$, $g(x)$ and $h(x)$ are nonzero convergent power series of x. In case (i) the regular singular point p is said to be *logarithmic*; in case (ii) it is said to be non-logarithmic or *apparent*. Remark that at an apparent singular point the difference of the characteristic exponents is necessarily an integer greater than or equal to 2.

DEFINITION 1.8. (i) Let $p \in M$ be an apparent singular point of a GL-operator L. If the difference of the characteristic exponents of L at p is $N+1$ ($N \in \boldsymbol{N}$), then L is said to have *multiplicity* N at p. Moreover, if the multiplicity N is the minimal possible value 1, than L is said to be of *ground state* at p.

(ii) Let $p_1, \cdots, p_k$ be the mutually different apparent singular points of a GL-operator L of multiplicities $N_1, \cdots, N_k$, respectively. Then the number $n := N_1 + \cdots + N_k$ is called the *total multiplicity* of L. A GL-operator is said to be of *ground state* if all of its apparent singular points are of ground state.

§2. Analytic spaces of SL-operators.

2.1. The space $E(l)$ of SL-operators.

As in Section 1, let M be a closed Riemann surface of genus $g \geqq 0$ and let ξ be a fixed holomorphic line bundle over M with $c_1(\xi) = 1-g$. We shall investigate the structure of the space

$$\hat{E}(l) := \{SL\text{-operators on } \xi \text{ with exactly } l \text{ regular singular points}\}$$

and its various subspace from the point of view of local analytic geometry. Let

$$\hat{B}(l) := \{\text{sets of } l \text{ points in } M\}.$$

A natural projection $\pi : \hat{E}(l) \to \hat{B}(l)$ is defined by sending an element of $\hat{E}(l)$ to the set of its singular points. Let

$$C(l) := \{(p_1, \cdots, p_l) \in M^l \,;\, \exists i, j \text{ such that } i \neq j,\ p_i = p_j\},$$

$$B(l) := M^l \setminus C(l).$$

$B(l)$ admits a free discontinuous action of the symmetric group $\mathfrak{S}_l$ of degree l such that $\hat{B}(l) = B(l)/\mathfrak{S}_l$. Let $\rho : B(l) \to \hat{B}(l)$ be the canonical projection. The fiber product $E(l) := B(l) \times_\rho \hat{E}(l)$ admits the following interpretation:

$$E(l) = \left\{ \begin{array}{l} SL\text{-operators on } \xi \text{ with exactly } l \\ \textit{ordered} \text{ regular singular points} \end{array} \right\}.$$

Let $\pi: E(l)\to B(l)$ be the restriction of the projection $B(l)\times\hat{E}(l)\to B(l)$ into $E(l)$. We are concerned with the space $E(l)$ rather than $\hat{E}(l)$ because $E(l)$ is easier to hundle than $\hat{E}(l)$ and considering $E(l)$ is the same thing as considering $\hat{E}(l)$ as far as one is concerned with the local deformation problem of differential equations. Only such problem will be discussed in the present paper. So we shall investigate the structure of $E(l)$ and its various subspaces from the point of view of the local analytic geometry.

2.2. The complex structure of $E(l)$.

First we provide $E(l)$ with a natural complex structure. To do this, we make the following observation: Remark 1.6 and Fuchs' criterion for regular singularity imply that, for $\boldsymbol{p}=(p_1,\cdots,p_l)\in B(l)$, $l\geqq 0$, the set of all SL-operators whose regular singular points are contained in $|\boldsymbol{p}|:=\{p_1,\cdots,p_l\}$ is identified with the linear space

$$F(l)_{\boldsymbol{p}}:=H^0(M,\mathcal{O}(\kappa^{\otimes 2}\otimes[2p_1+\cdots+2p_l])). \tag{2.1}$$

Here (and hereafter) $[D]$ denotes the line bundle associated with a divisor D. By the Riemann-Roch formula, we get

$$\dim F(l)_{\boldsymbol{p}}=(2l+3g-3)^+, \qquad \boldsymbol{p}\in B(l), \tag{2.2}$$

where

$$a^+:=\begin{cases} a & (a>0), \\ 1 & (a=0), \\ 0 & (a<0). \end{cases}$$

Let D_j $(j=1,\cdots,l)$ be dividors on $M\times B(l)$ associated with the hypersurface $\{(p;p_1,\cdots,p_l)\in M\times B(l);\ p=p_j\}$ and let $\varphi: M\times B(l)\to B(l)$ and $\psi: M\times B(l)\to M$ be the projections. Consider the sheaf $\mathcal{F}(l)$ over $B(l)$ defined by

$$\mathcal{F}(l)=\varphi_*\mathcal{O}_{M\times B(l)}(\psi^*\kappa^{\otimes 2}\otimes[2D_1+\cdots+2D_l]). \tag{2.3}$$

Since $\dim\mathcal{F}(l)_{\boldsymbol{p}}$ is independent of $\boldsymbol{p}\in B(l)$, Kodaira-Spencer's theorem [KS, I, Theorem 2.2] [GR, Chap. 10, §5, Theorem 5] implies that $\mathcal{F}(l)$ is a locally free analytic sheaf over $B(l)$. Let $F(l)$ be the associated holomorphic vector bundle. The fiber of $F(l)$ over $\boldsymbol{p}\in B(l)$ is identified with the linear space $F(l)_{\boldsymbol{p}}$. Hence the rank of $F(l)$ is $(2m+3g-3)^+$. Let $\pi_{l,j}$ $(j=1,\cdots,l)$ be the projections defined by

$$\pi_{l,j}: B(l)\longrightarrow B(l-1), \qquad (p_1,\cdots,p_l)\longrightarrow(p_1,\cdots,\hat{p}_j,\cdots,p_l),$$

where $\hat{p}_j$ stands for the ommision of the j-th entry p_j. We can easily see the following proposition.

PROPOSITION 2.1. *We have* $E(0)=F(0)$ *and*

$$E(l)=F(l)\setminus\bigcup_{j=1}^{l}\pi_{l,j}^{*}F(l-1)\,,\qquad l\geqq 1\,.$$

In particular, $E(l)$ *is a nonempty open complex submanifold of the* $\{(2l+3g-3)^{+}+l\}$*-dimensional complex manifold* $F(l)$ *except for the following two cases; In case* $g=0$, $l=1$, $E(l)$ *is empty; in case* $g=0$, $l=0$, $E(l)$ *consists of a single point.*

2.3. Various subspaces of $E(m+k)$.

Let $l=m+k$, $m\geqq 0$ and $k\geqq 1$. We define various subspaces of $E(m+k)$ which we need to consider in what follows. Let $p: B(m+k)\to B(m)$ be the projection into the first m factors; p is surjective. We have the diagram

$$\text{(2.4)}\qquad \begin{array}{ccc} & \overset{\pi}{\longleftarrow} & E(m+k) \\ B(m+k) & & \Big\downarrow \varpi \\ & \underset{p}{\longrightarrow} & B(m) \end{array}$$

$$E(m,k):=\left\{Q\in E(m+k)\,;\ \begin{matrix}\text{the last } k \text{ singular points}\\ \text{of } Q \text{ are apparent}\end{matrix}\right\}.$$

Given $N=(N_1,\cdots,N_k)\in \boldsymbol{N}^k$, let

$$E(m,k\,;N):=\left\{\begin{matrix} & \text{the } (m+j)\text{-th singular point of } Q\\ Q\in E(m+k)\,; & \text{is an apparent singular point of}\\ & \text{multiplicity } N_j\ (j=1,\cdots,k)\end{matrix}\right\}.$$

Clearly we have

$$E(m,k)=\coprod_{N\in \boldsymbol{N}^n}E(m,k\,;N)\,.$$

Given $\boldsymbol{p}\in B(m)$, let

$$E(\boldsymbol{p},k\,;N):=\{Q\in E(m,k\,;N)\,;\ \varpi(Q)=\boldsymbol{p}\}\,,\qquad \boldsymbol{p}\in B(m)\,.$$

Given $\theta=(\theta_1,\cdots,\theta_m)\in(\boldsymbol{C}_{+})^m$ (cf. (1.11)), let

$$E(m,k\,;\theta,N):=\left\{Q\in E(m,k\,;N)\,;\ \begin{array}{l}\text{the difference of characteristic}\\ \text{exponents of } Q \text{ at the } j\text{-th}\\ \text{singular point is } \theta_j\ (j=1,\cdots,m)\end{array}\right\}.$$

$$E(\boldsymbol{p},k\,;\theta,N):=E(\boldsymbol{p},k\,;N)\cap E(m,k\,;\theta,N)\,.$$

THEOREM 2.2. *Let* $m\geqq\max\{2-g,0\}$, $k\geqq 1$ *and* $N\in\boldsymbol{N}^k$. *Then,*
(i) $E(m,k\,;N)$ *is a purely dimensional analytic subspace of* $E(m+k)$;

$$\dim E(m,k\,;N)=3(m+g-1)+k\,.$$

(ii) $\pi: E(m,k\,;N)\to B(m+k)$ *is surjective.*

THEOREM 2.3. *Let* m, k *and* N *be as in Theorem* 2.2 *and let* $\boldsymbol{p}\in B(m)$. $E(\boldsymbol{p},k\,;N)$ *is a purely dimensional analytic space*;

$$\dim E(\boldsymbol{p},k\,;N)=2m+k+3g-3\,.$$

THEOREM 2.4. *Let* m, k *and* N *be as in Theorem* 2.2 *and let* $\theta\in(\boldsymbol{C}_+)^m$.
(i) $E(m,k\,;\theta,N)$ *is a purely dimensional analytic space*;

$$\dim E(m,k\,;\theta,N)=2m+k+3g-3\,.$$

(ii) $\pi: E(m,k\,;\theta,N)\to B(m+k)$ *is surjective.*

THEOREM 2.5. *Let* m, k, θ *and* N *be as in Theorem* 2.4 *and let* $\boldsymbol{p}\in B(m)$. $E(\boldsymbol{p},k\,;\theta,N)$ *is a purely dimensional analytic space*;

$$\dim E(\boldsymbol{p},k\,;\theta,N)=m+k+3g-3\,.$$

REMARK 2.6. (Part of) Theorem 2.2 is intuitively clear from the following heuristic argument: An SL-operator with $m+k$ regular singular points contains $3(m+k+g-1)$ parameters. (See Proposition 2.1. Here the exceptional case $a^+\neq a$ with $a=2(m+k)+3g-3$ is excluded.) On the other hand, Frobenius' method implies that there are two algebraic constraints in order that a regular singular point p be apparent, i.e., (i) the difference of characteristic exponents at p is an integer, (ii) there is no logarithmic solution at p. Thus the condition that the last k singular points be apparent lessen the number of parameters by $2k$. Hence $E(m,k\,;N)$ is expected to be a $3(m+g-1)+k$ dimensional analytic subspace of $E(m+k)$. However rigorous proof of this theorem needs some labor.

Theorem 2.2 will be proved in §§2.4. Theorems 2.3–2.5 can be established in a similar manner. So their proofs are omitted.

We conclude this subsection with introducing the space $E(\boldsymbol{p}|n)$, $\boldsymbol{p}\in B(m)$, $n\geqq 1$, defined by

$$E(\boldsymbol{p}|n):=\coprod_{(k,N)\in\Lambda(n)} E(\boldsymbol{p},k\,;N)\,, \tag{2.5}$$

where

$$\Lambda(n):=\{(k,N)\,;\,0\leqq k\leqq n,\ N\in\boldsymbol{N}^k,\ |N|=N_1+\cdots+N_k\leqq n\}\,. \tag{2.6}$$

$E(\boldsymbol{p}|n)$ is the space of SL-operators with the ordered regular singular points $\boldsymbol{p}$ and with apparent singular points of total multiplicity $\leqq n$. Notice that Theorem 2.3 implies

$$\dim E(\boldsymbol{p},k\,;N)<\dim E(\boldsymbol{p},n\,;\mathbf{1}_n)\qquad \text{for } (k,N)\in\Lambda(n)\backslash\{(n,\mathbf{1}_n)\}\,,$$

where $\mathbf{1}_n=(1,\cdots,1)$ (an n-vector). Hence "almost all" operators in $E(\boldsymbol{p}|n)$ are of ground state.

2.4. Proof of Theorem 2.2.

To prove Theorem 2.2 we consider the following subsheaves of the sheaf $\mathcal{F}(m+k)$ (cf. §2.2),

$$\mathcal{F}(m,k):=\varphi_*\mathcal{O}_{M\times B(m+k)}(\psi^*\kappa^{\otimes 2}\otimes[2D_1+\cdots+2D_m])\,,$$

$$\mathcal{F}^{j,\nu}_{m,k}=\varphi_*\mathcal{O}_{M\times B(m+k)}(\psi^*\kappa^{\otimes 2}\otimes[2D_1+\cdots+2D_m+\nu D_{m+j}])\,,$$

$$(j=1,\cdots,k,\ \nu=1,2)\,.$$

Given $\boldsymbol{p}=(p_1,\cdots,p_{m+k})\in B(m+k)$, let

$$F(m,k)_{\boldsymbol{p}}=H^0(M\,;\,\mathcal{O}(\kappa^{\otimes 2}\otimes[2p_1+\cdots+2p_m]))\,,$$

$$F^{j,\nu}_{m,k,\boldsymbol{p}}=H^0(M\,;\,\mathcal{O}(\kappa^{\otimes 2}\otimes[2p_1+\cdots+2p_m+\nu p_{m+j}]))\,.$$

Then, by the Riemann-Roch formula, we get

$$\dim F(m,k)_{\boldsymbol{p}}=(2m+3g-3)^+\,,\quad \dim F^{j,\nu}_{m,k,\boldsymbol{p}}=(2m+\nu+3g-3)^+ \tag{2.7}$$

for $\boldsymbol{p}\in B(m+k)$, where $a_+=\max\{a,0\}$. Since $\dim F(m,k)_{\boldsymbol{p}}$ and $\dim F^{j,\nu}_{m,k,\boldsymbol{p}}$ are independent of $\boldsymbol{p}\in B(m+k)$, a theorem of Kodaira-Spencer [KS, I, Theorem 2.2] [GR, Chap. 10, §5, Theorem 5] implies that $\mathcal{F}(m,k)$ and $\mathcal{F}^{j,\nu}_{m,k}$ are locally free analytic sheaves over $B(m+k)$. The associated holomorphic vector bundles $F(m,k)$ and $F^{j,\nu}_{m,k}$ are identified with $\coprod_{\boldsymbol{p}\in B(m+k)} F(m,k)_{\boldsymbol{p}}$ and $\coprod_{\boldsymbol{p}\in B(m+k)} F^{j,\nu}_{m,k,\boldsymbol{p}}$, respectively. Clearly we have

$$F(m,k)\subset F_{m,k}^{j,1}\subset F_{m,k}^{j,2}\subset F(m+k) \qquad (j=1,\cdots,k),$$

where $F'\subset F$ indicate that F' is a subbundle of a vector bundle F. Notice that (2.7) implies that

$$\text{(2.8)}\qquad \begin{aligned} &\operatorname{rank} F_{m,k}^{j,1}=\operatorname{rank} F(m,k)+1, \qquad \operatorname{rank} F_{m,k}^{j,2}=\operatorname{rank} F_{m,k}^{j,1}+1,\\ &\operatorname{rank} F(m+k)=\operatorname{rank} F(m,k)+2k \quad (j=1,\cdots,k)\end{aligned}$$

hold except for the two cases $g=0$, $m\leqq 1$ and $g=1$, $m=0$. Hence, for any point $\boldsymbol{p}^{\circ}=(p_1^{\circ},\cdots,p_{m+k}^{\circ})\in B(m+k)$, there exist an open product neighbourhood $U=U_1\times\cdots\times U_{m+k}$ of $\boldsymbol{p}^{\circ}$ in $B(m+k)$ (so U_j is an open neighbourhood of p_j° in M) and sections

$$\begin{aligned} &Q_{j,\nu}\in\mathcal{F}_{m,k}^{j,\nu}(U) \qquad (j=1,\cdots,k,\ \nu=1,2),\\ &R_l\in\mathcal{F}(m,k)(U) \qquad (l=1,\cdots,m':=2m+3g-3=\operatorname{rank} F(m,k)),\end{aligned}$$

such that the following properties hold for $\boldsymbol{p}=\boldsymbol{p}^{\circ}$.

$$\text{(2.9)}\qquad \begin{aligned} &Q_{j,1}(\boldsymbol{p})\in F_{m,k,\boldsymbol{p}}^{j,1}\backslash F(m,k)_{\boldsymbol{p}}, \qquad Q_{j,2}(\boldsymbol{p})\in F_{m,k,\boldsymbol{p}}^{j,2}\backslash F_{m,k,\boldsymbol{p}}^{j,1},\\ &R_1(\boldsymbol{p}),\cdots,R_{m'}(\boldsymbol{p}) \text{ is a basis of the vector space } F(m,k)_{\boldsymbol{p}}.\end{aligned}$$

Replacing U by a sufficiently small one, if necessary, we may assume that (2.9) holds for every $\boldsymbol{p}\in U$. Moreover multiplying $Q_{j,\nu}$ by a suitable holomorphic function in U, if necessary, we may assume that $Q_{j,\nu}(\boldsymbol{p})$ and $R_l(\boldsymbol{p})$ ($\boldsymbol{p}\in U$), regarded as meromorphic quadratic differentials on M, have the following local expressions

$$\text{(2.10)}\qquad \begin{aligned} &Q_{j,\nu}(\boldsymbol{p})=\left\{\frac{\delta_{ij}}{(x_j-t_j(\boldsymbol{p}))^{\nu}}+\sum_{\alpha=1-\nu}^{\infty}Q_{j,\nu}^{i,\alpha}(\boldsymbol{p})(x_i-t_i(\boldsymbol{p}))^{\alpha}\right\}(dx_i)^{\otimes 2},\\ &R_l(\boldsymbol{p})=\left\{\sum_{\alpha=0}^{\infty}R_l^{i,\alpha}(\boldsymbol{p})(x_i-t_i(\boldsymbol{p}))^{\alpha}\right\}(dx_i)^{\otimes 2}\end{aligned}$$

in U_{m+i} ($i,j=1,\cdots,k$, $\nu=1,2$, $l=1,\cdots,m'$), where x_i is a local coordinate of M in U_{m+i} and $t_i(\boldsymbol{p})=x_i(p_{m+i})$ for $\boldsymbol{p}=(p_1,\cdots,p_{m+k})\in U$. Note that t_i, $Q_{j,\nu}^{i,\alpha}$, $R_l^{i,\alpha}\in\mathcal{O}(U)$. It follows from (2.8)-(2.10) that $Q_{j,\nu}$, R_l ($j=1,\cdots,k$, $\nu=1,2$, $l=1,\cdots,m'$) form a frame over U of the vector bundle $F(m+k)$. Hence we have an isomorphism

$$\text{(2.11)}\qquad \begin{array}{ccc} F(m+k)|U & = & U\times \boldsymbol{C}^{2k+m'}\\ \Uparrow & & \Uparrow\\ Q & \longleftrightarrow & (\boldsymbol{p}\,;a,b,c)\end{array}$$

under the relation

$$Q=\sum_{j=1}^{k}\{a_jQ_{j,2}(\boldsymbol{p})+b_jQ_{j,1}(\boldsymbol{p})\}+\sum_{l=1}^{m'}c_lR_l(\boldsymbol{p}), \tag{2.12}$$

where $a=(a_1,\cdots,a_k)$, $b=(b_1,\cdots,b_k)$ and $c=(c_1,\cdots,c_{m'})$.

Now we shall obtain a useful description of the condition that a regular singular point become apparent. Consider the germ of differential equation at $x=0$,

$$-\frac{d^2f}{dx^2}+x^{-2}P(x)f=0, \qquad \text{with } P(x)=\sum_{\alpha=0}^{\infty}P_\alpha x^\alpha \tag{2.13}$$

where $P(x)$ is a convergent power series. The following lemma is an easy consequence of Frobenius' method.

LEMMA 2.7. *The origin $x=0$ is an apparent singular point of* (2.13) *of multiplicity N ($\geqq 1$) if and only if the following condition holds*:

$$P_0-\frac{1}{4}N(N+2)=0, \tag{2.14}$$

$$P_{N+1}+\sum_{\nu=1}^{N}P_{N+1-\nu}V_\nu=0, \tag{2.15}$$

where V_ν ($\nu=1,\cdots,N$) are defined recursively by

$$V_\nu=\frac{1}{\nu(N+1-\nu)}\sum_{j=0}^{\nu-1}P_{\nu-j}V_j \quad (\nu=1,\cdots,N), \qquad V_0=1.$$

By an induction argument, we can easily show the following lemma.

LEMMA 2.8. *V_ν ($1\leqq\nu\leqq N$), defined in Lemma 2.7, can be written as*

$$V_\nu=a_\nu(P_1)^\nu+f_\nu(P_1,\cdots,P_\nu) \quad (\nu=1,\cdots,N),$$

where $a_\nu=(N-\nu)!/\nu!N!\neq 0$ (with the convention $0!=1$) and $f_\nu\in\boldsymbol{Q}[P_1,\cdots,P_\nu]$, $\deg f_\nu\leqq\nu-1$.

Combining the above two lemmata, we obtain

PROPOSITION 2.9. *For any $N\in\boldsymbol{N}$, there exists a polynomial $H_N\in\boldsymbol{Q}[P_1,\cdots,P_{N+1}]$ with $\deg H_N\leqq N$ such that the origin $x=0$ is an apparent singular point of* (2.13) *of multiplicity N if and only if the following condition holds*:

$$P_0-\frac{1}{4}N(N+2)=0\,, \tag{2.16}$$

$$(P_1)^{N+1}+H_N(P_1,\cdots,P_{N+1})=0\,. \tag{2.17}$$

Given $N=(N_1,\cdots,N_k)\in \boldsymbol{N}^k$, let

$$F(m,k\,;N)=\left\{Q\in F(m+k)\,;\ \begin{array}{l}\text{the } (m+j)\text{-th singular point of } Q\\ \text{is an apparent singular point of}\\ \text{multiplicity } N_j\ (j=1,\cdots,k)\end{array}\right\}.$$

Note that $E(m,k\,;N)=F(m,k\,;N)\cap E(m+k)$. In the following lemma, we shall give a useful description of $F(m,k\,;N)|U$.

LEMMA 2.10. *There exist $G_j(\boldsymbol{p}\,;b,c)\in\mathcal{O}(U)[b_1,\cdots,b_k,c_1,\cdots,c_{m'}]$, polynomials in b, c with coefficients in $\mathcal{O}(U)$ $(j=1,\cdots,k)$ such that the following conditions hold:*

(i) $$\deg G_j(\boldsymbol{p}\,;\cdot)\leqq N_j \qquad (j=1,\cdots,k,\ \ \boldsymbol{p}\in U)\,,$$

(ii) *Under the identification $F(m+k)|U\cong U\times \boldsymbol{C}^{2k+m'}$ as in (2.11), $F(m,k\,;N)|U$ is the zero set $Z\subset U\times\boldsymbol{C}^{2k+m'}$ of the following $2k$ equations:*

$$a_j-\frac{1}{4}N_j(N_j+2)=0 \qquad (j=1,\cdots,k)\,, \tag{2.18}$$

$$(b_j)^{N_j+1}+G_j(\boldsymbol{p}\,;b,c)=0 \qquad (j=1,\cdots,k)\,. \tag{2.19}$$

In particular, $F(m,k\,;N)$ is an analytic subspace of $F(m+k)$.

PROOF. Consider $Q\in F(m+k)|U$ defined by (2.12). Put

$$(2.20)\qquad \begin{aligned} &x=x_j-t_j(\boldsymbol{p}),\quad N=N_j,\quad P_0=a_j,\quad P_1=b_j+\sum_{i=1}^{k}a_iQ_{i,2}^{j,-1}(\boldsymbol{p})\,,\\ &P_\alpha=\sum_{i=1}^{k}\{a_iQ_{i,2}^{j,\alpha-2}(\boldsymbol{p})+b_iQ_{i,1}^{j,\alpha-1}(\boldsymbol{p})\}+\sum_{l=1}^{m'}c_lR_l^{j,\alpha-2}(\boldsymbol{p}),\qquad \alpha\geqq 2\,,\end{aligned}$$

then Q becomes the equation (2.13), and Proposition 2.9 applies to Q. Thus Q has an apparent singular point of multiplicity N_j at p_{m+j} if and only if (2.16) and (2.17) hold under the assumption (2.20). Now (2.16) becomes (2.18). Moreover, substituting (2.18) and (2.20) into (2.17), we see that (2.17) is equivalent to (2.19). Hence the lemma is proved. ■

Let $V\subset\boldsymbol{C}^d$ be a domain with a coordinate z, and let $g_j(z\,;X)\in\mathcal{O}(V)[X_1,\cdots,X_k]$ $(j=1,\cdots,k)$ be polynomials in $X=(X_1,\cdots,X_k)$ with coefficients in $\mathcal{O}(V)$ such that $\deg g_j(z\,;\cdot)\leqq N_j$ $(z\in V)$ for a given $N=(N_1,\cdots,N_k)\in \boldsymbol{N}^k$. Consider the analytic set

$$Z=\{(z,x)\in V\times \boldsymbol{C}^k\,;\,(x_j)^{N_j+1}+g_j(z\,;x)=0\ \ (j=1,\cdots,k)\}\,.$$

Let $\pi:Z\to V$ be a natural map induced by the projection $\pi_1: V\times\boldsymbol{C}^k\to V$. The following is a key lemma of this section.

LEMMA 2.11. *$\pi:Z\to V$ is a finite map.*

PROOF. We provide $\boldsymbol{C}^k$ with the norm $|x|=\max\{|x_1|,\cdots,|x_k|\}$, and put $B(r)=\{x\in\boldsymbol{C}^k\,;\,|x|\leqq r\}$. Let K be any compact subset of V. Since $\deg g_j(z\,;\cdot)\leqq N_j$, there exists a constant r such that $|g_j(z\,;x)|\leqq r|x|^{N_j}$ for $z\in K$, $x\in\boldsymbol{C}^k$ and $|x|\geqq r$ $(j=1,\cdots,k)$. We assert $\pi^{-1}(K)=Z\cap\pi_1^{-1}(K)\subset K\times B(r)$. To see this, let $(z,x)\in\pi^{-1}(K)$. Then $|x|=|x_j|$ for some j, and $|x|^{N_j+1}=|x_j|^{N_j+1}=|g_j(z\,;x)|\leqq\max(r|x|^{N_j},r^{N_j+1})$. Hence $|x|\leqq r$, which proves the assertion. Since $\pi^{-1}(K)$ is a closed subset of the compact set $K\times B(r)$, $\pi^{-1}(K)$ is also compact. Hence π is a proper map. Moreover, $\pi^{-1}(z)$ $(z\in V)$ is a compact analytic subset of $z\times\boldsymbol{C}^k=\boldsymbol{C}^k$. So $\pi^{-1}(K)$ is necessarily finite. Hence the map π is finite. ■

LEMMA 2.12. *Under the assumption of Theorem 2.2, $E(m,k\,;N)_{\boldsymbol{p}}=E(m,k\,;N)\cap F(m+k)_{\boldsymbol{p}}$ is nonempty for each $\boldsymbol{p}\in B(m+k)$. In particular, $E(m,k\,;N)$ is nonempty.*

PROOF. We may assume $\boldsymbol{p}\in U$. By Lemma 2.10, $F(m,k\,;N)_{\boldsymbol{p}}=F(m,k\,;N)\cap F(m+k)_{\boldsymbol{p}}$ is identified with an affine algebraic subset Z of $\boldsymbol{C}^{k+m'}$ defined by

$$Z=\{(b,c)\in\boldsymbol{C}^k\times\boldsymbol{C}^{m'}\,;\,(b_j)^{N_j+1}+G_j(\boldsymbol{p}\,;b,c)=0\ \ (j=1,\cdots,k)\}\,.$$

Since Z is defined by k equations, we have $\operatorname{codim} Z\leqq k$ i.e. $\dim Z\geqq m'$. On the other hand, the assumption of Theorem 2.2 implies that $m'=2m+3g-3$ is positive. Hence Z is nonempty. Under the identification $F(m,k\,;N)_{\boldsymbol{p}}=Z$, Proposition 2.1 implies that $E(m,k\,;N)_{\boldsymbol{p}}=Z\cap\boldsymbol{C}^k\times(\boldsymbol{C}^{m'}\backslash H)$, where H is a finite union of proper linear subspaces of $\boldsymbol{C}^{m'}$. If $E(m,k\,;N)_{\boldsymbol{p}}$ were empty, then $Z\subset\boldsymbol{C}^k\times H$. Thus, if $\pi:Z\to\boldsymbol{C}^{m'}$ be a map induced by the projection $\boldsymbol{C}^k\times\boldsymbol{C}^{m'}\to\boldsymbol{C}^{m'}$, then $\pi(Z)\subset H$. By Lemma 2.11, π is a finite map, whence $\dim Z\leqq\dim H\leqq m'-1$. This is absurd, since $\dim Z\geqq m'$ was already established. Hence $E(m,k\,;N)_{\boldsymbol{p}}$ is nonempty. ■

Now it is easy to give a proof of Theorem 2.2.

PROOF OF THEOREM 2.2. We shall show that $E(m,k;N)$ is a $3(m+g-1)+k$ dimensional analytic subspace of the complex manifold $E(m+k)$. It is already known (Lemma 2.12) that $E(m,k;N)$ is nonempty. Recall that $E(m,k;N)=F(m,k;N)\cap E(m+k)$, $E(m+k)$ is an open submanifold of the complex manifold $F(m+k)$, and $F(m+k;N)$ is an analytic subspace of $F(m+k)$ (Lemma 2.10). Thus it suffices to show that $F(m,k;N)$ is of dimension $3(m+g-1)+k$ at each point $Q^\circ\in F(m,k;N)$. We may assume that $Q^\circ\in F(m+k)|U$ and $Q^\circ=(\boldsymbol{p}^\circ,a^\circ,b^\circ,c^\circ)$ under the isomorphism $F(m+k)|U\cong U\times \boldsymbol{C}^{2k+m'}$ stated in (2.11). In order to see $\dim F(m,k;N)\leqq 3(m+g-1)+k$ at Q°, it suffices to establish the following claim: *Q° is an isolated point of the zero set of the $3(m+g-1)+k$ holomorphic functions in $F(m,k;N)|U$ defined to be the restriction of $p_j-p_j^\circ$, $c_l-c_l^\circ\in\mathcal{O}(U\times\boldsymbol{C}^{2k+m'})$* ($j=1,\cdots,m+k$, $l=1,\cdots,m'=2m+3g-3$). Since $F(m,k;N)|U$ is defined by (2.18) and (2.19), the above claim is equivalent to saying that $b^\circ\in\boldsymbol{C}^k$ is an isolated point of the zero set of the functions $(b_j)^{N_j+1}+G_j(\boldsymbol{p}^\circ;b,c^\circ)$ ($j=1,\cdots,k$) in $\boldsymbol{C}^k$. Thus Lemma 2.11 implies that this claim is true. On the other hand, $\dim F(m,k;N)\geqq 3(m+g-1)+k$ is clear, since the ambient space $F(m+k)|U$ is a complex manifold of dimension $3(m+k+g-1)$ and $F(m,k;N)$ is defined to be the zero set of $2k$ holomorphic functions in $F(m+k)|U$. Hence the first half of the theorem is proved. The second half is already established in Lemma 2.12. ■

§3. Reducible and irreducible SL-operators.

3.1. Reducible SL-operators.

DEFINITION 3.1. An SL-operator $L=(L_j)$ with $L_j=-D_j^2+Q_j$ for a coordinate covering $\mathcal{U}=\{(U_j,x_j)\}$ is said to be *reducible* if there exist first order differential operators $M_j=-D_j+P_j$ with $P_j\in\mathcal{M}(U_j)$ such that $L_j=M_j^*M_j$. Here M_j^* is the formal adjoint of M_j, i.e., $M_j^*=D_j+P_j$. An SL-operator is said to be *reducible* if it has a reducible representative for some coordinate covering.

We put

$$E(l)_{\mathrm{red}}:=\{Q\in E(l)\,;\,Q\text{ is reducible}\},$$

$$E(l)_{\mathrm{irr}}:=\{Q\in E(l)\,;\,Q\text{ is irreducible}\}.$$

THEOREM 3.2. *Let $l\geqq\max\{1,2-g\}$.*

(i) *If $g=0$ and $l=2$, then $E(l)_{\mathrm{red}}=E(l)$.*

(ii) *Otherwise $E(l)_{\mathrm{red}}$ is a proper analytic subspace of $E(l)$ whose* *dimension and codimension are given by*

$$\dim E(l)_{\text{red}}=2l+g-1, \qquad \operatorname{codim} E(l)_{\text{red}}=l+2g-2 .$$

Let $l=m+k$, $m\geqq\max\{1,2-g\}$, $k\geqq 0$. For any analytic subspace D of $E(m+k)$, we put

$$D_{\text{red}}:=D\cap E(m+k)_{\text{red}}, \qquad D_{\text{irr}}:=D\cap E(m+k)_{\text{irr}} .$$

Theorem 3.2 implies that D_{red} is an analytic subspace of D.

THEOREM 3.3. *Let* $m\geqq\max\{1,2-g\}$, $k\geqq 0$ *and* $N\in\boldsymbol{N}^k$. $E(m,k;N)_{\text{red}}$ *is an analytic subspace of* $E(m,k;N)$ *such that*

$$\dim E(m,k;N)_{\text{red}}=2m+k+g-1, \quad \operatorname{codim} E(m,k;N)_{\text{red}}=m+2g-2 .$$

In particular, except for the case $g=0$, $m=2$, $E(m,k;N)_{\text{red}}$ *is a proper subspace of* $E(m,k;N)$.

