J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
38 (1991), 567-588.

The Cauchy problem for Hartree type Schrödinger equation in weighted Sobolev space

By Hitoshi HIRATA

§1. Introduction and the main results.

In this paper, we study the Cauchy problem for non-linear Schrödinger equations in $\boldsymbol{R}^n$ of Hartree-type:

$$\left\{\begin{array}{l} i\partial_t u=-\Delta u+u(W*|u|^2)+Vu \\ u(0,x)=\phi(x) \end{array}\right\}, \tag{1-1}$$

and prove the existence, uniqueness and the regularity of its solutions.

For stating the main results of the paper, we need some notations. For $p\in[1,\infty]$ and $k, m\in\overline{\boldsymbol{N}}=\{0\}\cup\boldsymbol{N}$, $W_m^{k,p}=\{u\in\mathcal{S}' : \|u\|_{W_m^{k,p}}=\sum_{|\alpha|\leq k}\|\partial_x^\alpha u\|_p+\sum_{|\beta|\leq m}\|x^\beta u\|_p<\infty\}$, where $\|\cdot\|_p$ is L^p-norm; $W^{k,p}=W_0^{k,p}$. For an interval I and a Banach space X, $C^k(I; X)$ is the space of X-valued C^k-functions on I, $k=0,1,2,\cdots$ and $L^p(I; X)$ is the space of L^p-functions. We equip these spaces with standard norms to make them Banach spaces. For Banach spaces X and Y, $B(X, Y)$ denotes the space of bounded linear operators from X to Y.

DEFINITION 1-1. We say that an exponent p is *admissible* if $1/2\geqq 1/p>1/2-1/n$. For an admissible exponent p, define $\theta=\theta(p)=4p/(n(p-2))$ and say (θ, p) is an *admissible pair*.

For an exponent p, p' denotes the conjugate of p: $p'=p/(p-1)$. For an admissible exponent p, we define function spaces $X_m^k(T; p)$ and $X_m'^k(T; p)$ as follows:

$$X_m^k(T;\ p)=C([0, T];\ W_m^{k,2})\cap L^\theta(0, T;\ W_m^{k,p}),$$

$$X_m'^k(T;\ p)=L^1(0, T;\ W_m^{k,2})+L^{\theta'}(0, T;\ W_m^{k,p'}).$$

Here (θ, p) is the admissible pair. These are Banach spaces with respective norms:

$$\|u\|_{X_m^k(T;\,p)}=\|u\|_{C([0,T];\,W_m^{k,2})}+\|u\|_{L^\theta(0,T;\,W_m^{k,p})},$$

$$\|u\|_{X'^k_m(T;\,p)}=\inf\{\|u_1\|_{L^1(0,T;\,W^{k,2}_m)}+\|u_2\|_{L^{\theta'}(0,T;\,W^{k,p'}_m)}:$$
$$u=u_1+u_2,\ u_1\in L^1(0,T;\ W^{k,2}_m),\ u_2\in L^{\theta'}(0,T;\ W^{k,p'}_m)\}.$$

We say $u\in X^k_{m,\mathrm{loc}}(T;\ p)$ if $u\in X^k_m(T';\ p)$ for every $T'<T$. $X^0_0=X, X^k_0=X^k$ and etc. $[x]_+\equiv x\vee 0$ for $x\neq 0$ and $[0]_+$ denotes an arbitrary small positive number.

We choose and fix an integer $k\geqq 0$ throughout this paper.

ASSUMPTION 1. For some $q, s\geqq 1$ with $q>n/(2k+2)$ and $s>n/2$ such that

$$3/2+1/2s-1/q\geqq 1/2+[1/2-1/2s-k/n]_+ +[1/2-k/n]_+, \tag{1-2}$$

$$1/2+1/2s-1/2q\geqq[1/2-k/n]_+, \tag{1-3}$$

we have

$$W=W_1+W_2,\quad W_1\in L^\infty,\quad W_2\in L^q, \tag{1-4}$$

$$V=V_1+V_2,\quad V_1\in W^{k,\infty},\quad V_2\in W^{k,s}. \tag{1-5}$$

REMARK 1-2. If W and V satisfy (1-4) and (1-5) for some $q, s\geqq 1$ with $q>n/(2k+2)$ and $s>n/2$, they satisfy Assumption 1 with q and s possibly different from the original ones. Hence (1-2) and (1-3) are not assumptions. We formulated Assumption 1 as above for stating the following Theorem 1 concisely.

Associated with (1-1) we consider the essentially equivalent integral equation:

$$u=\exp(it\Delta)\phi-i\int_0^t\exp\{i(t-s)\Delta\}u(s)(W*|u(s)|^2)+Vu(s)ds. \tag{1-6}$$

THEOREM 1-3. *Let Assumption* 1 *be satisfied,* $k\geqq m\in\overline{\boldsymbol{N}}$, $1/p_0=1/2-1/2s$ *and* $1/\theta_0=n/4s$. *Then, for every* $\phi\in W^{k,2}_m$, *there exists* $T^*=T^*(\phi)>0$ *such that* (1-6) *admits a unique solution* $u\in X^k_{m,\mathrm{loc}}(T^*;\ p_0)$. *This* u *satisfies the following properties.*

(1) $u\in X^k_{m,\mathrm{loc}}(T^*;\ p)$ *for all admissible exponent* p.

(2) *If* W *and* V *are real valued, then*

$$\|u(t)\|_2=\|\phi\|_2\qquad for\quad t\in[0, T^*).$$

(3) *If* W *and* V *are real valued and, in addition,* $k\geqq 1$, *then*

$$E(u(t))=E(\phi)\qquad for\quad t\in[0, T^*),$$

where $E(u)$ *is the energy of the system* (1-1):

$$E(v)=\|\nabla v\|_2^2+\frac{1}{2}\int_{R^n}|v|^2(x)(W*|v|^2)(x)dx+\int_{R^n}V(x)|v|^2(x)dx\,.$$

(4) *If* $T^*<\infty$, *then* $\lim_{t\to T^*}\|u(t)\|_{W_m^{k,2}}=\infty$ *and* $\limsup_{t\to T^*}\|u(t)\|_{W^{k,2}}=\infty$.

(5) *For any sequence* $\phi_j\in W_m^{k,2}$ *such that* $\phi_j\to\phi$ *in* $W_m^{k,2}$, $u_j\to u$ *in* $X_{m,\mathrm{loc}}^k(T^*;\,p_0)$, *where* u_j *is the solution of* (1-6) *with* ϕ *in place of* ϕ_j.

(6) *If* $k\geqq 1$, *we have* $u\in C^1([0,T^*);\,W^{k-2,2})$, *and* u *satisfies* (1-1) *in* $W^{k-2,2}$. *Here* $W^{-1,2}$ *stands for the dual space of* $W^{1,2}$. (*Remark of* (6): *When* $k=0$, *we have to assume* $q=s=2$ *if* $n\leqq 3$. *If* $n\geqq 4$, *this additional assumption is not necessary.*)

As for the study of global existence of the solution, we assume the following.

ASSUMPTION 2. W and V are real functions such that $W(x)=W(-x)$, and for some $k\in\overline{N}$, q and $s\geqq 1$ such that $q>n/4$ and $s>n/2$, $W=W_1+W_2$, $W_1\in L^\infty$, $W_2\in L^q$, $V=V_1+V_2$, $V_1\in W^{k,\infty}$, $V_2\in W^{k,s}$. Moreover for $r=1$ $(n=1)$, $r>1$ $(n=2)$ and $r=n/2$ $(n\geqq 3)$, $W_{2-}\equiv(-W_2)\vee 0\in L^r$.

THEOREM 1-4. *Let Assumption* 2 *be satisfied and* $m\in\overline{N}$ *be* $m\leqq k$. *Then the solution* u *of* (1-6) *of Theorem* 1-3 *becomes a global solution. i.e.* $T^*=\infty$.

REMARK 1-5. In our proofs of Theorem 1-4, using the equation (1-1), we can replace estimates for Δu by the ones for $\partial_t u$ and $u(W*|u|^2)+Vu$, and we may replace the assumptions on V by weaker ones, if we consider the integral equation (1-6) in $W^{2,2}$ only: if $W\in L^\infty+L^q$, $q>n/6$ and $q\geqq 1$, $V\in L^\infty+L^s$, $s>n/2$ and $s\geqq 1$, then for any initial data $\phi\in W^{2,2}$, there exists $T^*>0$ and a local solution u of (1-6) such that $u\in C([0,T],W^{2,2})\cap C^1([0,T];\,L^2)$ for all $T<T^*$.

This idea is essential due to Yajima [13] and Kato [10], and the proof may be given following their arguments.

