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On some nonlinear wave equations II: global existence and energy decay of solutions

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§ 1. Introduction.

This paper is concerned with the initial boundary value problem of the form

$$u_{tt}-M\left(\int_{\Omega}|\nabla u|^{2}dx\right)\Delta u+\delta|u|^{\alpha}u+\gamma u_{t}=f \quad \text{in} \quad \Omega\times[0,\infty), \tag{1.1}$$

$$u=0 \qquad \text{on} \quad \Gamma\times[0,\infty), \tag{1.2}$$

$$u(x,0)=u^{0}(x),\ u_{t}=u^{1}(x) \qquad \text{in} \quad \Omega, \tag{1.3}$$

where Ω is a bounded domain in R^{n} with smooth boundary Γ, $\delta>0$, $\alpha\geqq 0$ and $\lambda>0$ are given constants and $M(r)$ is a positive C^{1}-function on $[0,\infty)$. In our previous paper [4], we have discussed local existence and regularity properties for (1.1)-(1.3) in the case $\gamma=0$. Our interest of the present paper is to derive global existence and decay properties of solutions to (1.1)-(1.3) in the presence of restoring term $\delta|u|^{\alpha}u$ and damping term γu_{t}.

We mention here some related global existence results for (1.1). When initial data $\{u^{0}, u^{1}\}$ and Γ are analytic, there is a pioneering work of Pohozaev [11], who has established the global existence theory for (1.1)-(1.3) in the case $\delta=\gamma=0$. His result is extended by Arosio and Spagnolo [1] and Nishihara [8] in each direction (see also the paper of Nishihara [9], where the exponential decay of solutions is studied in the case $\delta=0$ and $\gamma>0$). When analyticity or sufficient smoothness of $\{u^{0}, u^{1}\}$ is not assumed, it seems very difficult to get the global existence for (1.1) in the case $\gamma=0$. Under the presence of a linear damping term (i.e., $\gamma>0$), some authors (see, e.g., Brito [2, 3], Ikehata [5] and Yamada [12]) have shown global existence results for (1.1) with $\delta=0$ by putting some smallness conditions on $\{u^{0}, u^{1}\}$.

The purpose of the present paper is to show that the restoring term does not give any serious effects on the global existence properties for (1.1)-(1.3) in the case $\gamma>0$ when $\{u^0, u^1\}\in H_0^1(\Omega)\cap H^2(\Omega)\times H_0^1(\Omega)$ is small in a sense. Moreover, we will give a simple proof for deriving decay rates of solutions. Finally we should refer the works of Nakao [7] and Nishihara [11], who intend to look for an unbounded set of $\{u^0, u^1\}$ which assures the global existence for (1.1)-(1.3) ($\delta=0$).

In the following sections we take $\delta=1$ without loss of generality. Section 2 contains main results; Theorem I (existence of global solution) and Theorem II (decay of global solution). In Sections 2 and 3 we give the proofs of Theorems I and II.

In the course of writing the manuscript the first author has not been able to continue the work because of illness. This paper is completed in the present style by the second author.

Notation. For any Banach space X, its norm is denoted by $\|\cdot\|_X$. Especially, for $X=L^2(\Omega)$, its norm and inner product are simply denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. By $B([0,\infty); X)$ we mean the space of all functions $u:[0,\infty)\to X$ such that u is bounded and continuous. Moreover, we denote by $B_w([0,\infty); X)$ the space of bounded functions u such that u is continuous in the weak topology of X with respect to $t\in[0,\infty)$.

§2. Assumptions and results.

Throughout this paper we impose the following assumptions on M, α, u^0, u^1 and f:

(A.1) $M\in C^1[0,\infty)$ and $M(r)\geqq m_0>0$ for $r\geqq 0$,

(A.2) $0\leqq\alpha\leqq 2/(n-4)$ if $n\geqq 5$ and $0\leqq\alpha<\infty$ if $n=1,2,3,4$,

(A.3) $u^0\in H_0^1(\Omega)\cap H^2(\Omega)$ and $u^1\in H_0^1(\Omega)$,

(A.4) $f\in L^1(0,\infty; H_0^1(\Omega))\cap L^\infty(0,\infty; L^2(\Omega))$.

