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On the well-posedness of the Cauchy problem for a class of linear partial differential equations of infinite order in Banach spaces

Dedicated to Professor G. Santagati, with my greatest esteem,
on his sixtieth birthday

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Introduction

Let $n \in \boldsymbol{N}$. In the sequel, when, with regard to a family $\{w_\alpha\}_{\alpha \in \boldsymbol{N}_0^n}$ ($\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$) of elements of a Banach space, we consider the symbol $\sum_{|\alpha|=0}^{\infty} w_\alpha$, it is understood that the family $\{w_\alpha\}_{\alpha \in \boldsymbol{N}_0^n}$ is absolutely summable and that $\sum_{|\alpha|=0}^{\infty} w_\alpha$ is its sum.

Let $(E, \|\cdot\|_E)$ be a real or complex Banach space. Let us introduce the basic function space of this paper. Namely, we denote by $V(\boldsymbol{R}^n, E)$ the space of all functions $u \in C^\infty(\boldsymbol{R}^n, E)$ such that, for every non-empty bounded set $\Omega \subseteq \boldsymbol{R}^n$, one has

$$\sup_{\alpha \in \boldsymbol{N}_0^n} \sup_{x \in \Omega} \|D^\alpha u(x)\|_E < +\infty$$

where $D^\alpha u = \dfrac{\partial^{\alpha_1 + \cdots + \alpha_n} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$.

As it is shown by Proposition 2 below, for each Ω as above, the mapping

$$u \longrightarrow \|u\|_{\Omega, E} = \sup_{\alpha \in \boldsymbol{N}_0^n} \sup_{x \in \Omega} \|D^\alpha u(x)\|_E$$

is a norm on $V(\boldsymbol{R}^n, E)$ and the space $(V(\boldsymbol{R}^n, E), \|\cdot\|_{\Omega, E})$ is complete.

We denote by $\mathcal{L}(E)$ the space of all continuous linear operators from E into itself, endowed with the usual norm:

$$\|A\|_{\mathcal{L}(E)} = \sup_{\|v\|_E \leq 1} \|A(v)\|_E .$$

The aim of this paper is to prove the following well-posedness result which was announced in [3]:

THEOREM 1. *Let $k\in \boldsymbol{N}$. For each $j=0,1,\cdots,k-1$ and each $\alpha\in \boldsymbol{N}_0^n$, let $A_{j,\alpha}\in \mathcal{L}(E)$ be given. Assume that*

$$\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty}\|A_{j,\alpha}\|_{\mathcal{L}(E)}<1 . \tag{1}$$

Then, for every $f\in V(\boldsymbol{R}^{n+1},E)$ and every $\varphi_0,\varphi_1,\cdots,\varphi_{k-1}\in V(\boldsymbol{R}^n,E)$, there exists a unique function $u\in V(\boldsymbol{R}^{n+1},E)$ such that, for all $t\in\boldsymbol{R}$, $x\in\boldsymbol{R}^n$, one has

$$D_t^k u(t,x)+\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty}A_{j,\alpha}(D_t^j D_x^\alpha u(t,x))=f(t,x)$$

$$D_t^j u(0,x)=\varphi_j(x) \qquad \text{for} \quad j=0,1,\cdots,k-1 .$$

Moreover, if $\lambda\in]0,1[$ is chosen in such a way that

$$\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty}\lambda^{j-k}\|A_{j,\alpha}\|_{\mathcal{L}(E)}\leqq 1 ,$$

this function u satisfies, for each $r\geqq 0$ and each non-empty bounded set $\Omega\subseteq\boldsymbol{R}^n$, the following inequality:

$$\begin{aligned}&\max_{0\le j\le k-1}\lambda^{-j}\|D_t^j u\|_{[-r,r]\times\Omega,E}\\ &\leqq\min\left\{e^r\left(\frac{\lambda^{1-k}}{1-\lambda}\|f\|_{\{0\}\times\Omega,E}+\max_{0\le j\le k-1}\lambda^{-j}\|\varphi_j\|_{\Omega,E}\right),\right.\\ &\qquad\left.\lambda^{1-k}\left(re^{\lambda r}+\frac{1}{1-\lambda}\right)\|f\|_{[-r,r]\times\Omega,E}+e^{\lambda r}\max_{0\le j\le k-1}\lambda^{-j}\|\varphi_j\|_{\Omega,E}\right\}.\end{aligned} \tag{2}$$

We wish to remark at once that the conclusion of Theorem 1 does not hold, in general, if condition (1) is violated. In this connection, the simplest example is provided by the equation

$$\frac{\partial u}{\partial t}-u=e^t .$$

Of course, there is no solution of this equation in $V(\boldsymbol{R}^{n+1},\boldsymbol{R})$, although the function $(t,x)\to e^t$ belongs to that space.

Nevertheless, the problem of finding a necessary and sufficient condition for the validity of (the first part of) the conclusion of Theorem 1, remains still open. In fact, we do not know any previous well-posedness result in the space $V(\boldsymbol{R}^{n+1},E)$.

The paper is arranged into two sections. In Section 1, we present a series of auxiliary results which lead to the proof of Theorem 1. Section 2 contains some consequences of Theorem 1. In particular, we put there a reformulation of it in purely algebraic terms (see Theorem 2) as well as a result on systems of infinitely many partial differential equations (see Theorem 5), and another on partial integro-differential equations (see Theorem 6).

1. Auxiliary results and proof of Theorem 1

Let us fix some notation. If $\alpha=(\alpha_1,\cdots,\alpha_n)\in \boldsymbol{N}_0^n$ and $x=(x_1,\cdots,x_n)\in \boldsymbol{R}^n$, as usual, we put $\alpha!=\alpha_1!\cdots\alpha_n!$ and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ (with the convention $0^0=1$). Moreover, we denote by $l^\infty(\boldsymbol{N}_0^n,E)$ (resp. $\tilde{l}^\infty(\boldsymbol{N}_0^n,E)$) the space of all families $\{v_\alpha\}_{\alpha\in\boldsymbol{N}_0^n}$ in E such that $\sup_{\alpha\in\boldsymbol{N}_0^n}\|v_\alpha\|_E<+\infty$ (resp. $\sup_{\alpha\in\boldsymbol{N}_0^n}\alpha!\|v_\alpha\|_E<+\infty$).

We now point out a characterization of $V(\boldsymbol{R}^n,E)$ used several times later.

PROPOSITION 1. *Let f: $\boldsymbol{R}^n\to E$ be a given function. Then, the following assertions are equivalent*:

(a) $f\in V(\boldsymbol{R}^n,E)$.

(b) *f is analytic in $\boldsymbol{R}^n$ and* $\{D^\alpha f(0)\}_{\alpha\in\boldsymbol{N}_0^n}\in l^\infty(\boldsymbol{N}_0^n,E)$.

(c) *There exists* $\{v_\alpha\}_{\alpha\in\boldsymbol{N}_0^n}\in\tilde{l}^\infty(\boldsymbol{N}_0^n,E)$ *such that*

$$f(x)=\sum_{|\alpha|=0}^{\infty}x^\alpha v_\alpha$$

for all $x\in\boldsymbol{R}^n$.