There are several results on the Cauchy problem (1-1) and (1-6) under various conditions on W and V. Chadam and Glassey ([2]) studied (1-6) the case that $n=3$ and W, V are Coulomb potentials. They showed the global existence of the solution of (1-6) in $W^{2,2}$ for $\phi\in W^{2,2}$. Dias and Figueira ([3]) studied (1-6) under the same conditions on W and V as in [2], and showed the existence of the global solution such that $ru\in L^2$ when $r\phi\in L^2$ and $\phi\in W^{2,2}$. Ginibre and Velo ([4]) showed that if $\phi\in W^{k,2}$, $q>n/2k$ and $V=0$, then there exists a local solution of (1-6) in $W^{k,2}$. Our result is an improvement of [4] in three points: 1) We allow non vanishing V; 2) we relax the condition $q>n/2k$ to $q>n/(2k+2)$ and; 3) we generalize the space of solutions $W^{k,2}$ to $W_m^{k,2}$ for $k\geqq m\geqq 0$.

Recently, Cazenave and Weissler ([1]) showed the existence of global solutions of (1-6) in $W^{1,2}$ for $\phi\in W^{1,2}$ when $q>n/4$ and $V=0$ in Assumption 2. This is a special case of our results.

Hayashi, Nakamitsu and M. Tsutsumi ([8]) studied the time dependent-Schrödinger equation with local and nonlocal nonlinearity in weighted Sobolev space. They showed the existence of local solution of (1-6) in $W_m^{k,2}$, for $\phi\in W_m^{k,2}$, $W\in L^\infty+L^q$, $q>n/2$ and $V=0$. We use some of their estimates in our paper, and the proof of Theorem 1-3 (4) is essentially due to them. Assumption $k\geqq m$ is essential for obtaining the existence of solutions of (1-3) in $C([0, T); W_m^{k,2})$. When $k<m$, the solution of (1-6) does not in general exist in $C([0, T); W_m^{k,2})$ even if $\phi\in W_m^{k,2}$. Note, however, when $W=\lambda_1|x|^{-\gamma_1}+\lambda_2|x|^{-\gamma_2}$, $V=\lambda_3|x|^{-\gamma_3}$, where $0<\gamma_1, \gamma_2<2\wedge n$ and $0<\gamma_3<2\wedge n/2$, there exists global solution $u\in C(\boldsymbol{R}; L^2)$, and $u\in C(\boldsymbol{R}\backslash\{0\}; L^p)$ for all $p\in[2, 2n/(n-2m)]$ if initial data $\phi\in W_m^{0,2}$ $(m=1,2)$ (Hayashi and Ozawa [7]).

We should also mention that the asymptotic behavior at $t=\infty$ of the global solutions has been studied by many authors. e.g. Dias and Figueira [3], Ginibre and Velo [4], Glassey [5], Hayashi and Ozawa [6], Hayashi and Y. Tsutsumi [9] etc. We shall not, however, consider this point in this paper.

§2. Estimates of linear operators.

In this section, we present several estimates for the operators:

$$U(t)\phi\equiv\exp(it\Delta)\phi\,,$$

and

$$Sv(t)\equiv\int_0^t U(t-s)v(s)ds\,,$$

in the weighted Sobolev spaces, which we will need in the following sections.

The following two lemmas are well-known.

LEMMA 2-1 (Kato [10]). *Let* $p\in[2, \infty]$ *and* $\theta(p)=4p/n(p-2)$. *Then, for any* $\phi\in L^{p'}$, $U\phi\in C(\boldsymbol{R}^1\backslash\{0\}; L^p)$ *and*

$$\|U(t)\phi\|_p\leqq(4\pi|t|)^{-2/\theta(p)}\|\phi\|_{p'}\,. \tag{2-1}$$

LEMMA 2-2 (Yajima [13]). *For admissible exponents* p *and* p_0,

$$\|U\phi\|_{X(T;\,p)}\leqq C\|\phi\|_2, \qquad \phi\in L^2\,, \tag{2-2}$$

$$\|Su\|_{X(T;\,p)} \leqq C\|u\|_{X'(T;\,p_0)}, \qquad u \in X'(T;\ p_0), \tag{2-3}$$

where the constant C is independent of ϕ, u and T.

In the weighted Sobolev space, we have the following lemmas.

LEMMA 2-3 (Hayashi, Nakamitsu and M. Tsutsumi [8]). *Let $k, m \in \overline{N}$ with $m \leqq k$ and (θ, p) be an admissible pair. Then for any s, $t \in \boldsymbol{R}$ and $\phi \in W_m^{k,p'}$,*

$$(x+2it\partial_x)^\alpha U(t-s)\phi = U(t-s)(x+2is\partial_x)^\alpha \phi. \tag{2-4}$$

We have for all $t \in \boldsymbol{R}\backslash\{0\}$,

$$\|U(t)\phi\|_{W_m^{k,p}} \leqq C(1+|t|)^m |t|^{-2/\theta} \|\phi\|_{W_m^{k,p'}}. \tag{2-5}$$

When $p=2$, (2-5) extends to the whole real line $\boldsymbol{R}$:

$$\|U(t)\phi\|_{W_m^{k,2}} \leqq C(1+|t|)^m \|\phi\|_{W_m^{k,2}}. \tag{2-6}$$

PROOF. We denote by $\mathcal{F}$ the Fourier transform on $\boldsymbol{R}^n$. Using $U(t)\phi = \mathcal{F}^{-1}\exp(-i(t-s)|\xi|^2)\mathcal{F}\phi$, we compute:

$$(x+2it\partial_x)^\alpha U(t-s)\phi = \mathcal{F}^{-1}(i\partial_\xi - 2t\xi)^\alpha \exp(-i(t-s)|\xi|^2)(\mathcal{F}\phi)(\xi)$$

$$= \mathcal{F}^{-1}\exp(-i(t-s)|\xi|^2)(i\partial_\xi - 2s\xi)^\alpha (\mathcal{F}\phi)(\xi) = U(t-s)(x+2is\partial_x)^\alpha \phi.$$

This is (2-4). Setting $t=0$ and $s=-t$ in (2-4) and applying (2-1), we obtain for $|\alpha| \leqq m$,

$$\|x^\alpha U(t)\phi\|_p \leqq C|t|^{-2/\theta}(1+|t|)^m \|\phi\|_{W_m^{m,p'}}. \tag{2-7}$$

Again by (2-1), we obtain

$$\|\partial_x^\alpha U(t)\phi\|_p \leqq C|t|^{-2/\theta} \|\partial_x^\alpha \phi\|_{p'}. \tag{2-8}$$

(2-7) and (2-8) imply (2-5). ■

LEMMA 2-4 (Hayashi, Nakamitsu and M. Tsutsumi [8]). *Let $k, m \in \overline{N}$ with $m \leqq k$, and p be an admissible exponent. Then for any $\phi \in W_m^{k,p'}$, we have*

$$U\phi \in C(\boldsymbol{R}\backslash\{0\};\ W_m^{k,p}). \tag{2-9}$$

When $p=2$,

$$U\phi \in C(\boldsymbol{R};\ W_m^{k,2}). \tag{2-10}$$

PROOF. Since $\partial_x^\alpha \phi \in L^{p'}$ for $|\alpha| \leqq k$, Lemma 2-1 implies

$$\|\partial_x^\alpha (U(t)-U(s))\phi\|_p = \|(U(t)-U(s))\partial_x^\alpha \phi\|_p \longrightarrow 0 \quad \text{as} \quad s \to t \neq 0. \tag{2-11}$$

On the other hand, since $(x-2is\partial_x)^\beta\phi\in L^{p'}$ for $|\beta|\leqq m$, (2-4) and Lemma 2-1 imply

$$\begin{aligned}&\|x^\beta(U(t)-U(s))\phi\|_p\\ &\leqq\|(U(t)-U(s))(x-2is\partial_x)^\beta\phi\|_p+\|U(t)\{(x-2it\partial_x)^\beta\phi-(x-2is\partial_x)^\beta\phi\}\|_p\\ &\leqq\|(U(t)-U(s))(x-2is\partial_x)^\beta\phi\|_p\\ &\quad+C|t|^{-2/\theta}\|\{(x-2it\partial_x)^\beta-(x-2is\partial_x)^\beta\}\phi\|_{p'}\longrightarrow 0\quad\text{as}\quad s\to t\neq 0\,. \quad (2\text{-}12)\end{aligned}$$

Summing up (2-11) and (2-12) for $|\alpha|\leqq k$ and $|\beta|\leqq m$ yields (2-9). If $p=2$, then $\theta=\infty$ and (2-12) holds for all t. This implies (2-10). ■

LEMMA 2-5. *Let $k, m\in\overline{N}$ with $m\leqq k$. Let p and p_0 be admissible exponents and $0<T<\infty$. Then,*

(1) $U(\cdot)\in B(W_m^{k,2}, X_m^k(T\,;\,p))$ *and*

$$\|U\phi\|_{X_m^k(T;\,p)}\leqq C(1+T)^m\|u\|_{W_m^{k,2}}\,. \quad (2\text{-}13)$$

(2) $S\in B(X_m'^k(T\,;\,p_0), X_m^k(T\,;\,p))$ *and*

$$\|Su\|_{X_m^k(T;\,p)}\leqq C(1+T)^m\|u\|_{X_m'^k(T;\,p_0)}\,. \quad (2\text{-}14)$$

