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An application of the Tsuji characteristic

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1. Introduction

We are concerned with the zeros of a function $f(z)$ meromorphic in the plane, and those of a linear differential polynomial in f, say

$$F(z)=L(f)=f^{(k)}(z)+\sum_{j=0}^{k-1} B_j(z)f^{(j)}(z) \tag{1.1}$$

where the coefficients $B_0, \cdots, B_{k-1}$ are rational functions. Our starting point is the following result of Frank and Hellerstein [3], in which the notation is that of [7].

THEOREM A. *Suppose that* $k\geqq 3$, *that* $B_{k-1}\equiv 0$, *and that* $B_0, \cdots, B_{k-2}$ *are polynomials. If* f *is meromorphic in the plane, and* $F=L(f)$ *is given by* (1.1) *and is not identically zero, then*

$$T(r, f'/f)=O(N(r, 1/f)+N(r, 1/F)+r^{\lambda}) \tag{1.2}$$

at least outside a set of finite measure, where

$$\lambda=\max\{1+\deg(B_j)/(k-j):\ j=0, \cdots, k-2\}.$$

If $B_j\equiv 0$ *for all* j, *then* r^{λ} *can be replaced by* $\log r$ *in* (1.2).

It follows that if f and F have only finitely many zeros then $T(r, f'/f)=O(r^{\lambda})$, and that if $N(r, 1/ff^{(k)})=o(T(r, f'/f))$ for some $k\geqq 3$ then f'/f is rational. Note that assuming that $B_{k-1}\equiv 0$ in this case amounts to multiplying f and F by a factor $\exp(P)$, with P a polynomial, and that the estimate (1.2) is not known for $k=2$, except when f is entire.

The problem then arises of characterizing those functions f for which f and F have no zeros, and this has been done by Steinmetz [12] for constant coefficients. For rational coefficients the following was proved in [9]:

THEOREM B. *Suppose that f is meromorphic in the plane and that $F(z)$ is given by* (1.1), *where $k\geqq 3$ and for $j=0,\cdots,k-1$, each B_j is rational and satisfies*

$$B_j(z)=O(|z|^{-(k-j)}) \qquad as\ z\to\infty. \tag{1.3}$$

If $f(z)$ and $F(z)$ have only finitely many zeros, then f'/f is rational.

Note that the assumption (1.3) implies that the homogeneous equation $L(w)=0$ has a regular singular point at infinity. With an additional assumption on the poles of f the following was proved in [10] for $k=2$.

THEOREM C. *Suppose that f is meromorphic in $|z|\geqq R$, and that f and F have no zeros there, where $F(z)=f''(z)+a_1(z)f'(z)+a_0(z)f(z)$, and a_1, a_0 are analytic in $|z|\geqq R$ such that, for $j=0,1$,*

$$a_j(z)=O(|z|^{j-2}) \qquad as\ z\to\infty. \tag{1.4}$$

If $\bar{N}(r,f)$ has finite lower order, then f'/f has at most a pole at infinity.

Here $\bar{N}(r,f)$ denotes the counting function of the points in $|z|\geqq R$ at which f has poles (see [2, p. 98]). Theorem C is sharp at least to the extent that the condition (1.4) cannot be weakened. The function $g(z)=\sec(\sqrt{z})$ has no zeros and nor has

$$g''+(1/2z)g'+(1/4z)g=g^3/2z.$$

On the other hand it seems likely that the hypothesis on $\bar{N}(r,f)$ is not necessary.

However Theorem B is not sharp. We shall prove:

THEOREM 1. *Suppose that $k\geqq 3$, that $f(z)$ is meromorphic in $|z|\geqq R_0$, and that $B_0,\cdots,B_{k-1}$ are analytic there, with*

$$B_{k-1}(z)=O(1/|z|) \qquad as\ z\to\infty, \tag{1.5}$$

and for some $\lambda\geqq 0$, and for $j=0,\cdots,k-2$,

$$B_j(z)=O(|z|^{(\lambda-1)(k-j)}) \qquad as\ z\to\infty. \tag{1.6}$$

Suppose further that $f(z)$ and $F(z)$ have no zeros in $|z|\geqq R_0$, where $F(z)$ is given by (1.1). *Then if $\lambda\geqq 1/2$,*

$$T(r,f'/f)=O(r^{\lambda}) \qquad as\ r\to\infty. \tag{1.7}$$

If $\lambda<1/2$ then f'/f has at most a pole at infinity.

Note that if $B_0, \cdots, B_{k-1}$ are any functions analytic in $|z|>R_0$ and each having at most a pole at infinity, then by means of a transformation $f=g\exp(P)$, $F=G\exp(P)$, with P a polynomial, we can always ensure that B_{k-1} satisfies (1.5). Notice also that for $k=2$ and $\lambda<1/2$, the condition (1.6) reduces to (1.4), and that Theorem B corresponds to $\lambda=0$.

Theorem 1 is sharp, at least for even k. Using the fact that the function $h_n(w)=\cos^{-n}(w)$ satisfies $h_n''+n^2h_n=(n^2+n)h_{n+2}$ and the change of variables $w=\sqrt{z}$ one can construct examples for any even k with $\lambda=1/2$. For example $f(z)=\sec(\sqrt{z})$ satisfies

$$f^{(4)}+(3/z)f^{(3)}+((10z+3)/4z^2)f''+(5/4z^2)f'+(9/16z^2)f=(3/2z^2)f^5\neq 0. \tag{1.8}$$

It seems possible however that the second conclusion of Theorem 1 holds for $\lambda<1$ if k is odd.

Our method is similar to that of [9], coupling the techniques of [3] with the Tsuji characteristic. However the change of independent variable used in [9] is dispensed with except in one subcase, and this makes the proof of (1.7) easier. The proof of the second conclusion of the theorem requires the $\cos\pi\rho$ theorem, Wiman-Valiron estimates, and a lower bound for the logarithmic derivative of an entire function of order less than 1/2 which satisfies a third order linear differential equation.