Our global existence result reads as follows.

THEOREM I. *Under assumptions* (A.1)-(A.4), *there exists a positive constant* ε_0 (*depending on* $\|\nabla u^0\|$, $\|u^0\|_{L^{\alpha+2}}$, $\|u^1\|$ *and* $\|f\|_{L^1(0,\infty;L^2(\Omega))}$) *such that, if* $\{u^0, u^1, f\}$ *satisfies*

$$\|\Delta u^0\|+\|\nabla u^1\|+\int_0^\infty\|\nabla f(t)\|dt<\varepsilon_0,$$

then there exists a unique solution u *for* (1.1)-(1.3) *in the class*

(2.1) $\qquad u\in B([0,\infty); H_0^1(\Omega))\cap B_w([0,\infty); H^2(\Omega))\cap L^2(0,\infty; H^2(\Omega)),$

(2.2) $u_t \in B([0,\infty);\ L^2(\Omega)) \cap B_w([0,\infty);\ H_0^1(\Omega)) \cap L^2(0,\infty;\ H_0^1(\Omega))$,

(2.3) $u_{tt} \in L^\infty(0,\infty;\ L^2(\Omega))$.

The solution u in Theorem I actually decays to zero (in a sense) as $t\to\infty$. Indeed, the mapping $t\to\|\nabla u(t)\|^2$ is integrable on $[0,\infty)$ by (2.1) and uniformly continuous on $[0,\infty)$ by (2.1) and (2.2); so that $\|\nabla u(t)\|\to 0$ as $t\to\infty$. In the similar manner we see from (2.2) and (2.3) that $\|u_t(t)\|\to 0$ as $t\to\infty$. We can also derive some decay rates for u.

THEOREM II. *Assume* (A.1)-(A.4) *and let u be the solution in Theorem* I.

(i) *Define*

$$E(u(t)) = \|u_t(t)\|^2 + \bar{M}(\|\nabla u(t)\|^2) + \frac{2}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \tag{2.4}$$

with $\bar{M}(r) = \int_0^r M(s)ds$. *Then there exist positive constants θ and C such that*

$$E(u(t))^{1/2} \leqq C\left\{E(u(0))^{1/2}e^{-\theta\gamma t} + \int_0^t e^{\theta\gamma(s-t)}\|f(s)\|ds\right\} \tag{2.5}$$

for all $t \geqq 0$.

(ii) *Define*

$$E^*(u(t)) = \|\nabla u_t(t)\|^2 + \|\Delta u(t)\|^2; \tag{2.6}$$

then there exist positive constants ω and C^* such that

$$E^*(u(t))^{1/2} \leqq C^*\left\{E^*(u(0))^{1/2}e^{-\omega\gamma t} + \int_0^t e^{\omega\gamma(s-t)}\|\nabla f(s)\|ds\right\} \tag{2.7}$$

for all $t \geqq 0$.

(iii) $\lim_{t\to\infty} E(u(t)) = \lim_{t\to\infty} E^*(u(t)) = 0$.

As a result of Theorem II, we see that, if $\|f(t)\|$ and $\|\nabla f(t)\|$ decay exponentially to zero as $t\to\infty$, then both $E(u(t))$ and $E^*(u(t))$ decay exponentially to zero as $t\to\infty$.

REMARK 2.1. It is possible to show (2.5) for every solution in the class (2.1)-(2.3) without any restrictions on $\{u^0, u^1, f\}$ (see also Remark 4.1).

§ 3. Proof of Theorem I.