Here the constants C in (2-13) *and* (2-14) *are independent of T.*

PROOF. 1) Let $\phi\in W_m^{k,2}$. It follows from (2-6) and (2-10) that

$$\|U\phi\|_{C([0,T];\,W_m^{k,2})}\leqq C(1+T)^m\|\phi\|_{W_m^{k,2}}\,. \quad (2\text{-}15)$$

On the other hand, for $|\alpha|\leqq k$, we obtain from (2-2)

$$\|\partial_x^\alpha U\phi\|_{L^\theta([0,T];\,L^p)}=\|U\partial_x^\alpha\phi\|_{L^\theta([0,T];\,L^p)}\leqq C\|\partial_x^\alpha\phi\|_2\leqq C\|\phi\|_{W_m^{k,2}}, \quad (2\text{-}16)$$

and, for $|\beta|\leqq m$, from (2-4) and (2-2)

$$\begin{aligned}\|x^\beta U\phi\|_{L^\theta([0,T];\,L^p)}&=\|U(t)(x-2it\partial_x)^\beta\phi\|_{L^\theta([0,T];L^p)}\\ &\leqq C\sum_{\beta_1\leq\beta}\|U(t)x^{\beta_1}(t\partial_x)^{\alpha-\beta_1}\phi\|_{L^\theta([0,T];\,L^p)}\\ &\leqq C\sum_{\beta_1\leq\beta}T^{\beta-\beta_1}\|x^{\beta_1}\partial_x^{\beta-\beta_1}\phi\|_2\\ &\leqq C(1+T)^m\|\phi\|_{W_m^{m,2}}\,. \quad (2\text{-}17)\end{aligned}$$

Summing up (2-16) and (2-17) for $|\alpha|\leqq k$ and $|\beta|\leqq m$ yields, since $k\geqq m$,

$$\begin{aligned}\|U\phi\|_{L^\theta(0,T;\,W_m^{k,p})}&\leqq\sum_{|\alpha|\leq k}\|\partial_x^\alpha U\phi\|_{L^\theta([0,T];\,L^p)}+\sum_{|\beta|\leq m}\|x^\beta U\phi\|_{L^\theta([0,T];\,L^p)}\\ &\leqq C(1+T)^m\|\phi\|_{W_m^{k,2}}\,. \quad (2\text{-}18)\end{aligned}$$

We obtain (2-13) from (2-15) and (2-18).

(2) Let $u\in X_m'^k(T;\ p_0)$. By (2-3), we have for $|\alpha|\leqq k$,

$$\|\partial_x^\alpha Su\|_{X(T,p)}=\|S\partial_x^\alpha u\|_{X(T,p)}\leqq C\|\partial_x^\alpha u\|_{X'(T,p_0)}\leqq C\|u\|_{X_m'^k(T;p_0)}. \tag{2-19}$$

On the other hand, for $|\beta|\leqq m$, (2-3) and (2-4) imply

$$\begin{aligned}\|x^\beta Su\|_{X(T,p)}&=\left\|\int_0^t U(t-s)\{x-2i(t-s)\partial_x\}^\beta u ds\right\|_{X(T,p)}\\&\leqq\sum_{\beta_1\leqq\beta}C\left\|\int_0^t U(t-s)x^\beta\{(t-s)\partial_x\}^{\beta-\beta_1}u ds\right\|_{X(T,p)}\\&\leqq C\sum_{\beta_1\leqq\beta}(1+T)^{\beta-\beta_1}\|S(x^{\beta_1}\partial_x^{\beta-\beta_1}u)\|_{X(T,p)}\\&\leqq C(1+T)^m\sum_{\beta_1\leqq\beta}\|x^{\beta_1}\partial_x^{\beta-\beta_1}u\|_{X'(T,p_0)}\\&\leqq C(1+T)^m\|u\|_{X_m'^m(T,p_0)}.\end{aligned} \tag{2-20}$$

Thus, we obtain the desired estimate (2-14):

$$\begin{aligned}\|Su\|_{X_m^k(T;p)}&\leqq C\Big(\sum_{|\alpha|\leqq k}\|\partial_x^\alpha Su\|_{X(T;p)}+\sum_{|\beta|\leqq m}\|x^\beta Su\|_{X(T;p)}\Big)\\&\leqq C(1+T)^m\|u\|_{X_m'^k(T;p_0)}.\end{aligned}$$

Finally, we show $Su\in C([0,T];\ W_m^{k,2})$ when $u\in X_m'^k(T;\ p)$. Take $u_j\in C_0([0,T];\ \mathcal{S})$ such that $u_j\to u$ in $X_m'^k(T;\ p)$. Since $Uu_j\in C([0,T];\ W_m^{k,2})$,

$$Su_j=\int_0^t U(t-s)u_j(s)ds\in C([0,T];\ W_m^{k,2}).$$

On the other hand, applying (2-14) to u_j-u, we obtain $Su_j\to Su$ in $L^\infty([0,T];\ W_m^{k,2})$. Hence, $Su\in C([0,T];\ W_m^{k,2})$, and this concludes the proof of Lemma 2-5. ■

At the end of this section, we record Sobolev's embedding theorem.

LEMMA 2-6. *Let* $s_1, s_2\in[1,\infty]$ *and* $j\in\overline{N}$. *If* $[1/s_1-j/n]_+\leqq 1/s_2\leqq 1/s_1$, $W^{j,s_1}\subset L^{s_2}$ *and* $\|\phi\|_{s_2}\leqq C\|\phi\|_{W^{j,s_1}}$.

§3. Proof of Theorem 1-3.

We first prove Remark 1-2.

PROPOSITION 3-1. *If W and V satisfy* (1-4) *and* (1-5) *for some* q, $s\geqq 1$ *with* $q>n/(2k+2)$ *and* $s>n/2$, *they satisfy following* (3-1) *and* (3-2) *with q and s possibly different from the original ones.*

$$3/2+1/2s-1/q\geqq 1/2+[1/2-1/2s-k/n]_{+}+[1/2-k/n]_{+}, \tag{3-1}$$

$$1/2+1/2s-1/2q\geqq[1/2-k/n]_{+}. \tag{3-2}$$

PROOF. Note that if V and W satisfy (1-4) and (1-5) for some $q, s\geqq 1$ with $q>n/(2k+2)$ and $s>n/2$, they do so for smaller $q, s\geqq 1$ with $q>n/(2k+2)$.

When $n=1$, (3-1) and (3-2) are obviously satisfied by $q=1$ and $s=1$. When $n=2$ and $2k+2>n$, we take $q=1$. Then (3-1) and (3-2) become

$$1/2s\geqq[1/2-1/2s-k/n]_{+}+[1/2-k/n]_{+}, \tag{3-3}$$

and

$$1/2s\geqq[1/2-k/n]_{+}, \tag{3-4}$$

respectively. However, $1/2-k/n-1/2s\leqq 0$ and $1/2-k/n\leqq 1/n$. Hence (3-3) and (3-4) are satisfied for s sufficiently close to but larger than $n/2$. Finally, when $n\geqq 2$ and $2k+2\leqq n$, (3-1) and (3-2) become

$$1/2s\geqq 1/2q-k/n, \tag{3-5}$$

$$1/q-1/s\leqq 2k/n, \tag{3-6}$$

respectively, since the quantities inside $[\cdot]_{+}$ are positive. Hence if we take q sufficiently close to but larger than $n/(2k+2)$, then (3-5) and (3-6) are satisfied since $1/s<2/n$. ∎

We set throughout this section

$$1/p_0=1/2-1/2s. \tag{3-7}$$

In what follows we assume V and W satisfy Assumption 1 and set non-linear operators $F_1(u)=u(W_1*|u|^2)$ and $F_2(u)=u(W_2*|u|^2)$, and

$$G(u)=F_1(u)+F_2(u)+V_1u+V_2u. \tag{3-8}$$

In the following lemmas, which are proved in Section 5, we estimate each terms of (3-8) separately. C denotes a constant which is independent of T, u and v, and may be different from line to line.