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2. Lemmas required for the Proof of Theorem 1

Our proof requires the Tsuji characteristic (see [11], [13]): if $f(z)$ is meromorphic in $\mathrm{Im}(z)\geqq 0$ we define

$$m_1(r,f)=\frac{1}{2\pi}\int_{\sin^{-1}(1/r)}^{\pi-\sin^{-1}(1/r)}\log^+|f(r\sin\theta\, e^{i\theta})|\frac{d\theta}{r\sin^2\theta} \tag{2.1}$$

and

$$N_1(r,f)=\int_1^r n_1(t,f)dt/t^2,$$

where $n_1(t,f)$ is the number of poles of f, counting multiplicities, in $\{z\colon |z-it/2|\leqq t/2,\ |z|\geqq 1\}$. (Obviously analogous functionals can be defined in any closed half-plane.) Setting $T_1(r, f) = m_1(r, f) + N_1(r, f)$ we have the following properties of this Tsuji characteristic. T_1 differs from a non-decreasing function by a term which is bounded, and if f is non-constant then for all complex numbers a,

$$T_1(r, 1/(f-a)) = T_1(r, f) + O(1). \tag{2.2}$$

Also importantly

$$m_1(r, f'/f) = O(\log^+ T_1(r, f) + \log r) \tag{2.3}$$

outside a set of r of finite measure. If α satisfies $0 < \alpha < \pi/2$ then there is a constant $A > 1$ such that if $n_2(r, f)$ is the number of poles of f in $1 \leqq |z| \leqq r$, $\alpha \leqq \arg z \leqq \pi - \alpha$, then

$$n_2(r, f) = O(rN_1(Ar, f)) \tag{2.4}$$

(see [9, p. 272]). Finally we need an estimate for the Tsuji characteristic of a function analytic in a closed half-plane.

LEMMA 1. *If $K > 0$, $M \geqq 0$, and $f(z)$ is analytic in* $\mathrm{Im}(z) \geqq 0$ *with* $\log^+ |f(z)| \leqq K(1 + \log^+ |z| + |z|^M)$ *there, then*

$$T_1(r, f) = O(r^{M-1} + \log r) \qquad as\ r \to \infty. \tag{2.5}$$

PROOF. If $M = 0$ then (2.5) is immediate. If M is positive then in (2.1),

$$\log^+ |f(r \sin\theta\, e^{i\theta})| \leqq Kr^M \sin^M \theta + O(\log r).$$

If $0 < M \leqq 1$ then since $r \sin\theta \geqq 1$ on the range of integration,

$$m_1(r, f) = O\left(\log r + \int_{\sin^{-1}(1/r)}^{\pi - \sin^{-1}(1/r)} \frac{d\theta}{\sin\theta}\right) = O(\log r).$$

On the other hand if $M > 1$ we have

$$m_1(r, f) = O\left(\log r + r^{M-1} \int_0^{\pi} \sin^{M-2}\theta\, d\theta\right) = O(r^{M-1}).$$

The next lemma is standard. Here the weight of a differential monomial $(h)^{i_0}(h')^{i_1} \cdots (h^{(m)})^{i_m}$ is defined as $i_0 + 2i_1 + \cdots + (m+1)i_m$.

LEMMA 2. *Suppose that $h = f'/f$. Then for $k \geqq 2$,*

$$f^{(k)}/f = h^k + Q_k(h)$$

where the differential polynomial Q_k *is a sum of monomials each of degree at most* $k-1$ *and weight at most* k.

The next lemma summarizes some information from the Wiman-Valiron theory (see [6] for details). For an entire function $g(z)=\sum_{n=0}^{\infty} a_n z^n$, we denote by $M(r, g)$ the maximum of $|g(z)|$ on $|z|=r$. If $\mu(r)=\max\{|a_n|r^n : n=0, 1, \cdots\}$ is the maximum term, then the central index $\nu(r)$ is the largest n such that $|a_n|r^n=\mu(r)$.

LEMMA 3. *Let* δ *be a positive constant and let* $g(z)$ *be a transcendental entire function with central index* $\nu(r)$. *Then there exists a set* E *of finite logarithmic measure such that if* r *is not in* E *and* $|z|=r$ *with* $|g(z)|=M(r, g)$ *then for* $j=1, 2, \cdots$

$$g^{(j)}(z)/g(z)=(\nu(r)/z)^j(1+o(1)) \tag{2.6}$$

and if $f=1/g$,

$$f^{(j)}(z)/f(z)=(\nu(r)/z)^j((-1)^j+o(1)). \tag{2.7}$$

Also if $j=1, 2, \cdots$, *then for* z *as above*

$$(g'/g)^{(j)}(z)=o(\nu(r)/r)^{j+1}. \tag{2.8}$$

In addition, if r *is not in* E,

$$\nu(r)\leqq(\log M(r, g))^{1+\delta}. \tag{2.9}$$

Further, if $\nu(r)=O(r^\lambda)$ *as* $r\to\infty$ *for some* $\lambda>0$, *then*

$$\log M(r, g)=O(r^\lambda) \qquad as\ r\to\infty. \tag{2.10}$$

Finally if g *has finite order then for any* $\varepsilon>0$ *there is a set* E_1 *of finite logarithmic measure such that if* z *is as above with, in addition,* $|z|$ *not in* E_1 *then*

$$(g'/g)'(z)=O(\nu(r)r^{\varepsilon-2}). \tag{2.11}$$

PROOF. For (2.6) to (2.9) see [6] and Lemma 3 of [9]. Defining (as in [6])

$$a(r)=r\frac{d}{dr}(\log M(r, g)) \quad \text{and} \quad b(r)=r\frac{d}{dr}(a(r))$$