PROOF. The implication (a)$\Rightarrow$(b) follows directly from a classical analyticity criterion. So, assume (b). By Abel's lemma, the power series $\left\{\frac{x^\alpha}{\alpha!}D^\alpha f(0)\right\}_{\alpha\in\boldsymbol{N}_0^n}$ is absolutely summable in $\boldsymbol{R}^n$. For each $x\in\boldsymbol{R}^n$, put

$$g(x)=\sum_{|\alpha|=0}^{\infty}\frac{x^\alpha}{\alpha!}D^\alpha f(0)\,.$$

Thus, the function g is analytic in $\boldsymbol{R}^n$ and, for each $\alpha\in\boldsymbol{N}_0^n$, one has $D^\alpha g(0)=D^\alpha f(0)$. Consequently, since f is assumed to be analytic in $\boldsymbol{R}^n$, one has $f(x)=g(x)$ for all $x\in\boldsymbol{R}^n$. Hence, (c) holds by taking $v_\alpha=\frac{D^\alpha f(0)}{\alpha!}$ for all $\alpha\in\boldsymbol{N}_0^n$. Finally, let (c) hold. Then, $f\in C^\infty(\boldsymbol{R}^n,E)$ and, for each $\alpha\in\boldsymbol{N}_0^n$, $x\in\boldsymbol{R}^n$, one has

$$D^\alpha f(x)=\sum_{|\beta|=0}^{\infty}\frac{(\alpha+\beta)!\,x^\beta}{\beta!}v_{\alpha+\beta}\,.$$

Consequently, if we put $M=\sup_{\alpha\in N_0^n}\alpha!\|v_\alpha\|_E$ (so, $M<+\infty$ by hypothesis), for every $x=(x_1,\cdots,x_n)\in R^n$, we get

$$\sup_{\alpha\in N_0^n}\|D^\alpha f(x)\|_E\leq Me^{|x_1|+\cdots+|x_n|}.$$

Hence, $f\in V(R^n,E)$. ■

Before stating the next proposition, it is useful to introduce the following notation. Namely, if Ω,Ω_1 are two non-empty bounded subsets of R^n, we put

$$c(\Omega,\Omega_1)=\sup_{(x_1,\cdots,x_n)\in\Omega}\ \inf_{(y_1,\cdots,y_n)\in\Omega_1}\sum_{i=1}^n|x_i-y_i|\,.$$

We have the following

PROPOSITION 2. *Let $\Omega\subseteq R^n$ be any non-empty bounded set. For each $u\in V(R^n,E)$, put*

$$\|u\|_{\Omega,E}=\sup_{\alpha\in N_0^n}\sup_{x\in\Omega}\|D^\alpha u(x)\|_E\,.$$

Then, the mapping $u\to\|u\|_{\Omega,E}$ is a norm on $V(R^n,E)$ and the space $(V(R^n,E),\|\cdot\|_{\Omega,E})$ is complete. Moreover, if $\Omega_1\subseteq R^n$ is another non-empty bounded set, one has

$$e^{-c(\Omega_1,\Omega)}\|u\|_{\Omega_1,E}\leq\|u\|_{\Omega,E}\leq e^{c(\Omega,\Omega_1)}\|u\|_{\Omega_1,E} \tag{3}$$

for all $u\in V(R^n,E)$.

PROOF. The first claim follows at once, taken into account that each function $u\in V(R^n,E)$ is analytic in R^n. In view of (3), the completeness of $(V(R^n,E),\|\cdot\|_{\Omega,E})$ follows, for instance, from that of $(V(R^n,E),\|\cdot\|_{0,E})$, where $\|\cdot\|_{0,E}$ stands briefly for $\|\cdot\|_{\{0\},E}$. Thus, to prove that $(V(R^n,E),\|\cdot\|_{0,E})$ is complete, let $\{f_h\}$ be any Cauchy sequence in that space. Hence, in particular, for each $\alpha\in N_0^n$, $\{D^\alpha f_h(0)\}$ is a Cauchy sequence in E. Let $w_\alpha=\lim_{h\to\infty}D^\alpha f_h(0)$. By a standard reasoning, it is seen that $\{w_\alpha\}_{\alpha\in N_0^n}\in l^\infty(N_0^n,E)$. For each $x\in R^n$, put

$$f(x)=\sum_{|\alpha|=0}^{\infty}\frac{x^\alpha}{\alpha!}w_\alpha\,.$$

Then, by Proposition 1, one has $f\in V(R^n,E)$, and it is immediate to check that

$$\lim_{h\to\infty}\sup_{\alpha\in N_0^n}\|D^\alpha f_h(0)-D^\alpha f(0)\|_E=0\,,$$

as desired. Thus, it remains only to prove (3). To this end, let $u \in V(\boldsymbol{R}^n, E)$ and $x^0 \in \Omega_1$. Since, for each $\alpha \in \boldsymbol{N}_0^n, x \in \boldsymbol{R}^n$, one has

$$D^\alpha u(x) = \sum_{|\beta|=0}^{\infty} \frac{(x-x^0)^\beta}{\beta!} D^{\alpha+\beta} u(x^0),$$

it follows that

$$\|D^\alpha u(x)\|_E \leqq e^{c(x^0,x)} \|u\|_{x^0,E} \leqq e^{c(x^0,x)} \|u\|_{\Omega_1,E}.$$

Hence, since x^0 is an arbitrary point of Ω_1, we get

$$\|u\|_{\Omega,E} \leqq e^{c(\Omega,\Omega_1)} \|u\|_{\Omega_1,E}.$$

The other inequality in (3) is obtained, of course, interchanging the roles of Ω and Ω_1. ■

Another basic result is the following

PROPOSITION 3. *Let $m \in \boldsymbol{N}$ and let $S \subseteqq \boldsymbol{R}^m$, $\Omega \subseteqq \boldsymbol{R}^n$ be two non-empty bounded sets. Consider on $V(\boldsymbol{R}^n, E)$, $V(\boldsymbol{R}^m, V(\boldsymbol{R}^n, E))$, $V(\boldsymbol{R}^{m+n}, E)$ the norms $\|\cdot\|_{\Omega,E}$, $\|\cdot\|_{S,V(\boldsymbol{R}^n,E)}$, $\|\cdot\|_{S\times\Omega,E}$, respectively. For each $u \in V(\boldsymbol{R}^m, V(\boldsymbol{R}^n, E))$, let $\Psi(u)$ be the function, from $\boldsymbol{R}^{m+n}$ into E, defined by putting*

$$\Psi(u)(s,x) = u(s)(x)$$

for all $s \in \boldsymbol{R}^m, x \in \boldsymbol{R}^n$.