LEMMA 3-2. *Let $u, v\in X_m^k(T;p)$. Then,*

$$\|F_1(u)-F_1(v)\|_{L^1(0,T;W_m^{k,2})}$$
$$\leqq CT\|W_1\|_\infty(\|u\|^2_{C([0,T];W^{k,2})}+\|v\|^2_{C([0,T];W_m^{k,2})})\|u-v\|_{C([0,T];W_m^{k,2})}, \tag{3-9}$$

and

$$\|F_1(u)\|_{L^1(0,T;W^{k,2}_m)} \leqq CT\|W_1\|_\infty \|u\|^2_{C([0,T];W^{k,2})}\|u\|_{C([0,T];W^{k,2}_m)}. \tag{3-10}$$

LEMMA 3-3. *Let $u \in X^k_m(T;p)$. Then,*

$$\|V_1(u)\|_{L^1(0,T;W^{k,2}_m)} \leqq CT\|V_1\|_{W^{k,\infty}}\|u\|_{C([0,T];W^{k,2}_m)}. \tag{3-11}$$

LEMMA 3-4. *Let $u, v \in X^k_m(T;p)$. Then,*

$$\|F_2(u)-F_2(v)\|_{L^{\theta_0'}(0,T;W^{k,r_0'}_m)}$$
$$\leqq CT^{1-2/\theta_0}\|W_2\|_q(\|u\|^2_{C([0,T];W^{k,2})}+\|v\|^2_{C([0,T];W^{k,2}_m)})\|u-v\|_{L^{\theta_0}(0,T;W^{k,p_0}_m)}, \tag{3-12}$$

and

$$\|F_2(u)\|_{L^{\theta_0'}(0,T;W^{k,r_0'}_m)} \leqq CT^{1-2/\theta_0}\|W_2\|_q\|u\|^2_{C([0,T];W^{k,2})}\|u-v\|_{L^{\theta_0}(0,T;W^{k,p_0}_m)}. \tag{3-13}$$

LEMMA 3-5. *Let $u \in X^k_m(T;p)$. Then,*

$$\|V_2(u)\|_{L^{\theta_0'}(0,T;W^{k,p_0'}_m)} \leqq CT^{1-2/\theta_0}\|V_2\|_{W^{k,s}}\|u\|_{L^{\theta_0}(0,T;W^{k,p_0}_m)}. \tag{3-14}$$

Note that several terms in the r.h.s. of (3-9) and (3-12) are not $\|\cdot\|_{C([0,T];W^{k,2}_m)}$, but $\|\cdot\|_{C([0,T];W^{k,2})}$. This is important in the proof of statement 4) of Theorem 1-3.

We are now ready to prove Theorem 1-3. We define the non-linear map Φ by

$$\Phi(u) = U\phi - iSG(u),$$

and consider Φ on the closed ball $B(T,M)=\{u \in X^k_m(T;p_0): \|u\|_{X^k_m(T;p_0)} \leqq M\}$.

LEMMA 3-6. *Fix $\phi \in W^{k,2}_m$. Then, there exist T and M, which depend only on $\|\phi\|_{W^{k,2}_m}$, such that Φ is a contraction in $B(T,M)$.*

PROOF. Since $k \geqq m$, we have from Lemma 2-5,

$$\|\Phi(u)\|_{X^k_m(T;p_0)} \leqq \|U\phi\|_{X^k_m(T;p_0)} + \|SG(u)\|_{X^k_m(T;p_0)}$$
$$\leqq C(1+T)^m(\|\phi\|_{W^{k,2}_m} + \|G(u)\|_{X'^k_m(T;p_0)})$$
$$\leqq C(1+T)^m(\|\phi\|_{W^{k,2}_m} + \|F_1(u)\|_{L^1(0,T;W^{k,2}_m)} + \|F_2(u)\|_{L^{\theta_0'}(0,T;W^{k,p_0'}_m)}$$
$$+\|V_1u\|_{L^1(0,T;W^{k,2}_m)} + \|V_2u\|_{L^{\theta_0'}(0,T;W^{k,p_0'}_m)}).$$

Applying (3-10), (3-11), (3-13) and (3-14) to the r.h.s., we obtain

$$\|\Phi(u)\|_{X_m^k(T;\,p_0)}$$
$$\leqq C(1+T)^m\{\|\phi\|_{W_m^{k,2}}+(T\|V_1\|_{W^{k,\infty}}+T^{1-2/\theta_0}\|V_2\|_{W^{k,s}})\|u\|_{X_m^k(T;\,p_0)}$$
$$+(T\|W_1\|_\infty+T^{1-2/\theta_0}\|W_2\|_q)\|u\|^2_{C([0,T];\,W^{k,2})}\|u\|_{X_m^k(T;\,p_0)}\}\,. \tag{3-15}$$

Similarly, by Lemma 2-5, we have

$$\|\Phi(u)-\Phi(v)\|_{X_m^k(T;\,p_0)}$$
$$\leqq C(1+T)^m\{\|F_1(u)-F_1(v)\|_{L^1(0,T;\,W_m^{k,2})}+\|V_1(u-v)\|_{L^1(0,T;\,W_m^{k,2})}$$
$$+\|F_2(u)-F_2(v)\|_{L^{\theta_0'}(0,T;\,W_m^{k,p_0'})}+\|V_2(u-v)\|_{L^{\theta_0'}(0,T;\,W_m^{k,p_0'})}\}\,.$$

Applying (3-9), (3-11), (3-12) and (3-14) to the r.h.s. yields

$$\|\Phi(u)-\Phi(v)\|_{X_m^k(T;\,p_0)}$$
$$\leqq C(1+T)^m\{(T\|W_1\|_\infty+T^{1-2/\theta_0}\|W_2\|_q)(\|u\|^2_{C([0,T];\,W_m^{k,2})}+\|v\|^2_{C([0,T];\,W_m^{k,2})})$$
$$+T\|V_1\|_{W^{k,\infty}}+T^{1-2/\theta_0}\|V_2\|_{W^{k,s}}\}\|u-v\|_{X_m^k(T;\,p_0)}\,. \tag{3-16}$$

We take $0<M<\infty$ such that

$$C\|\phi\|_{W_m^{k,2}}\leqq M/4\,, \tag{3-17}$$

and $T>0$ such that $(1+T)^m\leqq 2$ and

$$2(T\|W_1\|_\infty+T^{1-2/\theta_0}\|W_2\|_q)M^2+T\|V_1\|_{W^{k,\infty}}+T^{1-2/\theta_0}\|V_2\|_{W^{k,s}}\leqq 1/(4C)\,. \tag{3-18}$$

Then, (3-15)~(3-18) show that Φ is contraction in $B(T,M)$. ■

By the contraction mapping theorem, Φ has a unique fixed point $u\in B(T,M)$, which is a local solution of (1-6) in $X_m^k(T;\,p_0)=C([0,T];\,W_m^{k,2})\cap L^{\theta_0}(0,T;\,W_m^{k,p_0})$: $u=U\phi-iSG(u)$. In virtue of Lemma 2-5, for any admissible exponent p, u satisfies

$$\|u\|_{X_m^k(T;\,p)}\leqq\|U\phi\|_{X_m^k(T;\,p)}+\|SG(u)\|_{X_m^k(T;\,p)}$$
$$\leqq\|\phi\|_{W_m^{k,2}}+\|G(u)\|_{X'^k_m(T;\,p_0)}<\infty\,. \tag{3-19}$$

This proves 1).

We next prove 4), that is,

$$\limsup_{t\to T^*}\|u(t)\|_{W^{k,2}}=\infty\,.$$

By standard continuation argument, it is easy to see that there exists the supremum T^* of all $T>0$ for which there exists a solution u of (1-6) in $X_m^k(T;p_0)$. It follows by the well-known argument in the theory of ordinary differential equations that if $T^*<\infty$ then

$$\|u(t)\|_{W_m^{k,2}} \longrightarrow \infty \qquad \text{as} \quad t\to T^*. \tag{3-20}$$

Suppose

$$\|u(t)\|_{W^{k,2}} \leqq C_0<\infty \qquad 0\leqq t\leqq T^*. \tag{3-21}$$

We wish to show that this leads to

$$\|u(t)\|_{W_m^{k,2}} \leqq C_0'<\infty \qquad 0\leqq t\leqq T^*, \tag{3-22}$$

which contradicts with (3-20).

Fix $0\leqq t_0<T^*$ and let $T>0$ be sufficiently small. Then, for the solution $u(t)$ satisfying (3-21), we have

$$\begin{aligned}
&\|u(t_0+\cdot)\|_{X_m^k(T;p_0)}\\
&\leqq \|Uu(t_0)\|_{X_m^k(T;p_0)}+\|SG(u)\|_{X_m^k(T;p_0)}\\
&\leqq C(1+T)^m(\|u(t_0)\|_{W_m^{k,2}}+T\|u(t_0+\cdot)\|_{X_m^k(T;p_0)}+T^{1-2/\theta_0}\|u(t_0+\cdot)\|_{X_m^k(T;p_0)}).
\end{aligned} \tag{3-23}$$

Here it is important to notice that this constant C is determined by C_0 only. (Note that various constants depending on W and V remain bound on $\boldsymbol{R}$.) Thus, taking N large enough such that (3-23) is satisfied for $T=T^*/N>0$ and $C(1+T)^m(T+T^{1-2/\theta_0})<1/2$, we have for $t_0\leqq t\leqq t_0+T$, $\|u(t)\|_{W_m^{k,2}} \leqq \|u(t_0+\cdot)\|_{X_m^k(T;p_0)} \leqq 2C(1+T)^m\|u(t_0)\|_{W_m^{k,2}}$. Repeating this argument, we see that $\|u(t)\|_{W_m^{k,2}}\leqq\{2C(1+T)^m\}^N\|u(0)\|_{W_m^{k,2}}$ when $t\leqq TN$. This means (3-22).