THEOREM 3.4. *Let* m, k *and* N *be as in Theorem* 3.3 *and let* $\boldsymbol{p}\in B(m)$. $E(\boldsymbol{p},k;N)_{\text{red}}$ *is an analytic subspace of* $E(\boldsymbol{p},k;N)$ *such that*

$$\dim E(\boldsymbol{p},k;N)_{\text{red}}=m+k+g-1, \quad \operatorname{codim} E(\boldsymbol{p},k;N)_{\text{red}}=m+2g-2 .$$

In particular, except for the case $g=0$, $m=2$, $E(\boldsymbol{p},k;N)_{\text{red}}$ *is a proper subspace of* $E(\boldsymbol{p},k;N)$.

DEFINITION 3.5. Let $\theta=(\theta_1,\cdots,\theta_m)\in(\boldsymbol{C}_+)^m$ and $N=(N_1,\cdots,N_k)\in\boldsymbol{N}^k$ be given. The pair (θ,N) is said to be *generic* if

$$\sum_{i=1}^{m}\delta_i\theta_i+\sum_{j=1}^{k}\varepsilon_j(N_j+1)\neq 2g-2-m-k$$

for every δ_i, $\varepsilon_j\in\{\pm 1\}$ $(i=1,\cdots,m,\ j=1,\cdots,k)$. Otherwise (θ,N) is said to be *nongeneric*.

THEOREM 3.6. *Let* m *and* k *be as in Theorem* 3.3 *and let* $(\theta,N)\in(\boldsymbol{C}_+)^m\times\boldsymbol{N}^k$. *If* (θ,N) *is generic, then* $E(m,k;\theta,N)_{\text{red}}$ *is empty.*

This theorem implies that all SL-operators in $E(m,k;\theta,N)$ are irreducible for "almost all" (θ,N).

THEOREM 3.7. *Let* m, k *and* (θ,N) *be as in Theorem* 3.6. *If* (θ,N) *is nongeneric, then* $E(m,k;\theta,N)_{\text{red}}$ *is an analytic subspace of* $E(m,k;\theta,N)$ *such that*

$$\operatorname{codim} E(m,k;\theta,N)_{\text{red}}\geqq m+2g-3 .$$

THEOREM 3.8. *Let m, k and (θ, N) be as in Theorem 3.7 and let $\boldsymbol{p}\in B(m)$. If (θ, N) is nongeneric, then $E(\boldsymbol{p}, k; \theta, N)_{\mathrm{red}}$ is an analytic subspace of $E(\boldsymbol{p}, k; \theta, N)$ such that*

$$\operatorname{codim} E(\boldsymbol{p}, k; \theta, N)_{\mathrm{red}} \geqq m+2g-3 .$$

3.2. The auxiliary space $V(l)$.

If we put

$$\tau_{jk}=\frac{1}{2}D_j \log \kappa_{jk} \qquad \text{in } U_j \cap U_k ,$$

then we have $(\tau_{jk})\in Z^1(\mathcal{U}; \mathcal{O}(\kappa))$. Let us introduce an affine space $\mathcal{P}$ defined by

$$\mathcal{P}=\{P=(P_j)\in C^0(\mathcal{U}; \mathcal{M}(\kappa)) ;\ \delta(P_j)=(\tau_{jk})\} .$$

Under the notation of Definition 3.1, an easy calculation shows

LEMMA 3.9. *Let $L=(L_j)$ with $L_j=-D_j^2+Q_j$ be an SL-operator for a coordinate covering $\mathcal{U}=\{(U_j, x_j)\}$. L is reducible if and only if there exist $P_j\in \mathcal{M}(U_j)$ such that*

(i) $P=(P_j)\in \mathcal{P}$,

(ii) $dP_j/dx_j+P_j^2=Q_j$,

(iii) *P_j has at most simple poles in U_j.*

REMARK 3.10. After taking a refinement of the covering, if necessary, we may assume that each P_j has at most one simple pole in U_j and no pole in $U_j \cap U_k$ for every k $(\neq j)$. The Serre duality theorem yields an isomorphism $H^1(M; \mathcal{O}(\kappa))\cong H^0(M; \mathcal{O})^*=\boldsymbol{C}$. This isomorphism identifies $\tau\in H^1(M; \mathcal{O}(\kappa))$ defined by the 1-cocycle (τ_{jk}) with the complex number $\sum_j \operatorname{Res}(P_j dx_j)$, where $\operatorname{Res}(P_j dx_j)$ denotes the residue of the 1-form $P_j dx_j$ in U_j. On the other hand, an explicit calculation of the first Chern class $c_1(\kappa)=2g-2$ shows that the above isomorphism identifies τ with $(1/2)c_1(\kappa)$. Hence we have

$$\sum_j \operatorname{Res}(P_j dx_j)=g-1 . \tag{3.1}$$

In view of Lemma 3.9, (i), (iii), we shall investigate the structure of the space

$$V(l):=\{P\in \mathcal{P} ;\ P \text{ has exactly } l \textit{ ordered} \text{ simple poles}\} ,$$

for $l\geqq 1$. Let

$$A(l):=\{\alpha=(\alpha_1,\cdots,\alpha_l)\in \boldsymbol{C}^l\ ;\ \alpha_1+\cdots+\alpha_l=g-1, \alpha_1\cdots\alpha_l\neq 0\}.$$

Then we have a natural projection

$$\pi: V(l)\longrightarrow A(l)\times B(l), \qquad P\longmapsto(\alpha, \boldsymbol{p}),$$

where $\boldsymbol{p}=(p_1,\cdots,p_l)\in B(l)$ is the l-tuple of ordered simple poles of P, and $\alpha=(\alpha_1,\cdots,\alpha_l)\in A(l)$ is such that α_j is the residue of P at p_j.

THEOREM 3.11. *If $g\geqq 1$, then $V(l)$ carries a natural structure of complex manifold such that $\pi: V(l)\rightarrow A(l)\times B(l)$ is a holomorphic affine bundle of rank g. If $g=0$, then $V(l)$ carries a natural structure of complex manifold such that $\pi: V(l)\rightarrow A(l)\times B(l)$ is a biholomorphism. In particular,* $\dim V(l)=2l+g-1$.

PROOF. For any $\boldsymbol{p}^\circ=(p_1^\circ,\cdots,p_l^\circ)\in B(l)$, let $\mathcal{U}=\{(U_j, x_j)\}$ be a coordinate covering of M such that $p_j^\circ\in U_j$ and $U_j\cap U_k\neq\emptyset$ for $j, k=1,\cdots,l$. Then $U=U_1\times\cdots\times U_l$ is a neighbourhood of $\boldsymbol{p}^\circ$ in $B(l)$. For $\boldsymbol{p}=(p_1,\cdots,p_l)\in U$, we put $t_j(\boldsymbol{p})=x_j(p_j)$. Let $(\xi_j(\alpha,\boldsymbol{p}))\in C^0(\mathcal{U}\ ;\ \mathcal{M}(\kappa))$ with $(\alpha,\boldsymbol{p})\in A(l)\times U$ be a 0-cochain defined by

$$\xi_j(\alpha,\boldsymbol{p})=\begin{cases}\alpha_j d\log(x_j-t_j(\boldsymbol{p})) & \text{in } U_j\ \ (j=1,\cdots,l),\\ 0 & \text{in } U_j\ \ (j\neq 1,\cdots,l).\end{cases}$$

Then the 1-cocycle $\delta(\xi_j(\alpha,\boldsymbol{p}))\in Z^1(\mathcal{U}\ ;\ \mathcal{O}(\kappa))$ determines an element $\tau'\in H^1(M,\mathcal{O}(\kappa))$. By the definition of $A(l)$, τ' corresponds to the complex number $g-1$ under the isomorphism $H^1(M\ ;\ \mathcal{O}(\kappa))\cong\boldsymbol{C}$ mentioned in Remark 3.10. Hence $\tau=\tau'$ in $H^1(M\ ;\ \mathcal{O}(\kappa))$. Consider the 1-cocycle $\sigma(\alpha,\boldsymbol{p})=(\sigma_{jk}(\alpha,\boldsymbol{p}))\in Z^1(\mathcal{U}\ ;\ \mathcal{O}(\kappa))$ defined by

$$\sigma_{jk}(\alpha,\boldsymbol{p})=\tau_{jk}dx_j+\xi_j(\alpha,\boldsymbol{p})-\xi_k(\alpha,\boldsymbol{p}) \qquad \text{in } U_j\cap U_k.$$

Since $\sigma(\alpha,\boldsymbol{p})=0$ in $H^1(M\ ;\ \mathcal{O}(\kappa))$ and $\sigma_{jk}(\alpha,\boldsymbol{p})$ is holomorphic in $(\alpha,\boldsymbol{p})\in A(l)\times U$, a theorem of Kodaira-Spencer implies that (possibly after the coordinate covering $\mathcal{U}$ is replaced by a more refined one, which is still denoted by $\mathcal{U}$) for any $\alpha^\circ\in A(l)$ there exists $(\eta_j(\alpha,\boldsymbol{p}))\in C^0(\mathcal{U}\ ;\ \mathcal{O}(\kappa))$ such that

(i) $\sigma_{jk}(\alpha,\boldsymbol{p})=\eta_k(\alpha,\boldsymbol{p})-\eta_j(\alpha,\boldsymbol{p})$ in $U_j\cap U_k$,

(ii) $\eta_j(\alpha,\boldsymbol{p})$ is holomorphic in $(\alpha,\boldsymbol{p})\in\Delta\times U$, where Δ is a sufficiently small neighbourhood of α° in $A(l)$.

Let $P(\alpha,\boldsymbol{p})=(P_j(\alpha,\boldsymbol{p}))\in C^0(\mathcal{U}\ ;\ \mathcal{M}(\kappa))$ with $(\alpha,\boldsymbol{p})\in D=\Delta\times U$ be defined by

$$P_j(\alpha,\boldsymbol{p})=\xi_j(\alpha,\boldsymbol{p})+\eta_j(\alpha,\boldsymbol{p}) \qquad \text{in } U_j.$$

Emphasizing the dependence on D, we put $P(\alpha,\boldsymbol{p})=P(\alpha,\boldsymbol{p}\,;D)$. Then by the definition of $P(\alpha,\boldsymbol{p}\,;D)$, it is clear that

$$V(l)_{(\alpha,\boldsymbol{p})}=\pi^{-1}(\alpha,\boldsymbol{p})=P(\alpha,\boldsymbol{p}\,;D)+H^0(M\,;\mathcal{O}(\kappa))\,,\qquad (\alpha,\boldsymbol{p})\in D\,. \tag{3.2}$$

In particular, since $(\alpha^\circ,\boldsymbol{p}^\circ)\in D$ is an arbitrary point of $A(l)\times B(l)$, π is surjective. Moreover, if $g=0$, then $H^0(M\,;\mathcal{O}(\kappa))=0$, whence π is bijective. Let $\varphi_1: M\times A(l)\times B(l)\to M$ and $\varphi_2: M\times A(l)\times B(l)\to A(l)\times B(l)$ be the canonical projections. A theorem of Kodaira-Spencer [KS, I, Theorem 2.2] [GR, Chap. 10, § 5, Theorem 5] implies that $\varphi_{2*}\mathcal{O}_{M\times A(l)\times B(l)}(\varphi_1^*\kappa)$ is a locally free analytic sheaf over $A(l)\times B(l)$. Let $W(l)$ be the associated holomorphic vector bundle, then each fiber $W(l)_{(\alpha,\boldsymbol{p})}$ with $(\alpha,\boldsymbol{p})\in A(l)\times B(l)$ is identified with the g-dimensional vector space $H^0(M\,;\mathcal{O}(\kappa))$. Hence (3.2) implies that

$$V(l)_{(\alpha,\boldsymbol{p})}=P(\alpha,\boldsymbol{p}\,;D)+W(l)_{(\alpha,\boldsymbol{p})}\,,\qquad (\alpha,\boldsymbol{p})\in D\,.$$

If D' is another open set in $A(l)\times B(l)$ similar to D, then $P(\cdot\,;D)-P(\cdot\,;D')$ becomes a holomorphic section of $W(l)$ over $D\cap D'$. Thus the holomorphic vector bundle structure of $W(l)$ induces a holomorphic affine bundle structure of $V(l)$ over $A(l)\times B(l)$. Hence the theorem is proved. ■

3.3. Proof of Theorem 3.2.

We consider the holomorphic map

$$\begin{array}{rccc}\Phi: & V(l) & \longrightarrow & E(l)\\ & \Psi & & \Psi\\ & P=(P_j) & \longmapsto & Q=(D_jP_j+P_j^2)\,.\end{array}$$

REMARK 3.12. Lemma 3.9 implies

$$E(l)_{\mathrm{red}}=\Phi(V(l))\,.$$

It is easy to see that Φ is a closed map. Moreover, we have

PROPOSITION 3.13. *Each fiber of Φ contains at most 2^l points. In particular Φ is a finite holomorphic map.*

PROOF. Given any $Q\in E(l)$, let $\boldsymbol{p}=(p_1,\cdots,p_l)\in B(l)$ be the l-tuple of ordered singular points of Q and let $\theta_j\in\boldsymbol{C}_+$ $(j=1,\cdots,l)$ be the difference of the characteristic exponents of Q at p_j. For $\sigma=(\sigma_1,\cdots,\sigma_l)\in\{\pm1\}^l$, we put

$$\alpha(\sigma)=\left(\frac{1}{2}(1+\sigma_1\theta_1),\cdots,\frac{1}{2}(1+\sigma_l\theta_l)\right). \tag{3.3}$$

By the definition of Φ, it is easy to see that

$$\Phi^{-1}(Q)\subset\bigcup_{\sigma\in\{\pm 1\}^l}V(l)_{(\alpha(\sigma),\boldsymbol{p})}. \tag{3.4}$$

Thus, in order to show that $\Phi^{-1}(Q)$ contains at most 2^l points, it suffices to establish the following lemma. ∎

LEMMA 3.14. *The map Φ is injective on $V(l)_{(\alpha,\boldsymbol{p})}$ for each $(\alpha,\boldsymbol{p})\in A(l)\times B(l)$.*

PROOF. In case $g=0$, each $V(l)_{(\alpha,\boldsymbol{p})}$ consists of only one point, (Theorem 3.11). Hence the lemma is trivial. In case $g\geqq 1$, we shall establish the lemma by contradiction. Suppose that there exist $P=(P_j)$, $P'=(P_j')\in V(l)_{(\alpha,\boldsymbol{p})}$ such that $P\neq P'$ and $\Phi(P)=\Phi(P')$. By (3.2), $R:=P-P'$ is a (nonzero) holomorphic 1-form. Condition $\Phi(P)=\Phi(P')$ is then rewritten as

$$P_j=\frac{1}{2}\left(R_j-\frac{d}{dx_j}\log R_j\right) \quad \text{in } U_j. \tag{3.5}$$

This equality implies that

(3.6) (the sum of all residues of P)

$=-\frac{1}{2}$(the number of zeros of R counted with multiplicities).

By (3.1), the left-hand side of (3.6) is equal to $g-1\geqq 0$. However since $P\in V(l)$ with $l\geqq 1$, P has at least one pole. Hence, by (3.5), R has at least one zero. So the right-hand side of (3.6) is negative. This contradiction establishes the lemma. ∎

PROOF OF THEOREM 3.2. Since $V(l)$ is a complex manifold of dimension $2l+g-1$ (Theorem 3.11), Proposition 3.13 and the finite mapping theorem imply that $\Phi(V(l))$ is an analytic subspace of $E(l)$ such that $\dim\Phi(V(l))=\dim V(l)=2l+g-1$. By Remark 3.12 we have $E(l)_{\mathrm{red}}=\Phi(V(l))$. Hence $E(l)_{\mathrm{red}}$ is a $(2l+g-1)$-dimensional analytic subspace of $E(l)$. Let $l\geqq\max\{1,2-g\}$ and $(g,l)\neq(0,2)$. Then $\dim E(l)=3(l+g-1)$ (Proposition 2.1), whence $\operatorname{codim}E(l)_{\mathrm{red}}=l+2g-2$. This proves the second assertion (ii) of Theorem 3.2. Proof of the first assertion (i) is easy. ∎

3.4. Proof of Theorem 3.3—Theorem 3.8.

PROOF OF THEOREM 3.3. Given $N=(N_1, \cdots, N_k)\in \boldsymbol{N}^k$ and $\sigma=(\sigma_1, \cdots, \sigma_k)\in\{\pm 1\}^k$, let

$$A(m, k\,;N, \sigma):=\left\{\alpha=(\alpha_1, \cdots, \alpha_{m+k})\in A(m+k)\,;\ \begin{array}{c}\alpha_{m+j}=\beta_j(N, \sigma)\\ j=1, \cdots, k\end{array}\right\},$$

$$A(m, k\,;N):=\bigcup_{\sigma\in\{\pm 1\}^k} A(m, k\,;N, \sigma),$$

where

$$\beta_j(N, \sigma):=\frac{1}{2}\{1+\sigma_j(N_j+1)\}, \qquad j=1, \cdots, k. \tag{3.7}$$

In order to prove the theorem, we need the following lemma.

LEMMA 3.15. *Let $\Phi: V(m+k)\to E(m+k)$ be the map introduced in the first part of this subsection with l replaced by $m+k$. Then we have*

(i) $E(m, k\,;N)_{\mathrm{red}}\subset\Phi(V(m+k)|A(m, k\,;N)\times B(m+k))$,

(ii) $E(m, k\,;N)_{\mathrm{red}}\supset\Phi(V(m+k)|A(m, k\,;N, \mathbf{-1}_k)\times B(m+k))$,

where $\mathbf{-1}_k=(-1, \cdots, -1)\in\{\pm 1\}^k$.

Accepting this lemma for the moment, we continue the proof of the theorem. Since $C:=A(m,k\,;N)\times B(m+k)$ and $C':=A(m,k\,;N, \mathbf{-1})\times B(m+k)$, are $(2m+k-1)$-dimensional submanifolds of $A(m+k)\times B(m+k)$, $V(m+k)|C$ and $V(m+k)|C'$ are $(2m+k+g-1)$-dimensional submanifolds of $V(m+k)$ (cf. Theorem 3.11). In particular, $V(m+k)|C$ and $V(m+k)|C'$ are closed in $V(m+k)$. Since $\Phi: V(m+k)\to E(m+k)$ is a finite map (Proposition 3.13), we conclude that $\Phi(V(m+k)|C)$ and $\Phi(V(m+k)|C')$ are analytic subspaces of $E(m+k)$, whose dimensions are equal to those of $V(m+k)|C$ and $V(m+k)|C'$, respectively, i.e.,

$$\dim\Phi(V(m+k)|C)=\dim\Phi(V(m+k)|C')=2m+k+g-1.$$

Hence Theorem 3.3 follows from Lemma 3.15. ■

PROOF OF LEMMA 3.15. By the definition of $A(m, k\,;N)$, it is easy to see that

$$\Phi(V(m,k)|A(m,k;N)\times B(m+k))$$
$$=\left\{Q\in E(m+k)_{\mathrm{red}}\,;\ \begin{array}{l}\text{characteristic exponents of } Q \text{ at the } (m+j)\text{-th}\\ \text{singular point are } \dfrac{1}{2}\pm\dfrac{1}{2}(N_j+1),\ (j=1,\cdots,k)\end{array}\right\}.$$

This implies that the first assertion (i) holds. We proceed to the second assertion (ii). Let L be the SL-operator corresponding to an arbitrary element Q of $\Phi(V(m+k)|A(m,k;N,\mathbf{-1})\times B(m+k))$, (cf. (1.9)). Around its $(m+j)$-th singular point p_{m+j}, L has a local expression $L_j=M_j^*M_j$ with $M_j=-D_j+P_j$, x_j being a local coordinate of M around p_{m+j} such that $x_j(p_{m+j})=0$ (cf. Definition 3.1). Since P_j has a simple pole at p_{m+j} with residue $\mu:=1/2-(1/2)(N_j+1)$, the equation $M_jv=0$ has a local solution of the form $v=x_j^{\mu}f_2(x_j)$, where $f_2(x)\in\boldsymbol{C}\{x\}$, i.e., a convergent power series of x. Notice that $\lambda:=1/2+(1/2)(N_j+1)$ and μ are the characteristic exponents of L at p_{m+j} such that $\lambda-\mu>0$ and that v is a solution of the differential equation $Lf=0$ corresponding to the exponent μ. Since $\lambda-\mu>0$, Frobenius' method implies that $Lf=0$ has a solution of the form $u=x_j^{\lambda}f_1(x_j)$ with $f_1(x)\in\boldsymbol{C}\{x\}$. Since u and v contain no logarithmic term, L has an apparent singular point at p_{m+j} of multiplicity N_j. Hence $Q\in E(m,k;N)$. Combining the fact $Q\in E(m+k)_{\mathrm{red}}$, we have $Q\in E(m,k;N)_{\mathrm{red}}$. This establishes the lemma. ■

In a similar manner, we can prove Theorem 3.4. So we omit its proof.

PROOF OF THEOREM 3.6. Suppose $E(m,k;\theta,N)_{\mathrm{red}}$ is nonempty and let Q be an element of $E(m,k;\theta,N)_{\mathrm{red}}$. Let $\boldsymbol{p}\in B(m+k)$ be the ordered regular singular points of Q. For $\sigma=(\sigma_1,\cdots,\sigma_{m+k})\in\{\pm1\}^{m+k}$, we define $\alpha(\sigma):=(\alpha_1(\sigma),\cdots,\alpha_{m+k}(\sigma))\in A(m+k)$ by

$$\alpha_j(\sigma):=\begin{cases}\dfrac{1}{2}\{1+\sigma_j\theta_j\} & (j=1,\cdots,m),\\[2mm] \dfrac{1}{2}\{1+\sigma_j(N_{j-m}+1)\} & (j=m+1,\cdots,m+k),\end{cases}\tag{3.8}$$

(cf. (3.3) and (3.7)). In a similar manner as in (3.4), we have

$$\Phi^{-1}(Q)\in\bigcup_{\sigma\in\{\pm1\}^{m+k}}V(m+k)_{(\alpha(\sigma),\boldsymbol{p})}.$$

Hence there exists a $\sigma\in\{\pm1\}^{m+k}$ such that $\Phi^{-1}(Q)\in V(m+k)_{(\alpha(\sigma),\boldsymbol{p})}$. By the definition of $A(m+k)$, we have

$$\sum_{j=1}^{m+k} \alpha_j(\sigma) = g-1$$

for this σ. This contradicts the assumption that (θ, N) is generic, (cf. Definition 3.5). ■

PROOF OF THEOREM 3.7. Let $\alpha(\sigma)$, $\sigma \in \{\pm 1\}^{m+k}$, be the elements of $A(m+k)$ defined by (3.8). Since $C'' := \{\alpha(\sigma) ; \sigma \in \{\pm 1\}^{m+k}\} \times B(m+k)$ is an $(m+k)$-dimensional complex submanifold of $A(m+k) \times B(m+k)$, $V(m+k)|C''$ is an $(m+k+g)$-dimensional complex submanifold of $V(m+k)$. As in the proof of Lemma 3.15, we can easily show that

$$E(m, k ; \theta, N)_{\mathrm{red}} \subset \Phi(V(m+k)|C'') . \tag{3.9}$$

Since Φ is a finite holomorphic map, $\Phi(V(m+k)|C'')$ is an $(m+k+g)$-dimensional analytic subspace of $E(m, k ; \theta, N)$. Since $E(m, k ; \theta, N)$ is of dimension $2m+k+3g-3$ (Theorem 2.4 (i)), we have $\operatorname{codim} \Phi(V(m+k)|C'') = m+2g-3$. Hence (3.9) implies $\operatorname{codim} E(m, k ; \theta, N) \geqq m+2g-3$. ■

In a similar manner, we can prove Theorem 3.8. So we omit its proof.

§ 4. Gauge equivalence for SL-operators.

4.1. Gauge equivalence in $E(p ; k ; N)$.

As in Section 1, let ξ be a holomorphic line bundle over M with $c_1(\xi) = 1-g$ and let $(\xi_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ be a representative 1-cocycle on a projective coordinate covering $\mathcal{U} = \{(U_j, x_j)\}$ of M. We set $D_j = d/dx_j$ as before. If necessary, the covering $\mathcal{U}$ will be replaced by a finer one without comment. If $f=(f_j)$ is a local section of ξ, then $\boldsymbol{f} = ({}^t(f_j, D_j f_j))$ is a local section of the vector bundle Φ defined by the 1-cocycle

$$\Phi_{jk} = \xi_{jk} \begin{pmatrix} 1 & 0 \\ \eta_{jk} & \kappa_{jk} \end{pmatrix} \quad \text{in } U_j \cap U_k , \tag{4.1}$$

where $\eta_{jk} = D_j \log \xi_{jk}$. Since $c_1(\xi) = 1-g$ we have

$$\eta_{jk} = -\frac{1}{2} D_j \log \kappa_{jk} .$$

To each SL-operator $Q=(Q_j)$ on the line bundle ξ one can associate a meromorphic connection $\nabla(Q) : \mathcal{M}(\Phi) \to \mathcal{M}(\Phi \otimes \kappa)$ defined by

$$\nabla(Q)=d-\begin{pmatrix}0 & 1\\ Q_j & 0\end{pmatrix}dx_j \tag{4.2}$$

with respect to the local trivialization $\mathcal{M}(\Phi)|U_j\cong(\mathcal{M}|U_j)^2$. If $f=(f_j)$ is a local solution of the SL-equation $Lf=0$ associated with Q, then $\boldsymbol{f}=({}^t(f_j, D_jf_j))$ is a local $\nabla(Q)$-horizontal section of Φ.

Now we consider the *gauge equivalence* for connections on Φ. Here Φ is a holomorphic vector bundle over M not necessarily defined by (4.1). In the holomorphic category, things are standard; two holomorphic connections ∇ and ∇' on Φ are said to be *gauge-equivalent* if there exists a holomorphic section G of the bundle End Φ having the holomorphic inverse G^{-1} such that $\nabla'=G\nabla G^{-1}$. It is well known that ∇ *and* ∇' *are gauge-equivalent if and only if their monodromy representations are equivalent.* We call this property (GM). Here we recall that, for a group G and a vector space V, two linear representations $\rho, \rho'\in\mathrm{Hom}(G; GL(V))$ of G in V are said to be equivalent if there exists a $P\in GL(V)$ such that $\rho'(g)=P\rho(g)P^{-1}$ for every $g\in G$. For meromorphic connections, it is difficult to give a natural definition of the "gauge equivalence" having the property (GM). Let ∇ and ∇' be meromorphic connections on Φ with ordered regular singular points $\boldsymbol{p}=(p_1, \cdots, p_l)$ and $\boldsymbol{p}'=(p_1', \cdots, p_l')\in B(l)$, respectively. If anything, we can define a gauge equivalence in a similar manner as for the holomorphic connections by saying that ∇ and ∇' are gauge-equivalent if there exists a meromorphic section G of End Φ having the meromorphic inverse G^{-1} such that $\nabla'=G\nabla G^{-1}$. We call it the *meromorphic gauge equivalence.* In case $\boldsymbol{p}=\boldsymbol{p}'$, we can easily see that the meromorphic gauge equivalence has the property (GM). In case $\boldsymbol{p}\neq\boldsymbol{p}'$, things are more difficult. For $\boldsymbol{p}=(p_1, \cdots, p_l)\in B(l)$, we put $|\boldsymbol{p}|:=\{p_1, \cdots, p_l\}$ and $\mathrm{Rep}(|\boldsymbol{p}|):=\mathrm{Hom}(\pi_1(M\backslash|\boldsymbol{p}|); GL(2, \boldsymbol{C}))$. Then the monodromy representations of ∇ and ∇' are elements of $\mathrm{Rep}(|\boldsymbol{p}|)$ and $\mathrm{Rep}(|\boldsymbol{p}'|)$, respectively. There occur two problems: (i) In order for (GM) to make sense, we must make clear what we mean by saying that two elements $\rho\in\mathrm{Rep}(|\boldsymbol{p}|)$ and $\rho'\in\mathrm{Rep}(|\boldsymbol{p}'|)$ are equivalent. To do this, we must give a canonical identification of the fundamental group $\pi_1(M\backslash|\boldsymbol{p}|)$ with $\pi_1(M\backslash|\boldsymbol{p}'|)$. (ii) The meromorphic gauge equivalence of two meromorphic connections always implies the equivalence of their monodromy representations, but the converse is in general not true. Thus, in order to define a precise gauge equivalence having the property (GM), it might be neccessary to admit "multi-valued" meromorphic gauge transformations.

In the present paper we abandon to give a definition of the gauge equivalence in a general situation. We shall preferably define the gauge equivalence, roughly speaking only for connections $\nabla(Q)$ with $Q\in E(\boldsymbol{p}, k; N)$ for a fixed $\boldsymbol{p}\in B(m)$. We must make things more precise in connection with the problem (i). We proceed as follows: Let $\phi: \tilde{B}(m+k)\rightarrow B(m+k)$ be the universal covering of $B(m+k)$. Recall that we have the projection $\pi: E(m+k)\rightarrow B(m+k)$, (see §§ 2.3). Let $\tilde{E}(m+k)$ be the fiber product of $\tilde{B}(m+k)$ and $E(m+k)$ over $B(m+k)$ with respect to ϕ and π. We obtain the following diagram:

$$
(4.3)\qquad \begin{array}{ccc}
\tilde{E}(m+k) & \xrightarrow{\ \phi\ } & E(m+k) \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
\tilde{B}(m+k) & \xrightarrow[\ \phi\ ]{} & B(m+k)\,.
\end{array}
$$

Moreover, we put

$$\tilde{E}(m, k; N)=\phi^{-1}(E(m, k; N)), \quad \tilde{E}(m, k; \theta, N)=\phi^{-1}(E(m, k; \theta, N)),$$

$$\tilde{E}(\tilde{\boldsymbol{p}}, k; N)=\{\tilde{Q}\in\tilde{E}(m, k; N); \tilde{\pi}(\tilde{Q})=\tilde{\boldsymbol{p}}\} \qquad \text{and}$$

$$\tilde{E}(\tilde{\boldsymbol{p}}, k; \theta, N)=\{\tilde{Q}\in\tilde{E}(m, k; \theta, N); \tilde{\pi}(\tilde{Q})=\tilde{\boldsymbol{p}}\}$$

for $\tilde{\boldsymbol{p}}\in\tilde{B}(m)$, (cf. § 2.3). We shall define the gauge equivalence for connections $\nabla(\tilde{Q})$ with $\tilde{Q}\in\tilde{E}(\tilde{\boldsymbol{p}}, k; N)$ for any given $\tilde{\boldsymbol{p}}\in\tilde{B}(m)$.

Hereafter, we shall make the following abuse of notation: Even when we should use the notation $\tilde{\pi}$, $\tilde{\boldsymbol{p}}$, $\tilde{Q}$ etc., we shall use π, $\boldsymbol{p}$, Q etc. in place of them to simplify the notation.

First of all, remark that there exists a holomorphic map

$$
\begin{array}{ccc}
\tilde{B}(m+k)\times\tilde{B}(m+k) & \longrightarrow & \mathrm{Pic}\,(M)=\{\sigma\in H^1(M; \mathcal{O}^*); c_1(\sigma)=0\} \\
\Cup & & \Cup \\
(\boldsymbol{q}, \boldsymbol{r}) & \longmapsto & \sigma(\boldsymbol{q}, \boldsymbol{r})
\end{array}
$$

such that the following conditions hold:

(i) $$\sigma(\boldsymbol{q}, \boldsymbol{r})^{\otimes 2}=[q_1+\cdots+q_{m+k}-(r_1+\cdots+r_{m+k})]$$

if $\phi(\boldsymbol{q})=(q_1, \cdots, q_{m+k})$, $\phi(\boldsymbol{r})=(r_1, \cdots, r_{m+k})\in B(m+k)$.

(ii) $$\sigma(\boldsymbol{q}, \boldsymbol{q})=\text{the trivial line bundle.}$$

Indeed, it is well-known that, for a holomorphic family of line bundles $\{\tau(t); t\in T\}$ with an even Chern class, there exist 2^{2g} solutions of the equation $\sigma^{\otimes 2}=\tau(t)$ for each $t\in T$ depending holomorphically on t but possibly multi-valued. In our situation, since $T=\tilde{B}(m+k)\times\tilde{B}(m+k)$ is simply connected, the conditions (i) and (ii) determine the single-valued holomorphic function $\sigma(\boldsymbol{q}, \boldsymbol{r})$ uniquely.

Next we define a meromorphic connection $\nabla(\boldsymbol{q}, \boldsymbol{r})$ on $\sigma(\boldsymbol{q}, \boldsymbol{r})$ for each $(\boldsymbol{q}, \boldsymbol{r}) \in \tilde{B}(m+k) \times \tilde{B}(m+k)$ as follows: We put $\phi(\boldsymbol{q}) = (q_1, \cdots, q_{m+k})$ and $\phi(\boldsymbol{r}) = (r_1, \cdots, r_{m+k})$. Let $s(\boldsymbol{q}, \boldsymbol{r})$ be a meromorphic section of the line bundle $[q_1 + \cdots + q_{m+k} - (r_1 + \cdots + r_{m+k})]$ such that the associated divisor $\operatorname{div}(s(\boldsymbol{q}, \boldsymbol{r}))$ is $q_1 + \cdots + q_{m+k} - (r_1 + \cdots + r_{m+k})$. Such section $s(\boldsymbol{q}, \boldsymbol{r})$ is unique up to constant multiples. Let $D(\boldsymbol{q}, \boldsymbol{r})$ be the unique meromorphic connection on $[q_1 + \cdots + q_{m+k} - (r_1 + \cdots + r_{m+k})]$ such that $s(\boldsymbol{q}, \boldsymbol{r})$ is a $D(\boldsymbol{q}, \boldsymbol{r})$-horizontal section. $D(\boldsymbol{q}, \boldsymbol{r})$ is independent of the choice of $s(\boldsymbol{q}, \boldsymbol{r})$. There is a unique meromorphic connection on $\sigma(\boldsymbol{q}, \boldsymbol{r})$ whose connection forms are the half of those of $D(\boldsymbol{q}, \boldsymbol{r})$. We denote this connection by $\nabla(\boldsymbol{q}, \boldsymbol{r})$.