Then, the mapping $u \to \Psi(u)$ is a linear isometry from $V(\boldsymbol{R}^m, V(\boldsymbol{R}^n, E))$ onto $V(\boldsymbol{R}^{m+n}, E)$. Moreover, for each $u \in V(\boldsymbol{R}^m, V(\boldsymbol{R}^n, E))$, $\gamma \in \boldsymbol{N}_0^m$, one has

(4) $$D_s^\gamma \Psi(u) = \Psi(D^\gamma u).$$

PROOF. Let $u \in V(\boldsymbol{R}^m, V(\boldsymbol{R}^n, E))$. First, we prove that $\Psi(u) \in V(\boldsymbol{R}^{m+n}, E)$. Let $\{f_\gamma\}_{\gamma \in \boldsymbol{N}_0^m}$ be the family in $\bar{l}^\infty(\boldsymbol{N}_0^m, V(\boldsymbol{R}^n, E))$ such that

(5) $$u(s) = \sum_{|\gamma|=0}^{\infty} s^\gamma f_\gamma \qquad \text{for all } s \in \boldsymbol{R}^m.$$

So, we have

(6) $$\Psi(u)(s,x) = \sum_{|\gamma|=0}^{\infty} s^\gamma f_\gamma(x) \qquad \text{for all } s \in \boldsymbol{R}^m,\ x \in \boldsymbol{R}^n.$$

Next, for each $\gamma \in \boldsymbol{N}_0^m$, let $\{v_{\gamma,\alpha}\}_{\alpha \in \boldsymbol{N}_0^n}$ be the family in $\bar{l}^\infty(\boldsymbol{N}_0^n, E)$ such that

(7) $$f_\gamma(x) = \sum_{|\alpha|=0}^{\infty} x^\alpha v_{\gamma,\alpha} \qquad \text{for all } x \in \boldsymbol{R}^n.$$

Now, put $L=\sup_{\gamma\in N_0^m}\gamma!\|f_\gamma\|_{\Omega,E}$. Then, for every $\alpha\in N_0^n, \gamma\in N_0^m$, taking into account (3), one has

$$\gamma!\alpha!\|v_{\gamma,\alpha}\|_E=\gamma!\|D^\alpha f_\gamma(0)\|_E\leqq\gamma!e^{c(0,\Omega)}\|f_\gamma\|_{\Omega,E}\leqq Le^{c(0,\Omega)}.$$

Consequently, $\{v_{\gamma,\alpha}\}_{(\gamma,\alpha)\in N_0^{m+n}}\in\tilde{l}^\infty(N_0^{m+n},E)$. On the other hand, in view of (6) and (7), by associativity (see, for instance, [2], p. 96), we have

$$\Psi(u)(s,x)=\sum_{|(\gamma,\alpha)|=0}^{\infty}s^\gamma x^\alpha v_{\gamma,\alpha}\qquad\text{for all } s\in R^m, x\in R^n. \tag{8}$$

Hence, $\Psi(u)\in V(R^{m+n},E)$ by Proposition 1. Now, observe that, by (5) and (7), one has

$$D^\gamma u(s)(x)=\sum_{|\delta|=0}^{\infty}\frac{(\gamma+\delta)!s^\delta}{\delta!}\sum_{|\alpha|=0}^{\infty}x^\alpha v_{\gamma+\delta,\alpha} \tag{9}$$

and that (8) yields

$$D_s^\gamma\Psi(u)(s,x)=\sum_{|(\delta,\alpha)|=0}^{\infty}\frac{(\gamma+\delta)!s^\delta x^\alpha}{\delta!}v_{\gamma+\delta,\alpha}. \tag{10}$$

By associativity again, the right-hand sides of (9) and (10) are equal, and so (4) follows. Furthermore, we have

$$\begin{aligned}\|u\|_{S,V(R^n,E)}&=\sup_{\gamma\in N_0^m}\sup_{s\in S}\|D^\gamma u(s)\|_{\Omega,E}\\&=\sup_{\gamma\in N_0^m}\sup_{s\in S}\sup_{\alpha\in N_0^n}\sup_{x\in\Omega}\|D_s^\gamma D_x^\alpha\Psi(u)(s,x)\|_E\\&=\sup_{(\gamma,\alpha)\in N_0^{m+n}}\sup_{(s,x)\in S\times\Omega}\|D_s^\gamma D_x^\alpha\Psi(u)(s,x)\|_E=\|\Psi(u)\|_{S\times\Omega,E}.\end{aligned}$$

That is, Ψ is an isometry. Finally, the surjectivity of Ψ is proved by means of the same kind of reasoning adopted to show that $\Psi(V(R^m, V(R^n,E)))\subseteq V(R^{m+n},E)$. We leave the details to the reader. ■

The next two propositions deal with ordinary differential equations.

PROPOSITION 4. *Let $A\in\mathcal{L}(E)$ be such that $\|A\|_{\mathcal{L}(E)}<1$. Moreover, let $B\in V(R,E)$ and $v\in C^1(R,E)$ be such that*

$$v'(t)=A(v(t))+B(t)\qquad in\quad R.$$

Then, $v\in V(R,E)$ and, for every $r\geqq 0$, the following inequality holds:

$$\|v\|_{[-r,r],E}\leqq\min\left\{e^r\left(\|v(0)\|_E+\frac{\|B\|_{0,E}}{1-\|A\|_{\mathcal{L}(E)}}\right),\right. \tag{11}$$

$$\|v(0)\|_E e^{r\|A\|_{\mathcal{L}(E)}} + \left(re^{r\|A\|_{\mathcal{L}(E)}} + \frac{1}{1-\|A\|_{\mathcal{L}(E)}}\right)\|B\|_{[-r,r],E}\Big\}.$$

PROOF. First, observe that $v\in C^\infty(\boldsymbol{R}, E)$, since $B\in C^\infty(\boldsymbol{R}, \boldsymbol{E})$. Next, by induction, it is seen that

$$v^{(m)}(t) = A^m(v(t)) + \sum_{j=1}^{m-1} A^j(B^{(m-j-1)}(t)) + B^{(m-1)}(t) \tag{12}$$

for all $t\in\boldsymbol{R}, m\in\boldsymbol{N}, m\geqq 2$.

Fix $r\geqq 0$. From (12), we then get

$$\|v^{(m)}(t)\|_E \leqq \|v(t)\|_E + \frac{\|B\|_{[-r,r],E}}{1-\|A\|_{\mathcal{L}(E)}} \quad \text{for all } t\in[-r,r], m\in\boldsymbol{N}_0. \tag{13}$$

Consequently, $v\in V(\boldsymbol{R}, E)$. In particular, (13) yields

$$\|v\|_{0,E} \leqq \|v(0)\|_E + \frac{\|B\|_{0,E}}{1-\|A\|_{\mathcal{L}(E)}}$$

and so, by (3), one has

$$\|v\|_{[-r,r],E} \leqq e^r\left(\|v(0)\|_E + \frac{\|B\|_{0,E}}{1-\|A\|_{\mathcal{L}(E)}}\right). \tag{14}$$

On the other hand, being $v(t) = v(0) + \int_0^t A(v(\tau))d\tau + \int_0^t B(\tau)d\tau$, one has

$$\|v(t)\|_E \leqq \|v(0)\|_E + r\|B\|_{[-r,r],E} + \|A\|_{\mathcal{L}(E)}\left|\int_0^t \|v(\tau)\|_E d\tau\right|$$

for all $t\in[-r,r]$.