Next, we shall prove 5). Let $\phi_j\to\phi$ in $W_m^{k,2}$. Since $\|\phi\|_{W_m^{k,2}}\leqq 2\|\phi\|_{W_m^{k,2}}$ for sufficiently large j, Φ and $\Phi_j(\cdot)=U\phi_j-iSG(\cdot)$ are contraction maps on $B(T,M)$, and $\|u\|_{X_m^k(T;p_0)}$ and $\|u_j\|_{X_m^k(T;p_0)}\leqq M$, where T, M are the same as in Lemma 3-6. We have

$$\begin{aligned}
&\|u-u_j\|_{X_m^k(T;p_0)}\leqq\|U(\phi-\phi_j)\|_{X_m^k(T;p_0)}+\|SG(u-u_j)\|_{X_m^k(T;p_0)}\\
&\leqq C(1+T)^m(\|\phi-\phi_j\|_{W_m^{k,2}}+\|G(u)-G(u_j)\|_{X_m'^k(T;p_0)})\\
&\leqq C(1+T)^m\|\phi-\phi_j\|_{W_m^{k,2}}\\
&\quad +C(1+T)^m\{(T\|W_1\|_\infty+T^{1-2/\theta_0}\|W_2\|_q)(\|u\|^2_{C([0,T];W_m^{k,2})}+\|u_j\|^2_{C([0,T];W_m^{k,2})})\\
&\quad +T\|V_1\|_{W^{k,\infty}}+T^{1-2/\theta_0}\|V_2\|_{W^{k,s}}\}\times\|u-u_j\|_{X_m^k(T;p_0)}\\
&\leqq 2C\|\phi-\phi_j\|_{W_m^{k,2}}+1/2\|u-u_j\|_{X_m'^k(T;p_0)}.
\end{aligned}$$

From this, we conclude that

$$\|u-u_j\|_{X_m^k(T;p_0)} \leqq C\|\phi-\phi_j\|_{W_m^{k,2}} \longrightarrow 0 \quad \text{as} \quad j\to\infty .$$

Remark that above T is the same constant in Lemma 3-6. Thus, by the continuation argument, we obtain 5).

We turn now to the proof of statement 6). Since $u\in C([0,T];\ W_m^{k,2})$ for all $T<T^*$, we know that $\Delta u\in C([0,T];\ W^{k-2,2})$. Thus, it suffices to prove that $G(u)\in C([0,T];\ W^{k-2,2})$. ((1-1) follows by differentiating the integral equation (1-6) in $W^{k-2,2}$.) Now, we use the next lemma.

LEMMA 3-7. *Suppose Assumption* 1 *be satisfied and* $k\geqq 1$. *Then,* F_2 *and* V_2 *map* $W^{k,2}$ *into* $W^{k-2,2}$.

We also know F_1 and V_1 map $W^{k,2}$ into $W^{k,2}$. Since $u\in C([0,T^*);\ W^{k,2})$, this and above lemma imply $G(u)\in C([0,T^*);\ W^{k-1,2})$.

We now prove L^2-conservation laws 2). If $k\geqq 1$, this can be proved by taking the L^2-inner product of (1-1) with u. When $k=0$, we take a sequence $\phi_j\in W^{1,2}$ which converges to $\phi\in L^2$ in L^2. The solution $u_j(t)$ has the constant L^2-norm $\|\phi_j\|_2$ as long as it exists in $W^{1,2}$. By 5), it follows that $u_j\to u$ in L^2 uniformly on compacts in $[0,T^*)$ and $u(t)$ itself has the constant L^2-norm.

Before proving the statement 3), we remark next lemma, continuity of the energy in $W^{1,2}$.

LEMMA 3-8. *Suppose Assumption* 1 *be satisfied and let* $k=1$. *Then, the energy* $E(\cdot)$ *is continuous from* $W^{1,2}$ *to* $\boldsymbol{R}$.

Now, the energy conservation law 3) can be obtained as follows. If $k\geqq 2$, take the L^2-inner product of (1-1) with $\partial_t u$. Then the standard argument implies $E(u(t))=E(\phi)$. When $k=1$, we take a sequence $\phi_j\in W^{2,2}$ which converges to ϕ in $W^{1,2}$. The solution u_j has the constant energy $E(u_j)=E(\phi_j)$ as long as it exists in $W^{2,2}$. By 5), $u_j\to u$ in $W^{1,2}$ uniformly on compacts in $[0,T^*)$. It follows that

$$E(u)=\lim_{j\to\infty}E(u_j)=\lim_{j\to\infty}E(\phi_j)=E(\phi) .$$

This completes the proof of Theorem 1-3.

§ 4. Proof of Theorem 1-4.

First, we prepare a-priori estimate of $\|u\|_{W^{1,2}}$.

LEMMA 4-1. *Let Assumption 2 be satisfied. Then there exists a constant $C(\phi)$, depending only on $\|\phi\|_{W^{1,2}}$, such that the solution u of (1-2) satisfies*

$$\|u(t)\|_{W^{1,2}}\leqq C(\phi), \qquad \phi\in W^{1,2} \tag{4-1}$$

as long as it exists.

PROOF. Note that L^2-norm and the energy are conserved under Assumption 2.

We shall show that $\|\nabla u\|_2$ is bounded by $\|u\|_2$ and $E(u)$. Young's and Hölder's inequalities show

$$\int_{R^n}|u|^2(W_1*|u|^2)dx\geqq -\|W_1\|_\infty\|u\|_2^4, \tag{4-2}$$

and

$$\int_{R^n}V_1|u|^2dx\geqq -\|V_1\|_\infty\|u\|_2^2. \tag{4-3}$$

On the other hand, since Lemma 2-6 shows $\|u\|_{r/(r-1)}\leqq C\|u\|_{W^{1,2}}$, we have

$$\int_{R^n}|u|^2(W_2*|u|^2)dx\geqq -\int_{R^n}|u|^2(W_{2-}*|u|^2)dx$$

$$\geqq -\|W_{2-}\|_r\|u\|_2^2\|u\|_{r/(r-1)}^2\geqq -C\|W_{2-}\|_r\|u\|_2^2\|u\|_{W^{1,2}}. \tag{4-4}$$

By similar estimates as in Theorem 1-3, we have

$$\int_{R^n}V_2|u|^2dx\geqq -C\|V_2\|_s\|u\|_{W^{1,2}}^2. \tag{4-5}$$

Thus we obtain from (4-2)～(4-5),

$$E(u)\geqq\|\nabla u\|_2^2-1/2\|W_1\|_\infty\|u\|_2^4-C\|W_{2-}\|_r\|u\|_2^2(\|u\|_2^2+\|\nabla u\|_2^2)$$

$$-\|V_1\|_\infty\|u\|_2^2-C\|V_2\|_s(\|u\|_2^2+\|\nabla u\|_2^2)$$

$$=(1-C\|W_{2-}\|_r\|u\|_2^2-C\|V_2\|_s)\|\nabla u\|_2^2$$

$$-(1/2\|W_1\|_\infty+C\|W_{2-}\|_r)\|u\|_2^4-(\|V_1\|_\infty+C\|V_2\|_s)\|u\|_2^2.$$

Now let decomposition $W=W_1+W_2$ and $V=V_1+V_2$ in Assumption 2 in such a way that $\|W_{2-}\|_r$ and $\|V_2\|_s$ are sufficiently small so that $1-C\|W_{2-}\|_r\|u\|_2^2-C\|V_2\|_s\geqq 1/2$. Then, it follows that for some constants C_1 and $C_2<\infty$, $E(u)\geqq\|\nabla u\|_2^2-C_1\|u\|_2^4-C_2\|u\|_2^2$. Hence

$$\|\nabla u\|_2^2\leqq 2E(u)+C\|u\|_2^4+C\|u\|_2^2. \tag{4-6}$$

Since the r. h. s. of (4-6) is independent of t by conservation laws 2) and 3), (4-6) implies (4-1). ■

Now, we proceed to the proof of Theorem 1-4. Let $\phi\in W_m^{k,2}$, $k\geqq 1$. Then by the proof of Theorem 1-3, there exists a local solution u of (1-2) for the initial data ϕ such that $u\in C([0, T]; W^{1,2})$. Note that T depends only on $\|\phi\|_{W^{1,2}}$. Since $\|\phi\|_{W^{1,2}}$ is uniformly bounded by Lemma 4-1, u becomes global solution on $W^{1,2}$, i. e. $u\in C([0, \infty); W^{1,2})$. Next, we shall prove $\|u(t)\|_{W^{k,2}}<\infty$ for all $t\in \boldsymbol{R}$ under Assumption 2.

LEMMA 4-2. *Let Assumption 2 be satisfied. Then the global solution* $u\in W^{1,2}$ *of* (1-6) *has bounded* $\|u(t)\|_{W^{k,2}}$ *on every bounded interval* $[0, T]$.

PROOF. We shall suppose that

$$\|u(t)\|_{W^{l,2}}\leqq C \qquad \text{for} \quad 0\leqq t\leqq T, \tag{4-7}$$

for $1\leqq l\leqq k-1$, and show $\|u(t)\|_{W^{l+1,2}}\leqq C$ for $0\leqq t\leqq T$. This and Lemma 4-1 imply Lemma 4-2.