then Lemma 6 of [6] implies that, except for isolated values of r, if

$|z|=r$ and $|g(z)|=M(r,g)$, then

$$a(r)=zg'(z)/g(z) \tag{2.12}$$

and

$$|zg'(z)/g(z)+z^2(g'/g)'(z)|\leqq b(r). \tag{2.13}$$

Now (2.10) follows at once from (2.6) and (2.12). To prove (2.11) we note that for any positive constant d the fact that $\int_1^\infty b(r)a(r)^{-1-d}dr/r$ converges implies that $b(r)<a(r)^{1+d}$ outside a set of finite logarithmic measure. This and (2.13) give

$$z^2(g'/g)'(z)=O(a(r)^{1+d})=O(\nu(r)^{1+d})$$

if r is not in E. Now we choose d so that $\nu(r)^d<r^\varepsilon$.

LEMMA 4. *Suppose that* $k\geqq 2$, *and that* $a_0,\cdots,a_{k-1}$ *are analytic in* $|z|\geqq R$ *such that for some* $\lambda\geqq 0$,

$$a_j(z)=O(|z|^{(\lambda-1)(k-j)}) \qquad as\ z\to\infty. \tag{2.14}$$

Let $f(z)$ *be a solution of*

$$L(y)=y^{(k)}+a_{k-1}y^{(k-1)}+\cdots+a_0y=0 \tag{2.15}$$

in a sectorial region

$$S=\{z:\ |z|>R,\ \alpha<\arg z<\alpha+2\pi\} \tag{2.16}$$

where α *is real. Then as* $z\to\infty$ *in* S,

$$\log^+|f(z)|=O(|z|^\lambda+\log|z|). \tag{2.17}$$

PROOF. If $\lambda>0$ we take a subregion of S on which $Z=z^\lambda$ is analytic and one-one. Defining $F(Z)=f(z)$ transforms (2.15) into a differential equation

$$d^kF/dZ^k+A_{k-1}d^{k-1}F/dZ^{k-1}+\cdots+A_0F=0$$

in which the coefficients A_j are by (2.14) bounded. This gives $\log^+|F(Z)|=O(|Z|)$ and hence (2.17). The details are standard and we omit them. If $\lambda=0$ then (2.15) has a regular singular point at infinity and (2.17) can be obtained by setting $Z=\log z$.

LEMMA 5. *Suppose that* $M\geqq 0$, *that* $k\geqq 1$, *and that* $a_0,\cdots,a_{k-1}$ *are* *meromorphic in* $|z|>R$. *Suppose further that for any real* α *the equation* (2.15) *has in the sectorial region* (2.16) *a fundamental set of analytic solutions* $f_1, \cdots, f_k$ *each satisfying*

$$\log^+ |f_j(z)| = O(|z|^M + \log |z|) \tag{2.18}$$

as $z \to \infty$ *in* S. *Then we have, for each* j,

$$\log^+ |f_j'(z)| = O(|z|^M + \log |z|) \tag{2.19}$$

as $z \to \infty$ *in* S. *In addition, there exists a positive constant* M_1 *such that if* $|z|$ *lies outside a set of finite logarithmic measure, we have, for* $j = 0, \cdots, k-1$,

$$a_j(z) = O(|z|^{M_1}). \tag{2.20}$$

PROOF. To prove the first part we just note that, by (2.18) and Cauchy's integral formula, the estimate (2.19) holds for large z with $\alpha + \pi/2 < \arg z < \alpha + 3\pi/2$. To obtain (2.19) in the whole sector (2.16) we need only consider fundamental solution sets for different values of α.

To prove (2.20) we first observe that by analytic continuation, (2.15) has a solution $f(z) = z^d g(z)$, where d is a constant and g is analytic in $|z|>R$ (see [8, pp. 357-358]). By (2.18) g has finite order of growth and we have, for some positive M_2,

$$|f^{(j)}(z)/f(z)| \leqq |z|^{M_2} \tag{2.21}$$

for $j=1, \cdots, k$ and for $|z|$ lying outside a set of finite logarithmic measure. Now (2.20) is obvious for $k=1$. For $k \geqq 2$, we proceed by induction, assuming the proposition true for $k-1$.

Let y be an arbitrary solution of (2.15) in a sector (2.16) such that y and f are linearly independent. Then $u = y'f - yf'$ is analytic in S and by (2.19) satisfies an estimate of type (2.18). Locally we can write $y = f\int u/f^2 dz$ so that $y' = f'\int u/f^2 dz + u/f$. It is easy to prove by induction that for $j = 2, \cdots, k$,

$$y^{(j)} = f^{(j)} \int u/f^2 dz + \frac{1}{f}(u^{(j-1)} + Q_j(u)), \tag{2.22}$$

where each $Q_j(u)$ is a linear combination of $u, \cdots, u^{(j-2)}$, with coefficients which are differential polynomials in f'/f. By (2.22) and the fact that $L(y) = L(f) = 0$, we obtain a differential equation

$$u^{(k-1)}+B_{k-2}u^{(k-2)}+\cdots+B_0u=0,$$

where for $j=0,\cdots,k-2$, each function B_j-a_{j+1} is a linear combination of $a_{j+2},\cdots,a_k$, with coefficients which are differential polynomials in f'/f. Here for convenience we write $a_k=1$. The induction hypothesis gives $B_j(z)=O(|z|^N)$, say, for $|z|$ outside a set of finite logarithmic measure, so that we obtain (2.20) for $j=1,\cdots,k-1$, using (2.21). Now (2.21) again gives (2.20) for a_0.