As in our previous paper [4], we employ the Galerkin method to construct a global solution to (1.1)-(1.3). Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of eigenvalues for

$$-\Delta w=\lambda w \quad \text{in } \Omega \quad \text{and} \quad w=0 \quad \text{on } \Gamma.$$

Let $w_j \in H_0^1(\Omega)\cap H^2(\Omega)$ be the corresponding eigenfunction to λ_j and take $\{w_j\}_{j=1}^{\infty}$ as a completely orthonormal system in $L^2(\Omega)$. We construct approximate solutions u_m $(m=1, 2, 3, \cdots)$ in the form

$$u_m(t)=\sum_{j=1}^{m} g_{jm}(t)w_j,$$

where g_{jm} $(j=1, 2, \cdots, m)$ are determined by

$$\begin{aligned}(3.1)\qquad &(u_m''(t), w_j)+M(\|\nabla u_m(t)\|^2)(\nabla u_m(t), \nabla w_j)+(|u_m(t)|^{\alpha}u_m(t), w_j)\\ &+\gamma(u_m'(t), w_j)=(f(t), w_j), \qquad j=1, 2, \cdots, m,\end{aligned}$$

$('=\partial/\partial t$ and $''=\partial^2/\partial t^2)$ with initial conditions

$$(3.2)\qquad u_m(0)=u_m^0\equiv\sum_{j=1}^{m}(u^0, w_j)w_j \longrightarrow u^0 \quad \text{in} \quad H_0^1(\Omega)\cap H^2(\Omega) \quad \text{as} \quad m\to\infty,$$

$$(3.3)\qquad u_m'(0)=u_m^1\equiv\sum_{j=1}^{m}(u^1, w_j)w_j \longrightarrow u^1 \quad \text{in} \quad H_0^1(\Omega) \quad \text{as} \quad m\to\infty.$$

We concentrate our analysis in deriving some global estimates (independent of m) for u_m because the limiting procedure is a routine work (for details, see, e.g., [4]). In what follows, we sometimes drop the subscript of u_m for the sake of simplicity.

We use the following lemma.

LEMMA 3.1. *Suppose that a positive continuous function $X(t)$ satisfies*

$$X(t)^2\leqq A+2\int_0^t B(s)X(s)ds \qquad \textit{for } t\geqq 0,$$

where A is a positive constant and B is a nonnegative integrable function on $[0, \infty)$. Then

$$X(t)\leqq\sqrt{A}+\int_0^t B(s)ds \qquad \textit{for all } t\geqq 0.$$

The proof of Lemma 3.1 is elementary; so we omit it. Making use of this lemma we first derive the following estimate.

LEMMA 3.2. *Let u_m be defined by* (3.1)-(3.3). *Then there exists a positive constant C_1 depending on $\|u^0\|_{L^{\alpha+2}}$, $\|\nabla u^0\|$, $\|u^1\|$ and $\|f\|_{L^1(0,\infty;L^2(\Omega))}$ such that*

$$\|u'_m(t)\|^2+\|\nabla u_m(t)\|^2+\|u_m(t)\|^{\alpha+2}_{L^{\alpha+2}}+\int_0^t\|u'_m(s)\|^2 ds\leqq C_1^2 \tag{3.4}$$

for all $t\geqq 0$.

PROOF. Multiplying the j-th equation (3.1) by g'_{jm} and summing up with respect to j we have

$$\begin{aligned}(u''(t),u'(t))+M(\|\nabla u(t)\|^2)(\nabla u(t),\nabla u'(t))\\ +(|u(t)|^\alpha u(t),u'(t))+\gamma\|u'(t)\|^2=(f(t),u'(t)),\end{aligned} \tag{3.5}$$

where the subscript "m" is dropped. Define the functional $E(u)$ by (2.4). Then (3.5) leads to

$$\frac{d}{dt}E(u(t))+2\gamma\|u'(t)\|^2=2(f(t),u'(t)). \tag{3.6}$$

Integration of (3.6) over $[0,t]$ gives

$$\begin{aligned}E(u(t))+2\gamma\int_0^t\|u'(s)\|^2 ds&\leqq E(u(0))+2\int_0^t\|f(s)\|\,\|u'(s)\|ds\\ &\leqq E(u(0))+2\int_0^t\|f(s)\|E(u(s))^{1/2}ds.\end{aligned} \tag{3.7}$$