Recalling the estimate in Lemma 3-6, we obtain

$$\begin{aligned}\|u\|_{X^{l+1}(T, p_0)}&\leqq \|U\phi\|_{X^{l+1}(T, p_0)}+\|SF(u)\|_{X^{l+1}(T, p_0)}\\ &\leqq C(\|\phi\|_{W^{l+1,2}}+\|F(u)\|_{X'^{l+1}(T, p_0)}).\end{aligned}$$

For $|\alpha|\leqq l+1$, we estimate

$$\begin{aligned}\|\partial_x^\alpha F_1(u(t))\|_2&\leqq \sum_{\text{one of } \alpha_i \text{ is } \alpha}\|(\partial_x^{\alpha_1}u(t))W_1*(\partial_x^{\alpha_2}\overline{u(t)}\partial_x^{\alpha_3}u(t))\|_2\\ &\quad+\sum_{|\alpha_i|<|\alpha|, \alpha_1+\alpha_2+\alpha_3=\alpha}\|(\partial_x^{\alpha_1}u(t))W_1*(\partial_x^{\alpha_2}\overline{u(t)}\partial_x^{\alpha_3}u(t))\|_2\\ &\leqq C\|V_1\|_\infty(\|u(t)\|_{W^{l+1,2}}\|u(t)\|_2^2+\|u(t)\|_{W^{l,2}}^3).\end{aligned}$$

Thus by using (4-7), we deduce that $\|F_1(u(t))\|_{W^{l+1,2}}\leqq C\|u(t)\|_{W^{l+1,2}}+C$. Integrating both sides of the latter by t, we obtain

$$\|F_1(u)\|_{L^1(0, T; W^{l+1,2})}\leqq CT(\|u\|_{C([0, T]; W^{l+1,2})}+C). \tag{4-8}$$

By a similar argument as in the proof of Theorem 1-3, we have also

$$\|V_1u\|_{L^1(0, T; W^{l+1,2})}\leqq CT\|u\|_{C([0, T]; W^{l+1,2})}. \tag{4-9}$$

On the other hand, we obtain by Hölder's and Young's inequalities,

$$\|\partial_x^\alpha F_2(u(t))\|_{p_0'}\leqq C\|W_2\|\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\|\partial_x^{\alpha_1}u(t)\|_{q_1}\|\partial_x^{\alpha_2}u(t)\|_{q_2}\|\partial_x^{\alpha_3}u(t)\|_{q_3},$$

where $2-1/p_0-1/q=1/q_1+1/q_2+1/q_3$. If we take q_1, q_2 and q_3 such that

$$\left\{\begin{array}{c} [1/p_0-(l+1-|\alpha_1|)/n]_+\leqq 1/q_1\leqq 1/p_0 \\ [1/2-(l-|\alpha_2|)/n]_+\leqq 1/q_2\leqq 1/2 \\ [1/2-(l-|\alpha_3|)/n]_+\leqq 1/q_3\leqq 1/2 \end{array}\right\}. \tag{4-10}$$

Lemma 2-6 implies

$$\|\partial_x^\alpha F_2(u(t))\|_{p_0'}\leqq C\|W_2\|_q\|u(t)\|_{W^{l+1,p_0}}\|u(t)\|_{W^{l,2}}^2. \tag{4-11}$$

The choice of q_1, q_2 and q_3 is possible because the maximum of the sum of l. h. s. of (4-10) under the condition $|\alpha_1+\alpha_2+\alpha_3|\leqq l+1$ is $1/p_0+2[1/2-l/n]_+$, which is less than $2-1/p_0-1/q$ if $l\geqq 1$. Thus, we deduce from (4-11) that $\|F_2(u(t))\|_{W^{l+1,p_0'}}\leqq C\|u(t)\|_{W^{l+1,p_0}}$. Taking $L^{\theta_0'}$-norms of both sides with respect to t gives

$$\|F_2(u)\|_{L^{\theta_0'}(0,T;W^{l+1,p_0'})}\leqq CT^{1-2/\theta_0}\|u\|_{L^{\theta_0}(0,T;W^{l+1,p_0})}. \tag{4-12}$$

Similarly, we obtain

$$\|V_2u\|_{L^{\theta_0'}(0,T;W^{l+1,p_0'})}\leqq CT^{1-2/\theta_0}\|u\|_{L^{\theta_0}(0,T;W^{l+1,p_0})}. \tag{4-13}$$

It follows from (4-8), (4-9), (4-12) and (4-13) that

$$\|u\|_{X^{l+1}(T;p_0)}\leqq C\|\phi\|_{W^{l+1,2}}+C(T+T^{1-2/\theta_0})\|u\|_{X^{l+1}(T,p_0)}. \tag{4-14}$$

So taking $T_0>0$ such that $C(T_0+T_0^{1-2/\theta_0})\leqq 1/2$, we obtain from (4-14) that $\|u(t)\|_{W^{l+1,2}}\leqq\|u\|_{X^{l+1}(T_0;p_0)}\leqq C\|\phi\|_{W^{l+1,2}}$. Thus, $\|u(t)\|_{W^{l+1,2}}$ is bounded on $[0, T_0]$. Since T_0 is independent of $\|\phi\|_{W^{l+1,2}}$, repeating above argument, we obtain our desired estimate. ■

We return the proof of Theorem 1-4. By the argument in the proof of Theorem 1-3, above lemma implies $\sup_{t\in[0,T]}\|u(t)\|_{W_m^{k,2}}\leqq C$ for any compact interval $[0,T]$. This means above global solution u in $W^{1,2}$ becomes global solution in $W_m^{k,2}$.

This completes the proof of Theorem 1-4.

§5. The proofs of lemmas in Section 3.

PROOF OF LEMMA 3-2. Write

$$F_1(u)-F_1(v)=(u-v)(W_1*(\bar{u}u))+v(W_1*\{\overline{(u-v)}u\})+v(W_1*\{\bar{v}(u-v)\}),$$

and apply Leibniz formula. Then, by Young's and Hölder's inequalities, for $|\alpha|\leqq k$, we have

$$\|\partial_x^\alpha\{F_1(u(t))-F_1(v(t))\}\|_2$$
$$\leqq \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\{\|\partial_x^{\alpha_1}(u(t)-v(t))[W_1*\{\overline{\partial_x^{\alpha_2}u(t)}\partial_x^{\alpha_3}u(t)\}]\|_2$$
$$+\|\partial_x^{\alpha_2}v(t)[W_1*\{\partial_x^{\alpha_1}\overline{(u(t)-v(t))}\partial_x^{\alpha_3}u(t)\}]\|_2$$
$$+\|\partial_x^{\alpha_2}v(t)[W_1*\{\overline{\partial_x^{\alpha_3}v(t)}\partial_x^{\alpha_1}(u(t)-v(t))\}]\|_2\}$$
$$\leqq \|W_1\|_\infty \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\|\partial_x^{\alpha_1}(u(t)-v(t))\|_2(\|\partial_x^{\alpha_2}u(t)\|_2+\|\partial_x^{\alpha_2}v(t)\|_2)$$
$$\times(\|\partial_x^{\alpha_3}u(t)\|_2+\|\partial_x^{\alpha_3}v(t)\|_2)$$
$$\leqq C\|W_1\|_\infty\|u(t)-v(t)\|_{W^{k,2}}(\|u(t)\|_{W^{k,2}}+\|v(t)\|_{W^{k,2}})^2. \tag{5-1}$$

Similarly, an application of Young's and Hölder's inequalities implies, for $|\beta|\leqq m$,

$$\|x^\beta\{F_1(u(t))-F_1(v(t))\}\|_2$$
$$\leqq \|W_1\|_\infty\{\|x^\beta(u(t)-v(t))\|_2\|u(t)\|_2^2+\|x^\beta v(t)\|_2\|u(t)-v(t)\|_2(\|u(t)\|_2+\|v(t)\|_2)\}$$
$$\leqq \|W_1\|_\infty(\|u(t)\|_2^2+\|v(t)\|_{W_m^{0,2}}^2)\|u(t)-v(t)\|_{W_m^{0,k}}. \tag{5-2}$$

(5-1) and (5-2) yield

$$\|F_1(u(t))-F_1(v(t))\|_{W_m^{k,2}}$$
$$=\sum_{|\alpha|\leq k}\|\partial_x^\alpha\{F_1(u(t))-F_1(v(t))\}\|_2+\sum_{|\beta|\leq m}\|x^\beta\{F_1(u(t))-F_1(v(t))\}\|_2$$
$$\leqq C\|W_1\|_\infty(\|u(t)\|_{W^{k,2}}^2+\|v(t)\|_{W_m^{k,2}}^2)\|u(t)-v(t)\|_{W_m^{k,2}}. \tag{5-3}$$

Integrating both sides of (5-3) by t on $[0, T]$, we obtain the desired estimate (3-9). Letting $v=0$ in (3-9), we obtain (3-10). ■

PROOF OF LEMMA 3-3. Again by Leibniz formula and Hölder's inequality, we obtain for $|\alpha|\leqq k$,

$$\|\partial_x^\alpha V_1(u(t)-v(t))\|_2$$
$$\leqq C\sum_{\alpha_1\leq\alpha}\|\partial_x^{\alpha_1}V_1\|_\infty\|\partial_x^{\alpha-\alpha_1}(u(t)-v(t))\|_2$$
$$\leqq \|V_1\|_{W^{k,\infty}}\|u(t)-v(t)\|_{W^{k,2}}, \tag{5-4}$$

and for $|\beta|\leqq m$,

$$\|x^\beta V_1(u(t)-v(t))\|_2\leqq \|V_1\|_\infty\|u(t)-v(t)\|_{W_m^{0,2}}. \tag{5-5}$$