Therefore, by Gronwall's lemma, we get

$$\|v(t)\|_E \leqq (\|v(0)\|_E + r\|B\|_{[-r,r],E})e^{r\|A\|_{\mathcal{L}(E)}} \quad \text{for all } t\in[-r,r]. \tag{15}$$

At this point, (11) follows at once from (13), (14) and (15). ■

PROPOSITION 5. *Let $k\in\boldsymbol{N}$ and let $A_0, \cdots, A_{k-1}\in\mathcal{L}(E)$. Assume that*

$$\sum_{j=0}^{k-1} \|A_j\|_{\mathcal{L}(E)} < 1.$$

Then, for every $B\in V(\boldsymbol{R}, E)$ and every $w_0, w_1, \cdots, w_{k-1}\in E$, there exists a unique $v\in V(\boldsymbol{R}, E)$ such that

$$v^{(k)}(t) = \sum_{j=0}^{k-1} A_j(v^{(j)}(t)) + B(t) \qquad \text{in } \boldsymbol{R}$$

$$v^{(j)}(0)=w_j \qquad \textit{for } j=0, 1, \cdots, k-1.$$

Moreover, for every $r \geqq 0$ *and every* $\lambda \in]0, 1[$ *such that*

$$\sum_{j=0}^{k-1} \lambda^{j-k} \|A_j\|_{\mathcal{L}(E)} \leqq 1,$$

this function v *satisfies the following inequality*:

(16)
$$\max_{0 \leq j \leq k-1} \lambda^{-j} \|v^{(j)}\|_{[-r,r],E}$$
$$\leqq \min\left\{e^r\left(\max_{0 \leq j \leq k-1} \lambda^{-j}\|w_j\|_E + \frac{\lambda^{1-k}}{1-\lambda}\|B\|_{0,E}\right),\right.$$
$$\left. e^{\lambda r} \max_{0 \leq j \leq k-1} \lambda^{-j}\|w_j\|_E + \lambda^{1-k}\left(re^{\lambda r} + \frac{1}{1-\lambda}\right)\|B\|_{[-r,r],E}\right\}.$$

PROOF. If $k=1$, the conclusion follows at once from Picard-Lindelöf's theorem and Proposition 4. So, let us assume $k \geqq 2$. Consider the space E^k equipped with the norm

$$\|y\|_{E^k} = \max_{0 \leq j \leq k-1} \|y_i\|_E, \qquad \text{where } y=(y_0, y_1, \cdots, y_{k-1}).$$

Fix $\lambda \in]0, 1[$ in such a way that

(17)
$$\sum_{j=0}^{k-1} \lambda^{j-k} \|A_j\|_{\mathcal{L}(E)} \leqq 1.$$

Of course, this is possible since $\sum_{j=0}^{k-1} \|A_j\|_{\mathcal{L}(E)} < 1$. Next, consider the operator $A: E^k \to E^k$ defined by putting

$$A(y_0, y_1, \cdots, y_{k-1}) = \left(\lambda y_1, \lambda y_2, \cdots, \lambda y_{k-1}, \sum_{j=0}^{k-1} \lambda^{j-k+1} A_j(y_j)\right)$$
$$\text{for all } (y_0, y_1, \cdots, y_{k-1}) \in E^k.$$

Plainly, $A \in \mathcal{L}(E^k)$. Moreover, for each $(y_0, y_1, \cdots, y_{k-1}) \in E^k$, in view of (17), we have

$$\|A(y_0, y_1, \cdots, y_{k-1})\|_{E^k} = \max\left\{\lambda \max_{1 \leq j \leq k-1} \|y_i\|_E, \left\|\sum_{j=0}^{k-1} \lambda^{j-k+1} A_j(y_j)\right\|_E\right\}$$
$$\leqq \max\left\{\lambda, \sum_{j=0}^{k-1} \lambda^{j-k+1} \|A_j\|_{\mathcal{L}(E)}\right\} \max_{0 \leq j \leq k-1} \|y_i\|_E$$
$$= \lambda \|(y_0, y_1, \cdots, y_{k-1})\|_{E^k}.$$

Consequently, one has

(18)
$$\|A\|_{\mathcal{L}(E^k)} \leqq \lambda.$$

Now, let $B\in V(\boldsymbol{R}, E)$ and $w_0, w_1, \cdots, w_{k-1}\in E$. By Picard-Lindelöf's theorem, there exists a unique $v\in C^k(\boldsymbol{R}, E)$ such that

$$v^{(k)}(t)=\sum_{j=0}^{k-1} A_j(v^{(j)}(t))+B(t) \qquad \text{in } \boldsymbol{R}$$

$$v^{(j)}(0)=w_j \qquad \text{for } j=0, 1, \cdots, k-1.$$

Let Γ, ω: $\boldsymbol{R}\to E^k$ be the functions defined by putting

$$\Gamma(t)=(0, \cdots, 0, \lambda^{1-k}B(t))$$

and

$$\omega(t)=(v(t), \lambda^{-1}v'(t), \lambda^{-2}v''(t), \cdots, \lambda^{1-k}v^{(k-1)}(t)) \qquad \text{for all } t\in\boldsymbol{R}.$$

Of course, $\Gamma\in V(\boldsymbol{R}, E^k)$ and $\omega\in C^1(\boldsymbol{R}, E^k)$. Furthermore, observe that

$$\begin{aligned}
&A(\omega(t))+\Gamma(t)\\
&=\left(v'(t), \lambda^{-1}v''(t), \cdots, \lambda^{2-k}v^{(k-1)}(t), \lambda^{1-k}\sum_{j=0}^{k-1} A_j(v^{(j)}(t))+\lambda^{1-k}B(t)\right)\\
&=(v'(t), \lambda^{-1}v''(t), \cdots, \lambda^{2-k}v^{(k-1)}(t), \lambda^{1-k}v^{(k)}(t))=\omega'(t).
\end{aligned}$$

Then, since $\|A\|_{\mathcal{L}(E^k)}<1$ (by (18)), thanks to Proposition 4, we have $\omega\in V(\boldsymbol{R}, E^k)$, and so $v\in V(\boldsymbol{R}, E)$. Moreover, by (11) and (18), for each $r\geq 0$, we have

$$\begin{aligned}
\max_{0\leq j\leq k-1} \lambda^{-j}\|v^{(j)}\|_{[-r,r],E}=\|\omega\|_{[-r,r],E^k}\leq\min\Bigg\{&e^r\left(\|\omega(0)\|_{E^k}+\frac{\|\Gamma\|_{0,E^k}}{1-\|A\|_{\mathcal{L}(E^k)}}\right),\\
&\|\omega(0)\|_{E^k}e^{r\|A\|_{\mathcal{L}(E^k)}}+\left(re^{r\|A\|_{\mathcal{L}(E^k)}}+\frac{1}{1-\|A\|_{\mathcal{L}(E^k)}}\right)\Gamma_{[-r,r],E^k}\Bigg\}\\
\leq\min\Bigg\{&e^r\left(\max_{0\leq j\leq k-1} \lambda^{-j}\|w_j\|_E+\frac{\lambda^{1-k}}{1-\lambda}\|B\|_{0,E}\right),\\
&e^{\lambda r}\max_{0\leq j\leq k-1} \lambda^{-j}\|w_j\|_E+\lambda^{1-k}\left(re^{\lambda r}+\frac{1}{1-\lambda}\right)\|B\|_{[-r,r],E}\Bigg\}
\end{aligned}$$

that proves (16). ■

We also need the following

PROPOSITION 6. *Let $\{A_\alpha\}_{\alpha\in\boldsymbol{N}_0^n}$ be an absolutely summable family in $\mathcal{L}(E)$. For each $u\in V(\boldsymbol{R}^n, E), x\in\boldsymbol{R}^n$, put*

$$T(u)(x)=\sum_{|\alpha|=0}^{\infty} A_\alpha(D^\alpha u(x)).$$

Then, $T(u)(\cdot)\in V(\boldsymbol{R}^n, E)$, *the mapping* $u\to T(u)$ *belongs to* $\mathcal{L}(V(\boldsymbol{R}^n, E))$ *and one has*

$$\|T\|_{\mathcal{L}(V(\boldsymbol{R}^n,E))}\leqq\sum_{|\alpha|=0}^{\infty}\|A_\alpha\|_{\mathcal{L}(E)} \tag{19}$$

for any norm $\|\cdot\|_{\Omega,E}$ *on* $V(\boldsymbol{R}^n, E)$.