LEMMA 6 ([9]). *Let $f_1,\cdots,f_k$ be linearly independent solutions of* (2.15), *where $a_0,\cdots,a_{k-1}$ are analytic in* $\mathrm{Im}(z)\geqq 0$. *Then for $j=1,\cdots,k$ we have*

$$T_1(r,a_j)=O(\log^+(\max T_1(r,f_j))+\log r)$$

at least outside a set of finite measure.

LEMMA 7. *Suppose that $g(z)$ is a transcendental entire function of order less than* 1/2, *and that $g(z)$ satisfies an equation*

$$g^{(3)}(z)+A_2(z)g''(z)+A_1(z)g'(z)+A_0(z)g(z)=0,$$

where A_0, A_1 and A_2 are each analytic in $|z|\geqq R$ with at most a pole at infinity. Then for each $\varepsilon>0$ there exists a set L of finite logarithmic measure such that if $|z|$ is not in L we have

$$|z|^{-2/3-\varepsilon}\leqq|g'(z)/g(z)|\leqq|z|^{-2/3+\varepsilon}. \tag{2.23}$$

PROOF. We can write, for $j=1,2,3$,

$$A_j(z)=d_jz^{n_j}(1+o(1))$$

as $z\to\infty$, where d_j is a constant and n_j is an integer, and we denote the central index of g by $\nu(r)$. Then for large r outside a set of finite logarithmic measure, we have, if z is a maximum modulus point of g on $|z|=r$,

$$g^{(j)}(z)/g(z)=(1+o(1))(\nu(r)/z)^j \quad \text{for } j=1,2,3.$$

Also $\nu(r)$ is unbounded but satisfies $\nu(r)=O(r^{1/2-s})$ for some positive s. Let

$$m=\max\{n_j-j : 0\leqq j\leqq 2 \text{ and } A_j\not\equiv 0\}$$

and

$$D=\max\{j : 0\leqq j\leqq 2 \text{ and } A_j\not\equiv 0 \text{ and } n_j-j=m\}.$$

Then for z as above we have

$$A_2 g''/g + A_1 g'/g + A_0 = d_D z^m \nu(r)^D (1+o(1)).$$

For if $0 \leqq j \leqq 2$ and $n_j - j < m$, then

$$A_j(\nu(r)/r)^j = O(r^{m-1} r^{1-2s}).$$

Therefore we obtain

$$\nu(r)^{3-D} = -d_D z^{m+3}(1+o(1)).$$

This forces $D=0$ and $m=-2$, since otherwise we find that either $\nu(r)$ is bounded or g has order at least 1/2. So $n_2 \leqq -1$, $n_1 \leqq -2$, and $n_0 = -2$. Therefore, for $\varepsilon > 0$ and for $|z|$ outside a set of finite logarithmic measure we have, by Theorem 3 of [5], for $j=1, 2$,

$$g^{(j+1)}(z)/g'(z) = O(|z|^{-(2/3)j+\varepsilon/2})$$

and

$$g'(z)/g(z) = O(|z|^{-2/3+\varepsilon/2}).$$

This gives the required upper bound for g'/g. To obtain a lower bound we can write

$$\begin{aligned} -A_0 &= (g'/g)(O(|z|^{-(4/3)+\varepsilon/2}) + A_2 O(|z|^{-2/3+\varepsilon/2}) + A_1) \\ &= (g'/g) O(|z|^{-(4/3)+\varepsilon/2}) \end{aligned}$$

which gives the required estimate, since $A_0 \sim d_0 z^{-2}$.

The following lemma summarizes information from Lemmas 6 and 9 of [3].

LEMMA 8. *Suppose that $k \geqq 3$, that $a_0, \cdots, a_{k-2}$ and $A_0, \cdots, A_{k-2}$ are analytic in a domain D, and that $f_1, \cdots, f_k$ are linearly independent solutions in D of the equation*

$$w^{(k)} + a_{k-2} w^{(k-2)} + \cdots + a_0 w = 0. \tag{2.24}$$

Suppose further that for some h, g analytic in D, with $g \not\equiv 0$, the functions $f_j' g + f_j h$ are linearly independent solutions of

$$w^{(k)} + A_{k-2} w^{(k-2)} + \cdots + A_0 w = 0. \tag{2.25}$$

Then with the notation $b_j = A_j - a_j$, we have

$$-h' = ((k-1)/2) g'' + (b_{k-2}/k) g. \tag{2.26}$$

In addition, if $b_{k-2}\not\equiv 0$ *then* g *satisfies a homogeneous linear differential equation of order at most* 3 *over the field* E *generated by the* a_j, A_j *and their derivatives. Further, if* $b_{k-2}\not\equiv 0$, *either* h/g *is in* E *or* h/g *is a rational function of the* f_j *and their derivatives of first order.*

PROOF. As in Lemma 6 of [3], set $A_k=1$, $a_{-1}=a_{k-1}=0$, $M_{k,-1}=0$, and for $\nu=0, \cdots, k$,

$$M_{k,\nu}(w)=\sum_{\mu=\nu}^{k}\binom{\mu}{\nu}A_\mu w^{(\mu-\nu)}.$$

Then h and g satisfy, for $\nu=0, \cdots, k-1$, by Lemma 6 of [3],

$$M_{k,\nu}(h)-a_\nu h=-M_{k,\nu-1}(g)+a_\nu M_{k,k-1}(g)+(a'_\nu+a_{\nu-1})g. \tag{2.27}$$

Conversely if (2.27) holds for $\nu=0, \cdots, k-1$ with h, g replaced by H, G then the functions f'_jG+f_jH are solutions of (2.25). Now $\nu=k-1$ in (2.27) gives (2.26). Further $\nu=k-2$ in (2.27) and (2.26) give

$$b_{k-2}h=\frac{k(k^2-1)}{12}g^{(3)}+g'\left(\frac{-(k+1)}{2}b_{k-2}+2A_{k-2}\right)+c_1g, \tag{2.28}$$

where we use $c_1, c_2, \cdots$ to denote elements of the field E. In addition, $\nu=k-3$ in (2.27) and (2.26) give

$$\frac{2}{k-2}b_{k-3}h=\frac{k(k^2-1)}{12}g^{(4)}+g''\left(\frac{k-1}{3}b_{k-2}+2A_{k-2}\right)+c_2g'+c_3g. \tag{2.29}$$

We assume henceforth that $b_{k-2}\not\equiv 0$. Differentiating (2.28) and using (2.26) and (2.29) we obtain

$$\left(\frac{2}{k-2}b_{k-3}-b'_{k-2}\right)h=\frac{k+2}{3}b_{k-2}g''+c_4g'+c_5g. \tag{2.30}$$

From (2.28) and (2.30) we see that g satisfies a linear differential equation as required.