We deduce from (5-4) and (5-5) that

$$\| V_1(u(t)-v(t))\|_{W_m^{k,2}} \leqq C\| V_1\|_{W^{k,\infty}} \| u(t)-v(t)\|_{W_m^{k,2}}. \tag{5-6}$$

Integrating both sides of (5-6) by t on $[0, T]$, we obtain (3-11). ■

PROOF OF LEMMA 3-4. By Young's and Hölder's inequalities, we obtain as in (5-1),

$$\begin{aligned}&\|\partial_x^\alpha\{F_2(u(t))-F_2(v(t))\}\|_{p_0'}\\ &\leqq \| W_2\|_q \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \|\partial_x^{\alpha_1}(u(t)-v(t))\|_{q_1}(\|\partial_x^{\alpha_2}u(t)\|_{q_2}+\|\partial_x^{\alpha_2}v(t)\|_{q_2})\\ &\quad\times(\|\partial_x^{\alpha_3}u(t)\|_{q_3}+\|\partial_x^{\alpha_3}v(t)\|_{q_3}),\end{aligned} \tag{5-7}$$

where

$$2-1/p_0=1/q+1/q_1+1/q_2+1/q_3. \tag{5-8}$$

We choose q_1, q_2, q_3 such that

$$\left\{\begin{array}{l}[1/p_0-(k-|\alpha_1|)/n]_+ \leqq 1/q_1 \leqq 1/p_0\\ [1/2-(k-|\alpha_2|)/n]_+ \leqq 1/q_2 \leqq 1/2\\ [1/2-(k-|\alpha_3|)/n]_+ \leqq 1/q_3 \leqq 1/2\end{array}\right\}. \tag{5-9}$$

Since this choice is possible because the maximum of the sum of the l. h. s. of (5-9) under the condition $|\alpha_1+\alpha_2+\alpha_3|=k$ is

$$1/2+[1/p_0-k/n]_+ + [1/2-k/n]_+ ,$$

and, from (3-1) and (5-8),

$$\begin{aligned}1/2+[1/p_0-k/n]_+ + [1/2-k/n]_+ &\leqq 1/q_1+1/q_2+1/q_3\\ &=2-1/p_0-1/q.\end{aligned}$$

Then, by (5-7), (5-9) and Sobolev's inequality (Lemma 2-6), for $|\alpha| \leqq k$, we deduce that

$$\begin{aligned}&\|\partial_x^\alpha\{F_2(u(t))-F_2(v(t))\}\|_{p_0'}\\ &\leqq C\| W_2\|_q(\| u(t)\|_{W^{k,2}}^2+\| v(t)\|_{W^{k,2}}^2)\| u(t)-v(t)\|_{W^{k,p_0}}.\end{aligned} \tag{5-10}$$

Similarly, by Young's and Hölder's inequalities, we have

$$\|x^{\beta}\{F_2(u(t))-F_2(v(t))\}\|_{p_0'}$$
$$\leqq \|W_2\|_q\{\|x^{\beta}(u(t)-v(t))\|_{p_0}\|u(t)\|_{q_4}^2$$
$$+\|x^{\beta}v(t)\|_{q_4}(\|u(t)\|_{q_4}+\|v(t)\|_{q_4})\|u(t)-v(t)\|_{p_0}\}, \tag{5-11}$$

where $1-1/p_0=1/2q+1/q_4$. Since (3-5) is satisfied, we have $1/q_4=1-1/p_0-1/2q\geqq[1/2-k/n]_+$. Hence by Sobolev's inequality and Lemma 2-5, we have for $|\beta|\leqq m$,

$$\|x^{\beta}\{F_2(u(t))-F_2(v(t))\}\|_{p_0'}$$
$$\leqq C\|W_2\|_q(\|u(t)\|_{W^{k,2}}^2+\|v(t)\|_{W_m^{k,2}})\|u(t)-v(t)\|_{W_m^{k,p_0}}. \tag{5-12}$$

It follows from (5-10) and (5-12) that

$$\|F_2(u(t))-F_2(v(t))\|_{W_m^{k,p_0'}}$$
$$\leqq C\|W_2\|_q(\|u(t)\|_{W^{k,2}}^2+\|v(t)\|_{W_m^{k,2}}^2)\|u(t)-v(t)\|_{W_m^{k,p_0}}. \tag{5-13}$$

Taking $L^{\theta_0'}$-norms of both sides of (5-13) with respect to t, we obtain (3-12). Letting $v=0$ in (3-12), we obtain (3-13). ■

PROOF OF LEMMA 3-5. By Hölder's inequality, we obtain for $|\alpha|\leqq k$,

$$\|\partial_x^{\alpha}V_2(u(t)-v(t))\|_{p_0'}$$
$$\leqq C\sum_{\alpha_1\leqq\alpha}\|\partial_x^{\alpha_1}V_2\|_s\|\partial_x^{\alpha-\alpha_1}(u(t)-v(t))\|_{p_0}$$
$$\leqq C\|V_2\|_{W^{k,s}}\|u(t)-v(t)\|_{W^{k,p_0}}, \tag{5-14}$$

and,

$$\|x^{\beta}V_2(u(t)-v(t))_{p_0'}\leqq C\|V_2\|_{W^{k,s}}\|u(t)-v(t)\|_{W_m^{k,p_0}} \quad \text{for} \quad |\beta|\leqq m. \tag{5-15}$$

We deduce from (5-14) and (5-15) that,

$$\|V_2(u(t)-v(t))\|_{W_m^{k,p_0'}}\leqq C\|V_2\|_{W^{k,s}}\|u(t)-v(t)\|_{W_m^{k,p_0}}. \tag{5-16}$$

Taking $L^{\theta_0'}$-norms of both sides of (5-16) with respect to t, we obtain (3-14). ■

PROOF OF LEMMA 3.7. 1) First, we consider F_2. We assort following three cases according to n and k.

Case 1—The case that $2k+2\leqq n$.

Since $q>n/(2k+2)$, there exists (sufficiently small) $\varepsilon>0$ such that $1/q=(2k+2)/n-\varepsilon$. We define $1/p_1=1/2-2/n+\varepsilon$. Then, reminding that $k\geqq 1$ and $2k+2\leqq n$ mean $n\geqq 4$, we obtain $\|\cdot\|_{W^{k-2,2}}\leqq C\|\cdot\|_{W^{k,p_1'}}$. So, it is sufficient to prove that F_2 maps $W^{k,2}$ into $W^{k,p_1'}$.

Let $|\alpha|\leqq k$ and $\alpha_1+\alpha_2+\alpha_3=\alpha$. We set $1/a_j=1/2-(k-|\alpha_j|)/n$ $(j=1,2,3)$. Then by Sobolev's lemma, $\|\partial_x^{\alpha_j}v\|_{a_j}\leqq C\|v\|_{W^{|\alpha|,2}}$ and $\sum_{j=1}^3 1/a_j=3/2-2k/n$. Thus by Hölder's and Young's inequalities, we obtain

$$\|\partial_x^\alpha F_2(v)\|_{p_1'}\leqq \|W_2\|_q \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\|\partial_x^{\alpha_1}v\|_{a_1}\|\partial_x^{\alpha_2}v\|_{a_2}\|\partial_x^{\alpha_3}v\|_{a_3}\leqq C\|W_2\|_q\|v\|_{W^{k,2}}^3 . \tag{5-17}$$

This means the desired conclusion.

Case 2—The case that $2k+2>n\geqq 4$.

In this case $q=1$. We set $1/p_1=1/2-2/n+\varepsilon$, where $0<\varepsilon<\min\{(2k+2)/n-1, 4/n\}$. Then, it is sufficient to prove that F_2 maps $W^{k,2}$ into $W^{k,p_1'}$, because $\|\cdot\|_{W^{k-2,2}}\leqq C\|\cdot\|_{W^{k,p_1'}}$. Let $|\alpha|\leqq k$ and $\alpha_1+\alpha_2+\alpha_3=\alpha$. We shall choose $a_1\sim a_3$ such that

$$[1/2-(k-|\alpha_j|)/n]_+\leqq 1/a_j\leqq 1/2 , \tag{5-18}$$

and

$$1/a_1+1/a_2+1/a_3=1-1/p_1=1/2+2/n-\varepsilon .$$

This choice is possible because

$$\max_{|\alpha_1+\alpha_2+\alpha_3|\leqq k}\sum_{j=1}^3[1/2-(k-|\alpha_j|)/n]_+$$
$$\leqq\left\{\begin{array}{ll} 3/2-2k/n=(1/2+2/n)+(1-(2k+2)/n) & \text{if } 1/2>k/n, \\ 1/2+\varepsilon & \text{if } 1/2=k/n, \\ 1/2 & \text{if } 1/2<k/n. \end{array}\right\}$$
$$\leqq 1/2+2/n-\varepsilon .$$

Since (5-18) means $\|\partial_x^{\alpha_j}v\|_{a_j}\leqq C\|v\|_{W^{k,2}}$, we obtain

$$\|\partial_x^\alpha F_2(v)\|_{p_1'}\leqq \|W_2\|_1 \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\|\partial_x^{\alpha_1}v\|_{a_1}\|\partial_x^{\alpha_2}v\|_{a_2}\|\partial_x^{\alpha_3}v\|_{a_3} ,$$
$$\leqq C\|W_2\|_1\|v\|_{W_m^{k,2}}^3 .$$

Case 3—The case that $2k+2>n$ and $n\leqq 3$.