For $Q, R \in \tilde{E}(\boldsymbol{p}, k; N)$ with $\boldsymbol{p} \in \tilde{B}(m)$, we put

$$
(4.4) \qquad \begin{aligned} s(Q, R) &= s(\pi(Q), \pi(R)), & D(Q, R) &= D(\pi(Q), \pi(R)). \\ \sigma(Q, R) &= \sigma(\pi(Q), \pi(R)), & \nabla(Q, R) &= \nabla(\pi(Q), \pi(R)). \end{aligned}
$$

REMARK 4.1. For $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r} \in \tilde{B}(m+k)$, we have

$\sigma(\boldsymbol{p}, \boldsymbol{p})$: the trivial bundle, $\quad \sigma(\boldsymbol{p}, \boldsymbol{q}) \otimes \sigma(\boldsymbol{q}, \boldsymbol{r}) = \sigma(\boldsymbol{p}, \boldsymbol{r})$,

$\nabla(\boldsymbol{p}, \boldsymbol{p})$: the trivial connection, $\quad \nabla(\boldsymbol{p}, \boldsymbol{q}) \otimes \nabla(\boldsymbol{q}, \boldsymbol{r}) = \nabla(\boldsymbol{p}, \boldsymbol{r})$.

Now we can define the gauge equivalence for SL-operators in $\tilde{E}(\boldsymbol{p}, k; N)$ with $\boldsymbol{p} \in \tilde{B}(m)$.

DEFINITION 4.2. Two SL-operators Q and $R \in \tilde{E}(\boldsymbol{p}, k; N)$ with $\boldsymbol{p} \in \tilde{B}(m)$ are said to be *gauge-equivalent* if there exists a nontrivial meromorphic section G of the vector bundle $\mathrm{Hom}(\Phi; \sigma(Q, R) \otimes \Phi)$ having the meromorphic inverse $G^{-1} \in \Gamma(M; \mathcal{M}(\mathrm{Hom}(\sigma(Q, R) \otimes \Phi; \Phi)))$ such that $G \nabla(Q) G^{-1} = \nabla(Q, R) \otimes \nabla(R)$. G is called the *gauge transformation* sending Q to R.

If $\pi(Q) = \pi(R)$ in $\tilde{B}(m+k)$, the gauge equivalence in the sense of Definition 4.2 is nothing but the meromorphic gauge equivalence.

4.2. The $\sigma(Q, R)$-valued vector field associated with a gauge transformation.

Suppose two SL-operators $Q, R \in \tilde{E}(\boldsymbol{p}, k; N)$ are gauge-equivalent and let $G = (G_j) \in \Gamma(M; \mathcal{M}(\mathrm{Hom}(\Phi; \sigma(Q, R) \otimes \Phi)))$ be the gauge transformation sending Q to R. Let $\phi \cdot \pi(Q) = (p_1, \cdots, p_m, q_1, \cdots, q_k)$, $\phi \cdot \pi(R) = (p_1, \cdots, p_m, r_1, \cdots, r_k) \in B(m+k)$, (cf. (4.3)). We denote by $v_j(G)$ the $(1, 2)$-entry of the matrix G_j, i.e.

$$G_j=\begin{pmatrix} * & v_j(G) \\ * & * \end{pmatrix}. \tag{4.5}$$

LEMMA 4.3. *Under the above assumption,*

(i) $\det G$ *is a nontrivial* $D(Q,R)$*-horizontal meromorphic section of the line bundle* $\sigma(Q,R)^{\otimes 2}=[q_1+\cdots+q_k-(r_1+\cdots+r_k)]$. *In particular, the associated divisor* $\operatorname{div}(\det G)$ *is* $q_1+\cdots+q_k-(r_1+\cdots+r_k)$.

(ii) $\operatorname{tr} G$ *is a meromorphic section of the line bundle* $\sigma(Q,R)$.

(iii) $v(G)=(v_j(G))$ *is a* $\sigma(Q,R)$*-valued meromorphic vector field on* M.

We call $v(G)$ the *$\sigma(Q,R)$-valued vector field associated with* the gauge transformation G.

PROOF. We abbreviate $\sigma(Q,R)$ as $\sigma=(\sigma_{jk})$. Since G is a meromorphic section of the vector bundle $\operatorname{Hom}(\Phi,\sigma\otimes\Phi)$, we have $G_j=\sigma_{jk}\Phi_{jk}G_k\Phi_{jk}^{-1}$ in $U_j\cap U_k$. So $\det G_j=\sigma_{jk}^2\cdot\det G_k$ and $\operatorname{tr} G_j=\sigma_{jk}\operatorname{tr} G_k$ in $U_j\cap U_k$. Hence $\det G$ is a meromorphic section of the line bundle $\sigma^{\otimes 2}=[p_1+\cdots+p_k-(r_1+\cdots+r_k)]$ and $\operatorname{tr} G$ is a meromorphic section of the line bundle σ. Since G has the meromorphic inverse, $\det G$ is not identically zero. Furthermore, we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(\det\Phi) & \xrightarrow{\det G} & \mathcal{M}(\sigma^{\otimes 2}\otimes\det\Phi) \\
{\scriptstyle \operatorname{tr}\nabla(Q)}\downarrow & & \downarrow{\scriptstyle D(Q,R)\otimes\operatorname{tr}\nabla(R)} \\
\mathcal{M}(\det\Phi)\otimes\Omega^1 & \xrightarrow[\det G]{} & \mathcal{M}(\sigma^{\otimes 2}\otimes\det\Phi)\otimes\Omega^1
\end{array}$$

Denote by $D_j(Q,R)$ the connection form of $D(Q,R)$ in U_j. By (4.2), $\nabla(Q)$ and $\nabla(R)$ induce the trivial connection d on the determinant line bundle $\det\Phi$. So the above diagram implies

$$(\det G_j)d(\det G_j)^{-1}=d+D_j(Q,R),$$

which leads to $\{d+D_j(Q,R)\}(\det G_j)=0$. Namely, $\det G$ is a nontrivial meromorphic $D(Q,R)$-horizontal section of $\sigma^{\otimes 2}$. Hence the divisor $\operatorname{div}(\det G)$ is $q_1+\cdots+q_k-(r_1+\cdots+r_k)$. Finally, substituting (4.1) and (4.5) into the transition relation $G_j=\sigma_{jk}\Phi_{jk}G_k\Phi_{jk}^{-1}$ and looking at the (1, 2)-entry, we see that $v=(v_j)$ is a $\sigma(Q,R)$-valued meromorphic vector field on M. ∎

The following simple observation will be used later.

LEMMA 4.4. *Let* $\mathcal{U}=\{(U_j,x_j)\}$ *be a projective coordinate covering of*

M, D_j the differentiation with respect to x_j in U_j, σ a holomorphic line bundle, ∇ a meromorphic connection on σ. If $\theta_j dx_j$ is a connection form of ∇ in U_j, we put $\nabla_j = D_j + \theta_j$. For a σ-valued meromorphic vector field $v = (v_j)$ on M, let $w(v; \nabla) = (w_j(v; \nabla))$ be defined by

$$w_j(v; \nabla) := v_j(\nabla_j^2 v_j) - \frac{1}{2}(\nabla_j v_j)^2 . \tag{4.6}$$

Then $w(v; \nabla)$ is a meromorphic section of the line bundle $\sigma^{\otimes 2}$.

PROOF. Take a representative 1-cocycle $(\sigma_{jk}) \in Z^1(\mathcal{U}; \mathcal{O}^*)$ of σ. Then the section $v = (v_j)$ satisfies the transition relation $v_j = \sigma_{jk}\kappa_{jk}^{-1} v_k$ in $U_j \cap U_k$. Substituting this into (4.6) and taking into account the fact that $\mathcal{U}$ is a projective coordinate covering, we see that $w(v; \nabla)$ is a meromorphic section of $\sigma^{\otimes 2}$. ■

PROPOSITION 4.5. *Suppose two SL-operators Q, $R \in \tilde{E}(\boldsymbol{p}, k; N)$ are gauge-equivalent and let G be a gauge transformation sending Q to R. Then we have*

$$v(G) \otimes (R - Q) = \nabla(Q, R) \operatorname{tr} G , \tag{4.7}$$

$$v(G)^{\otimes 2} \otimes (R + Q) = \frac{1}{2}(\operatorname{tr} G)^{\otimes 2} + w(v(G); \nabla(Q, R)) - 2 \det G . \tag{4.8}$$

Remark that an element Q of $\tilde{E}(\boldsymbol{p}, k; N)$ is regarded as a "marked" meromorphic quadratic differentials on M, Q and R in the left-hand side of (4.7) and (4.8) should be understood as meromorphic quadratic differentials without marking.

COROLLARY 4.6. *Under the assumption of Proposition 4.5, suppose that Q and R are distinct as meromorphic quadratic differentials without marking. Then* $\operatorname{tr} G$ *and* $v(G)$ *does not vanish identically.*

PROOF OF PROPOSITION 4.5. Since $G = (G_j)$ is a gauge transformation sending Q to R, we have $G\nabla(Q) = \nabla(Q, R) \otimes \nabla(R) G$. If we put

$$G_j = \begin{pmatrix} a_j & v_j \\ b_j & c_j \end{pmatrix}, \tag{4.9}$$

this condition is rewritten entrywise as

$$v_j Q_j = b_j - \nabla_j a_j , \tag{4.10}$$

$$v_j R_j = b_j + \nabla_j c_j , \tag{4.11}$$

$$\nabla_j v_j = c_j - a_j\,, \tag{4.12}$$

$$c_j Q_j = a_j R_j - \nabla_j b_j\,. \tag{4.13}$$

Subtracting (4.10) from (4.11), we have $v_j(R_j - Q_j) = \nabla_j(a_j + c_j) = \nabla_j(\operatorname{tr} G_j)$, from which we obtain (4.7). To verify (4.8), we apply (4.12) to (4.10)+(4.11) to obtain $v_j(Q_j + R_j) = 2b_j + \nabla_j^2 v_j$. Hence,

$$\begin{aligned}
v_j^2(Q_j + R_j) &= 2v_j b_j + v_j \nabla_j^2 v_j \\
&= 2a_j c_j + v_j \nabla_j^2 v_j - 2(a_j c_j - v_j b_j) \\
&= \frac{1}{2}(a_j + c_j)^2 + v_j \nabla_j^2 v_j - \frac{1}{2}(a_j - c_j)^2 - 2\det G_j \\
&= \frac{1}{2}(\operatorname{tr} G_j)^2 + v_j \nabla_j^2 v_j - \frac{1}{2}(\nabla_j v_j)^2 - 2\det G_j\,,
\end{aligned}$$

where (4.12) is used again to obtain the fourth equality. Hence we obtain (4.8). ■

PROOF OF COROLLARY 4.6. First we shall show that $v(G)$ does not vanish identically. If otherwise, we have $w(v(G)\,;\nabla(Q, R)) = 0$. So (4.8) implies

$$(\operatorname{tr} G)^{\otimes 2} = 4\det G\,. \tag{4.14}$$

We put $\phi\cdot\pi(Q) = (p_1, \cdots, p_m, q_1, \cdots, q_k)$, $\phi\cdot\pi(R) = (p_1, \cdots, p_m, r_1, \cdots, r_k) \in B(m+k)$. By Lemma 4.3, $\operatorname{tr} G$ is a meromorphic section of the line bundle $\sigma(Q, R)$, $\det G$ is a meromorphic section of the line bundle $[q_1 + \cdots + q_k - (r_1 + \cdots + r_k)]$ such that $\operatorname{div}(\det G) = q_1 + \cdots + q_k - (r_1 + \cdots + r_k)$ and $\sigma(Q, R)^{\otimes 2} = [q_1 + \cdots + q_k - (r_1 + \cdots + r_k)]$. So (4.14) implies that there exist a permutation $\tau \in \mathfrak{S}_k$ such that $r_\alpha = q_{\tau(\alpha)}$ $(\alpha = 1, \cdots, k)$. Hence $\sigma(Q, R)^{\otimes 2}$ is the trivial line bundle, $\det G$ is a nonzero constant and $\operatorname{tr} G$ is a holomorphic section of $\sigma(Q, R)$. If $\sigma(Q, R)$ is nontrivial, we have $\operatorname{tr} G \equiv 0$ since $\sigma(Q, R)$ admits no nontrivial holomorphic section because of $c_1(\sigma(Q, R)) = 0$. So (4.14) implies $\det G \equiv 0$, which is a contradiction. Hence $\sigma(Q, R)$ is the trivial line bundle, $\operatorname{tr} G$ is a constant and $\nabla(Q, R)$ is the trivial connection. Since $v_j = 0$ is now assumed, (4.14) implies $a_j = c_j$ in (4.9). Since $\operatorname{tr} G_j = a_j + c_j$ is a constant, we see that $a_j = c_j$ is also a constant. Since $\nabla(Q, R)$ is the trivial connection, we have $\nabla_j = D_j$. Hence (4.10) implies $b_j = 0$. Summarizing the above argument, we conclude that, if $v(G)$ is identically zero, then $\sigma(Q, R)$ is the trivial line bundle and the gauge transformation $G \in \Gamma(M\,; \mathcal{M}(\operatorname{End}(\Phi))$ is of the form $G = aI$, where a is a nonzero constant and I is the identity .

If G is such a gauge transformation, the relation $G\nabla(Q)G^{-1}=\nabla(R)$ immediately implies that $Q=R$ as a meromorphic quadratic differential without marking, which is a contradiction. Hence $v(G)$ does not vanish identically.

Secondly, we shall show that $\operatorname{tr} G$ does not vanish identically. By the assumption of this Corollary and the first step of the proof, the left-hand side of (4.7) does not vanish identically. So (4.7) implies that $\operatorname{tr} G$ does not vanish identically. ■

4.3. Poles and zeros of the gauge transformation.

Let $m,\ k\in \boldsymbol{N},\ \theta=(\theta_1,\cdots,\theta_m)\in(\boldsymbol{C}_+)^m$ (see (1.11)), $N=(N_1,\cdots,N_k)\in \boldsymbol{N}^k$, $\boldsymbol{p}\in\tilde{B}(m+k)$. Let Q and R be SL-operators in $\tilde{E}(\boldsymbol{p},k\,;\theta,N)$ which are distinct but sufficiently close to one another. Assume Q and R are gauge-equivalent and let G be a gauge transformation sending Q to R. Since Q and R are sufficiently close in $\tilde{E}(\boldsymbol{p},k\,;\theta,N)$, they are distinct as meromorphic quadratic differentials without marking. So Corollary 4.6 implies that the trace $\operatorname{tr} G$ and the associated $\sigma(Q,R)$-valued vector field $v(G)$ are not identically zero. We shall discuss about the location and orders of poles and zeros of $\operatorname{tr} G$ and $v(G)$. In this subsection we assume

$$\theta_j\in \boldsymbol{C}_+\backslash \boldsymbol{Z}_+\,,\qquad (j=1,\cdots,m)\,, \tag{4.15}$$

i.e. the first m singular points are *generic*, (see § 1.2). We put $\phi\cdot\pi(Q)=(p_1,\cdots,p_m,q_1,\cdots,q_k)$, $\phi\cdot\pi(R)=(p_1,\cdots,p_m,r_1,\cdots,r_k)\in B(m+k)$. Since Q and R are assumed to be sufficiently close, we may assume that there exist sufficiently small coordinate open subsets U_i $(i=1,\cdots,k)$ of M such that

$$p_l\notin U_i,\quad q_i,r_i\in U_i,\quad U_i\cap U_j=\emptyset,\quad (l=1,\cdots,m,\ i,j=1,\cdots,k). \tag{4.16}$$

Let p be an arbitrary point of M, x a local coordinate at p with $x(p)=0$, $D=d/dx$ the differentiation with respect to x, σ a local frame of the line bundle $\sigma(Q,R)$ in a neighbourhood of p. We assume that $R\pm Q$ and $v(G)$ have the following local expressions:

$$R-Q=\{\alpha x^a+O(x^{a+1})\}(dx)^{\otimes 2}\,, \tag{4.17}$$

$$R+Q=\{\beta x^b+O(x^{b+1})\}(dx)^{\otimes 2}\,, \tag{4.18}$$

$$v(G)=\{\gamma x^c+O(x^{c+1})\}\sigma\otimes(d/dx)\,, \tag{4.19}$$

where α,β,γ are nonzero constants, a,b,c are integers with $a,\ b\geqq -2$ and O denotes Landau's symbol, i.e. $g=O(f)$ indicates g/f remains bounded as x tends to zero. We devide the set $\boldsymbol{k}:=\{1,\cdots,k\}$ into the following two subsets:

$$\boldsymbol{k}(0)=\{j\in\boldsymbol{k}\,;\,q_j\neq r_j\},\qquad \boldsymbol{k}(1)=\{j\in\boldsymbol{k}\,;\,q_j=r_j\}.$$

We have three cases according to the location of the point p:

Case (A): $p=q_j$, $(j\in\boldsymbol{k}(0))$,
Case (B): $p=r_j$, $(j\in\boldsymbol{k}(0))$,
Case (C): otherwise.

In Case (A) and in Case (B) $\det G$ has a simple zero and a simple pole at p, respectively (Lemma 4.3, (i)). So one can choose the local frame σ such that

$$\det G=\begin{cases}\{x+O(x^2)\}\sigma^{\otimes 2} & \text{in Case (A)},\\ \{x^{-1}+O(1)\}\sigma^{\otimes 2} & \text{in Case (B)}.\end{cases}\tag{4.20}$$

In Case (C) the connection $\nabla(Q,R)$ is holomorphic at p. So one can choose the local frame σ to be $\nabla(Q,R)$-*horizontal*. Moreover, since $\sigma^{\otimes 2}$ and $\det G$ are $D(Q,R)$-horizontal section of $\sigma(Q,R)^{\otimes 2}$, one can normalize σ such that

$$\det G=\sigma^{\otimes 2}\qquad \text{in Case (C)}.\tag{4.21}$$

If we put

$$\nabla(Q,R)\sigma=\eta\sigma\otimes(dx),$$

then, by the definition of the connection $\nabla(Q,R)$ (see § 4.1), we have

$$\eta=\begin{cases}\dfrac{1}{2}x^{-1}+O(1) & \text{in Case (A)},\\ -\dfrac{1}{2}x^{-1}+O(1) & \text{in Case (B)},\\ 0 & \text{in Case (C)}.\end{cases}\tag{4.22}$$

In Case (A) and in Case (B), taking (4.16) into account, we have

$$a=b=-2,\tag{4.23}$$

$$\alpha=\begin{cases}\dfrac{1}{4}N_j(N_j+2) & \text{in Case (A)},\\ -\dfrac{1}{4}N_j(N_j+2) & \text{in Case (B)},\end{cases}\tag{4.24}$$

$$\beta=\frac{1}{2}N_j(N_j+2).\tag{4.25}$$

We devide Case (A) and Case (B) into two subcases, respectively, and Case (C) into three subcases. To do so, we put

$$\boldsymbol{k}(0\,;\,o)=\{j\in\boldsymbol{k}(0)\,;\ N_j \text{ is odd}\}\,,$$

$$\boldsymbol{k}(0\,;\,e)=\{j\in\boldsymbol{k}(0)\,;\ N_j \text{ is even}\}\,.$$

Case (A$-\nu$), $(\nu=o,e)$: $p=q_j$ $(j\in\boldsymbol{k}(0\,;\,\nu))$,
Case (B$-\nu$), $(\nu=o,e)$: $p=r$ $(j\in\boldsymbol{k}(0\,;\,\gamma))$,
Case (C$-$i): $p=q_j=r_j$ $(j\in\boldsymbol{k}(1))$,
Case (C$-$ii): $p=p_i$ $(i\in\boldsymbol{m}:=\{1,\cdots,m\})$,
Case (C$-$iii): $p\neq p_i,\ q_j,\ r_j$ $(i\in\boldsymbol{m},\ j\in\boldsymbol{k})$.

Taking (4.16) into account, we obtain

(4.26) $\quad a\geqq-1,\ b=-2,\ \beta=\frac{1}{2}N_j(N_j+2)\quad$ in Case (C$-$i),

(4.27) $\quad a\geqq-1,\ b=-2,\ \beta=\frac{1}{2}(\theta_i^2-1)\quad$ in Case (C$-$ii),

(4.28) $\quad a\geqq 0,\ b\geqq 0\quad$ in Case (C$-$iii).

Substituting (4.17) and (4.19) into (4.7) and using (4.22) and (4.23), we obtain the following lemma.

LEMMA 4.7. *Case* (A): $\operatorname{tr} G=\left\{\frac{2\alpha\gamma}{2c-3}x^{c-1}+O(x^c)\right\}\otimes\sigma.$

Case (B): $\operatorname{tr} G=\left\{\frac{2\alpha\gamma}{2c-1}x^{c-1}+O(x^c)\right\}\otimes\sigma.$

Case (C): *We have* $a+c+1\neq 0$ *and*

$$\operatorname{tr} G=\left\{\frac{\alpha\gamma}{a+c+1}x^{a+c+1}+\delta+O(x^{a+c+2})\right\}\otimes\sigma\,,$$

where δ *is a suitable constant.*

By using (4.6), (4.19) and (4.22), we obtain

LEMMA 4.8.

$$w(v(G)\,;\,\nabla(Q,R))$$
$$=\left\{\frac{1}{2}\gamma^2\left(c-\frac{1}{2}\right)\left(c-\frac{5}{2}\right)x^{2c-2}+O(x^{2c-1})\right\}\sigma^{\otimes 2}\qquad \textit{in Case (A)},$$

$$=\left\{\frac{1}{2}\gamma^2\left(c+\frac{1}{2}\right)\left(c-\frac{3}{2}\right)x^{2c-2}+O(x^{2c-1})\right\}\sigma^{\otimes 2} \qquad \textit{in Case (B)},$$

$$=\left\{\frac{1}{2}\gamma^2 c(c-2)x^{2c-2}+O(x^{2c-1})\right\}\sigma^{\otimes 2} \qquad \textit{in Case (C)}.$$

Furthermore, by using (4.18) and (4.19), we obtain

$$v(G)^{\otimes 2}\otimes(R+Q)=\{\beta\gamma^2 x^{b+2c}+O(x^{b+2c+1})\}\sigma^{\otimes 2}. \tag{4.29}$$

With the above preliminaries, we can state the main theorem in this subsection.

THEOREM 4.9. *Let* $v(G)=\{\gamma x^c+O(x^{c+1})\}\sigma\otimes(d/dx)$ *at* p *with* $\gamma\neq 0$ *and* $c\in\boldsymbol{Z}$.

Case (A $-o$): $c=\frac{1}{2}(1-N_j)$, $\frac{1}{2}(3-N_j)$ *or* $c\geqq 2$.

Case (A $-e$): $c\geqq 2$.

Case (B $-o$): $c=\frac{1}{2}(\pm 1-N_j)$ *or* $c\geqq 1$.

Case (B $-e$): $c\geqq 1$.

Case (C $-$ i): $c=-N_j$ *or* $c\geqq 1$.

Case (C $-$ ii): $c\geqq 1$.

Case (C $-$ iii): $c\geqq 0$.

PROOF. We deduce the result from (4.8) by a case-by-case study.

Case (A): Applying Lemma 4.7, Lemma 4.8, (4.20), (4.23) and (4.29) to (4.8), we obtain

$$A\gamma^2 x^{2c-2}-2x+O(x^{2c-1})+O(x^2)=0, \tag{4.30}$$

where $A:=\dfrac{2\alpha^2}{(2c-3)^2}+\dfrac{1}{2}\left(c-\dfrac{1}{2}\right)\left(c-\dfrac{5}{2}\right)-\beta$.

If $c\leqq 1$, $A\gamma^2 x^{2c-2}$ is the lowest order term of (4.30). So we have $A=0$. Taking (4.24) and (4.25) into account, we see that this algebraic equation has the roots $c=(1/2)(1-N_j)$, $(1/2)(3\pm N_j)$, $(1/2)(5+N_j)$. Since c is an integer, we can not have $c\leqq 1$ in Case (A $-e$). In Case (A $-o$), if $c\leqq 1$, we have $c=(1/2)(1-N_j)$ or $(1/2)(3-N_j)$.

Case (B): Applying Lemma 4.7, Lemma 4.8, (4.20), (4.23) and (4.29) to (4.8), we obtain

$$B\gamma^2 x^{2c-2}-2x^{-1}+O(x^{2c-1})+O(1)=0, \tag{4.31}$$

where $B=\frac{2\alpha^2}{(2c-1)^2}+\frac{1}{2}\left(c+\frac{1}{2}\right)\left(c-\frac{3}{2}\right)-\beta$.

If $c\leqq 0$, $B\gamma^2x^{2c-2}$ is the lowest order term of (4.31). So we have $B=0$. Taking (4.24) and (4.25) into account, we see that this algebraic equation has the roots $c=-(1/2)(1+N_j)$, $(1/2)(1\pm N_j)$, $(1/2)(3+N_j)$. Since c is an integer, we can not have $c\leqq 0$ in Case (B$-e$). In Case (B$-o$), if $c\leqq 0$, we have $c=(1/2)(\pm 1-N_j)$.

Case (C$-$i) and Case (C$-$ii): Applying Lemma 4.7, Lemma 4.8, (4.21), (4.26), (4.27) and (4.29) to (4.8), we obtain

$$C\gamma^2x^{2c-1}+\frac{\alpha\gamma\delta}{a+c+1}x^{a+c+1}+\delta^2-2+O(x^{2c-1})+O(\delta x^{a+c+2})=0\,, \tag{4.32}$$

where $C=\frac{1}{2}c(c-2)-\beta$.

Since $a\geqq -1$ (see (4.26) and (4.27)), if $c\leqq 0$, $C\gamma^2x^{2c-2}$ is the lowest order term of (4.32). So we have $C=0$. By using (4.26) and (4.27), we see that this quadratic equation has the roots $c=-N_j$, N_j+2 in Case (C$-$1) and $c=1\pm\theta_i$ in Case ($C-$ii), respectively. Since c is an integer with $c\leqq 0$, we have $c=-N_j$ in Case ($C-$i). In Case ($C-$ii), by the assumption (4.15), we can not have $c\leqq 0$.

Case (C$-$iii): Applying Lemma 4.7, Lemma 4.8, (4.21), (4.28) and (4.29) to (4.8), we have

$$C'\gamma^2x^{2c-1}+\frac{\alpha\gamma\delta}{a+c+1}x^{a+c+1}+\delta^2-2+O(x^{2c-1})+O(\delta x^{a+c+2})=0\,, \tag{4.33}$$

where $C'=\frac{1}{2}c(c-2)$.

By (4.28), if $c\leqq 0$, then $C'\gamma^2x^{2c-1}$ is the lowest order term of (4.33). So we have $C'=0$ i.e. $c=0$ or 2. Hence, if $c\leqq 0$, we have $c=0$. ∎

Finally we state an important collorary of Theorem 4.9. We put

$$\boldsymbol{k}(o)=\{j\in\boldsymbol{k}\,;\ N_j \text{ is odd}\}\,,$$

$$\boldsymbol{k}(e)=\{j\in\boldsymbol{k}\,;\ N_j \text{ is even}\}\,.$$

We define a holomorphic line bundle $\zeta(Q,R)$ over M by

$$\zeta(Q,R)=\sigma(Q,R)\otimes\kappa^{\otimes-1}\otimes\Big[-\sum_{i=1}^{m}p_i+\sum_{j\in k(o)}\Big\{\frac{1}{2}(N_j-1)q_j+\frac{1}{2}(N_j+1)r_j\Big\}$$
$$+\sum_{j\in k(e)}\frac{1}{2}N_j(q_j+r_j)\Big].$$

Then $\{\zeta(Q,R)\}$ is a holomorphic family of line bundles parametrised by $(Q,R)\in\tilde{E}(\boldsymbol{p},k\,;\theta,N)\times\tilde{E}(\boldsymbol{p},k\,;\theta,N)$ such that

$$\zeta(Q,Q)=\kappa^{\otimes-1}\otimes\Big[-\sum_{i=1}^{m}p_i+\sum_{j=1}^{k}N_jq_j\Big]. \tag{4.34}$$

The following Theorem can be established easily by using Theorem 4.9 together with Corollary 4.6.

THEOREM 4.10. *Suppose Q and R are gauge-equivalent SL-Operators in $\tilde{E}(\boldsymbol{p},k\,;\theta,N)$ which are distinct but sufficiently close to one another. Let G be a gauge transformation sending Q to R. Then the associated $\sigma(Q,R)$-valued vector field $v(G)$ is a nontrivial holomorphic section of the line bundle $\zeta(Q,R)$.*

4.4. A Zariski open subset $B(m,k\,;N)$ of $B(m+k)$.

Let $m\geqq 1$, $k\geqq 0$ and $N\in\boldsymbol{N}^k$. We obtain from (2.4) and (4.3) the following diagram

$$\begin{array}{ccccc} & \tilde{E}(m,k\,;N) & \xrightarrow{\ \phi\ } & E(m,k\,;N) & \\ \swarrow^{\pi} & \downarrow\varpi & & \swarrow^{\pi}\quad \downarrow\varpi & \\ \tilde{B}(m+k) & \xrightarrow{\ \phi\ } & B(m+k) & & \\ \searrow_{p} & & \searrow_{p} & & \\ & \tilde{B}(m) & \xrightarrow{\ \phi\ } & B(m) & \end{array} \tag{4.35}$$

Notice that π, p and ϖ are surjective (see Theorem 2.2 (ii)). In this subsection, we introduce a Zariski open subset $B(m,k\,;N)$ of $B(m+k)$ and study its various properties as a preliminary of the next subsection, where we shall establish the main theorem of Section 4 which asserts *the discreteness of the gauge equivalence in $\tilde{\mathcal{E}}(\boldsymbol{p},k\,;N)$ for each $\boldsymbol{p}\in\tilde{\mathcal{B}}(m,k\,;N)$* (see Theorem 4.19). See (4.39) and (4.41) for the definition of $\tilde{\mathcal{B}}(m,k\,;N)$ and $\tilde{\mathcal{E}}(\boldsymbol{p},k\,;N)$.

For $\boldsymbol{r}=(p_1,\cdots,p_m,q_1,\cdots,q_k)\in B(m+k)$ and $N=(N_1,\cdots,N_k)\in\boldsymbol{N}^k$ let $\eta(\boldsymbol{r},N)$ be a holomorphic line bundle over M defined by

$$\eta(\boldsymbol{r}, N) := \kappa^{\otimes -1} \otimes [N_1 q_1 + \cdots + N_k q_k - (p_1 + \cdots + p_m)]\,. \tag{4.36}$$

We set

$$\begin{aligned} &D(m, k\,; N) := \{\boldsymbol{r} \in B(m+k)\,;\, \dim H^0(M\,; \mathcal{O}(\eta(\boldsymbol{r}, N))) \geqq 1\} \\ &B(m, k\,; N) := B(m+k) \backslash D(m, k\,; N)\,. \end{aligned} \tag{4.37}$$

Of course we can not avoid the possibility of the equality $D(m, k\,; N) = B(m+k)$ i.e. $B(m, k\,; N) = \emptyset$.

REMARK 4.11. (i) $c_1(\eta(\boldsymbol{r}, N)) = |N| - m + 2 - 2g$.

(ii) In particular, (i) implies that if $|N| \leqq m + 2g - 3$ then $D(m, k\,; N)$ is empty.

LEMMA 4.12. *$D(m, k\,; N)$ is an analytic subset of $B(m+k)$, i.e. $B(m, k\,; N)$ is a Zariski* ***open*** *subset of $B(m+k)$.*

This lemma is an immediate consequence of Grauert's theorem [Gra]. We introduce the following notation:

$$\begin{cases} \mathcal{E}(m, k\,; N) := \{Q \in E(m, k\,; N)\,;\, \pi(Q) \in B(m, k\,; N)\}\,, \\ \mathcal{E}(m, k\,; \theta, N) := \{Q \in E(m, k\,; \theta, N)\,;\, \pi(Q) \in B(m, k\,; N)\}\,, \\ \mathcal{B}(m, k\,; N) := p(B(m, k\,; N)) \subset B(m)\,. \end{cases} \tag{4.38}$$

$$\begin{cases} \tilde{\mathcal{E}}(m, k\,; N) := \{Q \in \tilde{E}(m, k\,; N)\,;\, \phi(Q) \in \mathcal{E}(m, k\,; N)\}\,, \\ \tilde{\mathcal{E}}(m, k\,; \theta, N) := \{Q \in \tilde{E}(m, k\,; \theta, N)\,;\, \phi(Q) \in \mathcal{E}(m, k\,; N)\}\,, \\ \tilde{B}(m, k\,; N) := \{\boldsymbol{r} \in \tilde{B}(m+k)\,;\, \phi(\boldsymbol{r}) \in B(m, k\,; N)\}\,, \\ \tilde{\mathcal{B}}(m, k\,; N) := \{\boldsymbol{p} \in \tilde{B}(m)\,;\, \phi(\boldsymbol{p}) \in \mathcal{B}(m, k\,; N)\}\,, \end{cases} \tag{4.39}$$

(see the diagram (4.35)). For any $\boldsymbol{p} \in \mathcal{B}(m, k\,; N)$, we put

$$\begin{aligned} &\mathcal{E}(\boldsymbol{p}, k\,; N) := \{Q \in \mathcal{E}(m, k\,; N)\,;\, \varpi(Q) = \boldsymbol{p}\}\,, \\ &\mathcal{E}(\boldsymbol{p}, k\,; \theta, N) := \{Q \in E(m, k\,; \theta, N)\,;\, \varpi(Q) = \boldsymbol{p}\}\,. \end{aligned} \tag{4.40}$$

Similarly, for any $\boldsymbol{p} \in \tilde{\mathcal{B}}(m, k\,; N)$, we put

$$\begin{aligned} &\tilde{\mathcal{E}}(\boldsymbol{p}, k\,; N) := \{Q \in \mathcal{E}(m, k\,; N)\,;\, \varpi(Q) = \boldsymbol{p}\}\,, \\ &\tilde{\mathcal{E}}(\boldsymbol{p}, k\,; \theta, N) := \{Q \in \tilde{\mathcal{E}}(m, k\,; \theta, N)\,;\, \varpi(Q) = \boldsymbol{p}\}\,. \end{aligned} \tag{4.41}$$

From (4.35) and (4.38)-(4.41) we obtain the diagram

$$
(4.42)\qquad
\begin{array}{ccccc}
 & \tilde{\mathscr{E}}(m,k;N) & \xrightarrow{\ \phi\ } & & \mathscr{E}(m,k;N) \\
 \swarrow^{\pi} & \downarrow \varpi & & \swarrow^{\pi} & \downarrow \varpi \\
\tilde{B}(m,k;N) & \xrightarrow{\qquad \phi \qquad} & B(m,k;N) & & \\
 \searrow_{p} & & & \searrow_{p} & \\
 & \tilde{\mathscr{B}}(m,k;N) & \xrightarrow[\ \phi\ ]{} & & \mathscr{B}(m,k;N)
\end{array}
$$

If $B(m,k;N)$ is nonempty, then $B(m,k;N)$, $\mathscr{E}(m,k;N)$ and $\mathscr{B}(m,k;N)$ are nonempty Zariski open subsets of $B(m+k)$, $E(m,k;N)$ and $B(m)$, respectively. Moreover the maps π, p and ϖ in (4.42) are subjective (cf. Theorem 2.2, Lemma 4.12). Now we give a sufficient condition on the holding of the equality $\mathscr{B}(m,k;N)=B(m)$.