By (A.2) and Sobolev's lemma, $H^2(\Omega)$ is embedded in $L^{\alpha+2}(\Omega)$; so that it follows from (3.2) and (3.3) that $E(u(0))$ is bounded by a positive constant independent of m. We apply Lemma 3.1 to (3.7) with

$$X(t)=\left\{E(u(t))+2\gamma\int_0^t\|u'(s)\|^2 ds\right\}^{1/2},$$
$$A=E(u(0))\quad\text{and}\quad B(t)=\|f(t)\|.$$

If we note $\bar{M}(\|\nabla u(t)\|^2)\geqq m_0\|\nabla u(t)\|^2$ by (A.1), then (3.4) easily follows. At the same time, this estimate implies that u_m exists globally in $[0,\infty)$. q.e.d.

We next derive estimates for $\|\Delta u_m(t)\|$ and $\|\nabla u'_m(t)\|$ by putting some restrictions on the size of the given data.

LEMMA 3.3. *Let u_m be defined by* (3.1)-(3.3). *Then there exists a constant ε_0 depending on $\|\nabla u^0\|$, $\|u^0\|_{L^{\alpha+2}}$, $\|u^1\|$ and $\|f\|_{L^1(0,\infty;L^2(\Omega))}$ such that, if*

$$\|\Delta u^0\|+\|\nabla u^1\|+\int_0^\infty \|f(t)\|dt<\varepsilon_0,$$

then

$$E^*(u_m(t))\leqq C_2 \qquad \textit{for all } t\geqq 0, \tag{3.8}$$

$$\int_0^t E^*(u_m(s))ds\leqq C_3 \qquad \textit{for all } t\geqq 0, \tag{3.9}$$

with some positive constants C_2 and C_3 independent of m, where $E^*(u)$ is defined by (2.6).

PROOF. Replace w_j in (3.1) by $-\Delta w_j/\lambda_j$; then

$$\begin{aligned}(\nabla u''(t), \nabla w_j)+M(\|\nabla u(t)\|^2)(\Delta u(t), \Delta w_j)-(|u(t)|^\alpha u(t), \Delta w_j)\\ +\gamma(\nabla u'(t), \nabla w_j)=(\nabla f(t), \nabla w_j), \qquad j=1, 2, \cdots, m.\end{aligned} \tag{3.10}$$

Multiplying the j-th equation of (3.10) by $g'_{jm}(t)$ and summing up with respect to j one can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u'(t)\|^2+\frac{1}{2}M(\|\nabla u(t)\|^2)\frac{d}{dt}\|\Delta u(t)\|^2+\gamma\|\nabla u'(t)\|^2\\ =(|u(t)|^\alpha u(t), \Delta u'(t))+(\nabla f(t), \nabla u'(t)).\end{aligned} \tag{3.11}$$

If we define

$$G(u(t))=\|\nabla u'(t)\|^2+M(\|\nabla u(t)\|^2)\|\Delta u(t)\|^2,$$

then it follows from (3.11) that

$$\begin{aligned}\frac{d}{dt}G(u(t))+2\gamma\|\nabla u'(t)\|^2=2M'(\|\nabla u(t)\|^2)(\nabla u(t), \nabla u'(t))\|\Delta u(t)\|^2\\ +2(|u(t)|^\alpha u(t), \Delta u'(t))+2(\nabla f(t), \nabla u'(t)).\end{aligned} \tag{3.12}$$

By virtue of Lemma 3.2, the first term in the right-hand side of (3.12) is bounded from above by

$$2M_1C_1\|\nabla u'(t)\|\,\|\Delta u(t)\|^2,$$

where $M_1=\max\{|M'(r)|;\ 0\leqq r\leqq C_1^2\}$. The second term can be estimated as in [4]; integration by parts yields

$$(|u(t)|^\alpha u(t), \Delta u'(t)) = -(\alpha+1)(|u(t)|^\alpha \nabla u(t), \nabla u'(t)).$$