In virtue of Sobolev's Lemma, $\|\cdot\|_{W^{k-2,2}}\leqq C\|\cdot\|_{W^{k,1}}$. So, it is sufficient to prove that F_2 maps $W^{k,2}$ into $W^{k,1}$.

Let $|\alpha|\leqq k$ and $\alpha_1+\alpha_2+\alpha_3=\alpha$. In this case, we shall choose $a_1\sim a_3$ such that

$$[1/2-(k-|\alpha_j|)/n]_+\leqq 1/a_j\leqq 1/2 \qquad \text{for } j=1\sim 3 , \tag{5-19}$$

and

$$1/a_1+1/a_2+1/a_3=1 .$$

For the same reason in Case 2, we can assume $1/2>k/n$. And since our assumption is $2k+2>n\geqq 2k$, we have $n=2,3$ and $k=1$. Then, $3/2-2k/n\leqq 1$ and,

$$\max_{|\alpha_1+\alpha_2+\alpha_3|\leq k}\sum_{j=1}^{3}[1/2-(k-|\alpha_j|)/n]_+=3/2-2k/n\leqq 1\,.$$

Thus, this choice is possible. Since (5-19) mean $\|\partial_x^{\alpha_j}v\|_{a_j}\leqq C\|v\|_{W^{k,2}}$, for $|\alpha|\leqq k$, we have

$$\|\partial_x^\alpha F_2(v)\|_1\leqq\|W_2\|_1\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\|\partial_x^{\alpha_1}v\|_{a_1}\|\partial_x^{\alpha_2}v\|_{a_2}\|\partial_x^{\alpha_3}v\|_{a_3}\,,$$
$$\leqq C\|W_2\|_1\|v\|_{W_m^{k,2}}\,. \tag{5-20}$$

2) Next, we consider V_2. When $n\geqq 2$, put $s-n/2=\varepsilon$ and $1/p_2=1/2-1/n+\varepsilon$. Then, by Sobolev's Lemma, it is sufficient to prove V_2 maps W^{k-1,p_2} into $W^{k-1,p_2'}$. This follows that for $|\alpha|\leqq k-1$, we have

$$\|\partial_x^\alpha(V_2v)\|_{p_2'}\leqq\sum_{\alpha_1\leq\alpha}\|\partial_x^{\alpha_1}V_2\|_s\|\partial_x^{\alpha-\alpha_1}v\|_{p^2}$$
$$\leqq\|V_2\|_{W^{k-1,s}}\|v\|_{W^{k-1,p_2}}\,.$$

When $n=1$ and $k\geqq 2$, we have for $|\alpha|\leqq k-2$,

$$\|\partial_x^\alpha(V_2v)\|_2\leqq\sum_{\alpha_1\leq\alpha}\|\partial_x^{\alpha_1}V_2\|_\infty\|\partial_x^{\alpha-\alpha_1}v\|_2$$
$$\leqq\|V_2\|_{W^{k-2,\infty}}\|v\|_{W^{k-2,2}}$$
$$\leqq C\|V_2\|_{W^{k,1}}\|v\|_{W^{k,2}}\,.$$

This means that V_2 maps $W^{k,2}$ into $W^{k-2,2}$.

Finally, when $n=1$ and $k=1$, in virtue of Sobolev's Lemma, V_2 maps L^∞ into L^1. This implies the statement.

This completes the proof. ■

PROOF OF LEMMA 3.8. It suffices to show the lemma under the following conditions.

$$\left\{\begin{array}{lll} q=1 & s=1 & \text{if } n=1,\\ q=1 & s>1 & \text{if } n=2,\\ q=1 & s>3/2 & \text{if } n=3,\\ q>n/4 & s>n/2 & \text{if } n\geqq 4.\end{array}\right\}$$

Let $u, v\in W^{1,2}$.

When $n\leqq 3$, Lemma 2-6 implies $\|\cdot\|_4\leqq C\|\cdot\|_{W^{1,2}}$. Hence by Hölder's and Young's inequalities, we have

$$\left|\int_{R^n}|u|^2(x)(W_2*|u|^2)(x)dx-\int_{R^n}|v|^2(x)(W_2*|v|^2)(x)dx\right|$$

$$\leqq\left|\int_{R^n}\overline{(u-v)}u(x)(W_2*|u|^2)(x)dx\right|+\left|\int_{R^n}\bar{v}(u-v)(x)(W_2*|u|^2)(x)dx\right|$$

$$+\left|\int_{R^n}(|v|^2(x)(W_2*\overline{(u-v)}u)(x)dx\right|+\left|\int_{R^n}|v|^2(x)(W_2*\bar{v}(u-v))(x)dx\right|$$

$$\leqq 4\|W_2\|_1(\|u\|_4^3+\|v\|_4^3)\|u-v\|_4$$

$$\leqq C\|W_2\|_1(\|u\|_{W^{1,2}}^3+\|v\|_{W^{1,2}}^3)\|u-v\|_{W^{1,2}}\,. \tag{5-21}$$

When $n\geqq 4$, $1/2-1/n<(2q-1)/4q<1/2$ and Lemma 2-6 imply $\|\cdot\|_{4q/(2q-1)}\leqq C\|\cdot\|_{W^{1,2}}$. Thus,

$$\left|\int_{R^n}|u|^2(x)(W_2*|u|^2)(x)dx-\int_{R^n}|v|^2(x)(W_2*|v|^2)(x)dx\right|$$

$$\leqq 4\|W_2\|_q(\|u\|_{4q/(2q-1)}^3+\|v\|_{4q/(2q-1)}^3)\|u-v\|_{4q/(2q-1)}$$

$$\leqq C\|W_2\|_q(\|u\|_{W^{1,2}}^3+\|v\|_{W^{1,2}}^3)\|u-v\|_{W^{1,2}}\,. \tag{5-22}$$

By Hölder's and Young's inequalities, we have

$$\left|\int_{R^n}|u|^2(x)(W_1*|u|^2)(x)dx-\int_{R^n}|v|^2(x)(W_1*|v|^2)(x)dx\right|$$

$$\leqq 4\|W_1\|_\infty(\|u\|_2^3+\|v\|_2^3)\|u-v\|_2\,. \tag{5-23}$$

(5-21)$\sim$(5-23) imply the lemma for $n\geqq 3$. The case $1\leqq n\leqq 2$ is similar and is omitted here. ■

References

[1] Cazenave, T. and B. Weissler, The Cauchy problem for the nonlinear Schrödinger equation in H^1, Manuscripta Math. **61** (1988), 477-494.

[2] Chadam, J.M. and R.T. Glassey, Global existence of solutions to the Cauchy problem for time-dependent Hartree equations, J. Math. Phys. **16** (1975), 1122-1130.

[3] Dias, J.P. and M. Figueira, Conservation laws and time decay for the solutions to some nonlinear Schrödinger-Hartree equations, J. Math. Anal. Appl. **84** (1981), 486-508.

[4] Ginibre, J. and G. Velo, On a class of nonlinear Schrödinger equations with non local interaction, Math. Z. **170** (1980), 109-136.

[5] Glassey, R.T., Asymptotic behavior of solutions to certain nonlinear Schrödinger-

Hartree equations, Comm. Math. Phys. 53 (1977), 9-18.
[6] Hayashi, N. and T. Ozawa, Time decay of solutions to the Cauchy problem for time-dependent Schrödinger-Hartree equations, Comm. Math. Phys. 110 (1987), 467-478.
[7] Hayashi, N. and T. Ozawa, Smoothing effect for some Schrödinger equations, J. Funct. Anal. 85 (1989), 307-348.
[8] Hayashi, N., Nakamitsu, K. and M. Tsutsumi, Nonlinear Schrödinger equations in weighted Sobolev Spaces, Funkcial Ekvac. 31 (1988), 363-381.
[9] Hayashi, N. and Y. Tsutsumi, Scattering theory for Hartree type equations, Ann. Inst. H. Poincaré Phys. Théor. 46 (1987), 187-213.
[10] Kato, T., On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. 46 (1987), 113-129.
[11] Strichartz, R. S., Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44 (1977), 705-714.
[12] Triebel, H., Spaces of distributions with weights: Multipliers in L^p-spaces with weights, Math. Nacher 78 (1977), 339-355.
[13] Yajima, K., Existence of solutions for Schrödinger evolution equations, Comm. Math. Phys. 110 (1987), 415-426.

(Received January 25, 1991)

Department of Pure and Applied Sciences
University of Tokyo
Komaba, Meguro-ku, Tokyo
153 Japan