THEOREM 4.13. *Let* $m\geqq 1$, $k\geqq 0$ *and* $N=(N_1,\cdots,N_k)\in \boldsymbol{N}^k$. *If*

(i) $|N|\leqq m+2g-3$ *or*

(ii) $m+2g-2\leqq|N|\leqq m+3g-3$, $\sharp\{j\,;\,N_j=1,\ j=1,\cdots,k\}\geqq g$,

then we have $\mathscr{B}(m,k;N)=B(m)$. *In particular*, $B(m,k;N)$ *is nonempty.*

We have the following particular but important cases to which Theorem 4.13 can be applied.

COROLLARY 4.14. *Suppose* (a) $g=0$, $|N|\leqq m-3$, *or* (b) $g\geqq 1$, $k\leqq m+3g-3$, $N=\mathbf{1}_k:=(1,\cdots,1)\in\boldsymbol{N}^k$. *Then we have* $\mathscr{B}(m,k;N)=B(m)$.

PROOF OF THEOREM 4.13. In case (i), Remark 4.11, (ii) implies that $\mathscr{B}(m,k;N)=B(m+k)$. Hence $\mathscr{B}(m,k;N)=p(B(m,k;N))=B(m)$. We turn to the case (ii). Since $m+2g-2\leqq|N|\leqq m+3g-3$, we have $g\geqq 1$. Since $\sharp\{j\,;\,N_j=1\}\geqq g$, we may assume, without loss of generality, that $N_j=1$ for $j=1,\cdots,g$. For any fixed $\boldsymbol{p}=(p_1,\cdots,p_m)\in B(m)$, we take mutually distinct points $q_{g+1},\cdots,q_k$ in M such that $q_j\neq p_1,\cdots,p_m$. Let σ be a holomorphic line bundle over M defined by

$$\sigma:=\kappa^{\otimes-1}\otimes[N_{g+1}q_{g+1}+\cdots+N_kq_k-(p_1+\cdots+p_m)]\,.$$

Let $L(d)$ be the set of holomorphic line bundles over M with the first Chern class $d\in\boldsymbol{Z}$. Notice that $L(d)=\zeta\otimes Pic(M)$ for any fixed $\zeta\in L(d)$. By the Riemann-Roch formula, any line bundle in $L(g)$ has a nontrivial holomorphic section, so the map

$$
\begin{array}{ccc}
M^g & \longrightarrow & L(g) \\
\Downarrow & & \Downarrow \\
\boldsymbol{q}=(q_1,\cdots,q_g) & \longmapsto & [\boldsymbol{q}]:=[q_1+\cdots+q_g]
\end{array}
$$

is surjective. Hence the following map is also surjective:

$$\begin{array}{ccc} M^g & \longrightarrow & L(c) \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ \boldsymbol{q}=(q_1, \cdots, q_g) & \longmapsto & [\boldsymbol{q}]\otimes\sigma , \end{array}$$

where $c:=|N|-m+2-2g$. By the assumption $|N|\leqq m+3g-3$, we have $c\leqq g-1$. Hence there exists a line bundle $\zeta_0\in L(c)$ such that $\dim H^0(M;\mathcal{O}(\zeta_0))=0$. By the semi-continuity theorem we have $\dim H^0(M;\mathcal{O}(\zeta))=0$ for ζ in a nonempty Zariski open subset of $L(c)$. So we have $\dim H^0(M;\mathcal{O}([\boldsymbol{q}]\otimes\sigma))=0$ for $\boldsymbol{q}$ in a nonempty Zariski open subset of M^g. In particular, there exists $\boldsymbol{q}=(q_1, \cdots, q_g)\in M^g$ such that $q_i\neq q_j$, $q_j\neq q_{g+1}, \cdots, q_k$, $p_1, \cdots, p_m$ $(i, j=1, \cdots, g)$ and $\dim H^0(M;\mathcal{O}([\boldsymbol{q}]\otimes\sigma))=0$. If we put $\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_k)$, then we have $\boldsymbol{r}\in B(m+k)$ and $\dim H^0(M;\mathcal{O}(\eta(\boldsymbol{r}, N)))=0$, i.e., $\boldsymbol{r}\in B(m, k;N)$. Moreover, we have $p(\boldsymbol{r})=\boldsymbol{p}\in B(m)$. Since $\boldsymbol{p}\in B(m)$ is arbitrary, the map $p: B(m, k;N)\to B(m)$ is surjective, i.e. $\mathscr{B}(m, k;N)=B(m)$. ∎

PROOF OF COROLLARY 4.14. Case (a) is a special case of Theorem 4.13, (i) in which $g=0$. Case (b) reduces to Theorem 4.13, (i) or (ii) according to $k\leqq m+2g-3$ or $k\geqq m+2g-2$. ∎

4.5. Discreteness of the gauge equivalence.

The following theorem is the main result in this section.

THEOREM 4.15. *Let $m, k\in\boldsymbol{N}$, $N=(N_1, \cdots, N_k)\in\boldsymbol{N}^k$ and $\theta=(\theta_1, \cdots, \theta_m)\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$. Suppose $B(m, k;N)$ is nonempty and $|N|\leqq m+3g-3$. Then, for any $\boldsymbol{p}\in\tilde{\mathscr{B}}(m, k;N)$, each gauge equivalence class forms a discrete subset of $\tilde{\mathscr{E}}(\boldsymbol{p}, k;\theta, N)$.*

PROOF. Let Q be an arbitrary SL-operators in $\tilde{\mathscr{E}}(\boldsymbol{p}, k;\theta, N)$. We have only to show that, if $R\in\tilde{\mathscr{E}}(\boldsymbol{p}, k;\theta, N)$ is gauge-equivalent and sufficiently close to Q, then R coincides with Q. Suppose Q and R are distinct and let G be a gauge transformation sending Q into R. By Theorem 4.10, the associated $\sigma(Q, R)$-valued vector field $v(G)$ is a nontrivial holomorphic section of the line bundle $\zeta(Q, R)$. On the other hand, we have $\zeta(Q, Q)=\eta(\phi\cdot\pi(Q);N)$ (see (4.34) and (4.36)). Since $Q\in\tilde{\mathscr{E}}(\boldsymbol{p}, k;\theta, N)$, (4.37) and (4.41) imply $\dim H^0(M;\mathcal{O}(\zeta(Q, Q)))=0$. Since $\zeta(Q, R)$ depends holomorphically on R, the semi-continuity theorem asserts that, if R is sufficiently close to Q then $\dim H^0(M;\mathcal{O}(\zeta(Q, R)))=0$. This contradicts the fact that $v(G)$ is nontrivial. Hence R must coincides with Q. ∎

We introduce the following spaces (cf. (2.5)).

$$\mathscr{E}(m|n)_{\mathrm{irr}}=\coprod_{(k,N)\in\Lambda(n)}\mathscr{E}(m,k\,;N)_{\mathrm{irr}}\,, \tag{4.43}$$

$$\mathscr{E}(m|n\,;\theta)_{\mathrm{irr}}=\coprod_{(k,N)\in\Lambda(n)}\mathscr{E}(m,k\,;\theta,N)_{\mathrm{irr}}\,, \tag{4.44}$$

where $\Lambda(n)$ is defined by (2.6) and $\coprod$ denotes the disjoint union as analytic spaces. We define $\tilde{\mathscr{E}}(m|n)_{\mathrm{irr}}$ and $\tilde{\mathscr{E}}(m|n\,;\theta)_{\mathrm{irr}}$ in a similar manner.

REMARK 4.16. Roughly speaking, $\mathscr{E}(m|n)_{\mathrm{irr}}$ is the space of irreducible SL-operators with m ordered regular singular points and ordered apparent singular points of total multiplicity $\leqq n$. $\mathscr{E}(m|n\,;\theta)_{\mathrm{irr}}$ admits a similar interpretation. If $m\geqq\max\{4-2g,1\}$ and $n\leqq m+3g-3$, then Corollary 4.14 asserts that $B(m,n\,;\mathbf{1}_n)$ with $\mathbf{1}_n=(1,\cdots,1)\in\boldsymbol{N}^n$ is nonempty. So Theorem 4.15 implies that $\mathscr{E}(m,n\,;\mathbf{1}_n)_{\mathrm{irr}}$ is an analytic space of pure dimension $3m+n+3g-3$. For any other component $\mathscr{E}(m,k\,;N)_{\mathrm{irr}}$ of $\mathscr{E}(m|n)_{\mathrm{irr}}$, we have

$$\dim\mathscr{E}(m,k\,;N)_{\mathrm{irr}}<\dim\mathscr{E}(m,n\,;\mathbf{1}_n)_{\mathrm{irr}}$$

for $(k,N)\in\Lambda(n)\backslash\{(n,\mathbf{1}_n)\}$. Hence "almost all" SL-operators in $\mathscr{E}(m|n)_{\mathrm{irr}}$ are of ground state. The same statement can be made for $\tilde{\mathscr{E}}(m|n\,;\theta)_{\mathrm{irr}}$.

§5. Moduli of projective representations of the fundamental group and the projective monodromy map.

5.1. The space of projective representations.

We consider the space of conjugacy classes of representations of the fundamental group of a Riemann surface of genus g with m punctures into the projective linear group $PSL(2\,;\boldsymbol{C})=\mathrm{Aut}(\boldsymbol{P}^1)$. Hereafter we denote $PSL(2\,;\boldsymbol{C})$ by G. Let M be a compact Riemann surface of genus $g\geqq 0$. For any ordered m-tuple $\boldsymbol{p}=(p_1,\cdots,p_m)\in B(m)$, we denote the unordering of $\boldsymbol{p}$ by $|\boldsymbol{p}|$ i.e. $|\boldsymbol{p}|=\{p_1,\cdots,p_m\}$. We define the real blow-up $[\boldsymbol{p}]$ of M at the points $p_1,\cdots,p_m$ as follows. Let (U_i,x_i) $(i=1,\cdots,m)$ be coordinate neighbourhoods of p_i in M such that $U_i\cap U_j=\emptyset$ for $i\neq j$ and $x_i(p_i)=0$. We put $V_i:=\{(p,t)\in U_i\times S^1\,;x_i(p)=t|x_i(p)|\}$, where S^1 is the unit circle in the complex plane. Putting $S_i^1:=\{p_i\}\times S^1$, we have diffeomorphisms $f_i: V_i\backslash S_i^1\to U_i\backslash\{p_i\}$, $(p,t)\mapsto p$. Then the *real blow-up* $[\boldsymbol{p}]$ of M at the points $p_1,\cdots,p_m$ is, by definition, the real smooth 2-dimensional manifold with boundary obtained by attaching V_i $(i=1,\cdots,m)$ to $M\backslash\{p_1,\cdots,p_m\}$ by the maps $f_i: V_i\backslash S_i^1\to U_i\backslash\{p_i\}$. Notice that the boundary $\partial[\boldsymbol{p}]$ of $[\boldsymbol{p}]$ is $\coprod_{i=1}^m S_i^1$, the disjoint union of m copies of the unit circle S^1.

Given $\boldsymbol{p}^*=(p_0, \boldsymbol{p})\in B(m+1)$, let $\hat{R}(\boldsymbol{p}^*)$ be the set of representations of the fundamental group $\pi_1(M\backslash|\boldsymbol{p}|, p_0)$ into the projective linear group G:

$$\hat{R}(\boldsymbol{p}^*):=\mathrm{Hom}\,(\pi_1(M\backslash|\boldsymbol{p}|, p_0)\,;\,G)\,.$$

Regarding $\pi_1(M\backslash|\boldsymbol{p}|, p_0)\,;\,G)$ as a discrete group, we topologize $\hat{R}(\boldsymbol{p}^*)$ with the compact-open topology. A representation $\rho\in\hat{R}(\boldsymbol{p}^*)$ is said to be *irreducible* if the subgroup $\rho(\pi_1(M\backslash|\boldsymbol{p}|, p_0))$ of $\mathrm{Aut}(\boldsymbol{P}^1)$ has no fixed point in $\boldsymbol{P}^1$. We denote by $\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}}$ the set of all irreducible elements in $\hat{R}(\boldsymbol{p}^*)$. The group $\pi_1(M\backslash|\boldsymbol{p}|, p_0)$ is isomorphic to the group generated by the elements α_i, β_i $(i=1,\cdots,g)$ and γ_j $(j=1,\cdots,m)$ satisfying the one relation

$$[\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]\gamma_1\cdots\gamma_m=1\,, \tag{5.1}$$

where $[\alpha, \beta]=\alpha\beta\alpha^{-1}\beta^{-1}$ is the commutator of α and β. If we fix a system of generators α_i, β_i $(i=1,\cdots,g)$ and γ_j $(j=1,\cdots,m)$ satisfying (5.1), then, by the definition of its topology, we see that $\hat{R}(\boldsymbol{p}^*)$ is homeomorphic to the complex submanifold of the complex Lie group G^{2g+m} defined by the equation

$$[A_1, B_1]\cdots[A_g, B_g]C_1\cdots C_m=1\,,$$

for $(A_1,\cdots,A_g,B_1,\cdots,B_g,C_1,\cdots,C_m)\in G^{2g+m}$. So $\hat{R}(\boldsymbol{p}^*)$ carries a complex manifold structure which makes the above homeomorphism a biholomorphism. We see that another choice of a system of generators of $\pi_1(M\backslash|\boldsymbol{p}|, p_0)$ satisfying (5.1) determines the same complex structure. Hence $\hat{R}(\boldsymbol{p}^*)$ is canonically a complex manifold. It is known that $\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}}$ is a nonempty Zariski open subset of $\hat{R}(\boldsymbol{p}^*)$.

The inner automorphism group $\mathrm{Ad}(G)$ of G acts on $\hat{R}(\boldsymbol{p}^*)$ by

$$\mathrm{Ad}\,(G)\times\hat{R}(\boldsymbol{p}^*)\longrightarrow\hat{R}(\boldsymbol{p}^*)\,,\qquad(\mathrm{Ad}\,(g),\rho)\longmapsto\mathrm{Ad}\,(g)\rho\,,$$

where $(\mathrm{Ad}\,(g)\rho)(\alpha):=g\rho(\alpha)g^{-1}$ for $\alpha\in\pi_1(M\backslash|\boldsymbol{p}|, p_0)$. $\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}}$ is invariant under the action of $\mathrm{Ad}\,(G)$. We see that the orbit space of $\hat{R}(\boldsymbol{p}^*)$ under this action is independent of the base point p_0. So we denote it by $R(\boldsymbol{p})$, i. e.

$$R(\boldsymbol{p}):=\hat{R}(\boldsymbol{p}^*)/\mathrm{Ad}\,(G)\,.$$

Similarly we denote the orbit space of $\hat{R}(\boldsymbol{p}^*)_{irr}$ by $R(\boldsymbol{p})_{irr}$, i. e.

$$R(\boldsymbol{p})_{irr}:=\hat{R}(\boldsymbol{p}^*)_{irr}/\mathrm{Ad}\,(G)\,.$$

Let $H^1([\boldsymbol{p}]\,;G)$ be the set of local G-systems (namely flat $\boldsymbol{P}^1$-bundles) over $[\boldsymbol{p}]$. Given $\rho\in R(\boldsymbol{p})$, let $L(\rho)$ be the local system over $[\boldsymbol{p}]$ whose characteristic homomorphism is given by ρ. Remark that there is the following bijection (=identification):

$$R(\boldsymbol{p}) = H^1([\boldsymbol{p}]; G), \qquad \rho \longleftrightarrow L(\rho).$$

We freely use this identification. We denote by $H^1([\boldsymbol{p}]; G)_{\mathrm{irr}}$ the set of irreducible local G-systems.

By Schur's lemma, the group $\mathrm{Ad}(G)$ acts on $\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}}$ freely. Furthermore the following theorem is rather standard (e.g. [Gu2]).

LEMMA 5.1. *$R(\boldsymbol{p})_{\mathrm{irr}}$ carries a natural complex manifold structure such that the canonical projection $\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}} \to R(\boldsymbol{p})_{\mathrm{irr}}$ becomes a holomorphic principal $\mathrm{Ad}(G)$-bundle. $R(\boldsymbol{p})_{\mathrm{irr}}$ is of dimension $3m+6g-6$. The tangent space of $R(\boldsymbol{p})_{\mathrm{irr}}$ at a point $\rho \in R(\boldsymbol{p})_{\mathrm{irr}}$ is canonically identified with the vector space $H^1([\boldsymbol{p}]; \mathrm{Ad}\, L(\rho))$ of the first cohomology group with coefficients in $L(\rho)$, i.e.*

$$T_\rho R(\boldsymbol{p})_{irr} = H^1([\boldsymbol{p}]; \mathrm{Ad}\, L(\rho)),$$

where $\mathrm{Ad}\, L(\rho)$ is the local system whose characteristic homomorphism is given by $\mathrm{Ad}(\rho: \pi_1(M \setminus |\boldsymbol{p}|, p_0) \to \mathrm{Aut}(\mathfrak{sl}(2, \boldsymbol{C})),\ \alpha \mapsto \mathrm{Ad}(\rho(\alpha))$.

Since the fundamental group of the unit circle S^1 is a cyclic group of infinite order, a local G-system on S^1 is identified with a conjugacy class of elements in $G = PSL(2, \boldsymbol{C})$. So a local G-system on S^1 is identified with a Jordan's canonical matrix $J(a)$ or J up to the multiplication of $\pm I$, where

$$J(a) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \quad (a \in \boldsymbol{C}_+ \setminus \{0\}), \qquad J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We call $[J(a)] = J(a) \bmod \pm I$ and $[J] = J \bmod \pm I$ the *type* of the local G-system on S^1. Notice that we obtain local G-systems on $S^1 \cong S^1_j \subset [\boldsymbol{p}]$ by restricting a local system $L(\rho)$ with $\rho \in R(\boldsymbol{p})$ to S^1_j $(j = 1, \cdots, m)$. For $\boldsymbol{p}^* = (p_0, \boldsymbol{p}^*) \in B(m+1)$ and $\theta = (\theta_1, \cdots, \theta_m) \in (\boldsymbol{C}_+)^m$, we define the subspace $\hat{R}(\boldsymbol{p}^*, \theta)_{\mathrm{irr}}$ of $\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}}$ as follows.

$$\hat{R}(\boldsymbol{p}^*; \theta)_{\mathrm{irr}} := \left\{ \rho \in \hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}} ; \begin{array}{r} \text{the local systems on } S^1_j \subset [\boldsymbol{p}] \\ \text{induced by } L(\rho) \text{ are of type} \\ [J(\exp(\pi i \theta_j))] \quad (j = 1, \cdots, m) \end{array} \right\}.$$

We readily see that $\hat{R}(\boldsymbol{p}^*; \theta)_{\mathrm{irr}}$ is $\mathrm{Ad}(G)$-invariant. We put

$$R(\boldsymbol{p}; \theta)_{\mathrm{irr}} := \hat{R}(\boldsymbol{p}^*; \theta)_{\mathrm{irr}} / \mathrm{Ad}(G).$$

THEOREM 5.2. *Suppose that* $\theta=(\theta_1, \cdots, \theta_m)\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$. *Then* $R(\boldsymbol{p};\theta)_{\mathrm{irr}}$ *is a* $2(m+3g-3)$*-dimensional complex submanifold of* $R(\boldsymbol{p})_{\mathrm{irr}}$. *The tangent space of* $R(\boldsymbol{p};\theta)_{\mathrm{irr}}$ *at a point* $\rho\in R(\boldsymbol{p};\theta)_{\mathrm{irr}}$ *is canonically identified with the kernel of the restriction homomorphism*

$$j^*: H^1([\boldsymbol{p}];\mathrm{Ad}\,L(\rho))\longrightarrow H^1(\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho)|\partial[\boldsymbol{p}]).$$

REMARK 5.3. For any $\rho\in R(\boldsymbol{p};\theta)_{\mathrm{irr}}$ with $\theta\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$, there is the following cohomology long exact sequence

$$\begin{aligned}0\longrightarrow H^0([\boldsymbol{p}];\mathrm{Ad}\,L(\rho))&\xrightarrow{j^*} H^0(\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho)|\partial[\boldsymbol{p}])\xrightarrow{\delta^*}\\ \longrightarrow H^1([\boldsymbol{p}],\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho))&\xrightarrow{i^*} H^1([\boldsymbol{p}];\mathrm{Ad}\,L(\rho))\xrightarrow{j^*}\\ \longrightarrow H^1(\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho)|\partial[\boldsymbol{p}])&\longrightarrow\cdots.\end{aligned}$$

Since ρ is irreducible, Schur's lemma implies

$$H^0([\boldsymbol{p}];\mathrm{Ad}\,L(\rho))=0.$$

If a local G-system L on S^1 is of type $[J(\alpha)]$ with $\alpha\neq 1$, then the sections of the local $\mathrm{Ad}(G)$-system $\mathrm{Ad}\,L$ on S^1 are precisely of the form cI, where $c\in\boldsymbol{C}$ and I is the identity endmorphism of L, i.e. $H^0(S^1;\mathrm{Ad}\,L)\cong\boldsymbol{C}$. Since $\theta\in\boldsymbol{C}_+\backslash\boldsymbol{Z}_+$, this observation applies to $\mathrm{Ad}\,L(\rho)|S^1_j$. Hence we obtain

$$H^0(\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho)|\partial[\boldsymbol{p}])=\bigoplus_{j=1}^{m}H^0(S^1_j;\mathrm{Ad}\,L(\rho)|S^1_j)\cong\boldsymbol{C}\oplus\cdots\oplus\boldsymbol{C}\ (m\text{-times}).$$

Theorem 5.2 and the above exact sequence implies that the following identification is available:

$$\begin{aligned}T_\rho R(\boldsymbol{p};\theta)_{\mathrm{irr}}&=\mathrm{Ker}\,[j^*: H^1([\boldsymbol{p}];\mathrm{Ad}\,L(\rho))\longrightarrow H^1(\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho)|\partial[\boldsymbol{p}])]\\ &=\frac{H^1([\boldsymbol{p}],\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho))}{\delta^*H^0(\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho)|\partial[\boldsymbol{p}])}.\end{aligned}\tag{5.2}$$

Let $\mathrm{Ad}\,L(\rho)\otimes\mathrm{Ad}\,L(\rho)\to\boldsymbol{C}$ be the multiplication map into the constant sheaf $\boldsymbol{C}$ obtained by the Killing form on the Lie algebra $\mathcal{L}ie\,G=\mathfrak{sl}(2,\boldsymbol{C})$. Then, by the Poincaré-Lefschetz duality theorem, we have the following nondegenerate bilinear form

$$H^1([\boldsymbol{p}];\mathrm{Ad}\,L(\rho))\otimes H^1([\boldsymbol{p}],\partial[\boldsymbol{p}];\mathrm{Ad}\,L(\rho))\longrightarrow\boldsymbol{C}.\tag{5.3}$$

Under the identification (5.2), we can show that the above form (5.3) induces a nondegenerate skew-symmetric bilinear form

$$\Theta(\boldsymbol{p}): T_\rho R(\boldsymbol{p}; \theta)_{\mathrm{irr}} \otimes T_\rho R(\boldsymbol{p}; \theta)_{\mathrm{irr}} \longrightarrow \boldsymbol{C},$$

on each tangent space at ρ. Furthermore we can show that this almost symplectic structure on $R(\boldsymbol{p}; \theta)_{\mathrm{irr}}$ defined by the form $\Theta(\boldsymbol{p})$ is integrable. Thus we get the canonical (complex) symplectic structure on $R(\boldsymbol{p}; \theta)_{\mathrm{irr}}$.

Fix a point $\boldsymbol{p}^*=(p_0, \boldsymbol{p}^\circ) \in B(m+1)$. We denote by $Br(m)$ the fundamental group of $B(m)$ with the base point $\boldsymbol{p}^\circ \in B(m)$, since it is a kind of *braid group*. We abbreviate the fundamental group $\pi_1(M \backslash |\boldsymbol{p}^\circ|, p_0)$ as π_1. There is a natural homomorphism h from $Br(m)$ into the outer automorphism group $\mathrm{Out}(\pi_1)=\mathrm{Aut}(\pi_1)/\mathrm{Inn}(\pi_1)$ of π_1, which is described as follows: Given any $l \in Br(m)$, let $\boldsymbol{p}(t)$ $(0 \leqq t \leqq 1)$ be a representative loop with the base point $\boldsymbol{p}^\circ$. We put $Y=\bigcup_{0 \leqq t \leqq 1} M(t)$, where $M(t)=t \times (M \backslash |\boldsymbol{p}(t)|)$. Identifying $(M \backslash |\boldsymbol{p}^\circ|, p_0)$ with $(M(\nu), \nu \times p_0)$, we get two inclusions $\iota_\nu: (M \backslash |\boldsymbol{p}^\circ|, p_0) \hookrightarrow (Y, \nu \times p_0)$, $(\nu=0, 1)$. It is easily seen that these inclusions induce isomorphisms $\iota_{\nu *}: \pi_1 \xrightarrow{\sim} \pi_1(\nu):=(Y, \nu \times p_0)$, $(\nu=0, 1)$. A homotopy class α of curves in Y with the initial point $0 \times p_0$ and the terminal point $1 \times p_0$ induces an isomorphism $\alpha_*: \pi_1(0) \simeq \pi_1(1)$, $\gamma \mapsto \alpha \cdot \gamma \cdot \alpha^{-1}$ and then an automorphism $\alpha_{**}:=(\iota_{1*})^{-1} \cdot \alpha_* \cdot \iota_{0*}$ of π_1. If β is another homotopy class of curves in Y with the initial point $0 \times p_0$ and the terminal point $1 \times p_0$, then we have $\beta \cdot \alpha^{-1} \in \pi_1(1)$ and $\beta_{**} \cdot (\alpha_{**})^{-1}=(\iota_{1*})^{-1}$. $(\beta \cdot \alpha^{-1})_* \cdot \iota_{1*}=\{(\iota_1)^{-1}(\beta \cdot \alpha^{-1})\}_*$. This implies that $\beta_{**} \cdot (\alpha_{**})^{-1}$ is an inner automorphism of π_1. Hence α_{**} determines a unique element of $\mathrm{Out}(\pi_1)$ depending only on $l \in Br(m)$, which is denoted by $h(l)$. It is easy to see that $h: Br(m) \to \mathrm{Out}(\pi_1)$ is a group homomorphism.

We define the space $R(m)_{\mathrm{irr}}$ by

$$R(m)_{\mathrm{irr}} := \coprod_{\boldsymbol{p} \in B(m)} R(\boldsymbol{p})_{\mathrm{irr}}.$$

Then $R(m)_{\mathrm{irr}}$ has a structure of a local system over $B(m)$ (i.e. a covariant functor of the fundamental groupoid of $B(m)$ into the category of complex manifolds) such that the corresponding characteristic homomorphism at the base point $\boldsymbol{p}^\circ \in B(m)$ is given by

$$\begin{array}{ccc} h_*: Br(m)=\pi_1(B(m), \boldsymbol{p}^\circ) & \longrightarrow & \mathrm{Aut}(R(\boldsymbol{p}^\circ)_{\mathrm{irr}}) \\ \Downarrow & & \Downarrow \\ l & \longmapsto & h_*(l) \end{array}$$

where $h_*(l)$ is defined as follows: Let α_{**} be an element of $\mathrm{Aut}(\pi_1)$ belonging to $h(l)$. Then there is the map $\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}} \to \hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}}$, $\rho \mapsto \rho \cdot (\alpha_{**})^{-1}$. Passing to the quotient space $R(\boldsymbol{p}^\circ)_{\mathrm{irr}}=\hat{R}(\boldsymbol{p}^*)_{\mathrm{irr}}/\mathrm{Ad}(G)$, we obtain an element of $\mathrm{Aut}(R(\boldsymbol{p}^\circ)_{\mathrm{irr}})$ depending only on l, which we denoted by $h_*(l)$. In particular, $R(m)_{\text{irr}}$ is naturally a complex manifold of dimension $4m+6g-6$, since $R(\boldsymbol{p}^{\circ})_{\text{irr}}$ is a complex manifold of dimension $3m+6g-6$ (see Lemma 5.1).

Given $\theta=(\theta_1, \cdots, \theta_m)\in(\boldsymbol{C}_+)^m$, let

$$R(m\,;\theta)_{\text{irr}}:=\coprod_{\boldsymbol{p}\in B(m)} R(\boldsymbol{p}\,;\theta)_{\text{irr}}.$$

One can check that $R(m\,;\theta)_{\text{irr}}$ is a local subsystem of $R(m)_{\text{irr}}$. If $\theta\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$, then $R(m\,;\theta)_{\text{irr}}$ is naturally a complex manifold of dimension $3m+9g-6$, since $R(\boldsymbol{p}^{\circ}\,;\theta)_{\text{irr}}$ is a complex manifold of dimension $2(m+3g-3)$, (see Lemma 5.2). By Remark 5.3, each $R(\boldsymbol{p}\,;\theta)_{\text{irr}}$ ($\boldsymbol{p}\in B(m)$) is a symplectic manifold. We summarize the above argument into the following theorem.

THEOREM 5.4. *$R(m)_{\text{irr}}$ is a complex manifold of dimension $6m+6g-6$ admitting a structure of a local system over $B(m)$. For any $\theta=(\theta_1, \cdots, \theta_m)\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$, $R(m\,;\theta)_{\text{irr}}$ is a local subsystem and a $(3m+6g-6)$-dimensional complex submanifold of $R(m)_{\text{irr}}$. Moreover, $R(m\,;\theta)_{\text{irr}}$ carries a natural Poisson manifold structure such that each fiber $R(\boldsymbol{p}\,;\theta)_{\text{irr}}$ of the projection $R(m\,;\theta)_{\text{irr}}\rightarrow B(m)$ is a symplectic leaf.*

5.2. The projective monodromy map.

Let $\mathcal{E}(m|n)_{\text{irr}}$ and $\mathcal{E}(m|n\,;\theta)_{\text{irr}}$ be the analytic spaces of SL-operators defined by (4.43) and (4.44), respectively. Now we define the projective monodromy map.

DEFINIITON 5.5. We define *the projective monodromy map* by

$$\begin{array}{rcl} PM:\mathcal{E}(m|n)_{\text{irr}} & \longrightarrow & R(m)_{\text{irr}} \\ \Downarrow & & \Downarrow \\ Q & \longmapsto & \begin{pmatrix}\text{the local } G\text{-system over} \\ M\backslash|\varpi(Q)| \text{ determined by } Q\end{pmatrix} \end{array}$$

More precise definition is given in the following manner: Let Q be any element of $\mathcal{E}(m|n)_{\text{irr}}$ and suppose that Q belongs to $\mathcal{E}(\boldsymbol{p}, k\,;N)_{\text{irr}}$ for some $\boldsymbol{p}\in\mathcal{B}(m, k, N)$ and $(k, N)\in\Lambda(n)$. Let $\pi(Q)=\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_k)\in B(m, k\,;N)$ be the ordered singular points of Q. Recall that the SL-operator Q is regarded as a second order linear Fuchsian differential operator $\mathcal{M}(\xi)\rightarrow\mathcal{M}(\xi\otimes\kappa^{\otimes 2})$, where ξ is a fixed line bundle with $c_1(\xi)=1-g$ (cf. Remark 1.4). So the solution sheaf of this operator gives rise to a rank two flat vector bundle over $M\backslash|\boldsymbol{r}|$. Since q_j $(j=1, \cdots, m)$ are apparent singular points, the circuit matrices around these points determined by that flat vector bundle belong to the centor $\{\pm I\}$ of $SL(2;\boldsymbol{C})$. Hence, associated with the canonical projection $SL(2;\boldsymbol{C})\rightarrow G$, we get a flat projective bundle over $M\backslash|\boldsymbol{p}|$, which we denote by $PM(Q)$. It can be easily seen that an SL-operator Q is irreducible (cf. Definition 3.1) if and only if the associated local system $PM(Q)$ is irreducible. Hence PM sends $\mathcal{E}(m|n)_{\mathrm{irr}}$ into $R(m)_{\mathrm{irr}}$. For $\theta=(\theta_1,\cdots,\theta_m)\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$, restricting PM to $\mathcal{E}(m|n;\theta)$, we obtain the map

$$PM:\mathcal{E}(m|n;\theta)_{\mathrm{irr}}\longrightarrow R(m;\theta)_{\mathrm{irr}},$$

which we also call *the projective monodromy map.*

The projective monodromy map PM is a holomorphic map. Given any subset $\mathcal{S}$ of $E(m+k)$, let

$$\mathcal{S}^{\dagger}:=\{Q\in\mathcal{S}\,;\ \text{the first } m \text{ singular points of } Q \text{ are generic}\}.$$

Notice that, for example, $\mathcal{E}^{\dagger}(m,k;N)=\coprod_{\theta\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m}\mathcal{E}(m,k;\theta,N)$. Let

n' := the total multiplicity of apparent singular points,
m := the number of generic singular points.