The last assertion of the Lemma now follows from the method of Lemma 9 of [3] (see also [9]). The equations (2.27) are, using (2.26), equivalent to the equations (2.26), (2.28) and $k-2$ further equations each of form either $h=L(g)$ or $N(g)=0$ where L and N denote linear differential operators with coefficients in E. Eliminating h by (2.28) we obtain a system S of linear differential equations in g, with coefficients in E, with the property that if G is a local common solution of these equations, then defining H by (2.28) the functions H and G solve (2.27).

Proceeding exactly as in [3, Lemma 9], if the system S has, up to multiplication by a constant, just the one common solution g, then by [8, p. 126] the function g solves a first order linear differential equation over E. Thus g'/g is in E and by (2.28) so is h/g. On the other hand, if the system S has another common (local) solution G with G/g non-constant then defining H by (2.28) there exist solutions g_j of (2.24) such that for $j=1, \cdots, k$,

$$f_j'G+f_jH-g_j'g-g_jh=0. \tag{2.31}$$

The argument on p. 424 of [3] shows that the rank of the coefficient matrix of the system (2.31) is 3, and we can solve for h/g by Cramer's rule.

We need finally a lemma of Tumura-Clunie type.

LEMMA 9. *Suppose that* $k\geqq 2$, *that* $\lambda, \mu\geqq 0$, *and that* f *is meromorphic in* $|z|\geqq R_0$ *such that* $fF\neq 0$ *there, where*

$$F(z)=f^{(k)}(z)+\sum_{j=0}^{k-1} a_j(z)f^{(j)}(z), \tag{2.32}$$

and the coefficients a_j *are analytic in* $|z|\geqq R_0$ *with*

$$a_j(z)=O(|z|^{(\lambda-1)(k-j)}) \qquad as\ z\to\infty. \tag{2.33}$$

Suppose further that

$$\bar{N}(r, f)=O(r^{\mu}+\log r) \qquad as\ r\to\infty, \tag{2.34}$$

and set $\nu=\max\{\lambda, \mu\}$. *Then as* $r\to\infty$,

$$T(r, f'/f)=O(r^{\nu}+\log r). \tag{2.35}$$

Also if $\mu=0$ *and* $\lambda<1/2$, f'/f *has at most a pole at infinity.*

PROOF. We set $u=f'/f$ and $U=F/f$, and clearly may assume without loss of generality that u has an essential singularity at infinity, since otherwise there is nothing to prove. We apply the Tumura-Clunie method, as in [7, pp. 69–73], but very slightly modified. We denote by $S_1(r, u)$ any term which satisfies

$$S_1(r, u)=O(r^{\mu}+\log(rT(r, u))) \tag{2.36}$$

as $r\to\infty$, at least outside a set of finite measure. Here we are using Nevanlinna functionals defined for $r>R_0$ (see [2, p. 98]). The same argument as in [7, pp. 69-73] now implies that either $T(r,u)=S_1(r,u)$, in which case both conclusions of the lemma hold at once, or

$$U=h^k \tag{2.37}$$

where

$$h=u+\frac{k-1}{2k}U'/U+a_{k-1}/k=u+\frac{k-1}{2}h'/h+a_{k-1}/k. \tag{2.38}$$

Assuming henceforth that (2.37) and (2.38) hold, a comparison of the Laurent series expansions of both sides of (2.38) at a pole of f of order m yields

$$m(m+1)\cdots(m+k-1)=(m+(k-1)/2)^k$$

which contradicts the arithmetic-geometric mean inequality. Therefore f has no poles in $|z|>R_0$ and h is analytic and non-zero there. Now if h has at most a pole at infinity then so has u, by (2.38). Therefore we can write, without loss of generality,

$$B=h'/h,\quad h''=(B'+B^2)h,\quad \text{etc.,} \tag{2.39}$$

where B is analytic in $|z|>R_0$ and $T(r,B)=S(r,h)$, with $S(r,h)$ the usual error term of Nevanlinna theory. We can write, using [7, p. 73], and (2.37),

$$\begin{aligned}h^k=&u^k+u^{k-2}u'k(k-1)/2+u^{k-3}u''k(k-1)(k-2)/6\\&+u^{k-4}(u')^2k(k-1)(k-2)(k-3)/8\\&+a_{k-1}(u^{k-1}+u^{k-3}u'(k-1)(k-2)/2)\\&+a_{k-2}u^{k-2}+Q_{k-3}(u)\end{aligned} \tag{2.40}$$

where Q_{k-3} is a differential polynomial in u, of degree at most $k-3$, whose coefficients are linear combinations of the a_j. Substituting (2.38) into (2.40) and using (2.39) we obtain

$$Dh^{k-2}=M_{k-3}(h) \tag{2.41}$$

where $M_{k-3}(h)$ is a polynomial in h, of degree at most $k-3$, whose coefficients have Nevanlinna characteristics which are $S(r,h)$. Here

$$D=\frac{k(k^2-1)}{24}B^2+c_1B'+c_2a_{k-1}B+c_3a'_{k-1}+c_4a^2_{k-1}+a_{k-2} \tag{2.42}$$

with $c_1, \cdots, c_4$ constants. Since $T(r, D)=S(r, h)$, (2.41) forces D to vanish identically. But then by the Wiman-Valiron theory, B can have at most a pole at infinity. Now (2.42) gives $B=O(|z|^{\lambda-1})$ as $z\to\infty$, which gives, by integration, $T(r, h)=O(r^{\lambda}+\log r)$ and (2.35) follows, using (2.38). Also if $\lambda<1/2$ then in (2.42) $a_j=O(|z|^{-(k-j)})$ which gives $B(z)=O(1/|z|)$ and $T(r, h)=O(\log r)$.