PROOF. Fix $u\in V(\boldsymbol{R}^n, E)$. Then, for each $x\in\boldsymbol{R}^n$, one has

$$\sum_{|\alpha|=0}^{\infty}A_\alpha(D^\alpha u(x))=\sum_{|\alpha|=0}^{\infty}A_\alpha\left(\sum_{|\beta|=0}^{\infty}\frac{x^\beta}{\beta!}D^{\alpha+\beta}u(0)\right)$$

$$=\sum_{|\alpha|=0}^{\infty}\sum_{|\beta|=0}^{\infty}\frac{x^\beta}{\beta!}A_\alpha(D^{\alpha+\beta}u(0))=\sum_{|\beta|=0}^{\infty}x^\beta\sum_{|\alpha|=0}^{\infty}\frac{A_\alpha(D^{\alpha+\beta}u(0))}{\beta!}.$$

Observe that $\left\{\sum_{|\alpha|=0}^{\infty}\frac{A_\alpha(D^{\alpha+\beta}u(0))}{\beta!}\right\}_{\beta\in\boldsymbol{N}_0^n}\in l^\infty(\boldsymbol{N}_0^n, E)$. Consequently, $T(u)\in V(\boldsymbol{R}^n, E)$ by Proposition 1. Also, a direct verification shows that $D^\beta T(u)=T(D^\beta u)$ for all $\beta\in\boldsymbol{N}_0^n$. Now, fix any norm $\|\cdot\|_{\Omega,E}$ on $V(\boldsymbol{R}^n, E)$ (Ω, of course, being a non-empty bounded subset of $\boldsymbol{R}^n$). Then, one has

$$\|T(u)\|_{\Omega,E}=\sup_{\beta\in\boldsymbol{N}_0^n}\sup_{x\in\Omega}\left\|\sum_{|\alpha|=0}^{\infty}A_\alpha(D^{\alpha+\beta}u(x))\right\|_E$$

$$\leqq\|u\|_{\Omega,E}\sum_{|\alpha|=0}^{\infty}\|A_\alpha\|_{\mathcal{L}(E)}$$

that yields (19), the linearity of T being obvious. ■

At this point, we are able to give the following

PROOF OF THEOREM 1. Fix $f\in V(\boldsymbol{R}^{n+1}, E)$ and $\varphi_0, \varphi_1, \cdots, \varphi_{k-1}\in V(\boldsymbol{R}^n, E)$. Consider $V(\boldsymbol{R}^n, E)$ equipped with any fixed norm $\|\cdot\|_{\Omega,E}$. For each $j=0, 1, \cdots, k-1$, $v\in V(\boldsymbol{R}^n, E), x\in\boldsymbol{R}^n$, put

$$T_j(v)(x)=-\sum_{|\alpha|=0}^{\infty}A_{j,\alpha}(D^\alpha v(x)).$$

So, by Proposition 6, each mapping $v\to T_j(v)$ belongs to $\mathcal{L}(V(\boldsymbol{R}^n, E))$ and, taken into account (1) and (19), one has

$$\sum_{j=0}^{k-1}\|T_j\|_{\mathcal{L}(V(\boldsymbol{R}^n,E))}<1.$$

Consequently, thanks to Proposition 5, there exists a unique function $\omega\in V(\boldsymbol{R}, V(\boldsymbol{R}^n, E))$ such that

$$(20)\qquad \begin{cases} \omega^{(k)}(t)=\sum_{j=0}^{k-1} T_j(\omega^{(j)}(t))+\Psi^{-1}(f)(t) & \text{in } \boldsymbol{R} \\ \omega^{(j)}(0)=\varphi_j & \text{for } j=0,1,\cdots,k-1, \end{cases}$$

where Ψ is the mapping, from $V(\boldsymbol{R}, V(\boldsymbol{R}^n, E))$ onto $V(\boldsymbol{R}^{n+1}, E)$, defined in the statement of Proposition 3.

Then, for every $t\in\boldsymbol{R}$, $x\in\boldsymbol{R}^n$, one has

$$\Psi(\omega^{(k)})(t,x)=\sum_{j=0}^{k-1}\Psi(T_j\circ\omega^{(j)})(t,x)+f(t,x)$$

$$=-\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty}A_{j,\alpha}(D^\alpha\omega^{(j)}(t)(x))+f(t,x)$$

$$\omega^{(j)}(0)(x)=\varphi_j(x) \qquad \text{for } j=0,1,\cdots,k-1.$$

Hence, if we put $u=\Psi(\omega)$, taking into account (4), we get

$$(21)\qquad \begin{cases} D_t^k u(t,x)+\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty}A_{j,\alpha}(D_t^j D_x^\alpha u(t,x))=f(t,x) \\ D_t^j u(0,x)=\varphi_j(x) \qquad \text{for } j=0,1,\cdots,k-1. \end{cases}$$

That is, the function u solves our problem. Conversely, if a function $\bar{u}\in V(\boldsymbol{R}^{n+1}, E)$ satisfies (21), then the function $\Psi^{-1}(\bar{u})$ satisfies (20), and so $\Psi^{-1}(\bar{u})=\omega$, that is $\bar{u}=u$. Finally, inequality (2) follows at once from (16) and Proposition 3, taking into account (18) again. ■

2. Some consequences of Theorem 1

As we said in the Introduction, Theorem 1 admits a reformulation in purely algebraic terms. Precisely, we have

THEOREM 2. *Let condition* (1) *be satisfied. Then, for every* $\{w_{h,\beta}\}_{(h,\beta)\in\boldsymbol{N}_0^{n+1}}\in l^\infty(\boldsymbol{N}_0^{n+1}, E)$ $(h\in\boldsymbol{N}_0, \beta\in\boldsymbol{N}_0^n)$ *and every* $\{\omega_{0,\beta}\}_{\beta\in\boldsymbol{N}_0^n}, \{\omega_{1,\beta}\}_{\beta\in\boldsymbol{N}_0^n}, \cdots, \{\omega_{k-1,\beta}\}_{\beta\in\boldsymbol{N}_0^n}\in l^\infty(\boldsymbol{N}_0^n, E)$, *there exists a unique* $\{v_{h,\beta}\}_{(h,\beta)\in\boldsymbol{N}_0^{n+1}}\in l^\infty(\boldsymbol{N}_0^{n+1}, E)$ *such that, for every* $h\in\boldsymbol{N}_0, \beta\in\boldsymbol{N}_0^n$, *one has*

$$(22)\qquad \begin{cases} v_{k+h,\beta}+\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty}A_{j,\alpha}(v_{j+h,\alpha+\beta})=w_{h,\beta} \\ v_{j,\beta}=\omega_{j,\beta} \qquad \textit{for } j=0,1,\cdots,k-1. \end{cases}$$