If $n'\leqq m+3g-3$, one can know more about the projective monodromy map. Indeed we have the following theorem.

THEOREM 5.6. *Let $n=m+3g-3$ and $\theta\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$. The projective monodromy maps $PM:\mathcal{E}^{\dagger}(m|n)_{\mathrm{irr}}\rightarrow R(m)_{\mathrm{irr}}$ and $PM:\mathcal{E}(m|n;\theta)\rightarrow R(m;\theta)_{\mathrm{irr}}$ are locally injective. In particular they are locally finite maps.*

To establish this theorem we shall make some preliminary arguments. Recall the property (GM) discussed in §§ 4.1 by which we mean that two connections are gauge-equivalent if and only if they give rise to equivalent monodromy representations. Now we shall establish a version of the property (GM) for our SL-operators.

We denote by H the special linear group $SL(2;\boldsymbol{C})$. For $\boldsymbol{r}\in B(m+k)$, we put

$$S(\boldsymbol{r})_{\mathrm{irr}}:=\mathrm{Hom}\,(\pi_1(M\backslash|\boldsymbol{r}|,p_0)\,;\,H)_{\mathrm{irr}}/\mathrm{Ad}\,(H),$$

where p_0 is a point in $M\backslash|\boldsymbol{r}|$. Just as in the case of $R(\boldsymbol{p})_{\mathrm{irr}}$, $S(\boldsymbol{r})_{irr}$ is independent of p_0 and is naturally a complex manifold of dimension $3(m+k)+6g-6$. We put

$$S(m+k)_{\mathrm{irr}} := \coprod_{r\in B(m+k)} S(\boldsymbol{r})_{\mathrm{irr}}.$$

Let $\tilde{S}(m+k)_{\mathrm{irr}}$ be the fiber product of $\tilde{B}(m+k)_{\mathrm{irr}}$ and $S(m+k)_{\mathrm{irr}}$ over $B(m+k)$, $\tilde{S}(\tilde{\boldsymbol{r}})_{\mathrm{irr}}$ the fiber of the projection $\tilde{S}(m+k)_{\mathrm{irr}}\to\tilde{B}(m+k)$ over $\tilde{\boldsymbol{r}}\in\tilde{B}(m+k)$. Fix a base point $\tilde{\boldsymbol{r}}^{\circ}\in\tilde{B}(m+k)$ and put $\boldsymbol{r}^{\circ}=\phi(\tilde{\boldsymbol{r}}^{\circ})$, where $\tilde{B}(m+k)\to B(m+k)$ is the natural projection. There is the canonical "trivialization" map T,

$$(5.4)\qquad \begin{array}{ccc} \tilde{S}(m+k)_{\mathrm{irr}} & \xrightarrow{\quad T \quad} & \tilde{B}(m+k)\times S(\tilde{\boldsymbol{r}}^{\circ})_{\mathrm{irr}} \\ & \searrow \quad \swarrow & \\ & \tilde{B}(m+k) & \end{array}$$

which is described as follows: Any point $\tilde{\boldsymbol{r}}\in\tilde{B}(m+k)$ is a homotopy class of curves in $B(m+k)$ with the initial point $\boldsymbol{r}^{\circ}$ and the terminal point $\boldsymbol{r}=\phi(\tilde{\boldsymbol{r}})$. Let $\boldsymbol{r}(t)$ $(0\leqq t\leqq 1)$ be a curve representing $\tilde{\boldsymbol{r}}$ and put $Y=\bigcup_{0\leqq t\leqq 1} M(t)$, where $M(t)=t\times(M\backslash|\boldsymbol{r}(t)|)$. Let p_{ν} $(\nu=0,1)$ be points in M such that $\nu\times p_{\nu}\in M(\nu)$. We identify $(M\backslash|\boldsymbol{r}^{\circ}|, p_0)$ and $(M\backslash|\boldsymbol{r}|, p_1)$ with $(M(0), 0\times p_0)$ and $(M(1), 1\times p_1)$, respectively. Let $\iota_{\nu}:(M(\nu),\nu\times p_{\nu})\to(Y,\nu\times p_{\nu})$ be inclusions $(\nu=0,1)$ and take a curve α in Y with the initial point $0\times p_0$ and the terminal point $1\times p_1$. Then, just as in the argument in §§ 5.1, we obtain an isomorphism

$$\alpha_{**}=(\iota_{1*})^{-1}\cdot\alpha_{*}\cdot\iota_{0*}:\pi_1(M\backslash|\boldsymbol{r}^{\circ}|, p_0)\longrightarrow\pi_1(M\backslash|\boldsymbol{r}|, p_1).$$

If β is another curve, then $\beta_{**}\cdot(\alpha_{**})^{-1}$ is an inner automorphism of $\pi_1(M\backslash|\boldsymbol{r}|, p_1)$. So α_{**} induces an isomorphism

$$(5.5)\qquad t[\tilde{\boldsymbol{r}}]:\tilde{S}(\tilde{\boldsymbol{r}})_{\mathrm{irr}}\longrightarrow\tilde{S}(\tilde{\boldsymbol{r}}^{\circ})_{\mathrm{irr}}$$

which is independent of p_{ν} $(\nu=0,1)$ and α. Now T is defined by $T(\rho)=(\tilde{\boldsymbol{r}}, t[\tilde{\boldsymbol{r}}](\rho))$ if $\rho\in\tilde{S}(\tilde{\boldsymbol{r}})$. Taking into account the fact that $\tilde{S}(\boldsymbol{r}^{\circ})_{\mathrm{irr}}$ is a complex manifold of dimension $3(m+k)+6g-6$, we equip $\tilde{S}(m+k)_{\mathrm{irr}}$ with a complex manifold structure which makes T a biholomorphism.

Let $\boldsymbol{r}^{\circ}=(p_1^{\circ},\cdots,p_m^{\circ},q_1^{\circ},\cdots,q_k^{\circ})$. Fix a system of generators $\alpha_1,\cdots,\alpha_g$, $\beta_1,\cdots,\beta_g,\gamma_1,\cdots,\gamma_m,\delta_1,\cdots,\delta_k$ of the fundamental group $\pi_1(M\backslash|\boldsymbol{r}^{\circ}|, p_0)$, where $\alpha_1,\cdots,\alpha_g,\beta_1,\cdots,\beta_g$ give rise to a system of generators of $\pi_1(M, p_0)$ and $\gamma_1,\cdots,\gamma_m$, $\delta_1,\cdots,\delta_k$ are loops surrounding $p_1^{\circ},\cdots,p_m^{\circ}$, $q_1^{\circ},\cdots,q_k^{\circ}$, respectively. Here, by a *loop surrounding a point* $p\in M$, we mean a loop l described as follows: Let (U,x) be a *sufficiently small* coordinate neighbourhood of p such that $x(p)=0$ and $x(U)=\{|z|<2\}$, l_1 the inverse image of the loop $\exp(2\pi\sqrt{-1}t)$ $(0\leqq t\leqq 1)$ under the map x, l_2 an arc in M starting from

$x^{-1}(1)$ and ending at p_0. Then l is given by $l_2\cdot l_1\cdot l_2^{-1}$. We put

$$\tilde{S}(\tilde{\boldsymbol{r}}^\circ, k)_{\rm irr}=\left\{\rho\in\tilde{S}(\tilde{\boldsymbol{r}}^\circ)_{\rm irr}\,;\;\begin{matrix}\text{For a representation } \hat{\rho} \text{ belonging}\\ \text{to } \rho,\ \hat{\rho}(\delta_j)=-I\ (j=1,\cdots,k)\end{matrix}\right\},$$

where I is the unit matrix in $SL(2\,;\boldsymbol{C})$. Furthermore, for $\theta=(\theta_1,\cdots,\theta_m)\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$, we put

$$\tilde{\mathcal{S}}^\circ(\tilde{\boldsymbol{r}}^\circ, k\,;\,\theta)_{\rm irr}=\left\{\rho\in\tilde{\mathcal{S}}(\boldsymbol{r}^\circ,\ k)_{\rm irr}\,;\;\begin{matrix}\hat{\rho}(\gamma_i) \text{ has eigenvalues}\\ \exp(\pm 2\pi\sqrt{-1}\theta_i),\ (i=1,\cdots,m)\end{matrix}\right\}.$$

We can show that $\tilde{\mathcal{S}}(\tilde{\boldsymbol{r}}^\circ, k)_{\rm irr}$ and $\tilde{\mathcal{S}}(\tilde{\boldsymbol{r}}^\circ, k\,;\theta)_{\rm irr}$ are $(3m+3g-6)$- and $2(m+3g-3)$-dimensional submanifold of $\tilde{S}(\tilde{\boldsymbol{r}}^\circ)_{\rm irr}$, respectively. For $\theta\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$, we put

$$\tilde{S}(m+k, k)_{\rm irr}=T^{-1}(\tilde{B}(m+k)\times\tilde{S}(\tilde{\boldsymbol{r}}^\circ, k)_{\rm irr}),$$

$$\tilde{S}(m+k, k\,;\,\theta)_{\rm irr}=T^{-1}(\tilde{B}(m+k)\times\tilde{S}(\tilde{\boldsymbol{r}}^\circ, k\,;\,\theta)_{irr}).$$

We define a holomorphic map $Md:\tilde{E}(m, k\,;N)_{\rm irr}\to\tilde{S}(m+k, k)_{\rm irr}$ over $\tilde{B}(m+k)$ by sending an element $\tilde{Q}\in\tilde{E}(m, k\,;N)_{irr}$ to its monodromy representation class. We call Md the *monodromy map*. There is the following commutative diagram, (recall §§ 4.4 for the notation).

$$\begin{array}{ccccc}\tilde{\boldsymbol{E}}(m, k\,;\,N)_{\rm irr} & \xrightarrow{Md} & \tilde{S}(m+k, k)_{\rm irr} & \xrightarrow{T} & \tilde{\boldsymbol{B}}(m+k)\times\tilde{S}(\tilde{\boldsymbol{r}}^\circ, k)_{\rm irr}\\ \downarrow\varpi & & & & \tilde{p}\times{\rm id}\downarrow\\ \tilde{\boldsymbol{B}}(m) & \longleftarrow & & & \tilde{\boldsymbol{B}}(m)\times\tilde{S}(\tilde{\boldsymbol{r}}^\circ, k)_{\rm irr}.\end{array}$$

We denote $(p\times{\rm id})\cdot T\cdot Md:\tilde{E}(m, k\,;N)\to\tilde{B}(m)\times\tilde{S}(\tilde{\boldsymbol{r}}^\circ, k)_{\rm irr}$ by MD and call also the *monodromy map*. In a standard manner, one can show the following lemma.

LEMMA 5.7. *For $\tilde{Q}$, $\tilde{R}\in\tilde{E}(m, k\,;N)_{\rm irr}$, we have $MD(\tilde{Q})=MD(\tilde{R})$ if and only if $\tilde{Q}$ and $\tilde{R}$ are gauge-equivalent in the sense of Definition* 4.2.

The canonical projection $H=SL(2\,;\boldsymbol{C})\to G=PSL(2\,;\boldsymbol{C})$ induces a 2^{2g+m-1}-fold covering $c:\tilde{S}(\tilde{\boldsymbol{r}}^\circ, k)_{\rm irr}\to\tilde{R}(\tilde{\boldsymbol{p}}^\circ)_{\rm irr}$, where $\tilde{\boldsymbol{p}}^\circ=p(\tilde{\boldsymbol{r}}^\circ)\in\tilde{B}(m)$. Let $\tilde{R}(m)_{\rm irr}$ be the fiber product of $\tilde{B}(m)$ and $R(m)_{irr}$ over $B(m)$. Since $R(m)_{\rm irr}$ is a local system over $B(m)$ (Theorem 5.4), there is the canonical trivialization map $T:\tilde{R}(m)_{\rm irr}\to\tilde{B}(m)\times\tilde{R}(\tilde{\boldsymbol{p}}^\circ)_{\rm irr}$ over $\tilde{B}(m)$. Let $(k, N)\in\Lambda(n)$ be such that $B(m, k\,;N)$ is nonempty (cf. §§ 4.4) and $\theta\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$. Then we have the commutative diagram:

$$
\begin{array}{ccccc}
\tilde{\mathcal{E}}(m,k;\theta,N)_{\mathrm{irr}} & & \xrightarrow{\quad MD \quad} & & \tilde{\mathcal{B}}(m,k;N)\times\tilde{S}(\tilde{r}^{\circ},k;\theta)_{\mathrm{irr}} \\
\Big\downarrow \varpi & \searrow^{PM} & & & \Big\downarrow \mathrm{id}\times c \\
 & & \tilde{R}(m;\theta)_{\mathrm{irr}} \mid \tilde{\mathcal{B}}(m,k;N) & \xleftarrow[T^{-1}]{} & \tilde{\mathcal{B}}(m,k;N)\times R(\tilde{p}^{\circ};\theta)_{\mathrm{irr}} \\
 & \swarrow & & & \\
\tilde{\mathcal{B}}(m,k;N)\,. & & & &
\end{array} \tag{5.6}
$$

With the above preliminary arguments we can easily establish Theorem 5.6.

PROOF OF THEOREM 5.6. It is sufficient to show that, for each $(k, N) \in \Lambda(n)$, the map $PM: \mathcal{E}^{\dagger}(m,k;N)_{\mathrm{irr}} \to R(m)_{\mathrm{irr}}|\mathcal{B}(m,k;N)$ is locally injective. Given $Q \in \mathcal{E}^{\dagger}(m,k;N)_{\mathrm{irr}}$, consider the closed subset $F=\{R\in \mathcal{E}^{\dagger}(m,k;N)_{\mathrm{irr}};\ PM(R)=PM(Q)\}$. If $Q\in\mathcal{E}^{\dagger}(m,k;\theta,N)_{\mathrm{irr}}$, then F is the disjoint union of the closed subsets $F(\theta')=F\cap\mathcal{E}(m,k;\theta',N)_{\mathrm{irr}}$, where θ' runs over all elements of $(\boldsymbol{C}_{+}\backslash\boldsymbol{Z}_{+})^{m}$ such that $\theta'-\theta\in\boldsymbol{Z}^{m}$. Hence, to establish the theorem, it is sufficient to show that PM: $\mathcal{E}(m,k;\theta,N)_{\mathrm{irr}}\to R(m;\theta)_{\mathrm{irr}}|\mathcal{B}(m,k;N)$ is injective for each $(k,N)\in\Lambda(n)$ and $\theta\in(\boldsymbol{C}_{+}\backslash\boldsymbol{Z}_{+})^{m}$. Furthermore, passing to the covering, we have only to show that $PM: \tilde{\mathcal{E}}(m,k;\theta,N)_{\mathrm{irr}}\to\tilde{R}(m;\theta)_{\mathrm{irr}}|\tilde{\mathcal{B}}(m,k;N)$ is locally injective. To show this, notice first that Theorem 4.19 and Lemma 5.7 imply that the map MD in the diagram (5.6) is locally injective. Since $\mathrm{id}\times c$ is a 2^{2g+m-1}-fold covering map, $PM=T^{-1}\cdot(\mathrm{id}\times c)\cdot MD$ is also a locally injective map (see Diagram (5.6)). This establish the theorem. ■

5.3. The Riemann-Hilbert problem and the number of apparent singular points.

Theorem 5.6 has many applications. As one of them, we shall consider the Riemann-Hilbert problem (RH) which is stated, roughly, as follows: Let $\mathcal{R}$ be a set of projective representations of the fundamental group $M\backslash S$, S being a finite subset of M. Let $\mathcal{E}$ be a set of SL-operators with (generic) singular points in S which may admit a certain number of apparent singular points outside S. Then the question is: Is the projective monodromy map $PM: \mathcal{E}\to\mathcal{R}$ surjective? If the answer is affirmative, then (RH) is said to be *solved*. If the Zariski closure of the image $PM(\mathcal{E})$ in $\mathcal{R}$ is $\mathcal{R}$, then (RH) is said to be *solved generically*. If M is of genus g and S consists of m points, then the Riemann-Hilbert problem is said to be of *conformal type* (g, m).

In this subsection, we assume $\theta\in(\boldsymbol{C}_{+}\backslash\boldsymbol{Z}_{+})^{m}$. Remark 4.16 and Theorem

5.4 immediately imply the following lemma.

LEMMA 5.8. *Let* $n=m+3g-3$. *For any* $(k, N)\in \Lambda(n)$, *we have*

$$\dim \mathcal{E}^{\dagger}(m, k\,;N)_{\mathrm{irr}}\left\{\begin{matrix}<\\ =\end{matrix}\right\}\dim R(m)_{\mathrm{irr}}\,,$$

$$\dim \mathcal{E}(m, k\,;\theta, N)_{\mathrm{irr}}\left\{\begin{matrix}<\\ =\end{matrix}\right\}\dim R(m\,;\theta)_{\mathrm{irr}}$$

according to $(k, N)\left\{\begin{matrix}\neq\\ =\end{matrix}\right\}(n, \mathbf{1}_n)$, *where* $\mathbf{1}_n=(1, \cdots, 1)\in \boldsymbol{N}^n$.

To state our first theorem in this subsection, we put

$$\mathcal{E}^{@}(m|n)_{\mathrm{irr}}:=\mathcal{E}^{\dagger}(m|n)_{\mathrm{irr}}\backslash\mathcal{E}^{\dagger}(m, n\,;\mathbf{1}_n)_{\mathrm{irr}}\,,$$

$$\mathcal{E}^{@}(m|n\,;\theta)_{\mathrm{irr}}:=\mathcal{E}(m, n\,;\theta)_{\mathrm{irr}}\backslash\mathcal{E}(m, n\,;\theta, \mathbf{1}_n)_{\mathrm{irr}}\,.$$

Theorem 5.6, Lemma 5.8 and the finite mapping theorem implies the following theorem.

THEOREM 5.9. *Let* $n=m+3g-3$.

(i) $PM(\mathcal{E}^{@}(m|n)_{\mathrm{irr}})$ *is nowhere dense in* $R(m)_{\mathrm{irr}}$. *Similarly,* $PM(\mathcal{E}^{@}(m, n\,;\theta)_{\mathrm{irr}})$ *is nowhere dense in* $R(m\,;\theta)_{\mathrm{irr}}$.

(ii) $PM:\mathcal{E}^{\dagger}(m, n\,;\mathbf{1}_n)_{\mathrm{irr}}\rightarrow R(m)_{\mathrm{irr}}$ *and* $PM:\mathcal{E}(m, n\,;\theta, \mathbf{1}_n)_{\mathrm{irr}}\rightarrow R(m\,;\theta)_{\mathrm{irr}}$ *are locally injective and open maps. Namely they are locally biholomorphic maps.*

One wants to find the minimum of the total multiplicities of apparent singular points with which the Riemann-Hilbert problem is solved (at least generically). By a similar argument as that of Otsuki [Ot], we can verify the following theorem.

THEOREM 5.10. *Let* $n=m+4g-3$. *Then the image of* $PM:\mathcal{E}^{\dagger}(m|n)_{\mathrm{irr}}\rightarrow R(m)_{\mathrm{irr}}$ *is Zariski open. For any* $\theta\in(\boldsymbol{C}_{+}\backslash\boldsymbol{Z}_{+})^{m}$, *the map* $PM:\mathcal{E}(m|n\,;\theta)_{\mathrm{irr}}\rightarrow R(m\,;\theta)_{\mathrm{irr}}$ *is surjective.*

Theorem 5.9 (i) implies that the SL-operators of total multiplicity $<m+3g-3$ together with those of total multiplicity $m+3g-3$ and of excited state are too limited to solve generically the Riemann-Hilbert problem of conformal type (g, m). On the other hand, Theorem 5.10 implies that the presence of apparent singular points of total multiplicity $m+4g-3$ is sufficient to solve the problem generically. Theorem 5.9 (ii) asserts that

the projective representations of ground-state SL-operators of total multiplicity $m+3g-3$ form an open subset of the space of representations in the usual topology. We propose the following problem.

Problem 5.11. Are the ground-state SL-operators of total multiplicity $m+3g-3$ sufficient to solve generically the Riemann-Hilbert problem of conformal type (g, m)?

Theorem 5.9, together with Corollary 4.14, immediately implies the following theorem.

THEOREM 5.12. *Let* $n=m+3g-3$ *and* $\theta\in(\boldsymbol{C}_{+}\backslash\boldsymbol{Z}_{+})^{m}$. *Then* $\mathcal{E}^{\dagger}(m, n;\mathbf{1}_{n})_{\mathrm{irr}}$ *and* $\mathcal{E}(m, n;\theta, \mathbf{1}_{n})_{\mathrm{irr}}$ *are complex manifolds. There exists the following commutative diagram*:

$$\begin{array}{ccccc} & & \mathcal{E}(m,n;\theta,\mathbf{1}_n)_{\mathrm{irr}} & \xrightarrow{PM} & R(m;\theta)_{\mathrm{irr}} \\ & \overset{\pi}{\swarrow} & & & \\ B(m,n;\mathbf{1}_n) & & \downarrow \varpi & \swarrow & \\ & \underset{p}{\searrow} & & & \\ & & B(m), & & \end{array} \tag{5.7}$$

where PM *is a locally biholomorphic map and* π, p, ϖ *are surjections.*

In Section 6, the space $\mathcal{E}(m, n;\theta, \mathbf{1}_{n})_{\mathrm{irr}}$ with $n=m+3g-3$ will be studied more deeply in connection with a kind of *Cousin problem*. Making use of Theorem 5.12, we can define the monodromy preserving deformation on $\mathcal{E}(m, n;\theta, \mathbf{1}_{n})_{\mathrm{irr}}$.

DEFINITION 5.13. Since $R(m;\theta)_{\mathrm{irr}}$ is a local system over $B(m)$ (Theorem 5.4), its local horizontal sections give rise to a foliation $\mathfrak{F}$ on $R(m;\theta)_{\mathrm{irr}}$. Since PM is locally biholomorphic, $\mathfrak{F}$ induces a foliation $PM^{*}\mathfrak{F}$ of dimension m on $\mathcal{E}(m, n;\theta, \mathbf{1}_{n})_{\mathrm{irr}}$ via PM. We call $PM^{*}\mathfrak{F}$ *the monodromy preserving deformation* (or *foliation*) *on* $\mathcal{E}(m, n;\theta, \mathbf{1}_{n})_{\mathrm{irr}}$.

The monodromy preserving deformation on $\mathcal{E}(m, n;\theta, \mathbf{1}_{n})_{\mathrm{irr}}$ will be studied in Section 8. We shall derive there the deformation equation, i.e. a system of nonlinear differential equations which describes the *infinitesimal* monodromy preserving deformation. Miwa-Jimbo-Ueno [MJU] and Okamoto [Ok5, 6] had to verify that their deformation equations are completely integrable, because they considered the monodromy preserving deformation from the infinitesimal point of view. In our approach, however, the complete integrability of the deformation equation is evident, because it is known *a priori* that the integral submanifold exists thanks to Theorem 5.12. Moreover it is evident that the monodromy preserving foliation is transverse to each fiber of the projection $\varpi:\mathcal{E}(m,n;\theta,\mathbf{1}_n)_{\mathrm{irr}}\to B(m)$. Hence we can take the location $\boldsymbol{p}\in B(m)$ of generic singular points as deformation parameters. This is another advantage of our approach. In fact Okamoto [Ok5] *assumed* to take $\boldsymbol{p}$ as deformation parameters, but did not verify to *be able to* do so.

§ 6. Cousin's problem and the complex manifold of SL-operators of ground state.

6.1. The space $X(m)$ of regular singular points.

In view of the discussion in Section 5, it is very important to understand more deeply the space of ground-state SL-operators with m generic singular points and $m+3g-3$ apparent singular points. So hereafter we assume

(6.1) $n=m+3g-3=$ the moduli number of Riemann surfaces of genus g with m punctures.

Under this assumption, to simplify the notation, we put

$$\mathcal{E}(m;\ \theta):=\mathcal{E}(m,\ n;\ \theta,\ \mathbf{1}_n).$$

In Section 5 we considered the irreducible part $\mathcal{E}(m;\ \theta)_{\mathrm{irr}}$ of $\mathcal{E}(m;\ \theta)$ for $\theta\in(\boldsymbol{C}_+\backslash\boldsymbol{Z}_+)^m$. But in this section we shall consider the whole space $\mathcal{E}(m;\ \theta)$ for any $\theta\in(\boldsymbol{C}_+)^m$.

Given $\boldsymbol{r}=(p_1,\ \cdots,\ p_m,\ q_1,\ \cdots,\ q_n)\in B(m+n)$, let $\xi(\boldsymbol{r})$ be a holomorphic line bundle over M defined by

$$\xi(\boldsymbol{r}):=\kappa^{\otimes 2}\otimes[p_1+\cdots+p_m-(q_1+\cdots+q_n)].$$

For a line bundle φ, the line bundle $\kappa\otimes\varphi^{\otimes(-1)}$ is said to be *Serre-dual* to φ.

REMARK 6.1. (i) $\xi(\boldsymbol{r})$ is Serre-dual to the line bundle $\eta(\boldsymbol{r},\ \mathbf{1}_n)$ defined by (4.36).

(ii) By the assumption (6.1), we have $c_1(\xi(\boldsymbol{r}))=g-1$. Thus the Riemann-Roch formula for $\xi(\boldsymbol{r})$ reads

(6.2) $$\dim H^0(M;\ \mathcal{O}(\xi(\boldsymbol{r})))=\dim H^1(M;\ \mathcal{O}(\xi(\boldsymbol{r})))\qquad\text{for}\quad \boldsymbol{r}\in B(m+n).$$

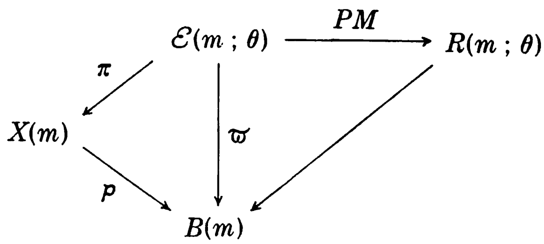
This is one of major advantages of the assumption (6.1). We propose to call (6.2) the *Fredholm alternative.* For, (6.2) implies that the vanishing of $H^0(M;\ \mathcal{O}(\xi(\boldsymbol{r})))$ is equivalent to that of $H^1(M;\ \mathcal{O}(\xi(\boldsymbol{r})))$. Later we shall consider a kind of Cousin problem associated with the line bundles $\xi(\boldsymbol{r})$. Notice that the vanishing of $H^1(M;\ \mathcal{O}(\xi(\boldsymbol{r})))$ leads to the solvability of the Cousin problem, while the vanishing of $H^0(M;\ \mathcal{O}(\xi(\boldsymbol{r})))$ leads to the uniqueness of solution to it. Hence (6.2) suggests that the solvability and the uniqueness of solution are equivalent in our Cousin problem. Thus, interpreting cohomology as an extension of the theory of integral equations, we may call (6.2) the Fredholm alternative.

We define $D(m)$ and $X(m)\subset B(m+n)$ as follows.

$$
\begin{cases} D(m) := \{\boldsymbol{r}\in B(m+n)\,;\ \dim H^0(M;\ \mathcal{O}(\xi(\boldsymbol{r})))\geqq 1\}\,, \\ X(m) := B(m+n)\backslash D(m)\,. \end{cases} \tag{6.3}
$$

Since $\xi(\boldsymbol{r})$ is Serre-dual to $\eta(\boldsymbol{r},\ \mathbf{1}_n)$ (Remark 6.1, (i)), the Serre duality theorem and (4.37) implies $D(m)=D(m,\ n\,;\ \mathbf{1}_n)$ and $X(m)=B(m,\ n\,;\ \mathbf{1}_n)$. Hence $X(m)$ is a nonempty Zariski open subset of $B(m+n)$ and there is the following commutative diagram (see Corollary 4.14 and Diagram (5.7)).

(6.4)



Notice that π, p and ϖ are surjective; among other things we insist on the surjectivity of p.

In low genus case $g=0$ or 1, we can demonstrate what the analytic subset $D(m)$ is.

Example 6.2. (i) In case $g=0$, $D(m)$ is empty and hence $X(m)=B(m+n)$. Indeed, since $c_1(\xi(\boldsymbol{r}))=-1$, $\xi(\boldsymbol{r})$ has no nontrivial holomorphic section for every $\boldsymbol{r}\in B(m+n)$.

(ii) In case $g=1$, we have $m=n$ and $c_1(\xi(\boldsymbol{r}))=0$. So $\xi(\boldsymbol{r})$ has a nontrivial holomorphic section if and only if $\xi(\boldsymbol{r})$ is trivial. Hence we have

$$D(m)=\{\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_m)\in B(m+m)\ ;$$

$$p_1+\cdots+p_m\sim q_1+\cdots+q_m \text{ (linearly equivalent)}\}$$

$$=\left\{\boldsymbol{r}\in B(m+m)\ ;\ \sum_{j=1}^{m}\int_{p_j}^{q_j}\omega\equiv 0 \text{ (mod periods)}\right\},$$

where ω is a nontrivial holomorphic 1-form on M (unique up to constant multiples) and the second equality in the above formula is a consequence of Abel's theorem.

We call $X(m)$ *the space of regular singular points*. We want to know more about the space $\mathcal{E}(m\,;\,\theta)$ and the map $\pi:\mathcal{E}(m\,;\,\theta)\to X(m)$ with an emphasis on its fiber bundle structure. For this purpose we have to introduce the notion of accessory parameters of an SL-operator (cf. Definition 6.9), which plays a role of fiber coordinates, and formulate a kind of Cousin problem (which is sometimes abbreviated as (CP)) associated with the line bundles $\xi(\boldsymbol{r})$ $(\boldsymbol{r}\in X(m))$. These will be done in the next subsection. The main purpose of this section is to give a solution of the following problem.

Problem 6.3. The pair of a location $\boldsymbol{r}\in X(m)$ of singular points and a system of accessory parameters $\nu\in\boldsymbol{C}^n$ is called a *Cousin datum*. Given a Cousin datum $(\boldsymbol{r}, \nu)$, construct an SL-operator $Q\in\mathcal{E}(m\,;\,\theta)$ whose ordered singular points are $\boldsymbol{r}$ i.e. $\pi(Q)=\boldsymbol{r}$ and whose accessory parameters are ν as a solution to the Cousin problem. Furthermore, investigate the dependence of the solution Q on the Cousin datum $(\boldsymbol{r}, \nu)$. Understand the structure of the space $\mathcal{E}(m\,;\,\theta)$ by considering these problems.

The following lemma is evident from the definition (6.3) of the space $X(m)$, but plays a key role in solving our Cousin problem.

KEY LEMMA 6.4 (The following statement consists of one sentence).
(i) $H^1(M\,;\,\mathcal{O}(\xi(\boldsymbol{r})))=0$ *and*
(ii) $H^0(M\,;\,\mathcal{O}(\xi(\boldsymbol{r})))=0$ *hold*
(iii) *for every* $\boldsymbol{r}\in X(m)$.

REMARK 6.5. In Lemma 6.4, (i) guarantees the solvability of (CP) and (ii) guarantees the uniqueness of solution to (CP). Moreover (iii) and a Kodaira-Spencer's theorem guarantees the holomorphic dependence of the solution to (CP) on the Cousin data.

6.2. Accessory parameters of SL-operators.

To define accessory parameters and to make the later argument precise, we need to introduce a special type of local coordinate neighbourhoods of an arbitrary point in $X(m)$. So we make the following rather technical definition.

DEFINITION 6.6. Let $\boldsymbol{r}^{\circ}=(p_1^{\circ}, \cdots, p_m^{\circ}, q_1^{\circ}, \cdots, q_n^{\circ})$ be an arbitrary point in $X(m)$ and let $\mathcal{U}=\{(U_i, x_i)\}_{i\in I}$ be a coordinate covering of M. We put

$$U_i' := U_i \backslash \overline{(\bigcup_{j\neq i} U_j)}\,.$$

We say that $\mathcal{U}$ is *$\boldsymbol{r}^{\circ}$-admissible* if and only if

(i) The index set I is a subset of $\boldsymbol{N}$ containing $\{1, 2, \cdots, m+n\}$,

(ii) U_j' $(j=1, \cdots, m)$ and U_{m+k}' $(k=1, \cdots, n)$ are nonempty open neighbourhoods of p_j° and q_k°, respectively, and

(iii) $W := U_1\times \cdots \times U_{m+n}\subset X(m)$ i. e. W is a open (product) neighbourhood of $\boldsymbol{r}^{\circ}$ in $X(m)$.

In this situation we call W an *admissible neighbourhood* of $\boldsymbol{r}^{\circ}$ (associated with the $\boldsymbol{r}^{\circ}$-admissible coordinate covering $\mathcal{U}$ of M). It is easy to see that there is an $\boldsymbol{r}^{\circ}$-admissible coordinate covering of M such that the associated admissible neighbourhood is as small as one likes. We put $W' := U_1'\times \cdots \times U_{m+n}'$. W' $(\subset W)$ is also an open neighbourhood of $\boldsymbol{r}^{\circ}$ in $X(m)$. In the above situation, to simplify the notation, we put

$$V_k := U_{m+k}, \quad V_k' := U_{m+k}', \quad y_k := x_{m+k} \qquad (k=1, \cdots, n)\,.$$

For $\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_n)\in W$ let

$$t_j(\boldsymbol{r}) := x_j(p_j) \qquad (j=1, \cdots, m)\,,$$

$$\lambda_k(\boldsymbol{r}) := y_k(q_k) \qquad (k=1, \cdots, n)\,.$$

We see that $(t_1, \cdots, t_m, \lambda_1, \cdots, \lambda_n)$ is a local coordinate of $X(m)$ defined in W. We call $(W; t_1, \cdots, t_m, \lambda_1, \cdots, \lambda_n)$ an *admissible coordinate neighbourhood* of $\boldsymbol{r}^{\circ}$ in $X(m)$.