3. Proof of Theorem 1

In view of Lemma 9, we need only obtain an estimate for $\bar{N}(r, f)=N(r, f'/f)$. We take a sectorial region

$$S=\{z\colon |z|>R_0, -\pi<\arg z<\pi\} \tag{3.1}$$

and (as in [3] or [10]) define $g(z)$ in S by

$$g(z)^k=f(z)/F(z). \tag{3.2}$$

We also define, in S, linearly independent analytic solutions $u_1, \cdots, u_k$ of the homogeneous equation $L(u)=0$, and define

$$W(z)=W(u_1, \cdots, u_k) \tag{3.3}$$

to be their Wronskian, so that

$$W'(z)/W(z)=-B_{k-1}(z)=O(1/|z|). \tag{3.4}$$

We define a branch of $W(z)^{-1/k}$ in S, and obtain from (3.3)

$$W(f_1, \cdots, f_k)=1 \tag{3.5}$$

where

$$f_j=u_j W^{-1/k}. \tag{3.6}$$

Writing

$$\phi(z)=f(z)W(z)^{-1/k}, \quad \Phi(z)=F(z)W(z)^{-1/k}, \tag{3.7}$$

we have

$$W(f_1, \cdots, f_k, \phi)=\Phi=\phi g^{-k}$$

so that

$$W(v_1, \cdots, v_k)=(-1)^k \tag{3.8}$$

where

$$v_j=f_j'g+f_jh \tag{3.9}$$

and

$$h=(-\phi'/\phi)g=(-(f'/f)-B_{k-1}/k)g. \tag{3.10}$$

From (3.8), (3.9) and (3.10) it follows that the v_j are, in S, linearly independent analytic solutions of an equation

$$v^{(k)}+A_{k-2}v^{(k-2)}+\cdots+A_0v=0 \tag{3.11}$$

where $A_0, \cdots, A_{k-2}$ are analytic in S.

We assert that the coefficients $A_0, \cdots, A_{k-2}$ are in fact analytic in $|z|>R_0$. To see this, let S^* be the sectorial region obtained by rotating S through an angle π. In S^* we can define, exactly as in S, corresponding functions g^*, h^*, W^*, f_j^* and v_j^*, so that the v_j^* solve an equation

$$v^{(k)}+A_{k-2}^*v^{(k-2)}+\cdots+A_0^*v=0,$$

where the A_j^* are analytic in S^*. But it is clear that in each component of the intersection of S and S^*, each v_j^* is a linear combination of $v_1, \cdots, v_k$. Therefore $A_j^*=A_j$ in each such component. This proves the assertion above.

From (3.5) and (3.6) we observe that the functions f_j are linearly independent solutions in S of an equation

$$w^{(k)}+a_{k-2}w^{(k-2)}+\cdots+a_0w=0 \tag{3.12}$$

where the a_j are analytic in $|z|>R_0$, and each satisfy $\log^+|a_j(z)|=O(\log|z|)$ there. We now apply Lemma 8, to obtain

$$-h'=((k-1)/2)g''+(b_{k-2}/k)g \tag{3.13}$$

where $b_{k-2}=A_{k-2}-a_{k-2}$. Again using Lemma 8 we can distinguish 3 possible cases.

Case 1, in which $b_{k-2}\equiv 0$.

In this case we obtain from (3.13)

$$-h=((k-1)/2)g'+c_1 \tag{3.14}$$

for some constant c_1, so that using (3.10) we have

$$f'/f=((k-1)/2)(g'/g)+c_1/g-B_{k-1}/k. \tag{3.15}$$

If $c_1=0$ then (3.15) implies that f has no poles in $|z|>R_0$, and in this case both conclusions of the theorem follow at once from Lemma 9. If

$c_1 \neq 0$ then g is analytic in $|z|>R_0$ and we apply the Wiman-Valiron theory to estimate the growth of g and hence of $\bar{N}(r, f)$.

To this end we can write

$$g(z)=z^{m_1}g_1(z)(1+O(1/|z|)) \tag{3.16}$$

as $z \to \infty$, where m_1 is an integer and g_1 is entire (see e.g. [14, p. 15]). If g_1 is a polynomial then clearly f has only finitely many poles in $|z| \geqq R_0$ and the conclusions of the theorem follow from Lemma 9. If g_1 is transcendental then by Lemma 3 there exists a set L_1 of finite logarithmic measure such that if z is a maximum modulus point of g_1 and $|z|$ is not in L_1 then for $j=1, \cdots, k$, using (2.6) and (3.16),

$$g^{(j)}(z)/g(z) \sim (\nu(|z|)/z)^j \tag{3.17}$$

where $\nu(r)$ is the central index of g_1. For such z we also have (Lemma 3), for $j=1, \cdots, k-1$,

$$(g'/g)^{(j)}(z)=o(\nu(|z|)/|z|)^{j+1}.$$

Using (3.15) and Lemma 2 we therefore have, for such z, and for $j=1, \cdots, k$,