PROOF. In view of Theorem 1 and Proposition 1, we know that there exists a unique function $u\in(\boldsymbol{R}^{n+1}, E)$ such that, for every $t\in\boldsymbol{R}, x\in\boldsymbol{R}^n$, we have

$$(23)\quad \begin{cases} D_t^k u(t,x)+\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty} A_{j,\alpha}(D_t^j D_x^\alpha u(t,x))=\sum_{|(h,\beta)|=0}^{\infty}\frac{t^h x^\beta}{h!\,\beta!}w_{h,\beta} \\ D_t^j u(0,x)=\sum_{|\beta|=0}^{\infty}\frac{x^\beta}{\beta!}\omega_{j,\beta} \quad \text{for } j=0,1,\cdots,k-1. \end{cases}$$

Hence, taking into account that

$$D_t^j D_x^\alpha u(t,x)=\sum_{|(h,\beta)|=0}^{\infty}\frac{t^h x^\beta}{h!\,\beta!}D_t^{j+h}D_x^{\alpha+\beta}u(0),$$

we have

$$(24)\quad \begin{cases} \sum_{|(h,\beta)|=0}^{\infty}\frac{t^h x^\beta}{h!\,\beta!}\Big(D_t^{k+h}D_x^\beta u(0)+\sum_{j=0}^{k-1}\sum_{|\alpha|=0}^{\infty}A_{j,\alpha}(D_t^{j+h}D_x^{\alpha+\beta}u(0))\Big) \\ =\sum_{|(h,\beta)|=0}^{\infty}\frac{t^h x^\beta}{h!\,\beta!}w_{h,\beta} \\ \sum_{|\beta|=0}^{\infty}\frac{x^\beta}{\beta!}D_t^j D_x^\beta u(0)=\sum_{|\beta|=0}^{\infty}\frac{x^\beta}{\beta!}\omega_{j,\beta} \quad \text{for } j=0,1,\cdots,k-1. \end{cases}$$

Then, if we put $v_{h,\beta}=D_t^h D_x^\beta u(0)$, we have $\{v_{h,\beta}\}_{(h,\beta)\in N_0^{n+1}}\in l^\infty(N_0^{n+1},E)$ and, by a classical result (see, for instance, [2], p. 195), we get (22) directly from (24). Conversely, if a family $\{\bar{v}_{h,\beta}\}_{(h,\beta)\in N_0^{n+1}}$ in $l^\infty(N_0^{n+1},E)$ satisfies (22), then it is seen that the function $(t,x)\to\bar{u}(t,x)=\sum_{|(h,\beta)|=0}^{\infty}\frac{t^h x^\beta}{h!\,\beta!}\bar{v}_{h,\beta}$ (which belongs to $V(\boldsymbol{R}^{n+1},E)$) satisfies (23), and so $\bar{u}=u$, that is $\bar{v}_{h,\beta}=v_{h,\beta}$ for all $h\in N_0$, $\beta\in N_0^n$. ■

In the sequel, it is understood that the space $V(\boldsymbol{R}^n,E)$ is considered with any fixed norm $\|\cdot\|_{\Omega,E}$. We now state

THEOREM 3. *Let $k_1,\cdots,k_n$ be n ($n\geq 2$) positive integers and let $A_1,\cdots,A_n$ be n linear homeomorphisms from E onto itself. Let T be the element of $\mathcal{L}(V(\boldsymbol{R}^n,E))$ defined by putting*

$$T(u)(x)=\sum_{j=1}^{n}A_j\Big(\frac{\partial^{k_j}u(x)}{\partial x_j^{k_j}}\Big) \qquad \textit{for all } u\in V(\boldsymbol{R}^n,E),\, x\in\boldsymbol{R}^n.$$

Assume that

$$\min_{1\leq i\leq n}\sum_{\substack{j=1\\ j\neq i}}^{n}\|A_i^{-1}\circ A_j\|_{\mathcal{L}(E)}<1.$$

Then, there exists a linear subspace F of $V(\boldsymbol{R}^n,E)$ such that $T_{|F}$ is a linear homeomorphism from F onto $V(\boldsymbol{R}^n,E)$.

PROOF. Let $h \in \boldsymbol{N}$ $(1 \leqq h \leqq n)$ be such that

$$\sum_{\substack{j=1 \\ j \neq h}}^{n} \|A_h^{-1} \circ A_j\|_{\mathcal{L}(E)} = \min_{1 \leq i \leq n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \|A_i^{-1} \circ A_j\|_{\mathcal{L}(E)} .$$

Then, applying Theorem 1 in an obvious manner, for every $f \in V(\boldsymbol{R}^n, E)$, we get a unique $u \in V(\boldsymbol{R}^n, E)$ such that

$$\frac{\partial^{k_h} u(x)}{\partial x_h^{k_h}} + \sum_{\substack{j=1 \\ j \neq h}}^{n} A_h^{-1}\left(A_j\left(\frac{\partial^{k_j} u(x)}{\partial x_j^{k_j}}\right)\right) = A_h^{-1}(f(x)) \qquad \text{in } \boldsymbol{R}^n$$

$$\frac{\partial^j u(x_1, \cdots, x_{h-1}, 0, x_{h+1}, \cdots, x_n)}{\partial x_h^j} = 0 \quad \text{in } \boldsymbol{R}^{n-1}, \text{ for } j = 0, 1, \cdots, k_h - 1 .$$

Therefore, $T(u) = f$. To get our conclusion, it suffices to take

$$F = \{v \in V(\boldsymbol{R}^n, E) : D_{x_h}^j v(x_1, \cdots, x_{h-1}, 0, x_{h+1}, \cdots, x_n) = 0 \quad \text{in } \boldsymbol{R}^{n-1},$$
$$\text{for } \ j = 0, 1, \cdots, k_h - 1\} . \quad \blacksquare$$

In particular, we have

THEOREM 4. *Let h, $m \in \boldsymbol{N}$ and a, $b \in \boldsymbol{R} \backslash \{0\}$. Then, the differential operator*

$$u \longrightarrow a \frac{\partial^h u}{\partial x^h} + b \frac{\partial^m u}{\partial y^m}$$

from $V(\boldsymbol{R}^2, \boldsymbol{R})$ into itself, is surjective if and only if $|a| \neq |b|$.