REMARK 6.7. It is clear from the above definition that U_l' $(l=1, \cdots, m+n)$ and U_i $(i\in I\backslash\{l\})$ do not intersect. Hence we have

$$x_j(p)-t_j(\boldsymbol{r})\neq 0 \qquad \text{for} \quad (p, \boldsymbol{r})\in (U_i\cap U_j)\times W'\,,$$

$$y_k(p)-\lambda_k(\boldsymbol{r})\neq 0 \qquad \text{for} \quad (p, \boldsymbol{r})\in (V_k\cap U_i)\times W'\,,$$

for $i\in I$, $j=1, \cdots, m$, $k=1, \cdots, n$ with $i\neq j$ and $i\neq m+k$. This observation will be used in the proof of Theorem 6.10 and Theorem 6.11 in the next subsection.

PROPOSITION 6.8. *Let $(W; t_1, \cdots, t_m, \lambda_1, \cdots, \lambda_n)$ be an admissible coordinate neighbourhood of an arbitrary point $\boldsymbol{r}^{\circ}\in X(m)$, and let Q be any element of $\mathcal{E}(m;\theta)$ such that $\boldsymbol{r}:=\pi(Q)\in W=U_1\times\cdots\times U_m\times V_1\times\cdots\times V_n$. Regarded as a meromorphic quadratic differentials on M (cf. (1.9)), Q admits the following local expressions in U_j and V_k:*

(i) $$Q=\left\{\frac{\alpha_j}{(x_j-t_j)^2}+O\left(\frac{1}{x_j-t_j}\right)\right\}(dx_j)^{\otimes 2} \qquad in \quad U_j$$

for $j=1, \cdots, m$, where $\alpha_j:=(1/4)(\theta_j^2-1)$.

(ii) *There uniquely exists $\nu(Q)=(\nu_1, \cdots, \nu_n)\in \boldsymbol{C}^n$ such that*

$$Q=\left\{\frac{3}{4(y_k-\lambda_k)^2}-\frac{\nu_k}{y_k-\lambda_k}+\nu_k^2+O(y_k-\lambda_k)\right\}(dy_k)^{\otimes 2} \qquad in \quad V_k$$

for $k=1, \cdots, n$, where Landau's symbol O is employed.

PROOF. Let R be an SL-operator with a regular singular point at $p\in M$. Then R is regarded as a meromorphic quadratic differential on M with at most double pole at p. Using a local coordinate x at p with $x(p)=0$, we write R as $R=\{\alpha x^{-2}+O(x^{-1})\}(dx)^{\otimes 2}$. We see that α does not depend on the choice of the local coordinate. Hence we can define the residue $\mathrm{Res}(R;p)$ of R at p by $\mathrm{Res}(R;p)=\alpha$. If σ is the difference of the characteristic exponents of R at p, then it is easy to see that $\mathrm{Res}(R;p)=(1/4)(\sigma^2-1)$. Let $\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_n)\in B(m+n)$. Applying the above observation to Q, we find that the residues of Q at p_j and q_k are α_j and $3/4$, respectively. This establishes the assertion (i) and a part of the assertion (ii) of the proposition. To complete the proof of the assertion (ii), we have only to verify the assertion that the coefficient of the $(y_k-\lambda_k)^0$-term is equal to the square of the coefficient of the $(y_k-\lambda_k)^{-1}$-term. Using Frobenius' method, we can show that this assertion is equivalent to the assumption that $q_1, \cdots, q_n$ are apparent singular points of ground state. See also Okamoto [Ok5, p. 587]. ∎

DEFINITION 6.9. $\nu(Q)=(\nu_1, \cdots, \nu_n)\in \boldsymbol{C}^n$ appearing in the above proposition is called the *accessory parameters* of Q (with respect to the admissible coordinate neighbourhood $(W; t_1, \cdots, t_m, \lambda_1, \cdots, \lambda_n)$ of $\boldsymbol{r}^{\circ}$).

Now we shall establish the converse of Proposition 6.6, by which we shall give an answer to Problem 6.3. Namely, for an ordered $(m+n)$-tuple

$\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_n)\in W$ of singular points and a system of accessory parameters $\nu=(\nu_1, \cdots, \nu_n)\in \boldsymbol{C}^n$, we shall show that there uniquely exists an SL-operator $Q\in\mathcal{E}(m\,;\,\theta)$ which admits the local expressions stated in Proposition 6.6 and the correspondence

$$W\times \boldsymbol{C}^n \longrightarrow \mathcal{E}(m\,;\,\theta), \qquad (\boldsymbol{r}, \nu)\longmapsto Q \tag{6.5}$$

is a biholomorphic map.

To do this we introduce some notation. For $\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_n)\in X(m)$, let D_j $(j=1, \cdots, m)$ be a divisor on $M\times X(m)$ defined by the hypersurface $\{(p, \boldsymbol{r})\in M\times X(m)\,;\, p=p_j\}$ and let D'_k $(k=1, \cdots, n)$ be a divisor on $M\times X(m)$ defined by the hypersurface $\{(p, \boldsymbol{r})\in M\times X(m)\,;\, p=q_k\}$. Let u and v be the canonical projections of $M\times X(m)$ into the first factor M and into the second factor $X(m)$, respectively.

$$\begin{array}{ccc} & M\times X(m) & \\ {}^{u}\swarrow & & \searrow^{v} \\ M & & X(m) \end{array} \tag{6.6}$$

Since

$$\dim H^0(M\,;\,\mathcal{O}(\kappa^{\otimes 2}\otimes[2p_1+\cdots+2p_m+2q_1+\cdots+2q_n]))=2(m+n)+3g-3$$

is independent of $\boldsymbol{r}\in X(m)$, a sheaf over $X(m)$ defined by

$$v_*\mathcal{O}_{M\times X(m)}(u^*\kappa^{\otimes 2}\otimes[2D_1+\cdots+2D_m+2D'_1+\cdots+2D'_n])$$

is a locally free analytic sheaf. Let $F\rightarrow X(m)$ be the associated holomorphic vector bundle over $X(m)$. To construct the map (6.5), we need to construct auxiliary local holomorphic sections of F which satisfy certain special properties. This will be done in the next subsection by formulating and solving a kind of Cousin problem.

6.3. Cousin's problem and auxiliary local sections of the vector bundle F.

Fix an arbitrary point $\boldsymbol{r}^\circ$ in $X(m)$ and let $(W\,;\, t_1, \cdots, t_m, \lambda_1, \cdots, \lambda_n)$ be the admissible coordinate neighbourhood of $\boldsymbol{r}^\circ$ associated with an $\boldsymbol{r}^\circ$-admissible coordinate covering $\mathcal{U}$ of M. Recall that an open neighbourhood $W'(\subset W)$ of $\boldsymbol{r}^\circ$ is also associated with $\mathcal{U}$ (see Definition 6.6). If necessary, $\mathcal{U}$ will be replaced by another one without any comment so that the associated $\mathcal{U}$ and $\mathcal{U}'$ are as small as one likes. Even in such a case the notation W and W' will be kept throughout this subsection. For $\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_n)\in W$, let $\eta(\boldsymbol{r})$ be a line bundle defined by

$$\eta(\boldsymbol{r}):=[2p_1+\cdots+2p_m+2q_1+\cdots+2q_n].$$

Note that the value $f(\boldsymbol{r})$ of a holomorphic section $f\in\Gamma(W; \mathcal{O}_{X(m)}(F))$ at a point $\boldsymbol{r}\in W$ is regarded as a holomorphic section of the line bundle $\kappa^{\otimes 2}\otimes\eta(\boldsymbol{r})$ over M and hence as a meromorphic section of the line bundle $\kappa^{\otimes 2}$ over M. The main purpose in this subsection is to establish the following two theorems.

THEOREM 6.10. *There exist one and only one holomorphic section* $\varphi_k^{(a)}\in\Gamma(W'; \mathcal{O}_{X(m)}(F))$ *for each* $a=0, 1, 2$ *and* $k=1, \cdots, n$ *such that the following condition holds: For any* $\boldsymbol{r}\in W'$, *regarding* $\varphi_k^{(a)}$ *as*

$$\varphi_k^{(a)}(\boldsymbol{r})\in\Gamma(M; \mathcal{O}(\kappa^{\otimes 2}\otimes\eta(\boldsymbol{r})))\subset\Gamma(M; \mathcal{M}(\kappa^{\otimes 2})),$$

$\varphi_k^{(a)}(\boldsymbol{r})$ *is holomorphic in* $M\backslash\{p_1, \cdots, p_m, q_1, \cdots, q_n\}$,

(i) $\varphi_k^{(a)}(\boldsymbol{r})$ *has at most simplepoles at the points* $p_1, \cdots, p_m$ *and*

(ii) $$\varphi_k^{(a)}(\boldsymbol{r})=\left\{\frac{\delta_{kl}}{(y_l-\lambda_l)^a}+O(y_l-\lambda_l)\right\}(dy_l)^{\otimes 2} \quad in \quad V_l$$

for $l=1, \cdots, n$.

THEOREM 6.11. *There exists one and only one holomorphic section* $\psi\in\Gamma(W'; \mathcal{O}_{X(m)}(F))$ *such that the following condition holds: For any* $\boldsymbol{r}\in W'$, *regarding* $\psi(\boldsymbol{r})$ *as*

$$\psi(\boldsymbol{r})\in\Gamma(M; \mathcal{O}(\kappa^{\otimes 2}\otimes\eta(\boldsymbol{r})))\subset\Gamma(M; \mathcal{M}(\kappa^{\otimes 2})),$$

$\psi(\boldsymbol{r})$ *is holomorphic in* $M\backslash\{p_1, \cdots, p_m, q_1, \cdots, q_n\}$,

(i) $$\psi(\boldsymbol{r})=\left\{\frac{\alpha_j}{(x_j-t_j)^2}+O\left(\frac{1}{x_j-t_j}\right)\right\}(dx_j)^{\otimes 2} \quad in \quad U_j$$

for $j=1, \cdots, m$, *where* $\alpha_j=(1/4)(\theta_j^2-1)$, *and*

(ii) $\psi(\boldsymbol{r})$ *has zeros at the points* $q_1, \cdots, q_n$.

Lemma 6.4 plays an important role in the proof of these theorems. To establish these theorems, we need to attack the problem of constructing meromorphic sections of a line bundle which admit a prescribed singularities. This is a kind of Cousin problem. Consider the sheaf $\mathcal{B}$ over $M\times X(m)$ defined by

$$\mathcal{B}:=\mathcal{O}_{M\times X(m)}(u^*\kappa^{\otimes 2}\otimes[D_1+\cdots+D_m-(D_1'+\cdots+D_n')]),$$

(cf. (6.6)). Lemma 6.4 and Kodaira-Spencer's theorem [KS, I, Theorem 2.2] [GR, Chap. 10, §5, Theorem 5] imply that the zeroth and first direct image sheaves of $\mathscr{B}$ for the projection $u: M\times X(m)\to X(m)$ are the zero sheaves:

$$\mathscr{R}^i v_*(\mathscr{B})=0 \qquad (i=0,1)\,. \tag{6.7}$$

PROOF OF THEOREM 6.10. First we shall prove the existence of the sections $\varphi_k^{(a)}$. Let ϕ be the meromorphic section of the line bundle $[D_1+\cdots+D_m-(D_1'+\cdots+D_n')]$ such that the associated divisor is $D_1+\cdots+D_m-(D_1'+\cdots+D_n')$. We denote by $\mathcal{U}\times W'$ the open covering $\{U_i\times W'\,;\ i\in I\}$ of $M\times W'$. We define 1-cocycles $\sigma_k^{(a)}\in Z^1(\mathcal{U}\times W'\,;\ \mathscr{B})$ $(k=1,\cdots,n)$ by letting $\sigma_{k,ij}^{(a)}\in\mathscr{B}((U_i\cap U_j)\times W')$ $(i,j\in I)$ to be

$$\sigma_{k,ij}^{(a)}=\begin{cases} \dfrac{(dy_k)^{\otimes 2}}{(y_k-\lambda_k)^a}\otimes\phi & \text{if } \ i=m+k\neq j \\ -\dfrac{(dy_k)^{\otimes 2}}{(y_k-\lambda_k)^a}\otimes\phi & \text{if } \ i\neq m+k=j \\ 0 & \text{otherwise}\,. \end{cases}$$

This is well-defined. For, the cocycle condition is easily checked and Remark 6.7 implies that $y_k-\lambda_k\neq 0$ in $(U_i\cap U_j)\times W'$ for $i=m+k\neq j$ and for $i\neq m+k=j$. Hence $\sigma_{k,ij}^{(a)}$ is holomorphic in $(U_i\cap U_j)\times W'$. The 1-cocycle $\sigma_k^{(a)}$ determines an element of $\mathscr{R}^1v_*(\mathscr{B})$ at $\boldsymbol{r}^{\circ}$, which is in fact zero element (cf. (6.7)). Hence, replacing W' by a sufficiently small one if necessary, we conclude that there exists a 0-cochain $(\tau_{k,i}^{(a)})\in C^0(\mathcal{U}\times W'\,;\ \mathscr{B})$ such that

$$\sigma_{k,ij}^{(a)}=\tau_{k,j}^{(a)}-\tau_{k,i}^{(a)} \qquad \text{in} \quad (U_i\cap U_j)\times W'\,.$$

Using this 0-cochain, we can define a meromorphic section $\varphi_k^{(a)}=(\varphi_{k,i}^{(a)})\in\Gamma(M\times W'\,;\ \mathscr{M}_{M\times X(m)}(u^*\kappa^{\otimes 2}))$ by

$$\varphi_{k,i}^{(a)}:=\tau_{k,i}^{(a)}\otimes\phi^{\otimes(-1)}+\frac{\delta_{i,m+k}(dy_k)^{\otimes 2}}{(y_k-\lambda_k)^a} \qquad \text{in} \quad U_i\times W'\,,$$

where δ_{ij} is Kronecker's delta. It follows immediately from the definition that $\varphi_k^{(a)}$ is regarded as a holomorphic section of the vector bundle F over W' and satisfy the results of Theorem 6.10. Thus we have established the existence assertion of the theorem.

We proceed to the proof of the uniqueness assertion. Suppose that $\varphi_k^{(a)}$ and $\varphi_k'^{(a)}$ are two holomorphic sections of F over W' satisfying the conditions of the theorem. We consider the difference $\Phi=\varphi_k^{(a)}-\varphi_k'^{(a)}$. For any $\boldsymbol{r}\in W'$, if $\Phi(\boldsymbol{r})$ is regarded as a meromorphic quadratic differential on

M, then $\Phi(\boldsymbol{r})$ has at most simple poles at $p_1, \cdots, p_m$ and zeros at $q_1, \cdots, q_n$, i. e. $\Phi(\boldsymbol{r}) \in H^0(M; \mathcal{O}(\xi(\boldsymbol{r})))$. Since $H^0(M; \mathcal{O}(\xi(\boldsymbol{r})))=0$ (Lemma 6.4), we have $\Phi(\boldsymbol{r})=0$ for any $\boldsymbol{r} \in W'$, which establishes the uniqueness assertion of the theorem. ■

For $\boldsymbol{r}=(p_1, \cdots, p_m, q_1, \cdots, q_n) \in X(m)$, let

$$\zeta_j(\boldsymbol{r}) := \xi(\boldsymbol{r}) \otimes [p_j] \qquad (j=1, \cdots, m).$$

To prove Theorem 6.11, we need the following lemma.

LEMMA 6.12. *We have* $\dim H^0(M; \mathcal{O}(\zeta_j(\boldsymbol{r})))=1$ *for* $\boldsymbol{r} \in X(m)$ *and* $j=1, \cdots, m$.

PROOF. Since $c_1(\xi(\boldsymbol{r}))=g-1$, we have $c_1(\zeta_j(\boldsymbol{r}))=g$. So the Riemann-Roch formula implies $\dim H^0(M; \mathcal{O}(\zeta_j(\boldsymbol{r}))) \geqq 1$. Assuming that $\dim H^0(M; \mathcal{O}(\zeta_j(\boldsymbol{r}))) \geqq 2$, we shall deduce a contradiction. Let φ_1 and φ_2 be two linearly independent holomorphic sections of $\zeta_j(\boldsymbol{r})$. These are regarded as meromorphic sections of $\xi(\boldsymbol{r})$ which have at most simple pole at p_j adn are holomorphic in $M \backslash \{p_j\}$. So we can take a nonzero vector $(c_1, c_2) \in \boldsymbol{C}^2$ such that the linear combination $\varphi := c_1\varphi_1 + c_2\varphi_2$ is a holomorphic section of $\xi(\boldsymbol{r})$. Since φ_1 and φ_2 are linearly independent, φ does not vanish identically. Hence we obtain $\dim H^0(M; \mathcal{O}(\xi(\boldsymbol{r}))) \geqq 1$, which contradicts Lemma 6.4, (i). ■

PROOF OF THEOREM 6.11. Consider the sheaves $\mathcal{C}_j := \mathcal{B} \otimes \mathcal{O}_{M \times X(m)}([D_j])$ $(j=1, \cdots, m)$ over $M \times X(m)$. Since $\dim H^0(M; \mathcal{O}(\zeta(\boldsymbol{r})))=1$ for every $\boldsymbol{r} \in X(m)$, Kodaira-Spencer's theorem [KS, I, Theorem 2.2][GR. Chap. 10, § 5, Theorem 5] implies that the direct image sheaves $v_*(\mathcal{C}_j)$ are locally free analytic sheaves of rank 1 over $X(m)$. Let L_j be the associated holomorphic line bundles. If W' is sufficiently small, then there exist local frames ψ_j of L_j over W'. For any $\boldsymbol{r} \in W'$, $\psi_j(\boldsymbol{r})$ are regarded as nontrivial holomorphic sections of $\zeta_j(\boldsymbol{r})$ over M. Moreover, if $\psi_j(\boldsymbol{r})$ are regarded as meromorphic quadratic differentials on M, then

(i) $\psi_j(\boldsymbol{r})$ are holomorphic in $M \backslash \{p_1, \cdots, p_m, q_1, \cdots, q_n\}$,

(ii) $$\psi_j(\boldsymbol{r}) = \left\{ \frac{h_j(\boldsymbol{r})}{(x_i-t_i)^2} \delta_{ij} + O\left(\frac{1}{x_i-t_i}\right) \right\} (dx_i)^{\otimes 2} \qquad \text{in} \quad U_i$$

for $i=1, \cdots, m$, where $h_j(\boldsymbol{r})$ are holomorphic functions in W' and δ_{ij} is Kronecker's symbol, and

(iii) $\psi_j(\boldsymbol{r})$ have zeros at $q_1, \cdots, q_n$.

We notice that $h_j(\boldsymbol{r})\neq 0$ for $\boldsymbol{r}\in W'$. For, otherwise, (ii) and (iii) implies that $\psi_j(\boldsymbol{r})$ is a holomorphic section of $\xi(\boldsymbol{r})$. Hence Lemma 6.4, (ii) implies that $\psi_j(\boldsymbol{r})$ is identically zero on M. This contradicts the fact that ψ_j is a frame of the line bundle L_j over W'. Hence we have $h_j(\boldsymbol{r})\neq 0$ for $\boldsymbol{r}\in W'$. Replacing ψ_j by ψ_j/h_j and rewriting ψ_j/h_j as ψ_j, we see that the new ψ_j satisfy (i), (iii) and

$$\text{(ii)}' \qquad \psi_j(\boldsymbol{r})=\left\{\frac{\delta_{ij}}{(x_i-t_i)^2}+O\left(\frac{1}{x_i-t_i}\right)\right\}(dx_i)^{\otimes 2} \qquad \text{in} \quad U_i$$

for $i=1,\cdots,m$. ψ_j are regarded as holomorphic sections of the vector bundle F over W'. Now we put

$$\psi:=\alpha_1\psi_1+\cdots+\alpha_m\psi_m\in\Gamma(W'\,;\,\mathcal{O}_{X(m)}(F))\,.$$

It is clear that ψ satisfies the desired properties of Theorem 6.11. Next we shall show the uniqueness of ψ. If there are two such sections ψ and ψ', then the difference $\Psi(\boldsymbol{r}):=\psi(\boldsymbol{r})-\psi'(\boldsymbol{r})$ belongs to $H^0(M\,;\,\mathcal{O}(\xi(\boldsymbol{r})))$. Hence, by Lemma 6.4, (ii), we obtain $\Psi(\boldsymbol{r})=0$ for $\boldsymbol{r}\in W'$, from which the uniqueness follows. ■

6.4. Affine bundle structure of $\mathcal{E}(m\,;\,\theta)$.

Theorem 6.10 and Theorem 6.11 imply immediately the following theorem.

THEOREM 6.13. *For any* $\boldsymbol{r}^\circ\in X(m)$ *let* $(W\,;\,t_1,\cdots,t_m,\lambda_1,\cdots,\lambda_n)$ *be a sufficiently small admissible coordinate neighbourhood of* $\boldsymbol{r}^\circ$ *in* $X(m)$. *For any* $\nu=(\nu_1,\cdots,\nu_n)\in\boldsymbol{C}^n$ *there exists the unique section* $Q\in\Gamma(W\,;\,\mathcal{O}_{X(m)}(F))$ *such that the following condition holds: For any* $\boldsymbol{r}=(p_1,\cdots,p_m,q_1,\cdots,q_n)\in W$, *regarding* $Q(\boldsymbol{r})$ *as*

$$Q(\boldsymbol{r})\in\Gamma(M\,;\,\mathcal{O}(\kappa^{\otimes 2}\otimes\eta(\boldsymbol{r})))\subset\Gamma(M\,;\,\mathcal{M}(\kappa^{\otimes 2}))\,,$$

$Q(\boldsymbol{r})$ *is holomorphic in* $M\backslash\{p_1,\cdots,p_m,q_1,\cdots,q_n\}$,

$$\text{(i)} \qquad Q(\boldsymbol{r})=\left\{\frac{\alpha_j}{(x_j-t_j)^2}+O\left(\frac{1}{x_j-t_j}\right)\right\}(dx_j)^{\otimes 2} \qquad \textit{in} \quad U_j$$

for $j=1,\cdots,m$, *and*

$$\text{(ii)} \quad Q(\boldsymbol{r})=\left\{\frac{3}{4(y_k-\lambda_k)^2}-\frac{\nu_k}{y_k-\lambda_k}+\nu_k^2+O(y_k-\lambda_k)\right\}(dy_k)^{\otimes 2} \qquad \textit{in} \quad V_k$$

for $k=1,\cdots,n$. *In terms of auxiliary sections* $\varphi_k^{(a)}$ *and* ψ, *the meromor-*

phic quadratic differential Q *is given by*

$$Q=\sum_{k=1}^{n}\left\{\frac{3}{4}\varphi_k^{(2)}-\nu_k\varphi_k^{(1)}+\nu_k^2\varphi_k^{(0)}\right\}+\psi. \tag{6.8}$$

COROLLARY 6.14. *$\mathcal{E}(m;\theta)$ is a complex manifold of dimension $m+2n=m+2(m+3g-3)$. Let $(W;t_1,\cdots,t_m,\lambda_1,\cdots,\lambda_n)$ be as in Theorem 6.10. We can take $(t,\lambda,\nu)=(t_1,\cdots,t_m,\lambda_1,\cdots,\lambda_n,\nu_1,\cdots,\nu_n)$ as a local coordinate of $\mathcal{E}(m;\theta)$ in $\mathcal{E}(m;\theta)|W:=\pi^{-1}(W)$.*

Notice that the local coordinate (t,λ,ν) of $\mathcal{E}(m;\theta)$ in $\mathcal{E}(m;\theta)|W$ depends on the choice of an admissible coordinate neighbourhood $(W;t,\lambda)$. For another admissible coordinate neighbourhood $(\tilde{W};\tilde{t},\tilde{\lambda})$ such that $W\cap\tilde{W}\neq\emptyset$, let $\tilde{\nu}$ be the associated accessory parameters. We ask how two local coordinates (t,λ,ν) and $(\tilde{t},\tilde{\lambda},\tilde{\nu})$ of $\mathcal{E}(m;\theta)$ are related. To answer this question, we need the following lemma, which is proved by an easy direct calculation.

LEMMA 6.15. *Let φ be a meromorphic quadratic differential in a open subset D of M and suppose that φ has at most a double pole at a point $q\in D$. Let x and y are local coordinates of M in D such that the transition function $y=f(x)$ is a fractional linear transformation. We put $\xi=x(q)$ and $\eta=y(q)$. Suppose that φ is expressed in terms of the local coordinates x and y as*

$$\varphi=\sum_{k=-2}^{\infty}a_k(x-\xi)^k(dx)^{\otimes 2}=\sum_{k=-2}^{\infty}b_k(y-\eta)^k(dy)^{\otimes 2}.$$

Then a_k and b_k $(k=-2,-1)$ satisfy the following relations:

$$a_{-2}=b_{-2},\qquad b_{-1}=\kappa'(\xi)a_{-2}+\kappa(\xi)a_{-1}, \tag{6.9}$$

where $\kappa(\xi)=1/f'(\xi)$.

Let $\mathcal{U}=\{(U_i,x_i)\}$ and $\tilde{\mathcal{U}}=\{(\tilde{U}_i,\tilde{x}_i)\}$ be two $\boldsymbol{r}^\circ$-admissible projective coordinate covering of M and let $(W;t,\lambda)$ and $(\tilde{W};\tilde{t},\tilde{\lambda})$ be the associated admissible coordinate neighbourhoods of $\boldsymbol{r}^\circ$ in $X(m)$, respectively. Moreover let ν and $\tilde{\nu}$ be the accessory parameters associated with $(W;t,\lambda)$ and $(\tilde{W};\tilde{t},\tilde{\lambda})$, respectively. We write the transition relations as $\tilde{x}_j=f_j(x_j)$ $(j=1,\cdots,m)$ and $\tilde{y}_k=g_k(y_k)$ $(k=1,\cdots,n)$, where f_j and g_k are fractional linear transformations. Then the two local coordinates (t,λ,ν) and $(\tilde{t},\tilde{\lambda},\tilde{\nu})$ on $\mathcal{E}(m;\theta)|(W\cap\tilde{W})$ satisfy the following relations:

$$(6.10)\qquad \begin{cases} \tilde{t}_j = f_j(t_j) & (j=1, \cdots, m), \\ \tilde{\lambda}_k = g_k(\lambda_k) & (k=1, \cdots, n), \\ \tilde{\nu}_k = \kappa_k(\lambda_k)\nu_k + (3/4)\kappa_k'(\lambda_k) & (k=1, \cdots, n), \end{cases}$$

where $\kappa_k(\lambda) := 1/g_k'(\lambda)$. The third formula of (6.10) follows from Lemma 6.15. This observation leads us to the following theorem.

THEOREM 6.16. *$\mathcal{E}(m\,;\,\theta)$ admits a natural complex manifold structure such that the projection $\pi : \mathcal{E}(m\,;\,\theta) \to X(m)$ is a holomorphic affine bundle of rank $n = m + 3g - 3$.*

6.5. Fundamental 2-form on $\mathcal{E}(m\,;\,\theta)$ and Hamiltonians.

In this subsection we shall show that there exists a closed 2-form defined canonically on the complex manifold $\mathcal{E}(m\,;\,\theta)$. For this purpose we shall first introduce the "Hamiltonian functions" $H_j\,(j=1, \cdots, m)$ defined locally on $\mathcal{E}(m\,;\,\theta)$.

Let $(W;\, t, \lambda)$ be an admissible coordinate neighbourhood of $X(m)$, $\nu = (\nu_1, \cdots, \nu_n)$ the accessory parameters associated with $(W;\, t, \lambda)$. Then we can express $Q \in \mathcal{E}(m\,;\,\theta)|W$ as

$$(6.11)\qquad Q = \left\{ \frac{\alpha_j}{(x_j - t_j)^2} + \frac{H_j}{x_j - t_j} + O(1) \right\} (dx_j)^{\otimes 2} \qquad \text{in} \quad U_j$$

for $j = 1, \cdots, m$, where $H_j = H_j(Q)$ is holomorphic functions of Q in $\mathcal{E}(m\,;\,\theta)|U$.

DEFINITION 6.17. We call $H_j\,(j=1, \cdots, m)$ the *Hamiltonian functions* (with respect to an admissible coordinate neighbourhood $(W;\, t, \lambda)$ of a point $\boldsymbol{r}^\circ \in X(m)$).

Consider the closed 2-form Ω on $\mathcal{E}(m\,;\,\theta)|W$ defined by

$$(6.12)\qquad \Omega := \sum_{k=1}^{n} d\nu_k \wedge d\lambda_k - \sum_{j=1}^{m} dH_j \wedge dt_j\,.$$

For another admissible coordinate neighbourhood $(\tilde{W}\,;\, \tilde{t}, \tilde{\lambda})$ we define the closed 2-form $\tilde{\Omega}$ on $\mathcal{E}(m\,;\,\theta)|\tilde{W}$ in a similar manner. We shall show that, if $W \cap \tilde{W} \neq \emptyset$, then $\Omega = \tilde{\Omega}$ on $\mathcal{E}(m\,;\,\theta)|(W \cap \tilde{W})$. Let $\tilde{H}_j$ be the Hamiltonian functions with respect to $(\tilde{W}\,;\, \tilde{t}, \tilde{\lambda})$. Then Lemma 6.15 implies that the following transition relations hold:

$$(6.13)\qquad \tilde{H}_j = \alpha_j K_j'(t_j) + K_j(t_j) H_j \qquad (j=1, \cdots, m),$$

where $K_j(t):=1/f_j'(t)$ (cf. Lemma 6.15). Making use of (6.8) and (6.10), we can show that $\Omega=\tilde{\Omega}$ holds on $\mathcal{E}(m;\theta)|(W\cap\tilde{W})$. Hence we obtain the following theorem.

THEOREM 6.18. *There exists a closed 2-form Ω on $\mathcal{E}(m;\theta)$ which is locally expressed as* (6.12).

DEFINITION 6.19. The closed 2-form Ω is called the *fundamental* 2-*form on* $\mathcal{E}(m;\theta)$.

§ 7. Auxiliary vector fields and residue calculus for Hamiltonians.

7.1. Auxiliary meromorphic vector fields on M.

The Hamiltonian functions H_j ($j=1,\cdots,m$) depend on the choice of an admissible coordinate neighbourhood $(W;t,\lambda)$ in $X(m)$ and they are holomorphic functions in $\mathcal{E}(m;\theta)|W$, (cf. Section 6). Since (t,λ,ν) can be taken as a local coordinate in $\mathcal{E}(m;\theta)|W, H_j$ are holomorphic functions of (t,λ,ν). We shall consider the monodromy preserving deformation on $\mathcal{E}(m;\theta)_{\mathrm{irr}}$ in the next section, in which we need to know the partial derivatives $\dfrac{\partial H_j}{\partial\lambda_k}$ and $\dfrac{\partial H_j}{\partial\nu_k}$ of the Hamiltonian functions H_j in some detail. To do this we shall introduce certain auxiliary meromorphic vector fields on M, in terms of which we can express these derivatives in a useful fashion. The reason why it is convenient to consider meromorphic vector fields on M is stated as follows: Recall that an SL-operator is identified with a meromorphic quadratic differential on M. By the residue theorem, the following sesquilinear form vanishes identically.

$$(7.1)\qquad \begin{array}{ccc} \Gamma(M;\mathcal{M}(\kappa^{\otimes 2}))\times\Gamma(M;\mathcal{M}(\kappa^{\otimes(-1)})) & \longrightarrow & \boldsymbol{C} \\ \Cup & & \Cup \\ (Q,v) & \longmapsto & \langle Q\otimes v\rangle\,, \end{array}$$

where $\langle Q\otimes v\rangle$ denotes the total sum of the residues of the meromorphic differential $Q\otimes v$ on M. Of course we have $\langle Q\otimes v\rangle=0$. Applying this trivial pairing to SL-operators and auxiliary vector fields, we can obtain various formulae which are useful when discussing the monodromy preserving deformation. See also Remark 7.6 below.

For $\boldsymbol{r}=(p_1,\cdots,p_m,q_1,\cdots,q_n)\in X(m)$, let $\sigma_j(\boldsymbol{r})$ ($j=1,\cdots,m$) be the holomorphic line bundles over M defined by

$$\sigma_j(\boldsymbol{r}) := \kappa^{\otimes(-1)} \otimes [q_1 + \cdots + q_n - (p_1 + \cdots + \hat{p}_j + \cdots + p_m)],$$

where $\hat{p}_j$ stands for the omittion of p_j. Notice that $\sigma_j(\boldsymbol{r})$ is the Serre-dual to $\xi(\boldsymbol{r}) \otimes [-p_j]$. Hence, by the Serre duality theorem and the Riemann-Roch formula, we obtain

$$\begin{aligned}\dim H^0(M; \mathcal{O}(\sigma_j(\boldsymbol{r}))) &= \dim H^1(M; \mathcal{O}(\xi(\boldsymbol{r}) \otimes [-p_j])) \\ &= \dim H^0(M; \mathcal{O}(\xi(\boldsymbol{r}) \otimes [-p_j])) - c_1(\xi(\boldsymbol{r}) \otimes [-p_j]) + g - 1 \\ &= \dim H^0(M; \mathcal{O}(\xi(\boldsymbol{r}) \otimes [-p_j])) + 1 .\end{aligned}$$

Since $H^0(M; \mathcal{O}(\xi(\boldsymbol{r}))) = 0$ for $\boldsymbol{r} \in X(m)$ (cf. Lemma 6.4), we have $H^0(M; \mathcal{O}(\xi(\boldsymbol{r}) \otimes [-p_j])) = 0$. Hence we obtain

(7.2) $\qquad \dim H^0(M; \mathcal{O}(\sigma_j(\boldsymbol{r}))) = 1 \quad$ for $\quad \boldsymbol{r} \in X(m) \quad (j = 1, \cdots, m)$.