$$f^{(j)}(z)/f(z) \sim ((k-1)/2)^j(\nu(|z|)/z)^j. \tag{3.18}$$

Substituting (3.18) into the equation $F/f=g^{-k}$ and using (1.6) we obtain $\nu(r)=O(r^\lambda)$ outside a set of finite logarithmic measure and hence for all large r. This gives $\log M(r, g)=O(r^\lambda)$ as $r \to \infty$ (see Lemma 3), so that $\bar{N}(r, f)=O(r^\lambda)$. If $\lambda \geqq 1/2$, then the conclusion of the theorem follows from Lemma 9. If $\lambda<1/2$ then by (1.5), (3.15) and the $\cos \pi\rho$ theorem [1] applied to g_1 there exist arbitrarily large r such that on the circle $|z|=r$ we have

$$f'(z)/f(z)=((k-1)/2)g'(z)/g(z)+O(1/r)$$

which implies, using the argument principle, that f has only finitely many poles in $|z| \geqq R_0$, and by Lemma 9 f'/f has at most a pole at infinity.

Case 2, in which h/g is a rational function of the A_j, a_j and their derivatives.

In this case we estimate the function $\bar{N}(r, f)$ as follows. We can take a half-plane $S_1=\{z\colon \mathrm{Re}(e^{i\alpha}z) \geqq R_1\}$ for some fixed large R_1 and any real α, and in S_1 define functions $\hat{g}, \hat{f}_j, \hat{h}$ etc. all exactly as in (3.2)-(3.10). The functions

$$\hat{v}_j = \hat{f}_j \hat{h} + \hat{f}'_j \hat{g}$$

must satisfy the equation (3.11) in S_1. Since the $\hat{f}_j$ satisfy (3.12) in S_1, the Tsuji characteristics $T_1(r, \hat{f}_j)$ are bounded by a power of r as $r \to \infty$, using Lemmas 1 and 4. Also from $(\hat{g})^k = f/F$ and $\hat{h} = (-\phi'/\phi)\hat{g}$ we obtain

$$T_1(r, \hat{g}) + T_1(r, \hat{h}) = O(T_1(r, f'/f) + \log r) \quad \text{n.e.}$$

where as usual "n.e." denotes "outside a set of finite measure". Therefore for some positive m_2,

$$T_1(r, \hat{v}_j) = O(T_1(r, f'/f) + r^{m_2}) \quad \text{n.e.}$$

But then Lemma 6 gives

$$T_1(r, A_j) = O(\log^+ T_1(r, f'/f) + \log r) \quad \text{n.e.}$$

Since by (3.10) and the assumption of case 2 f'/f is a rational function of the A_j, a_j, their derivatives and B_{k-1}, this gives $T_1(r, f'/f) = O(\log r)$. It follows that the number $n_1(r)$ of poles of f'/f in $\mathrm{Re}(e^{i\alpha}z) \geqq R_1$, $|z| \leqq r$, $|\arg(e^{i\alpha}z)| \leqq \pi/4$ satisfies $n_1(r) = O(r \log r)$. Applying this reasoning for different values of α we conclude that $N(r, f'/f)$, the standard Nevanlinna counting function, has finite order. Therefore by Lemma 9 f'/f has finite order of growth in $|z| \geqq R_0$, and so have g^k and h^k. Therefore in any half-plane of form S_1 the solutions $\hat{v}_j$ have finite order of growth, and the A_j have at most a pole at infinity, by Lemma 5. But then f'/f has at most a pole at infinity, again since f'/f is a rational function of the A_j, a_j, their derivatives and B_{k-1}.

Case 3, in which $h/g = -\phi'/\phi$ is a rational function of the f_j and their derivatives.

Here we first use a transformation to enable us to apply the Tsuji characteristic. We take a small positive ε, and define a branch of $z^{2+\varepsilon}$ on $\mathrm{Re}(z) > 0$, defined to be positive on the positive real axis. It is clear that the function $\psi(z) = \phi'(z^{2+\varepsilon})/\phi(z^{2+\varepsilon})$ is meromorphic on $\mathrm{Re}(z) > 0$, $|z| > R_2$, say. The functions $w_j(z) = f_j(z^{2+\varepsilon})$, $y_j(z) = f'_j(z^{2+\varepsilon})$ are initially defined on $|z| > R_2$, $|\arg z| < \pi/(2+\varepsilon)$. But we can use the solutions f_j^* of (3.12) which were defined on S^* to analytically continue w_j and y_j to all of $\mathrm{Re}(z) > 0$, $|z| > R_2$, and by (1.5), (1.6), (3.4), (3.6) and Lemmas 4 and 5, we have

$$\log^+ |w_j(z)| + \log^+ |y_j(z)| = O(|z|^{\lambda(2+\varepsilon)} + \log |z|)$$

as $z\rightarrow\infty$ in this region. It follows using Lemma 1 that in the half-plane $\mathrm{Re}(z)\geqq R_2+1$, we have the estimate

$$T_1(r, w_j)+T_1(r, y_j)=O(r^{\lambda(2+\varepsilon)-1}+\log r) \tag{3.19}$$

for the Tsuji characteristics of w_j and y_j. But by the identity theorem, ϕ is a rational function of the w_j and y_j in $\mathrm{Re}(z)\geqq R_2+1$, so that (3.19) leads to

$$T_1(r, \phi)=O(r^{\lambda(2+\varepsilon)-1}+\log r). \tag{3.20}$$

Therefore for any positive δ_1, the number $n_2(r)$ of poles of ϕ in $|z-R_2-1|<r$, $|\arg(z-R_2-1)|<(\pi/2)-\delta_1$, satisfies, using (2.4),

$$n_2(r)=O(r^{\lambda(2+\varepsilon)}+r\log r). \tag{3.21}$$

But if δ_1 is small enough then for any large pole ζ of f, which is in turn a pole of ϕ'/ϕ, the function ϕ has a pole at a point z with $|\arg(z-R_2-1)|<(\pi/2)-\delta_1$, and with $z^{2+\varepsilon}=\zeta$. Therefore by (3.21),

$$\bar{N}(r, f)=N(r, f'/f)=O(r^{\lambda}+r^{1/(2+\varepsilon)}\log r). \tag{3.22}$$

If $\lambda\geqq 1/2$, then (1.7) follows from (3.22) and Lemma 9.