Therefore, on account of (A.2), Hölder's inequality combined with Sobolev's inequality implies that the second term is bounded by

$$C_4\|\Delta u(t)\|^{\alpha+1}\|\nabla u'(t)\| \leqq \frac{\gamma}{2}\|\nabla u'(t)\|^2 + \frac{1}{2\gamma}C_4^2\|\Delta u(t)\|^{2\alpha+2}$$

with some positive constant C_4 (see the proof of (3.12) in [4]). In this proof we denote by C_i various positive constants independent of m. Making use of these bounds we rearrange (3.12) to get

$$\begin{aligned}(3.13)\qquad &\frac{d}{dt}G(u(t)) + \frac{3\gamma}{2}\|\nabla u'(t)\|^2 \\ &\leqq 2M_1C_1\|\nabla u'(t)\|\,\|\Delta u(t)\|^2 + \frac{1}{2\gamma}C_4^2\|\Delta u(t)\|^{2\alpha+2} + 2\|\nabla f(t)\|\,\|\nabla u'(t)\|.\end{aligned}$$

We next multiply the j-th equation of (3.10) by g_{jm} and sum up with respect to j;

$$\begin{aligned}(3.14)\qquad &(\nabla u''(t), \nabla u(t)) + M(\|\nabla u(t)\|^2)\|\Delta u(t)\|^2 \\ &+ (\nabla(|u(t)|^\alpha u(t)), \nabla u(t)) + \gamma(\nabla u'(t), \nabla u(t)) = (\nabla f(t), \nabla u(t)).\end{aligned}$$

Since

$$(\nabla(|u(t)|^\alpha u(t)), \nabla u(t)) = (\alpha+1)(|u(t)|^\alpha \nabla u(t), \nabla u(t)) \geqq 0,$$

it follows from (A.1) and (3.14)

$$\begin{aligned}(3.15)\qquad &\frac{d}{dt}\left\{(\nabla u'(t), \nabla u(t)) + \frac{\gamma}{2}\|\nabla u(t)\|^2\right\} + m_0\|\Delta u(t)\|^2 \\ &\leqq \|\nabla u'(t)\|^2 + \|\nabla f(t)\|\,\|\nabla u(t)\|.\end{aligned}$$

Addition of (3.13) and (3.15)$\times\gamma$ gives

$$\begin{aligned}(3.16)\qquad &\frac{d}{dt}H(u(t)) + \frac{\gamma}{2}\|\nabla u'(t)\|^2 \\ &+ \left\{m_0\gamma - 2M_1C_1\|\nabla u'(t)\| - \frac{1}{2\gamma}C_4^2\|\Delta u(t)\|^{2\alpha}\right\}\|\Delta u(t)\|^2 \\ &\leqq \|\nabla f(t)\|(2\|\nabla u'(t)\| + \gamma\|\nabla u(t)\|),\end{aligned}$$

where $H(u(t)) = G(u(t)) + \gamma(\nabla u'(t), \nabla u(t)) + \gamma^2\|\nabla u(t)\|^2/2$. We observe here that

(3.17)
$$H(u(t))\geqq\frac{1}{2}\|\nabla u'(t)\|^2+m_0\|\Delta u(t)\|^2.$$

Furthermore, since $\|\Delta u\|\geqq\sqrt{\lambda_1}\|\nabla u\|$ (λ_1=the least eigenvalue of $-\Delta$ with zero Dirichlet condition) for every $u\in H_0^1(\Omega)\cap H^2(\Omega)$, we see

(3.18)
$$\begin{aligned}H(u(t))&\leqq\|\nabla u'(t)\|^2+M_2\|\Delta u(t)\|^2+\gamma\|\nabla u'(t)\|\,\|\nabla u(t)\|+\frac{\gamma^2}{2}\|\nabla u(t)\|^2\\&\leqq\frac{3}{2}\|\nabla u'(t)\|^2+\left(M_2+\frac{\gamma^2}{\lambda_1}\right)\|\Delta u(t)\|^2,\end{aligned}$$

where $M_2=\max\{M(r);\ 0\leqq r\leqq C_1^2\}$. By (3.17) and (3.18),

(3.19)
$$C_5E^*(u(t))\leqq H(u(t))\leqq C_6E^*(u(t))$$

with some $C_5, C_6>0$.