Consider the sheaves $\mathcal{D}_j$ over $M \times X(m)$ defined by

$$\mathcal{D}_j := \mathcal{O}_{M \times X(m)}(u^* \kappa^{\otimes(-1)} \otimes [D'_1 + \cdots + D'_n - (D_1 + \cdots + \hat{D}_j + \cdots + D_m]),$$

(see Section 6 as for the notation). Kodaira-Spencer's theorem [KS, I, Theorem 2.2] [GR, Chap. 10, §5, Theorem 5], together with (7.2), implies that the direct image sheaves $v_*(\mathcal{D}_j)$ are locally free analytic sheaves of rank 1.

THEOREM 7.1. *For any $\boldsymbol{r}^\circ \in X(m)$, let $(W; t, \lambda)$ be a sufficiently small admissible coordinate neighbourhood of $\boldsymbol{r}^\circ$ in $X(m)$. There exist unique holomorphic sections $V_j \in \Gamma(W; v_*(\mathcal{D}_j))$ $(j=1, \cdots, m)$ such that the following conditions fold: For any $\boldsymbol{r} \in W$, if $V_j(\boldsymbol{r})$ are regarded as meromorphic vector fields on M, then*

(i) $\qquad$ *$V_j(\boldsymbol{r})$ are holomorphic in $M \backslash \{p_1, \cdots, p_m, q_1, \cdots, q_n\}$,*

(ii) $\qquad V_j(\boldsymbol{r}) = \{-\delta_{ij} + O(x_i - t_i)\} \dfrac{\partial}{\partial x_i} \quad$ *in* $\quad U_i$

for $i = 1, \cdots, m$, where δ_{ij} is Kronecker's symbol, and

(iii) $\qquad$ *$V_j(\boldsymbol{r})$ have at most simple poles at $q_1, \cdots, q_n$.*

PROOF. Let L_j be the holomorphic line bundles over $X(m)$ associated with the locally free analytic sheaves $v_*(\mathcal{D}_j)$ of rank 1. If W is sufficiently small, then there exist local frames W_j of L_j over W. If $W_j(\boldsymbol{r})$ $(\boldsymbol{r} \in W)$ are regarded as meromorphic vector fields on M, then

(i)′ $\qquad W_j(\boldsymbol{r})$ are holomorphic in $M \backslash \{p_1, \cdots, p_m, q_1, \cdots, q_n\}$, (ii)′ $W_j(\boldsymbol{r})=\{-a_j(\boldsymbol{r})\delta_{ij}+O(x_i-t_i)\}\dfrac{\partial}{\partial x_i}$ in U_i

for $i=1,\cdots,m$, where $a_j(\boldsymbol{r})$ are holomorphic functions in W,

(iii)′ $W_j(\boldsymbol{r})$ have at most simple poles at $q_1,\cdots,q_n$.

We shall show $a_j(\boldsymbol{r})\neq 0$ for $\boldsymbol{r}\in W$. Suppose that $a_j(\boldsymbol{r})=0$ for some $\boldsymbol{r}\in W$, then (i)′-(iii)′ implies that $W_j(\boldsymbol{r})\in H^0(M;\mathcal{O}(\sigma_j(\boldsymbol{r})\otimes[-p_j]))$. Since $\sigma_j(\boldsymbol{r})\otimes[-p_j]$ are Serre-dual to $\xi_j(\boldsymbol{r})$, Lemma 6.4, (i) and the Serre duality theorem imply $H^0(M;\mathcal{O}(\sigma_j(\boldsymbol{r})\otimes[-p_j]))=0$. So $W_j(\boldsymbol{r})$ vanishes identically on M, which contradict the fact that W_j is a frame of L_j over W. Hence $a_j(\boldsymbol{r})\neq 0$ for any $\boldsymbol{r}\in W$. Now it is clear from (i)′-(iii)′ that $V_j:=W_j/a_j$ are the desired sections of $v_*(\mathcal{D}_j)$ over W. Also the uniqueness of V_j is easily established. Indeed, if there are two such sections V_j and V_j', then the difference $V_j(\boldsymbol{r})-V_j'(\boldsymbol{r})$ ($\boldsymbol{r}\in W$) belongs to $H^0(M;\mathcal{O}(\zeta_j(\boldsymbol{r})\otimes[-p_j]))=\{0\}$. Hence $V_j(\boldsymbol{r})=V_j'(\boldsymbol{r})$ for $\boldsymbol{r}\in W$, which establishes the uniqueness. ■

7.2. Residue calculus for the Hamiltonians.

Let $(W;t,\lambda)$ be a sufficiently small admissible coordinate neighbourhood of a point in $X(m)$, $\nu=(\nu_1,\cdots,\nu_n)$ the associated accessory parameters, H_j $(j=1,\cdots,m)$ the associated Hamiltonian functions on $\mathcal{E}(m;\theta)|W$. Our purpose in this subsection is express the partial derivatives $\dfrac{\partial H_j}{\partial t_i}, \dfrac{\partial H_j}{\partial \lambda_k}(\boldsymbol{r})$ and $\dfrac{\partial H_j}{\partial \nu_k}(\boldsymbol{r})$ $(i=1,\cdots,m,\ k=1,\cdots,n)$ of the Hamiltonian functions H_j $(j=1,\cdots,m)$ at $\boldsymbol{r}=(p_1,\cdots,p_m,q_1,\cdots,q_n)\in W$ in terms of the coefficients of the Laurent expansions of $Q(\boldsymbol{r})$ *and* $V_j(\boldsymbol{r})$ at the *apparent singular points* $q_1,\cdots,q_n$.

Let $\varphi_k^{(a)},\phi\in\Gamma(W;\mathcal{O}_{X(m)}(F))$ $(a=0,1,2,\ k=1,\cdots,n)$ be the holomorphic families of auxiliary meromorphic quadratic differentials on M introduced in Theorems 6.10 and 6.11 and let $V_j\in\Gamma(W;v_*(\mathcal{D}_j))$ $(j=1,\cdots,m)$ the holomorphic families of auxiliary meromorphic vector fields on M introduced in Theorem 7.1. We assume that the auxiliary vector fields V_i admits the following local expressions:

$$V_i=\{-\delta_{ij}+\xi_{ij}^{(1)}(x_j-t_j)+\xi_{ij}^{(2)}(x_j-t_j)^2+O(x_j-t_j^3)\}\frac{d}{dx_j} \quad \text{in } U_j,$$

$$(7.3)\qquad V_i=\left\{\sum_{l=-1}^{\infty}\eta_{ik}^{(l)}(y_k-\lambda_k)^l\right\}\frac{d}{dy_k} \quad \text{in } V_k,$$

where the coefficients $\xi_{ij}^{(b)}$ and $\eta_{ik}^{(l)}$ $(i,j=1,\cdots,m,\ k=1,\cdots,n,\ b=1,2)$ are holomorphic functions in W. We shall obtain expressions of the auxiliary quadratic differentials $\varphi_k^{(a)}$ in terms of these coefficients.

LEMMA 7.2. *$\varphi_k^{(a)}$ $(k=1,\cdots,n,\ a=1,2,3)$ and ψ admit the following local expressions*:

$$\varphi_k^{(a)}=\left\{\frac{\eta_{jk}^{(a-1)}}{x_j-t_j}+O(1)\right\}(dx_j)^{\otimes 2} \qquad in \quad U_j,$$

$$\psi=\left\{\frac{\alpha_j}{(x_j-t_j)^2}+\frac{1}{x_j-t_j}\sum_{i=1}^{m}\alpha_i\xi_{ji}^{(1)}+O(1)\right\}(dx_j)^{\otimes 2} \qquad in \quad U_j$$

for $j=1,\cdots,m$.

PROOF. As was noted in (7.3), the tensor product $R\otimes V$ of a meromorphic quadratic differentials R and a meromorphic vector field V on M is a meromorphic differential on M, whence the residue theorem implies that the total sum $\langle R\otimes V\rangle$ of its residues equals to 0. We apply this observation to $R=\varphi_k^{(a)},\psi$ (the auxiliary quadratic differentials) and $V=V_j$ (the auxiliary vector fields). We see that $\langle\varphi_k^{(a)}\otimes V_j\rangle=0$ leads to the first assertion of the theorem and $\langle\psi\otimes V_j\rangle=0$ leads to the second assertion. ■

We shall give useful expressions of the Hamiltonian functions.

LEMMA 7.3. *The Hamiltonian functions H_j $(j=1,\cdots,m)$ admit the following expressions*:

$$H_j=\sum_{i=1}^{m}\alpha_i\xi_{ji}^{(1)}+\sum_{k=1}^{n}\left(\frac{3}{4}\eta_{jk}^{(1)}-\nu_k\eta_{jk}^{(0)}+\nu_k^2\eta_{jk}^{(-1)}\right).$$

In particular, H_j are quadratic polynomials of the accessory parameters $\nu=(\nu_1,\cdots,\nu_n)$ with coefficients holomorphic in (t,λ).

PROOF. Substituting the formulae in Lemma 7.2 into (6.8) and using (6.11), we obtain Lemma 7.3 immediately. ■

We assume that $\varphi_k^{(a)}$ and ψ admit the following local expressions around the apparent singular points $q_1,\cdots,q_n$:

$$\varphi_k^{(a)}=\left\{\frac{\delta_{kl}}{(y_l-\lambda_l)^a}+C_{kl}^{(a)}(y_l-\lambda_l)+O((y_l-\lambda_l)^2)\right\}(dy_l)^{\otimes 2} \qquad \text{in} \quad V_l,$$

$$\psi=\{D_l(y_l-\lambda_l)+O(y_l-\lambda_l)\}(dy_l)^{\otimes 2} \qquad \text{in} \quad V_l$$

for $k,l=1,\cdots,n$ and $a=0,1,2$.

LEMMA 7.4. *We have*

$$\text{(i)}\qquad \frac{\partial \eta_{jk}^{(a-1)}}{\partial \lambda_l} = -C_{kl}^{(a)}\eta_{jl}^{(-1)} + a\delta_{kl}\eta_{jk}^{(a)},$$

$$\text{(ii)}\qquad \sum_{i=1}^{m} \alpha_i \frac{\partial \xi_{ji}^{(1)}}{\partial \lambda_l} = -D_l \eta_{jl}^{(-1)},$$

$$\text{(iii)}\qquad \frac{\partial \eta_{jk}^{(a-1)}}{\partial t_i} = \xi_{ji}\eta_{jk}^{(a-1)},$$

$$\text{(iv)}\qquad \sum_{l=1}^{m} \alpha_l \frac{\partial \xi_{jl}^{(1)}}{\partial t_i} = 2\alpha_i \xi_{ji}^{(2)} + \xi_{ji}^{(1)} \sum_{l=1}^{m} \alpha_l \xi_{il}^{(1)}$$

for $j=1,\cdots,m$ *and* $a=0,1,2.$

PROOF. As in the proof of Lemma 7.2, the first formula of the lemma follows from $\left\langle \frac{\partial V_j}{\partial \lambda_l} \otimes \varphi_k^{(a)} \right\rangle = 0$, the second one follows from $\left\langle \frac{\partial V_j}{\partial \lambda_l} \otimes \psi \right\rangle = 0$, the third one follows from $\left\langle \frac{V_j}{\partial t_i} \otimes \varphi_k^{(a)} \right\rangle = 0$ and the last one follows from $\left\langle \frac{V_j}{\partial t_i} \otimes \psi \right\rangle = 0$. ■

Furthermore we assume that Q admits the following local expression around the apparent singular points:

$$Q = \left\{ \frac{3}{4(y_k-\lambda_k)^2} - \frac{\nu_k}{y_k-\lambda_k} + \nu_k^2 + u_k(y_k-\lambda_k) + O((y_k-\lambda_k)^2) \right\} (dy_k)^{\otimes 2}$$
$$\text{in} \quad V_k \ (k=1,\cdots,n).$$

The following theorem is the main theorem in this subsection.

THEOREM 7.5 (Partial derivatives of the Hamiltonian functions).

$$\begin{cases} \dfrac{\partial H_j}{\partial \nu_k} = -\eta_{jk}^{(0)} + 2\nu_k \eta_{jk}^{(-1)}, \\ \dfrac{\partial H_j}{\partial \lambda_k} = \dfrac{3}{2}\eta_{jk}^{(2)} - \nu_k \eta_{jk}^{(1)} + u_k \eta_{jk}^{(-1)}, \\ \dfrac{\partial H_j}{\partial t_i} = 2\alpha_i \xi_{ji}^{(2)} + \xi_{ji}^{(1)} H_i, \end{cases}$$

for $i,j=1,\cdots,m$ *and* $k=1,\cdots,n.$

PROOF. The first formula follows immediately from Lemma 7.2 and the second and the third formula follow from Lemma 7.2 and Lemma 7.4. ∎

REMARK 7.6. Notice that the formulae in Theorem 7.5 express the partial derivatives $\frac{\partial H_j}{\partial \lambda_k}$ and $\frac{\partial H_j}{\partial \nu_k}$ of H_j in terms of ν_k, $\eta_{jk}^{(l)}$ and u_k, which are a part of the coefficients of the Laurent expansions of the SL-operator Q and those of the auxiliary vector fields V_j *at the apparent singular points* $q_1, \cdots, q_n$. Namely the partial derivatives of H_j can be expressed only in terms of the quantities determined by the germs of Q and V_j *at the apparent singular points* $q_1, \cdots, q_n$. This fact is remarkable, because H_j are determined by the germs of Q at the generic singular points $p_1, \cdots, p_m$, and so are their partial derivatives. Thus we can say that *the auxiliary vector fields* V_j *on* M *are introduced to convert some of the information about the germ of* Q *at the generic singular points into those about the germs of* Q *and* V_j *at the apparent singular points by way of the residue calculus.*

§ 8. Monodromy preserving deformation.

8.1. Infinitesimal monodromy preserving deformation.

Let $m \in \boldsymbol{N}$, $\theta \in (\boldsymbol{C}_+ \backslash \boldsymbol{Z}_+)^m$. As in the last section we put $n = m + 3g - 3$ (cf. (6.1)). In this section we shall study the monodromy preserving deformation on $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$, which is, by definition, a foliation on $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$ (the monodromy preserving foliation) induced from that on $R(m\,;\,\theta)_{\mathrm{irr}}$ associated with its local system structure over $B(m)$ via the locally biholomorphic map PM (cf. Definition 5.13). The monodromy preserving foliation $\mathfrak{F}$ is of codimension $2n = 2(m + 3g - 3)$ and transverse to each fiber $\mathcal{E}(\boldsymbol{p}\,;\,\theta)_{\mathrm{irr}}$ ($\boldsymbol{p} \in B(m)$) of the projection $\varpi : \mathcal{E}(m\,;\,\theta)_{\mathrm{irr}} \to B(m)$. Thus we can take the space $B(m)$ of generic singular points as the space of deformation parameters (cf. Comments after Definition 5.13). We call each leaf of $\mathfrak{F}$ an *isomonodromic leaf*.

DEFINITION 8.1. The foliation $\mathfrak{F}$ determines a distribution $\mathfrak{D}$ of dimension m on $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$ such that each leaf of $\mathfrak{F}$ is an integral submanifold of $\mathfrak{D}$. We call $\mathfrak{D}$ the *infinitesimal monodromy preserving deformation* on $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$.

In this section we shall derive the *deformation equation*, i. e. a system of nonlinear differential equations which describes the infinitesimal monodromy preserving deformation $\mathfrak{D}$.

Given any $Q^\circ \in \mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$, let $\boldsymbol{r}^\circ=\pi(Q^\circ)$ and $\boldsymbol{p}^\circ=\varpi(Q^\circ)$. Consider the isomonodromic leaf $\mathfrak{l}$ through Q°. Let $\mathcal{U}=\{(U_i, x_i)\}$ be an $\boldsymbol{r}^\circ$-admissible projective coordinate covering of M such that the associated admissible neighbourhood $W=U_1\times\cdots\times U_m\times V_1\times\cdots\times V_n$ of $\boldsymbol{r}^\circ$ is sufficiently small (cf. Definition 6.6). Then $U:=U_1\times\cdots\times U_m$ is a sufficiently small neighbourhood of $\boldsymbol{p}^\circ$ in $B(m)$. Since $\mathfrak{l}$ is transverse to each fiber of $\varpi:\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}\to B(m)$, we may assume that $\varpi:\mathfrak{l}\|U:=\mathfrak{l}\cap\varpi^{-1}(U)\to U$ is biholomorphic. Let $U\ni \boldsymbol{p}\mapsto Q(\boldsymbol{p})\in\mathfrak{l}\|U$ be the inverse correspondence. We put $\pi(Q(\boldsymbol{p}))=(\boldsymbol{p}, \boldsymbol{q}(\boldsymbol{p}))\in X(m)$. Replacing U by a smaller one, if neccessary, we may assume $\boldsymbol{q}(\boldsymbol{p})\in V:=V_1\times\cdots\times V_n$ for $\boldsymbol{p}\in U$. Let $(t,\lambda,\nu)=(t_1,\cdots,t_m\,;\,\lambda_1,\cdots,\lambda_n\,;\,\nu_1,\cdots,\nu_n)$ be the admissible coordinate in $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}|W$ associated with $\mathcal{U}$. On $\mathfrak{l}\|U$ one can take t as local coordinates. Hence λ and ν are holomorphic functions of t i.e. $\lambda=\lambda(t)$ and $\nu=\nu(t)$. Let $\dfrac{\partial}{\partial t_j}, \dfrac{\partial}{\partial\lambda_k}$ and $\dfrac{\partial}{\partial\nu_k}$ $(j=1,\cdots,m,$ $k=1,\cdots,n)$ be the vector fields on $\mathcal{E}(m\,;\,\theta)|W$ associated with the local coordinates (t,λ,ν), whereas $\left(\dfrac{\partial}{\partial t_j}\right)$ $(j=1,\cdots,m)$ the vector fields on $\mathfrak{l}\|U$ associated with the local coordinates t. Then, we have

$$\left(\frac{\partial}{\partial t_j}\right)=\frac{\partial}{\partial t_j}+\sum_{k=1}^{n}\left(\frac{\partial\lambda_k}{\partial t_j}(t)\frac{\partial}{\partial\lambda_k}+\frac{\partial\nu_k}{\partial t_j}(t)\frac{\partial}{\partial\nu_k}\right). \tag{8.1}$$

For a meromorphic quadratic differential $Q=(Q_i)$ on M, we introduce locally defined differential operators

$$A_i(Q)=-\frac{1}{2}D_i^3+2Q_iD_i+D_iQ_i\,,$$

where $D_i=\dfrac{\partial}{\partial x_i}$. For a meromorphic vector field $v=(v_i)$ on M, we put $A(Q)v=(A_i(Q)v_i)$. The following lemma is easy to see.

LEMMA 8.2. *$A(Q):\mathcal{M}(\kappa^{\otimes(-1)})\to\mathcal{M}(\kappa^{\otimes 2})$ is a well-defined differential operator.*

Let V_j $(j=1,\cdots,m)$ be the auxiliary vector fields on M associated with the admissible coordinate neighbourhood $(W\,;\,t,\lambda)$ (cf. Theorem 7.1). Given $Q\in\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}|W$, let $V_j(Q)=V_j(\pi(Q))$. For $Q\in\mathfrak{l}\|U$, consider the meromorphic quadratic differentials

$$\Theta_j(Q)=A(Q)V_j(Q)-\left(\frac{\partial}{\partial t_j}\right)Q \qquad (j=1,\cdots,m)$$

on M. The following theorem is fundamental.

THEOREM 8.3. (i) *For any* $Q\in \mathfrak{l}\| U$, $\Theta_j(Q)$ $(j=1,\cdots,m)$ *vanish identically.*

(ii) $\left(\frac{d}{dt_j}\right)-V_j$ $(j=1,\cdots,m)$ *are commuting vector fields on* $M\times(\mathfrak{l}\| U)$.

The result follows from the complete integrability condition for the so-called extended system of Q (cf. [Ok5, Proposition 1.1]). This theorem will be established in the next subsection. By the unicity theorem for analytic functions, it is sufficient to prove the assertion of the theorem only for $Q\in \mathfrak{l}\| U'$.

8.2. Complete integrability condition for the extended system.

In this subsection we shall introduce the extended system of an SL-operator and consider what follow from the complete integrability condition for it. The main result is Lemma 8.4 stated in the last part.

Let $\mathcal{U}=\{(U_i, x_i)\}$ be the $\boldsymbol{r}^{\circ}$-admissible coordinate covering of M in the last subsection. We may assume that the intersection of any finite number of open sets in $\mathcal{U}$ is simply connected. We put $B(U'):=\bigcup_{\boldsymbol{p}\in U'}|\boldsymbol{r}(\boldsymbol{p})|\times\{\boldsymbol{p}\}$. By the definition of U' (Definition 6.6), $B(U')$ has intersection at most with $U_j\times U'$ $(j=1,\cdots,m+n)$. Given $Q=(Q_j)\in \mathfrak{l}\| U'$, let $L_j=-D_j^2+Q_j$. Consider the differential equations

$$L_j f_j=0 \quad \text{in} \quad U_j. \tag{8.2}$$

There exist fundamental systems of solutions $\boldsymbol{f}_j=(f_j, g_j)$ to (8.2) such that (i) $\boldsymbol{f}_j$ $(j=1,\cdots,m+n)$ are multi-valued holomorphic functions in $U_j\times U'\backslash B(U')$ admitting singularities of regular singular type along $(U_j\times U')\cap B(U')$, (ii) $\boldsymbol{f}_j$ $(j>m+n)$ are single-valued holomorphic functions in $U_j\times U'$. Taking appropriate branches of $\boldsymbol{f}_j$ $(j=1,\cdots,m+n)$, we obtain a 1-cocycle $(C_{jk})\in Z^1(\mathcal{U}\,;\, SL(2\,;\,\boldsymbol{C}))$ such that each C_{jk} is holomorphic in U' and

$$\xi_{jk}\boldsymbol{f}_k=\boldsymbol{f}_j C_{jk} \quad \text{in} \quad (U_j\cap U_k)\times U'. \tag{8.3}$$

Since Q lies on an isomonodromic leaf $\mathfrak{l}\| U'$ and U' is assumed to be sufficiently small, there exist $SL(2\,;\,\boldsymbol{C})$-valued holomorphic functions P_j in U' and constant matrices C_{jk}° such that $C_{jk}=P_j C_{jk}^{\circ} P_k^{-1}$. Replacing $\boldsymbol{f}_j$ by $P_j\boldsymbol{f}_k$ and rewriting $P_j\boldsymbol{f}_j$ as $\boldsymbol{f}_j$, we may assume that C_{jk} in (8.3) are constants.

Taking $c_1(\xi)=1-g$ into account, we differentiate (8.3) with respect to x_j to obtain

$$\xi_{jk}\left\{\kappa_{jk}D_k\boldsymbol{f}_k-\frac{1}{2}(D_j\log\kappa_{jk})\boldsymbol{f}_k\right\}=D_j\boldsymbol{f}_j C_{jk}. \tag{8.4}$$

Moreover, differentiation of (8.3) with respect to t_i yields

$$\xi_{jk}\left(\frac{\partial}{\partial t_i}\right)\boldsymbol{f}_k=\left(\frac{\partial}{\partial t_i}\right)\boldsymbol{f}_j C_{jk}\,. \tag{8.5}$$

We put

$$\left(\frac{\partial}{\partial t_i}\right)\boldsymbol{f}_j=u_j^{(i)}\boldsymbol{f}_j+v_j^{(i)}D_j\boldsymbol{f}_j \qquad (i=1,\cdots,m)\,. \tag{8.6}$$

For $j=1,\cdots,m+n$, $u_j^{(i)}$ and $v_j^{(i)}$ are meromorphic functions in $U_j\times U'$ with poles along $(U_j\times U')\cap B(U')$; for $j>m+n$ they are holomorphic functions in $U_j\times U'$. Note that $u_j^{(i)}$ and $v_j^{(i)}$ are given by

$$u_j^{(i)}=\frac{(\partial/\partial t_i)W(f_j,g_j\,;\,D_j)-D_jW(f_j,g_j\,;\,(\partial/\partial t_i))}{2W(f_j,g_j\,;\,D_j)}, \tag{8.7}$$

$$v_j^{(i)}=\frac{W(f_j,g_j\,;\,(\partial/\partial t_i))}{W(f_j,g_j\,;\,D_j)}, \tag{8.8}$$

where $W(f,g\,;\,D)$ denotes the Wronskian of f and g with respect to a derivation D i.e. $W(f,g\,;\,D)=f(Dg)-(Df)g$. By (8.3)-(8.5) we obtain the following transition relations

$$\begin{cases} u_j^{(i)}=u_k^{(i)}+\dfrac{1}{2}(D_k\log\kappa_{jk})v_k^{(i)}\,, \\ v_j^{(i)}=\kappa_{jk}^{-1}v_k^{(i)}\,. \end{cases} \tag{8.9}$$

In particular $v^{(i)}=(v_j^{(i)}D_j)$ are meromorphic vector fields on M. The system

$$\begin{cases} \text{(a)} \quad D_j^2f_j=Q_jf_j\,, \\ \text{(b-}i\text{)} \quad \left(\dfrac{\partial}{\partial t_i}\right)f_j=u_j^{(i)}f_j+v_j^{(i)}D_jf_j \qquad (i=1,\cdots,m)\,, \end{cases} \tag{8.10}$$

is called *the extended system* of the SL-equation (8.2) (cf. (8.2) and (8.6)). From the above argument, (8.10) must be completely integrable.

We shall consider what follows from the complete integrability condition for (8.10). The compatibility condition for (a) and (b-i) in (8.10) is given by

$$D_j^2u_j^{(i)}+2Q_jD_jv_j^{(i)}+(D_jQ_j)v_j^{(i)}-\left(\frac{\partial}{\partial t_i}\right)Q_j=0\,,$$

$$2D_ju_j^{(i)}+D_j^2v_j^{(i)}=0\,.$$

These can be rewritten as

$$A_j(Q)v_j^{(i)}-\left(\frac{\partial}{\partial t_i}\right)Q_j=0\,, \tag{8.11}$$

$$u_j^{(i)} = -\frac{1}{2} D_j v_j^{(i)} + c_j^{(i)}\,, \tag{8.12}$$

where $c_j^{(i)}$ are suitable holomorphic functions of t which are independent of x_j. Substitution of (8.12) into the first formula of (8.9) shows that $c_j^{(i)}$ are independent of j i.e. $c_j^{(i)} = c^{(i)}$. Next we consider the compatibility condition for (a–i) and (a–i') in (8.10). To describe it we introduce meromorphic vector fields $X^{(i)}$ $(i=1, \cdots, m)$ on $M \times U'$ defined by

$$X^{(i)} = \left(\frac{\partial}{\partial t_i}\right) - v^{(i)}\,. \tag{8.13}$$

Then the compatibility condition for (a–i) and (a–i') is given by

$$[X^{(i)}, X^{(i')}] = 0\,, \tag{8.14}$$

$$X^{(i)} u_j^{(i')} = X^{(i')} u_j^{(i)} \tag{8.15}$$

for $i, i' = 1, \cdots, m$. Substituting (8.12) into (8.15) and applying (8.14), we see that (8.15) is equivalent to the following condition:

$$c := \sum_{i=1}^{m} c^{(i)} dt_i \quad \text{is a closed 1-form in } U'\,. \tag{8.15$'$}$$

Since U' is simply connected, there exists a nonzero holomorphic function h in U' such that $c = -d \log h$, where d is the exterior differential with respect to t. If f_j are replaced by hf_j and hf_j are rewritten as f_j, then the extended system (8.10) is changed into the following form:

$$\begin{cases} \text{(a)} \quad D_j^2 f_j = Q_j f_j\,, \\ \text{(b-}i\text{)} \quad X^{(i)} f_j = \left\{-\dfrac{1}{2} D_j v_j^{(i)}\right\} f_j \qquad (i=1, \cdots, m)\,. \end{cases} \tag{8.16}$$

We call (8.16) *the normalized extended system* of (8.2). For (8.16) we have $u_j^{(i)} = -(1/2) D_j v_j^{(i)}$. Hence (8.7) and (8.8) imply $(\partial/\partial t_i) W(f_j, g_j\,; D) = 0$, i.e. $W(f_j, g_j\,; D)$ are independent on (x_j, t). So there exist fundamental systems (f_j, g_j) of solutions of (8.16) such that $W(f_j, g_j\,; D_j)$ is identically one. The complete integrability condition for (8.16) is given by (8.11) and (8.14). We summarize the above argument into the following lemma.

LEMMA 8.4. *Given an isomonodromic leaf* $\mathfrak{L}|U'$, *there exists meromorphic vector fields* $v^{(i)}$ $(i=1, \cdots, m)$ *on* $M \times U'$ *with poles along* $B(U')$ *such that*

(i) *for each fixed* $p \in U'$, $v^{(i)}$ *are meromorphic vector fields on* M,

(ii) $A(Q)v^{(i)}-\left(\frac{\partial}{\partial t_i}\right)Q=0$ *for* $Q\in\mathfrak{l}\|U'$, *and*

(iii) $\left(\frac{\partial}{\partial t_i}\right)-v^{(i)}$ $(i=1,\cdots,m)$ *are commuting vector fields on* $M\times U'$.

There exist fundamental systems (f_j, g_j) *of solutions to the normalized extended system* (8.16) *of* Q *such that* $W(f_j, g_j\,;\,D_j)\equiv 1$. *In terms of them, the vector fields* $v^{(i)}=(v_j^{(i)})$ *are given by*

$$v_j^{(i)}=W(f_j, g_j\,;\,(\partial/\partial t_i))\,. \tag{8.17}$$

Conditions (ii) *and* (iii) *are the complete integrability condition for* (8.16).

8.3. Auxiliary vector fields.

The next question to be considered is: What are the vector fields $v^{(i)}$ $(i=1,\cdots,m)$ in the last subsection. The answer is given by the following lemma.

LEMMA 8.5. *The vector fields* $v^{(i)}$ *in Lemma* 8.4 *are just the auxiliary vector fields* V_i *introduced in Theorem* 7.1.

Notice that Theorem 8.3 follows immediately from Lemma 8.4 and Lemma 8.5. The key to this lemma is the formula (8.17).

PROOF OF LEMMA 8.5. By Lemma 8.4, we know that the meromorphic vector fields $v^{(i)}$ $(i=1,\cdots,m)$ on $M\times U'$ are holomorphic in $(M\times U')\backslash B(U')$. We shall find the orders of singularities (i.e. poles or zeros) of $v^{(i)}$ along $B(U')$. First we consider $v_j^{(i)}$ in $U_j\times U'$ $(j=1,\cdots,m)$. Let (f_j, g_j) be a fundamental system of solutions to the normalized extended system (8.16) of Q such that $W(f_j, g_j\,;\,D_j)\equiv 1$ (cf. Lemma 8.4). Then $v_j^{(i)}$ are ex pressed as

$$v_j^{(i)}=W(f_j, g_j\,;\,\partial_i)\,, \tag{8.18}$$

where $\partial_i=\left(\frac{\partial}{\partial t_i}\right)$. Let (F, G) be a fundamental system of solutions to $D_j^2 f=Q_j f$ which admits the following expression:

$$\begin{cases} F=(x_j-t_j)^{(1/2)(1+\theta_j)}\Phi(x_j\,;\,t)\,, \\ G=(x_j-t_j)^{(1/2)(1-\theta_j)}\Psi(x_j\,;\,t)\,, \end{cases} \tag{8.19}$$

where Φ and Ψ are holomorphic functions in $U_j\times U'$ and

$$\Phi(t_j\,;\,t)=\Psi(t_j\,;\,t)=1\,. \tag{8.20}$$

We compute the Wronskian of F and G. By using (8.19), we have $W(F, G ; D_j) = -\theta_j\Phi\Psi+(x_j-t_j)W(\Phi, \Psi ; D_j)$. Since the lefthand side is independent of x_j, we put $x_j=t_j$ and apply (8.20) to obtain $W(F, G ; D_j)=-\theta_j$. Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the $GL(2 ; \boldsymbol{C})$-valued holomorphic function in U' such that $(f_j, g_j)=(F, G)A$. Note that $\det A=-1/\theta_j$. An easy computation shows

$$\begin{aligned} W(f_j, g_j ; \partial_i)=(\det A)W(F, G ; \partial_i) \\ +W(a, c ; \partial_i)F^2+W(b, d ; \partial_i)G^2 \\ +\{W(a, d ; \partial_i)+W(b, c ; \partial_i)\}FG. \end{aligned}$$

Moreover, by using (8.19), we have

$$W(F, G ; \partial_i)=\delta_{ij}\theta_j\Phi\Psi+(x_j-t_j)W(\Phi, \Psi ; \partial_i),$$

where δ_{ij} denotes Kronecker's symbol. Combining these formulae and using (8.18), we obtain

$$\begin{aligned} v_j^{(i)} &= -\delta_{ij}\Phi\Psi \\ &\quad +(x_j-t_j)\Big(-\frac{1}{\theta_j}W(\Phi, \Psi ; \partial_i)+\{W(a, d ; \partial_i)+W(b, c ; \partial_i)\}\Phi\Psi\Big) \\ &\quad +(x_j-t_j)^{1+\theta_j}W(a, c ; \partial_i)\Phi^2+(x_j-t_j)^{1-\theta_j}W(b, d ; \partial_i)\Psi^2 \\ &= -\delta_{ij}+(x_j-t_j)\alpha(x_j ; t) \\ &\quad +(x_j-t_j)^{1+\theta_j}\beta(x_j, t)+(x_j-t_j)^{1-\theta_j}\gamma(x_j ; t), \end{aligned}$$

where α, β and γ are suitable holomorphic functions in $U_j\times U'$. Since $v^{(i)}$ is meromorphic in $U_j\times U'$, the assumption that θ_j is not an integer implies that β and γ vanish identically. Hence we have

$$(8.21)\qquad \begin{cases} v^{(i)} \text{ has zeros at the generic singular} \\ \text{points } p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_m \text{ and} \\ v^{(i)}=\{-\delta_{ij}+O(x_j-t_j)\}\dfrac{d}{dx_j} \text{ at } p_i. \end{cases}$$

Next we consider $v_{m+k}^{(i)}$ in $V_k\times U'$ for $k=1, \cdots, n$. In a similar manner as in (8.19) and (8.20), we introduce a fundamental system (F, G) of solutions to $D_{m+k}^2 f=Q_{m+k}f$ such that

$$F=(y_k-\lambda_k(t))^{3/2}\Phi(y_k ; t), \qquad G=(y_k-\lambda_k(t))^{-1/2}\Psi(y_k, t)$$

with $\Phi(\lambda_k(t)\,;\,t)=\Psi(\lambda_k(t)\,;\,t)=1$. In a similar manner as for $v_j^{(i)}$ $(j=1,\cdots,m)$, we can express $v_{m+k}^{(i)}$ as

$$v_{m+k}^{(i)}=-(\partial/\partial t_i)\lambda_k+(y_k-\lambda_k)\alpha(y_k\,;\,t)$$
$$+(y_k-\lambda_k)^3\beta(y_k\,;\,t)+(y_k-\lambda_k)^{-1}\gamma(y_k\,;\,t),$$

where α,β and γ are holomorphic functions in $V_k\times U'$. Hence,

(8.22) $v^{(i)}$ has at most a simple pole at the apparent singular points $q_1,\cdots,q_n$.