If $\lambda<1/2$, we show that f'/f has at most a pole at infinity as follows. We first observe that by (3.22) and Lemma 9 the functions f'/f and f/F both have order of growth less than 1/2. Now the function $H=f^{-2k}g^{k(k-1)}$ is analytic in $|z|>R_0$ and every pole of f is a zero of H. If H has only finitely many zeros then the conclusion of the theorem follows at once from Lemma 9. To complete the proof therefore we need only show that the assumption that H has infinitely many zeros in $|z|\geqq R_0+1$, say, leads to a contradiction.

Now (3.10) and (3.13) give

$$(H'/H)'+(g'/g)(H'/H)=2B_{k-1}(g'/g)+2B'_{k-1}-2b_{k-2}. \tag{3.23}$$

We first estimate the coefficient b_{k-2}, noting that

$$B_{k-1}(z)=O(1/|z|) \quad\text{and}\quad B'_{k-1}(z)=O(1/|z|^2) \tag{3.24}$$

as $z\rightarrow\infty$. Now by Lemma 8 we can assume that g satisfies, in S, an equation

$$g^{(3)}+C_2g''+C_1g'+C_0g=0 \tag{3.25}$$

where the coefficients C_j are analytic in $|z|>R_0$ each with at most a pole at infinity. Since all poles of F/f have multiplicity k, we can write

$$g(z)=z^{m_3}(1+O(1/|z|))g_2(z) \qquad \text{as } z\to\infty, \tag{3.26}$$

where m_3 is real (possibly fractional) and g_2 is a transcendental entire function, which in turn satisfies an equation of type (3.25). Applying Lemma 7 and using (3.26) we see that for any positive ε_1, since g_2 has order less than 1/2,

$$|z|^{-2/3-\varepsilon_1}\leqq|g'(z)/g(z)|\leqq|z|^{-2/3+\varepsilon_1} \tag{3.27}$$

for $|z|$ lying outside a set of finite logarithmic measure. (Here we note that g'/g is meromorphic in $|z|>R_0$.) Since H'/H has order less than 1/2 we can apply Theorem 3 of [5] to write (3.23) in the form

$$2b_{k-2}+O(|z|^{-1/2-\varepsilon_2})H'/H=O(|z|^{-5/3+\varepsilon_1}) \tag{3.28}$$

for some positive ε_2. (3.28) holds for all z with $|z|$ outside a set of finite logarithmic measure. To estimate b_{k-2} from (3.28) we require an estimate for H'/H. We can write

$$f(z)=z^{m_4}(1+O(1/|z|))F_1(z)^{-1} \tag{3.29}$$

where m_4 is an integer and F_1 is an entire function. Denoting by $\nu_1(r)$ the central index of F_1, (3.29) and Lemma 3 imply that if z is a maximum modulus point of F_1 and $|z|$ lies outside a set of finite logarithmic measure, then

$$f^{(j)}(z)/f(z)\sim(-1)^j(\nu_1(|z|)/z)^j \tag{3.30}$$

for $j=1,\cdots,k$. But f/F has order less than 1/2 so that for r lying in a set of positive lower logarithmic density we have $F(z)/f(z)=O(r^{-2k})$ on $|z|=r$. This follows from the $\cos\pi\rho$ theorem. Substituting (3.30) into the formula for F/f we deduce that for r in a set of positive lower logarithmic density we have

$$\nu_1(r)=O(r^{\lambda}). \tag{3.31}$$

Now (3.27), (3.30) and (3.31) imply that we can find arbitrarily large z such that (3.28) holds with

$$H'(z)/H(z)=O(|z|^{\lambda-1}+|z|^{-2/3+\varepsilon_1}),$$

so that

$$b_{k-2}(z)=O(|z|^{-2}) \qquad \text{as } z\to\infty. \tag{3.32}$$

We are now in a position to obtain a contradiction to the assumption that H has infinitely many zeros in $|z|\geqq R_0+1$. We can write

$$H(z)=z^{m_5}(1+O(1/|z|))H_1(z),$$

where m_5 is an integer and H_1 is a transcendental entire function. Since (3.31) holds on a set of positive lower logarithmic density we deduce that F_1, f and H_1 have finite order. Denoting by $\nu_2(r)$ the central index of H_1, it follows from Lemma 3 that if z is a maximum modulus point of H_1 with $|z|$ lying outside a set of finite logarithmic measure, we have

$$H'(z)/H(z)\sim\nu_2(|z|)/z,\quad (H'/H)'(z)=O(\nu_2(|z|)|z|^{\varepsilon_1-2}) \tag{3.33}$$

and we can assume that (3.27) still holds. Substituting (3.33) into (3.23) and using (3.27) and (3.32) we obtain, for such z,

$$(\nu_2(|z|)/z)(g'/g)(1+o(1))=O(|z|^{-5/3+\varepsilon_1})$$

and $\nu_2(|z|)=O(|z|^{2\varepsilon_1})$. Since ε_1 may be chosen arbitrarily small, this implies that $T(r, H_1)$ has order zero, and therefore so have $N(r, f)$ and $T(r, f/F)$. But this in turn implies that for any $\varepsilon_2>0$,

$$g'(z)/g(z)=O(|z|^{\varepsilon_2-1}) \tag{3.34}$$

for all z with $|z|$ lying outside a set of finite logarithmic measure, using Theorem 3 of [5] again. But (3.34) contradicts (3.27), and the proof is complete.

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