We are ready to deduce a priori estimates for $E^*(u(t))$ with the aid of (3.19). Take $\{u_m^0, u_m^1\}$ satisfying

(3.20)
$$2M_1C_1\|\nabla u_m^1\|+\frac{1}{2\gamma}C_4^2\|\Delta u_m^0\|^{2\alpha}\leqq\frac{1}{4}m_0\gamma.$$

We will show

(3.21)
$$2M_1C_1\|\nabla u'(t)\|+\frac{1}{2\gamma}C_4^2\|\Delta u(t)\|^{2\alpha}<\frac{1}{2}m_0\gamma,$$

for all $0\leqq t<\infty$ by putting some additional size conditions on $\{u_m^0, u_m^1, f\}$. Suppose that there exists a positive number τ such that (3.21) holds for $0\leqq t<\tau$ and

(3.22)
$$2M_1C_1\|\nabla u'(\tau)\|+\frac{1}{2\gamma}C_4^2\|\Delta u(\tau)\|^{2\alpha}=\frac{1}{2}m_0\gamma.$$

Then it follows from (3.16) together with (3.17) and (3.18) that

(3.23)
$$\frac{d}{dt}H(u(t))+2\omega\gamma H(u(t))\leqq 2C_7\|\nabla f(t)\|H(u(t))^{1/2},\qquad 0\leqq t\leqq\tau,$$

with some positive constants ω and C_7. Since (3.23) is rewritten in the following differential inequality

$$\frac{d}{dt}\{e^{2\omega\gamma t}H(u(t))\}\leqq 2C_7e^{\omega\gamma t}\|\nabla f(t)\|\{e^{2\omega\gamma t}H(u(t))\}^{1/2},$$

it is easy to deduce

$$e^{\omega\gamma t}H(u(t))^{1/2}\leqq H(u(0))^{1/2}+C_7\int_0^t e^{\omega\gamma s}\|\nabla f(s)\|ds \tag{3.24}$$

for $0\leqq t\leqq\tau$. Especially, (3.24) together with (3.19) implies

$$\begin{aligned}\max\{\|\nabla u'(\tau)\|,\|\Delta u(\tau)\|\}&\leqq E^*(u(\tau))^{1/2}\leqq\{H(u(\tau))/C_5\}^{1/2}\\ &\leqq H_0\equiv\left\{H(u(0))^{1/2}+C_7\int_0^\infty\|\nabla f(t)\|dt\right\}^{1/2}.\end{aligned} \tag{3.25}$$

We make H_0 sufficiently small so that

$$2M_1C_1H_0+\frac{1}{2\gamma}C_4^2H_0^{2\alpha}<\frac{1}{2}m_0\gamma \tag{3.26}$$

holds. Then it follows from (3.25) that

$$2M_1C_1\|\nabla u'(\tau)\|+\frac{1}{2\gamma}C_4^2\|\Delta u(\tau)\|^{2\alpha}<\frac{1}{2}m_0\gamma,$$

which contradicts to (3.22). Thus we have shown (3.21).

If $\|\Delta u^0\|,\|\nabla u^1\|$ and $\|\nabla f\|_{L^1(0,\infty;L^2(\Omega))}$ are sufficiently small, one can easily see that (3.20) and (3.26) are valid if m is sufficiently large; so that (3.8) follows from (3.21). In order to show (3.9), we use (3.16) and (3.21) to derive

$$\frac{d}{dt}H(u(t))+\frac{\gamma}{2}\{\|\nabla u'(t)\|^2+m_0\|\Delta u(t)\|^2\}\leqq C_8\|\nabla f(t)\|$$

with some C_8. Integration of the above inequality yields (3.9). q.e.d.