Comparing (8.21) and (8.22) with Theorem 7.1, we can conclude that $v^{(i)}$ are nothing but the auxiliary vector fields V_i. ■

8.4. Derivation of the Garnier system of genus g.

Let (t,λ,ν) be the admissible coordinates in $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}|W$ mentioned in Subsection 8.1. Since the isomonodromic leaf $\mathrm{I}\|U$ is transverse to each fiber of $\varpi:\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}|W\to U$, λ and ν are holomorphic functions of t, i.e. $\lambda=\lambda(t)$, $\nu=\nu(t)$. Now we shall derive a system of nonlinear differential equations satisfied by $\lambda(t)$ and $\nu(t)$, by using Theorem 8.3, (i).

Recall that an SL-operator Q and the auxiliary vector fields $V_i=V_i(Q)$ are expressed around their singular points in the following manner (see Sections 6 and 7):

$$(8.23)\quad \begin{cases} Q=\left\{\dfrac{\alpha_j}{(x_j-t_j)^2}+\dfrac{H_j}{x_j-t_j}+O(1)\right\}(dx_j)^{\otimes 2} & \text{in } U_j,\\[2mm] Q=\left\{\dfrac{3}{4(y_k-\lambda_k)^2}-\dfrac{\nu_k}{y_k-\lambda_k}+\nu_k^2+u_k(y_k-\lambda_k)+O((y_k-\lambda_k)^2)\right\}(dy_k)^{\otimes 2} & \text{in } V_k,\end{cases}$$

where $\alpha_j=(1/4)(\theta_j^2-1)$.

$$(8.24)\quad \begin{cases} V_i=\{-\delta_{ij}+\xi_{ij}^{(1)}(x_j-t_j)+O((x_j-t_j)^2)\}\dfrac{d}{dy_k} & \text{in } U_j,\\[2mm] V_i=\left\{\displaystyle\sum_{l=-1}^{\infty}\eta_{ik}^{(l)}(y_k-\lambda_k)^l\right\}\dfrac{d}{dy_k} & \text{in } V_k.\end{cases}$$

Using these expressions, we investigate the singularities of the meromorphic quadratic differentials $\Theta_i(Q)=A_i(Q)V_i(Q)-(\partial/\partial t_i)Q$, (see §8.1). Substituting (8.23) and (8.24) into the definition of $\Theta_i(Q)$, we obtain the following lemma.

LEMMA 8.6. *Let* $\pi(Q)=(p_1,\cdots,p_m,q_1,\cdots,q_n)\in W$.

(i) $\Theta_i(Q)$ *are holomorphic outside* $\{p_1,\cdots,p_m,q_1,\cdots,q_n\}$.

(ii) $\Theta_i(Q)$ *have at most simple poles at* $p_1,\cdots,p_m$.

(iii) $\Theta_i(Q)$ *have at most triple poles at* $q_1,\cdots,q_n$. *More precisely, we have*

$$\Theta_i(Q)=\Big\{-\frac{3}{2}A_{ik}(y_k-\lambda_k)^{-3}+\nu_kA_{ik}(y_k-\lambda_k)^{-2}+B_{ik}(y_k-\lambda_k)^{-1}$$
$$+(u_kA_{ik}-2\nu_kB_{ik})+O(y_k-\lambda_k)\Big\}(dy_k)^{\otimes 2}\qquad in\ V_k,$$

where A_{ik} *and* B_{ik} *are given by*

$$A_{ik}=\frac{\partial\lambda_k}{\partial t_i}+\eta_{ik}^{(0)}-2\nu_k\eta_{ik}^{(-1)},$$

$$B_{ik}=\frac{\partial\nu_k}{\partial t_i}+\frac{3}{2}\eta_{ik}^{(2)}-\nu_k\eta_{ik}^{(1)}+u_k\eta_{ik}^{(-1)}.$$

Remark that Theorem 7.5 immediately implies

$$A_{ik}=\frac{\partial\lambda_k}{\partial t_i}-\frac{\partial H_i}{\partial\nu_k},\qquad B_{ik}=\frac{\partial\nu_k}{\partial t_i}+\frac{\partial H_i}{\partial\lambda_k}.\tag{8.25}$$

Since $\Theta_i(Q)$ vanish identically (Theorem 8.3, (i)), Lemma 8.6 together with (8.25) immediately implies the following theorem.

THEOREM 8.7. *Each isomonodromic leaf* $\mathrm{I}\|U$ *is described by the completely integrable Hamiltonian system*

$$\begin{cases}\dfrac{\partial\lambda_k}{\partial t_i}=\dfrac{\partial H_i}{\partial\nu_k}\\[2mm]\dfrac{\partial\nu_k}{\partial t_i}=-\dfrac{\partial H_i}{\partial\lambda_k}\end{cases}\qquad(i=1,\cdots,m,\ k=1,\cdots,n).\tag{8.26}$$

The system (8.26) is a generalization of the Garnier system (G) mentioned in Theorem 0.2. We call (8.26) the *Garnier system of genus g*. The previous authors ([MJU] [Ok5, 6]) derived the deformation equations from the infinitesimal point of view. So they had to verify the complete integrability of the deformation equations. In our approach, however, the complete integrability of our deformation equations (8.26) is obvious, because the existence of integral submanifolds (=isomonodromic leaves) is known *a priori*, (see Section 5).

§ 9. Poisson structure on the moduli space.

9.1. Poisson structure on $\mathcal{E}(m\,;\,\theta)$.

A *Poisson structure* on a complex manifold X is a Lie algebra structure $\{\cdot,\cdot\}$ on the sheaf $\mathcal{O}_X$ of germs of holomorphic functions in X such that $\{F, GH\}=\{F, G\}H+G\{F, H\}$ holds for $F, G, H\in\mathcal{O}_X$ (see e.g. [LM]). We shall see that the moduli space $\mathcal{E}(m\,;\,\theta)$ of SL-operators admits a natural Poisson structure. In this section we shall freely use the notation employed in the last section.

At least locally we can define a Poisson structure on $\mathcal{E}(m\,;\,\theta)$. Indeed the bracket $\{\cdot,\cdot\}$ defined by

$$\{F, G\}:=\sum_{k=1}^{n}\left(\frac{\partial F}{\partial\nu_k}\frac{\partial G}{\partial\lambda_k}-\frac{\partial F}{\partial\lambda_k}\frac{\partial G}{\partial\nu_k}\right)\tag{9.1}$$

is a Poisson structure on $\mathcal{E}(m\,;\,\theta)|W$. Let $(\tilde{W}\,;\,\tilde{t},\tilde{\lambda},\tilde{\nu})$ be another admissible coordinate neighbourhood of $\mathcal{E}(m\,;\,\theta)$ such that $W\cap\tilde{W}\neq\emptyset$. Then a Poisson structure $\{\cdot,\cdot\}^{\sim}$ on $\mathcal{E}(m\,;\,\theta)|\tilde{W}$ is defined in a similar manner. By using the transition relations (6.10) and (6.13), we can easily see that $\{\cdot,\cdot\}=\{\cdot,\cdot\}^{\sim}$ on $\mathcal{E}(m\,;\,\theta)|(W\cap\tilde{W})$. Hence the local Poisson structure $\{\cdot,\cdot\}$ defined by (9.1) is in fact a global Poisson structure on $\mathcal{E}(m\,;\,\theta)$. The following proposition is easy to see.

PROPOSITION 9.1. *The Poisson structure $\{\cdot,\cdot\}$ on $\mathcal{E}(m\,;\,\theta)$ has the constant rank $2n$ and its symplectic leaves consist of all fibers of the projection $\varpi:\mathcal{E}(m\,;\,\theta)\to B(m)$.*

As for the definition of the rank of a Poisson structure and symplectic leaves, see [LM].

9.2. Commuting vector fields.

In view of (8.1) and Theorem 8.7, it is natural to consider the vector fields $\mathcal{H}_j=\mathcal{H}_j(Q)$, $(j=1,\cdots,m)$ on $\mathcal{E}(m\,;\,\theta)|W$ defined by

$$\mathcal{H}_j=\frac{\partial}{\partial t_j}+\sum_{k=1}^{n}\left\{\frac{\partial H_j}{\partial\nu_k}\frac{\partial}{\partial\lambda_k}-\frac{\partial H_j}{\partial\lambda_k}\frac{\partial}{\partial\nu_k}\right\}.$$

To put Theorem 8.7 in other words, we have the following theorem.

THEOREM 9.2. *The vector fields $\mathcal{H}_i$ $(i=1,\cdots,m)$ form a frame over $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}|W$ of the infinitesimal monodromy preserving deformation on $\mathcal{E}(m\,;\,\theta)_{\mathrm{irr}}$, (see Definition 8.1).*

Motivated by the definition of $\Theta_j(Q)$, we define

$$\hat{\Theta}_j(Q):=A(Q)V_j(Q)-\mathcal{H}_j(Q)Q \qquad (j=1,\cdots,m).$$

Notice that Θ_j are defined only on each isomonodromic leaf $\mathfrak{l}\| U$, while $\hat{\Theta}_j$ are defined on the whole $\mathcal{E}(m;\theta)|W$. As expected from Theorem 7.5 and Lemma 8.6, the following lemma is valid.

LEMMA 9.3. *Given $Q\in\mathcal{E}(m;\theta)|W$, let $\pi(Q)=\boldsymbol{r}=(p_1,\cdots,p_m,q_1,\cdots,q_n)\in W$. The meromorphic quadratic differentials $\hat{\Theta}_j(Q)$ on M have at most simple poles at $p_1,\cdots,p_m$, zeros at $q_1,\cdots,q_n$, and are holomorphic outside $\{p_1,\cdots,p_m,q_1,\cdots,q_n\}$. Namely, $\hat{\Theta}_j(Q)$ are regarded as holomorphic sections of the line bundle $\xi(\boldsymbol{r})$, (see §§6.1).*

Proof of this lemma is similar to that of Lemma 8.6. Since $W\subset X(m)$, Lemma 6.4 and Lemma 9.3 immediately imply that $\hat{\Theta}_j(Q)$ vanish identically.

THEOREM 9.4. *For each $Q\in\mathcal{E}(m;\theta)|W$, we have*

$$\mathcal{H}_j(Q)Q=A_j(Q)V_j(Q), \qquad (j=1,\cdots,m).$$

To state an important consequence of the above theorem, we put

$$\begin{aligned}\langle H_i,H_j\rangle &:=\frac{\partial H_j}{\partial t_i}-\mathcal{H}_jH_i\\ &=\frac{\partial H_j}{\partial t_i}-\frac{\partial H_i}{\partial t_j}+\sum_{k=1}^{n}\left\{\frac{\partial H_i}{\partial \nu_k}\frac{\partial H_j}{\partial \lambda_k}-\frac{\partial H_i}{\partial \lambda_k}\frac{\partial H_j}{\partial \nu_k}\right\}.\end{aligned}$$

Then we obtain the following lemma.

LEMMA 9.5. $\qquad \langle H_i,H_j\rangle=0 \qquad (i,j=1,\cdots,m).$

PROOF. We compare the coefficients of the Laurent expansions of the both sides of the equality $\mathcal{H}_jQ=AV_j$ (Theorem 9.4) at the generic singular point p_i. We obtain

$$\mathcal{H}_jH_i=2\alpha_i\xi_{ji}^{(2)}+\xi_{ji}^{(1)}H_i\,.$$

Theorem 7.5 implies that the right-hand side equals to $\partial H_j/\partial t_i$. This establish the lemma. ■

Now we consider the commutator $[\mathcal{H}_i,\mathcal{H}_j]$ of the vector fields $\mathcal{H}_i$ and $\mathcal{H}_j$. An easy calculation shows

$$[\mathcal{H}_i,\mathcal{H}_j]=\sum_{k=1}^{n}\left(\frac{\partial}{\partial \nu_k}\langle H_i,H_j\rangle\cdot\frac{\partial}{\partial \lambda_k}-\frac{\partial}{\partial \lambda_k}\langle H_i,H_j\rangle\cdot\frac{\partial}{\partial \nu_k}\right). \tag{9.2}$$

Combining Lemma 9.5 with (9.2), we obtain the following theorem.

THEOREM 9.6. *$\mathcal{H}_i$ $(i=1, \cdots, m)$ are commuting vector fields on $\mathcal{E}(m;\theta)|W$.*

Let X and Φ be a vector field and a differential form on $\mathcal{E}(m;\theta)$, respectively. We denote the interior product of Φ with respect to X by $\iota_X\Phi$ and the Lie derivative of Φ with respect to X by $L_X\Phi$. The role of the fundamental 2-form Ω on $\mathcal{E}(m;\theta)$ introduced in Definition 6.18 is well understood by the following theorem.

THEOREM 9.7. $\iota_{\mathcal{H}_i}\Omega=0, \qquad L_{\mathcal{H}_i}\Omega=0 \qquad (i=1, \cdots, m)$.

PROOF. Recall that Ω is written as

$$\Omega=\sum_{k=1}^{n} d\nu_k\wedge d\lambda_k-\sum_{j=1}^{m} dH_j\wedge dt_j,$$

(see §§6.5). Using this expression, we can easily see

$$\iota_{\mathcal{H}_i}\Omega=-\frac{1}{2}\sum_{j=1}^{m}\langle H_i, H_j\rangle dt_j.$$

Now Lemma 8.11 immediately implies $\iota_{\mathcal{H}_i}\Omega=0$. Since Ω is a closed form, E. Cartan's formula $L_X=d\cdot\iota_X+\iota_X\cdot d$ implies $L_{\mathcal{H}_i}\Omega=0$. ■

Combining Theorem 9.2 and Theorem 9.7, we obtain

THEOREM 9.8. *The monodromy preserving foliation on $\mathcal{E}(m;\theta)_{\mathrm{irr}}$ is the Ω-Lagrangian foliation which is transverse to each fiber of ϖ: $\mathcal{E}(m;\theta)_{\mathrm{irr}}\to B(m)$.*

9.3. Sheaf of monodromy changing Hamiltonians.

First we discuss about a relation between the Poisson structure on $\mathcal{E}(m;\theta)$ and the vector fields $\mathcal{H}_i$. For a holomorphic function F in $\mathcal{E}(m;\theta)$, let $\mathcal{X}(F)$ be the Hamiltonian vector field generated by F, i.e. for a holomorphic function G,

$$\mathcal{X}(F)G=\{F, G\}. \tag{9.3}$$

Note that $\mathcal{H}_i=(\partial/\partial t_i)+\mathcal{X}(H_i)$. The following theorem is easy to see.

LEMMA 9.9. *For a holomorphic functions F and G, we have*

$$\mathcal{H}_i\{F,G\}=\{\mathcal{H}_iF,G\}+\{F,\mathcal{H}_iG\} \qquad (i=1,\cdots,m).$$

The vector fields $\mathcal{H}_i$ depends on the admissible coordinate neighbourhood $(W;\ t,\lambda,\nu)$. Let $(\tilde{W};\ \tilde{t},\tilde{\lambda},\tilde{\nu})$ be another admissible coordinate neighbourhood such that $W\cap\tilde{W}\neq\emptyset$, $\tilde{\mathcal{H}}_i$ the associated vector fields. The transition relation between (t,λ,ν) and $(\tilde{t},\tilde{\lambda},\tilde{\nu})$ is given by (6.10). We use the notation in Section 6. Using (6.10) and (6.13) we can easily show the following lemma.

LEMMA 9.10. $\tilde{\mathcal{H}}_i=K_i(t_i)\mathcal{H}_i$ *on* $\mathcal{E}(m;\theta)|(W\cap\tilde{W})$.

In other words, Lemma 9.10 asserts that the locally defined (1, 1)-tensor

$$D=\sum_{i=1}^{m}\mathcal{H}_i\otimes dt_i \tag{9.4}$$

makes sense globally on $\mathcal{E}(m;\theta)$. Let $\Omega^{(i)}$ $(i=0,\cdots,m,\cdots)$ be a subsheaf of the sheaf of holomorphic i-forms on $\mathcal{E}(m;\theta)$ defined by

$$\Omega^{(i)}=\mathcal{O}_{\mathcal{E}(m;\theta)}(\varpi^*\wedge^iT^*B(m)). \tag{9.5}$$

For a local section $f=\sum_{j_1<\cdots<j_i}f_{j_1\cdots j_i}dt_{j_1}\wedge\cdots\wedge dt_{j_i}$ of $\Omega^{(i)}$, we put $Df=\sum_k\sum_{j_1<\cdots<j_i}(\mathcal{H}_kf_{j_1,\cdots,j_i})dt_k\wedge dt_{j_1}\wedge\cdots\wedge dt_{j_i}$. Lemma 9.10 implies that $D:\Omega^{(i)}\to\Omega^{(i+1)}$ is a well-defined sheaf mapping. Moreover Theorem 9.6 implies that $D\cdot D=0$. Namely, the sequence

$$\Omega^{(\cdot)}:\ \Omega^{(0)}\xrightarrow{D}\Omega^{(1)}\longrightarrow\cdots\xrightarrow{D}\Omega^{(m)}\longrightarrow 0 \tag{9.6}$$

is a complex.

DEFINITION 9.11. We put

$$\begin{aligned}\mathcal{P}&=\mathrm{Ker}[D:\ \Omega^{(0)}\longrightarrow\Omega^{(1)}]\\&=\{F\in\mathcal{O}_{\mathcal{E}(m,\theta)};\ \mathcal{H}_iF=0\ \ (i=1,\cdots,m)\}.\end{aligned}$$

We call $\mathcal{P}$ the *sheaf of monodromy changing Hamiltonians,* and a (local) section of $\mathcal{P}$ a *monodromy changing Hamiltonian.*

Lemma 9.9 implies that $\mathcal{P}$ is a subalgebra of the (sheaf of) Poisson algebra $\mathcal{O}_{\mathcal{E}(m;\theta)}$ with the bracket $\{\cdot,\cdot\}$. Moreover we can easily verify the following theorem by using Lemma 9.9.

THEOREM 9.12. *The Hamiltonian vector field generated by a monodromy changing Hamiltonian commutes with* $\mathscr{H}_i$ $(i=1, \cdots, m)$.

LEMMA 9.13. *For any* $Q^\circ \in \mathcal{E}(m\,;\,\theta)$, *there exist a neighbourhood* $\mathcal{N}$ *of* Q° *and monodromy changing Hamiltonians* $s_1, \cdots, s_{2n}$ *in* $\mathcal{N}$ *such that* $(t_1, \cdots, t_m, s_1, \cdots, s_{2n})$ *is a local coordinate system of* $\mathcal{E}(m\,;\,\theta)$ *in* $\mathcal{N}$.

PROOF. Since $\mathscr{H}_i$ are transverse to each fiber of $\varpi: \mathcal{E}(m\,;\,\theta) \to B(m)$, there exists a sufficiently small neighbourhood $\mathcal{N}$ of Q° such that the integral submanifold of $\mathscr{H}_i$ $(i=1, \cdots, m)$ through an arbitrary point $Q \in \mathcal{N}$ intersects with $\mathcal{E}(\varpi(Q)\,;\,\theta) \cap \mathcal{N}$ only once. Let $\boldsymbol{Q}(Q)$ be such an intersection point. We put $s_j(Q) = \lambda_j(\boldsymbol{Q}(Q))$ and $s_{j+n}(Q) = \nu_j(\boldsymbol{Q}(Q))$, $(j=1, \cdots, m)$. Then it is easy to see that $s_1, \cdots, s_{2n}$ are desired functions. ■

Now we can establish a "Poincaré's lemma".

THEOREM 9.14. *The complex* $\Omega^{(\cdot)}$ *defined by* (9.6) *is exact. Namely,* $0 \to \mathcal{P} \to \Omega^{(\cdot)}$ *is a resolution of the sheaf of monodromy changing Hamiltonians.*

PROOF. We may consider things locally. The vector fields $\mathscr{H}_i$ and 1-forms dt_i in the old coordinates (t, λ, ν) correspond to $\partial/\partial t_i$ and dt_i in the new coordinates (t, s), respectively. The neighbourhood $\mathcal{N}$ in Lemma 9.13 may be identified with a product space $T \times S$, where T is a domain in $\boldsymbol{C}^m$ with the coordinates t and S is a domain in $\boldsymbol{C}^{2n}$ with the coordinates s. Then D is nothing but the exterior differential operator d_t with respect to t. Now the theorem follows from the ordinary Poincaré's lemma which asserts that

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\ i\ } \mathcal{O}_{T\times S}(\wedge^0 T^*T) \xrightarrow{\ d_t\ } \cdots \xrightarrow{\ d_t\ } \mathcal{O}_{T\times S}(\wedge^m T^*T) \longrightarrow 0$$

is exact. ■

REMARK 9.15. In case the genus $g=0$, we can show that the moduli space $\mathcal{E}(m\,;\,\theta)$ is a Stein manifold. Hence the resolution of Theorem 9.14 is acyclic.

REMARK 9.16. We expect that the Hamiltonian flow generated by a monodromy changing Hamiltonians describes a "nice" monodromy changing deformation in a certain sense. The sheaf $\mathcal{P}$ and the monodromy changing deformation must be considered further in the future.

§ 10. Open problems.

We conclude the present paper by proposing some open problems to be considered in the future.

Problem 10.1. Recall that the moduli space $\mathcal{E}(m;\theta)$ of SL-operators admits a Poisson structure (see §§ 9.1). This Poisson structure is *mysterious* in the sense that, although it is originally defined locally, fortunately, it is patched together to make sense globally on $\mathcal{E}(m;\theta)$. On the other hand, the moduli space $R(m;\theta)_{\mathrm{irr}}$ of representations of the fundamental group also admits a Poisson structure (see Theorem 5.4). This Poisson structure is *natural* in the sense that it comes from a duality of cohomologies. We have the projective monodromy map

$$PM: \mathcal{E}(m;\theta)_{\mathrm{irr}} \longrightarrow R(m;\theta)_{\mathrm{irr}}.$$

We have a conjecture: The mysterious Poisson structure on $\mathcal{E}(m;\theta)_{\mathrm{irr}}$ is nothing but the pull-back of the natural Poisson structure on $R(m;\theta)_{\mathrm{irr}}$. Prove this conjecture!

Problem 10.2. In the present paper, we are concerned only with Fuchsian SL-operators. A problem in the next step is to consider SL-operators of irregular singular type obtained from Fuchsian SL-operators by the confluence of their regular singular points. This problem is related to the "compactification" of the moduli space of SL-operators. To understand this problem, it is also helpful to recall that the Painlevé equations (I)-(V) are obtained from the Painlevé equation (VI) by the step-by-step degeneration procedures which correspond to the confluence procedures of singular points of linear differential equations ([Gar1], see also [KH1]).

Problem 10.3. This problem is related to Riemann's program mentioned in §§ 0.1. Theta functions are seems to have a connection with the deformation of SL-operators "degenerate" in a certain sense. Make clear the meaning of degeneracy and develop a suitable deformation theory for such SL-operators. By doing so, investigate theta functions from the point of view of Hamiltonian dynamics.

Problem 10.4. Consider the deformation of SL-operators of excited state. Some results in the case of genus 0 are obtained by T. Kimura [KT] and Matsuda [Mat].

There are many other problems to be considered which are unsettled even in the low genus case. We omit them here and advice the reader to find them in the references.

Note. Problem 10.1 was solved affirmatively after the first version of the manuscript had been completed. This result will be reported upon in a separate paper [Iw2].

References

[BMW] Barouch, E., McCoy, B. M. and T. T. Wu, Zero-field susceptibility of the two-dimensional Ising model near T_c, Phys. Rev. Lett. **31** (1973), 1409-1411.

[D] Deligne, P., Equations différentielles à points singuliers réguliers, Lecture Notes in Math. vol. 163, Springer-Verlag, Berlin-New York, 1970.

[FN] Flaschka, H. and A. C. Newell, Monodromy- and spectrum-preserving deformations I, Comm. Math. Phys. **76** (1980), 65-116.

[F] Fuchs, R., Uber lineare homogene Differentialgleichungen zweiter ordnung mit drei im Endlich gelegende wesentlich singulären Stellen, Math. Ann. **63** (1907), 301-321.

[Gam] Gambier, B., Sur les équations differentielles du second ordre et du premier degré dont l'intégrale générale est à points cririques fixes, Acta Math. **33** (1910), 1-55.

[Gar1] Garnier, R., Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale génerale a ses points critiques fixes, Ann. Sci. École Norm. Sup. (3) **29** (1912), 1-126.

[Gar2] Garnier, R., Solution du problème de Riemann pour les systèmes différentiels linéaires du second ordre, Ann. Sci. École Norm. Sup. (3) **43** (1926), 177-307.

[GS] Gérard, R. and A. Sec, Feuilletages de Painlevé, Bull. Soc. Math. France **100** (1972), 47-72.

[Gof] Goffar-Lombet, J., Sur des systèmes polynomiaux d'équations différentielles dont l'intégrale générale est à points critiques fixes, Acad. Roy. Belg. Cl. Sci. Mém. Collect 8° (2) **41** (1974).

[Gol] Goldman, W. M., The symplectic nature of fundamental groups of surfaces, Adv. in Math. **54** (1984), 200-225.

[Gra] Grauert, H., Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, Publ. Inst. Hautes Études Sci. N°**5** (1960), 233-292.

[GR] Grauert, H. and R. Remmert, Coherent analytic sheaves, Grundlehren Math. Wiss. vol. 265, Springer-Verlag, New York-Berlin, 1984.

[Gro] Gromak, V. I., Theory of Painlevé's equation, Differential'nye Uravneniya **11** (1975), 373-376.

[Gu1] Gunning, R. C., Lectures on Riemann surfaces, Princeton Math. Notes, Princeton Univ. Press, Princeton and London, 1966.

[Gu2] Gunning, R. C., Lectures on vector bundles over Riemann surfaces, Princeton Math. Notes, Princeton Univ. Press, Princeton and London, 1967.

[H] Hejhal, D. A., Monodromy groups and linear polymorphic functions, Acta Math. **135** (1975), 1-55.

[In] Ince, E. L., Ordinary Differential Equations, Dover, New York, 1956.

[Iw1] Iwasaki, K., Moduli and deformation for Fuchsian projective connections on a

Riemann surface, Preprint Ser. UTYO-MATH 89-16, Univ. Tokyo, Tokyo, 1989.

[Iw2] Iwasaki, K., Fuchsian moduli on Riemann surfaces—its Poisson structure and the Poincaré-Lefschetz duality, to appear in Pacific J. of Math.

[IKSY] Iwasaki, K., Kimura, H., Shimomura, S. and M. Yoshida, From Gauss to Painlevé—A modern theory of special functions, Aspects of Math. vol. E16, F. Vieweg and Sohn, Braunschweig, 1991.

[JMU] Jimbo, M., Miwa, T. and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I—General theory and τ-function—, Physica **2D** (1981), 306-352.

[JM1] Jimbo, M. and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II, Physica **2D** (1981), 407-448.

[JM2] Jimbo, M. and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients III, Physica **4D** (1981), 26-46.

[JMSM] Jimbo, M., Miwa, T., Sato, M. and Y. Môri, Holonomic Quantum Fields—The unanticipated link between deformation theory of differential equations and quantum fields, Mathematical problems in theoretical physics 5, Lecture Notes in Phys. vol. 126, Springer, Berlin-New York, 1980, 119-142.

[JMMS] Jimbo, M., Miwa, T., Môri, Y. and M. Sato, Density matrix of an inpenetrable Bose gas and the fifth Painlevé transcendent, Physica **1D** (1980), 80-158.

[KH1] Kimura, H., Degeneration of two dimensional Garnier system, Proc. Japan Acad. Ser. A Math. Sci. **57** (1981), 446-449.

[KH2] Kimura, H., Feuilletage uniforme associé au système de Garnier, preprint.

[KO1] Kimura, H. and K. Okamoto, On the isomonodromic deformation of linear ordinary differential equations of higher order, Funkcial. Ekvac. **26** (1983), 37-50.

[KO2] Kimura, H. and K. Okamoto, On the polynomial Hamiltonian structure of the Garnier system, J. Math. Pures Appl. **63** (1984), 129-146.

[KT] Kimura, T., On the isomonodromic deformation for linear ordinary differential equations of the second order I, Proc. Japan Acad. Ser. A Math. Sci. **57** (1981), 285-290; II, ibid. **58** (1982), 294-297.

[KN] Kobayashi, S. and K. Nomizu, Foundation of differential geometry I-II, Wiley-Intersci. Publ., Wiley, New York, 1963.

[KS] Kodaira, K. and D.C. Spencer, On deformations of complex analytic structures I, Ann. of Math. **67** (1958), 328-401; II, ibid. **67** (1958), 403-466; III, ibid. **71** (1960), 43-76.

[LM] Lieberman, P. and C.M. Marle, Symplectic Geometry and Analytical Mechanics, Reidel Publ., Dordrecht, 1987.

[Mal1] Malgrange, B., Sur les deformations isomonodromiques, I. Singularites regulieres, Mathématique et Physique, Seminaire E.N.S. (1981-1982) 4 n°8, Birkhäuser, Boston-Basel-Stuttgart, 1983, 401-426.

[Mal2] Malgramge, B., Sur les deformation isomonodromiques, II. Singularites irregulieres, Mathématique et Physique, Seminaire E.N.S. (1981-1982), 4 n°8, Birkhäuser, Boston-Basel-Stuttgart, 1983, 427-438.

[Man] Manin, Y., Moduli Fuchsiani, Ann. Scuola Norm. Sup. Pisa Ser III **19** (1965), 113-126.

[Mat] Matuda, T., On the isomonodromic deformation related to the sixth Painlevé equation, Natur. Sci. Rep. Ochanomizu Univ. **39** (1988), 15-24.

[Na] Nag, S., The complex analytic theory of Teichmüller spaces, Wiley-Intersci. Publ., Wiley, New York, 1988.

[Ni] Nishioka, K., A note on the transcendency of Painlevé's first transcendent, Nagoya Math. J. **109** (1988), 63-67.

[Ok1] Okamoto, K., On Fuchs's problem on a torus I, Funkcial. Ekvac. **14** (1971), 137-152.

[Ok2] Okamoto, K., Sur le problème de Fuchs sur un tore II, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 357-372.

[Ok3] Okamoto, K., Déformation d'une équation différentielle lineàre avec une singularité irrégulière sur un tore, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **26** (1979), 501-518.

[Ok4] Okamoto, K., Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Japan. J. Math. **5** (1979), 1-79.

[Ok5] Okamoto, K., Isomonodromic deformation and Painlevé equations, and the Garnier system, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 575-618.

[Ok6] Okamoto, K., Elliptic Garnier system, preprint, Univ. of Tokyo, Tokyo, 1987.

[Ok7] Okamoto, K., Introduction to the Painlevé equations (in Japanese), Sophia kokyuroku **19** (1985), Sophia Univ.

[Ot] Otsuki, M., On the number of apparent singularities of a linear differential equation, Tokyo J. Math. **5** (1982), 23-29.

[P] Painlevé, P., Sur les équations différentielles du second ordre dont l'intégrale générale est uniforme, Oeuvre t. III, 187-271.

[Ri] Riemann, B., Gesammelte Mathematische Werke, Teubner-verlag, Leipzig, 1892.

[Rö] Röhrl, H., Das Riemann-Hilbertsche problem der Theorie der linearen Differentialgleichungen, Math. Ann. **133** (1957), 1-25.

[SMJ] Sato, M., Miwa, T. and M. Jimbo, Aspects of holonomic quantum fields—Isomonodromic deformation and Ising model, Complex analysis, Microlocal Calculus and Relativistic Quantum Theory, Lecture Notes in Phys. vol. 126, Springer, Berlin-New York, 1980, 429-491.

[Sc] Schlesinger, L., Über eine klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten, J. für Math. **141** (1912), 96-145.

[Sp] Spanier, E. H., Algebraic Topology, McGraw-Hill, New York, 1966.

[Ue] Ueno, K., Monodromy preserving deformation of linear differential equations with irregular singular points, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), 97-102.

[Um1] Umemura, H., Birational automorphism groups and differential equations, to appear in the Proc. Franco-Japanese colloquium on differential equations at Strasbourg in 1985.

[Um2] Umemura, H., On the irreducibility of the first differential equations of Painlevé, Algebraic Geometry and Commutative Algebra in honor of Masayoshi Nagata, 1987, 101-119.

[Um3] Umemura, H., Transcendency of the Painlevé equations (in Japanese), Sûgaku **40** (1988), 451-475.

[WMTB] Wu, T. T., McCoy, B. M., Tracy, C. A. and E. Brouch, Spin-spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region, Phys. Rev. **B13** (1976), 316-374.

[Y] Yoshida, M., Fuchsian differential equations, Aspects of Math. **E11**, F. Vieweg & Sohn, Braunschweig, 1987.

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Department of Mathematics
Faculty of Science
University of Tokyo
7-3-1 Hongô, Bunkyo-ku
Tokyo
113 Japan