PROOF. The sufficiency of the condition follows directly from Theorem 3. To prove necessity, assume $|a| = |b|$. Then, we have to show that both the operators

$$u \longrightarrow \frac{\partial^h u}{\partial x^h} + \frac{\partial^m u}{\partial y^m}$$

and

$$u \longrightarrow \frac{\partial^h u}{\partial x^h} - \frac{\partial^m u}{\partial y^m}$$

considered as acting in $V(\boldsymbol{R}^2, \boldsymbol{R})$, are not surjective. For instance, let $u \in C^\infty(\boldsymbol{R}^2, \boldsymbol{R})$ be such that

$$\frac{\partial^h u(x, y)}{\partial x^h} + \frac{\partial^m u(x, y)}{\partial y^m} = e^x g(y) \qquad \text{in } \boldsymbol{R}^2 ,$$

where g is any non-null function belonging to $V(\boldsymbol{R}, \boldsymbol{R})$ such that

$$g^{(m)}(y)+g(y)=0 \qquad \text{in } \boldsymbol{R}.$$

Then, it is easy to check that, for each $p\in\boldsymbol{N}$, one has

$$\frac{\partial^{ph}u(x,y)}{\partial x^{ph}}+(-1)^{p+1}\frac{\partial^{pm}u(x,y)}{\partial y^{pm}}=pe^{x}g(y) \qquad \text{in } \boldsymbol{R}^{2}.$$

Consequently, one has $u\notin V(\boldsymbol{R}^2,\boldsymbol{R})$. Analogously, it is seen that, for instance, the equation

$$\frac{\partial^h u}{\partial x^h}-\frac{\partial^m u}{\partial y^m}=e^{x+y}$$

has no solution in $V(\boldsymbol{R}^2,\boldsymbol{R})$. ■

The next application of Theorem 1 deals with infinite differential systems. But before it is useful to point out the following proposition, where, as usual, l^∞ stands for $l^\infty(\boldsymbol{N},\boldsymbol{R})$, with the norm $\|\xi\|_{l^\infty}=\sup_{h\in\boldsymbol{N}}|\xi_h|$ $(\xi=\{\xi_h\})$.

PROPOSITION 7. *Let* $f\colon \boldsymbol{R}^n\to l^\infty$, *and let* $f(x)=\{f_h(x)\}$ *for each* $x\in\boldsymbol{R}^n$. *Then, the following are equivalent*:

(i) $f\in V(\boldsymbol{R}^n, l^\infty)$.

(ii) $\{f_h\}$ *is a bounded sequence in* $V(\boldsymbol{R}^n,\boldsymbol{R})$.

PROOF. Let (i) hold. For each $h\in\boldsymbol{N}$, $\xi=\{\xi_h\}\in l^\infty$, let $T_h(\xi)=\xi_h$. Then, since $T_h\in(l^\infty)^*$, the function $T_h(f(\cdot))$ (that is f_h) belongs to $V(\boldsymbol{R}^n,\boldsymbol{R})$. Also, for each $\alpha\in\boldsymbol{N}_0^n, x\in\boldsymbol{R}^n$, we have

$$|D^\alpha T_h(f(x))|=|T_h(D^\alpha f(x))|\leqq\|D^\alpha f(x)\|_{l^\infty}$$

that yields (ii). Conversely, let (ii) hold. We then have

$$\sup_{h\in\boldsymbol{N}}\sup_{\alpha\in\boldsymbol{N}_0^n}|D^\alpha f_h(0)|<+\infty\,.$$

Consequently, if we put $w_\alpha=\left\{\frac{D^\alpha f_h(0)}{\alpha!}\right\}_{h\in\boldsymbol{N}}$, we have $\{w_\alpha\}_{\alpha\in\boldsymbol{N}_0^n}\in l^\infty(\boldsymbol{N}_0^n, l^\infty)$. On the other hand, for each $h\in\boldsymbol{N}, x\in\boldsymbol{R}^n$, one has

$$T_h\Big(\sum_{|\alpha|=0}^{\infty}x^\alpha w_\alpha\Big)=\sum_{|\alpha|=0}^{\infty}\frac{x^\alpha}{\alpha!}D^\alpha f_h(0)=f_h(x)$$

that is to say

$$f(x)=\sum_{|\alpha|=0}^{\infty}x^\alpha w_\alpha\,.$$

Hence, $f \in V(\boldsymbol{R}^n, l^\infty)$ by Proposition 1. ■

We now state

THEOREM 5. *Let $k \in \boldsymbol{N}$. For each $j=0,1,\cdots,k-1$, $\alpha \in \boldsymbol{N}_0^n$, h, $p \in \boldsymbol{N}$, let $a_{j,\alpha,h,p} \in \boldsymbol{R}$ be given. Assume that*

$$\sum_{j=0}^{k-1} \sum_{|\alpha|=0}^{\infty} \sup_{h \in \boldsymbol{N}} \sum_{p=1}^{\infty} |a_{j,\alpha,h,p}| < 1 .$$

Then, for every bounded sequence $\{f_h\}$ in $V(\boldsymbol{R}^{n+1}, \boldsymbol{R})$ and every k-tuple of bounded sequences $\{\varphi_{0,h}\}, \{\varphi_{1,h}\}, \cdots, \{\varphi_{k-1,h}\}$ in $V(\boldsymbol{R}^n, \boldsymbol{R})$, there exists a unique bounded sequence $\{u_h\}$ in $V(\boldsymbol{R}^{n+1}, \boldsymbol{R})$ such that, for every $t \in \boldsymbol{R}$, $x \in \boldsymbol{R}^n$, $h \in \boldsymbol{N}$, one has

$$D_t^k u_h(t,x) + \sum_{j=0}^{k-1} \sum_{|\alpha|=0}^{\infty} \sum_{p=1}^{\infty} a_{j,\alpha,h,p} D_t^j D_x^\alpha u_p(t,x) = f_h(t,x)$$

$$D_t^j u_h(0,x) = \varphi_{j,h}(x) \qquad \textit{for} \quad j=0,1,\cdots,k-1 .$$

PROOF. Let $f(t,x)=\{f_h(t,x)\}$, $\varphi_j(x)=\{\varphi_{j,h}(x)\}$. So, by Proposition 7, we have $f \in V(\boldsymbol{R}^{n+1}, l^\infty)$, $\varphi_j \in V(\boldsymbol{R}^n, l^\infty)$. For each $j=0,1,\cdots,k-1, \alpha \in \boldsymbol{N}_0^n$, let $A_{j,\alpha}$ be the continuous linear operator, from l^∞ into itself, given by

$$A_{j,\alpha}(\xi) = \left\{ \sum_{p=1}^{\infty} a_{j,\alpha,h,p} \xi_p \right\}_{h \in \boldsymbol{N}} \qquad \text{for } \xi = \{\xi_h\} \in l^\infty .$$

As it is known (see, for instance, [4], p. 223), one has

$$\|A_{j,\alpha}\|_{\mathcal{L}(l^\infty)} = \sup_{h \in \boldsymbol{N}} \sum_{p=1}^{\infty} |a_{j,\alpha,h,p}| .$$

Consequently, by Theorem 1, there exists a unique $u \in V(\boldsymbol{R}^{n+1}, l^\infty)$ such that, for each $t \in \boldsymbol{R}$, $x \in \boldsymbol{R}^n$, one has

$$D_t^k u(t,x) + \sum_{j=0}^{k-1} \sum_{|\alpha|=0}^{\infty} A_{j,\alpha}(D_t^j D_x^\alpha u(t,x)) = f(t,x)$$

$$D_t^j u(0,x) = \varphi_j(x) \qquad \text{for} \quad j=0,1,\cdots,k-1 .$$

Let $u(t,x)=\{u_h(t,x)\}$. Then, it is seen at once that the sequence $\{u_h\}$ satisfies our conclusion. ■