Finally we multiply (3.1) by g''_{jm} and sum up with respect to j. Then we obtain

$$\|u''_m(t)\|\leqq M_2\|\Delta u_m(t)\|+C_9\|u_m(t)\|^{\alpha+1}_{L^{2\alpha+2}}+\gamma\|u'_m(t)\|+\|f(t)\|;$$

so that it follows with the aid of Lemmas 3.2, 3.3 and Sobolev's inequality that

$$\|u''_m(t)\|\leqq C_{10}$$

with some C_{10}. Thus we have established all estimates which enable us to carry out the limiting procedure. The rest of proof can be done in the standard manner (for details, see, e.g., [4]).

§ 4. Proof of Theorem II.

(i) Although we can study the rate of decay for u with use of Nakao's technique [6] (see also [7] and [9]), we will give another simple proof.

As in the proof of Theorem I, define u_m by (3.1)-(3.3). To show (2.5), it suffices to derive the corresponding rate of decay for u_m. In what follows we drop the subscript. Clearly, we have

$$(4.1)\qquad (u''(t), u(t)) + M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 + \|u(t)\|_{L^{\alpha+2}}^{\alpha+2} + \gamma(u'(t), u(t)) = (f(t), u(t)),$$

which is obtained by multiplying (3.1) by g_{jm} and summing up with respect to j. We define the functional $F(u)$ by

$$F(u(t)) = E(u(t)) + k\{2(u'(t), u(t)) + \gamma\|u(t)\|^2\},$$

where k is a positive number to be determined later. Then it follows from (3.6) and (4.1) that

$$(4.2)\qquad \frac{d}{dt}F(u(t)) + 2(\gamma-k)\|u'(t)\|^2 + 2km_0\|\nabla u(t)\|^2 + 2k\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \leqq 2\|f(t)\|(\|u'(t)\| + 2k\|u(t)\|).$$

Since $\|\nabla u\|^2 \geqq \lambda_1\|u\|^2$ for $u \in H_0^1(\Omega)$ and $M(\|\nabla u(t)\|^2)$ is uniformly bounded in $[0, \infty)$, there exists a positive constant D_1 such that

$$(4.3)\qquad F(u(t)) \leqq D_1(\|u'(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|_{L^{\alpha+2}}^{\alpha+2}).$$

On the contrary one can also show

$$(4.4)\qquad F(u(t)) \geqq \left(1 - \frac{k}{\gamma}\right)\|u'(t)\|^2 + m_0\|\nabla u(t)\|^2 + \frac{2}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2}.$$

Hence, by taking $k = \gamma/2$, (4.2) leads us to

$$(4.5)\qquad \frac{d}{dt}F(u(t)) + 2\theta\gamma F(u(t)) \leqq 2D_2\|f(t)\|F(u(t))^{1/2}$$

with some $D_2 > 0$. Therefore, solving (4.5) we see that $F(u)$ satisfies (2.5) with $E(u)$ replaced by $F(u)$. Since (4.3) and (4.4) imply

$$D_3E(u(t)) \leqq F(u(t)) \leqq D_4E(u(t)), \qquad t \geqq 0,$$

with some $D_3, D_4 > 0$, (2.5) easily follows with the aid of the limiting procedure $m \to \infty$.

(ii) The proof is the same as (i). We use (3.23) in place of (4.5). Since (3.19) is valid, (2.7) is derived as (2.5).

(iii) In order to show (iii), it suffices to make use of (A.4), (2.5) and (2.7). q.e.d.

REMARK 4.1. As is seen from the proof of Theorem II, any solution in the class (2.1)-(2.3) satisfies (2.5), while smallness conditions on $\{u^0, u^1, f\}$ are required to get (2.7).

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