Before stating the last results, we introduce another function space. Namely, let Y be a given compact topological space. We denote by $V_0(\boldsymbol{R}^n \times Y, E)$ the space of all functions $u: \boldsymbol{R}^n \times Y \to E$ such that, for each $\alpha \in \boldsymbol{N}_0^n$, the function $(x,y) \to D_x^\alpha u(x,y)$ is (defined and) continuous in $\boldsymbol{R}^n \times Y$

and, for each non-empty bounded set $\Omega \subseteq \boldsymbol{R}^n$, one has

$$\sup_{\alpha \in \boldsymbol{N}_0^n} \sup_{(x,y)\in\Omega\times Y} \|D_x^\alpha u(x,y)\|_E < +\infty .$$

As usual, let $C^0(Y,E)$ denote the space of all continuous functions from Y into E, with the sup-norm. Then, we have the following proposition, whose proof (based on Proposition 1 and, in principle, similar to that of Proposition 3) is left to the reader.

PROPOSITION 8. *For each $u \in V(\boldsymbol{R}^n, C^0(Y,E))$, let $\Psi_n(u)$ be the function, from $\boldsymbol{R}^n \times Y$ into E, defined by putting*

$$\Psi_n(u)(x,y) = u(x)(y) \qquad \text{for all } x \in \boldsymbol{R}^n,\ y \in Y.$$

Then, $\Psi_n(u) \in V_0(\boldsymbol{R}^n \times Y, E)$ and the mapping $u \to \Psi_n(u)$ is surjective. Moreover, for each $\alpha \in \boldsymbol{N}_0^n$, one has $D_x^\alpha \Psi_n(u) = \Psi_n(D^\alpha u)$.

In the theorem which follows, Y is a non-empty compact subset of $\boldsymbol{R}^m$ $(m \in \boldsymbol{N})$, and the integrals there appearing are understood in the sense of Bochner (with respect to the Lebesgue measure).

THEOREM 6. *Let $k \in \boldsymbol{N}$. For each $j=0,1,\cdots,k-1$ and each $\alpha \in \boldsymbol{N}_0^n$, let $T_{j,\alpha} \in C^0(Y, \mathcal{L}(E))$ and $\Phi_{j,\alpha} \in C^0(Y \times Y, \boldsymbol{R})$ be given. Assume that*

$$\sum_{j=0}^{k-1} \sum_{|\alpha|=0}^{\infty} \max_{y\in Y}\left(\|T_{j,\alpha}(y)\|_{\mathcal{L}(E)} + \int_Y |\Phi_{j,\alpha}(y,\xi)| d\xi \right) < 1 .$$

Then, for every $f \in V_0(\boldsymbol{R}^{n+1} \times Y, E)$ and every $\varphi_0, \varphi_1, \cdots, \varphi_{k-1} \in V_0(\boldsymbol{R}^n \times Y, E)$, there exists a unique function $u \in V_0(\boldsymbol{R}^{n+1} \times Y, E)$ such that, for every $t \in \boldsymbol{R}$, $x \in \boldsymbol{R}^n$, $y \in Y$, one has

$$D_t^k u(t,x,y) + \sum_{j=0}^{k-1} \sum_{|\alpha|=0}^{\infty} \Big(T_{j,\alpha}(y)(D_t^j D_x^\alpha u(t,x,y)) + \int_Y \Phi_{j,\alpha}(y,\xi) D_t^j D_x^\alpha u(t,x,\xi) d\xi \Big) = f(t,x,y)$$

$$D_t^j u(0,x,y) = \varphi_j(x,y) \qquad \text{for} \quad j=0,1,\cdots,k-1 .$$

PROOF. For each $j=0,1,\cdots,k-1$, $\alpha \in \boldsymbol{N}_0^n$, let $A_{j,\alpha}$ be the continuous linear operator, from $C^0(Y,E)$ into itself, defined by putting

$$A_{j,\alpha}(v)(y) = T_{j,\alpha}(y)(v(y)) + \int_Y \Phi_{j,\alpha}(y,\xi) v(\xi) d\xi$$

for all $v \in C^0(Y,E)$, $y \in Y$. Obviously, we have

$$\|A_{j,\alpha}\|_{\mathcal{L}(C^0(Y,E))} \leq \max_{y\in Y}\left(\|T_{j,\alpha}(y)\|_{\mathcal{L}(E)} + \int_Y |\Phi_{j,\alpha}(y,\ \xi)|d\xi\right).$$

Consequently, by Theorem 1, in view of Proposition 8, there exists a unique function $w \in V(\boldsymbol{R}^{n+1},\ C^0(Y,\ E))$ such that, for every $t \in \boldsymbol{R}$, $x \in \boldsymbol{R}^n$, one has

$$D_t^k w(t,\ x) + \sum_{j=0}^{k-1} \sum_{|\alpha|=0}^{\infty} A_{j,\alpha}(D_t^j D_x^\alpha w(t,\ x)) = \Psi_{n+1}^{-1}(f)(t,\ x)$$

$$D_t^j w(0,\ x) = \Psi_n^{-1}(\varphi_j)(x) \qquad \text{for} \quad j=0,\ 1,\ \cdots,\ k-1\,.$$

Now, put $u=\Psi_{n+1}(w)$. Then, proceeding in a by now evident manner, it is seen that the function u satisfies our conclusion. ■

Finally, we want to stress the following very particular case of Theorem 6 which could be of interest in linear transport theory (see, for instance, [1], chapter III):

THEOREM 7. *Let Y be a non-empty compact subset of $\boldsymbol{R}^n$. For each $i=1,\ \cdots,\ n$, let $\delta_i = \max_{(y_1,\cdots,y_n)\in Y} |y_i|$. Moreover, let $\sigma \in \boldsymbol{R}$ and $\Phi \in C^0(Y\times Y,\ \boldsymbol{R})$ be given. Assume that*

$$|\sigma| + \sum_{i=1}^{n} \delta_i + \max_{y\in Y} \int_Y |\Phi(y,\ \xi)|d\xi < 1\,.$$

Then, for every $f \in V_0(\boldsymbol{R}^{n+1}\times Y,\ \boldsymbol{R})$ and every $\varphi \in V_0(\boldsymbol{R}^n \times Y,\ \boldsymbol{R})$, there exists a unique function $u \in V_0(\boldsymbol{R}^{n+1}\times Y,\ \boldsymbol{R})$ such that, for every $t\in \boldsymbol{R}$, $x \in \boldsymbol{R}^n$, $y=(y_1,\ \cdots,\ y_n) \in Y$, one has

$$\frac{\partial u(t,\ x,\ y)}{\partial t} + \sum_{i=1}^{n} y_i \frac{\partial u(t,\ x,\ y)}{\partial x_i} + \sigma u(t,\ x,\ y) + \int_Y \Phi(y,\ \xi) u(t,\ x,\ \xi) d\xi$$

$$= f(t,\ x,\ y)$$

$$u(0,\ x,\ y) = \varphi(x,\ y)\,.$$

For other recent results on linear partial differential equations of infinite order, we refer to [5] and to the bibliography quoted there.

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