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Application of interpolation spaces with a function parameter to the eigenvalue distribution of compact operators

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0. Introduction.

The theory of interpolation spaces $(X_0, X_1)_{\varphi,p}$ with a function parameter $\varphi(t)$ is the extension of the theory of interpolation spaces $(X_0, X_1)_{\theta,p}$ (with a numerical parameter θ) which originates from Lions and Peetre [7]. $(X_0, X_1)_{\theta,p}$ coincides with $(X_0, X_1)_{\varphi,p}$ when $\varphi(t)=t^{-\theta}$.

In this paper we apply the theory of interpolation spaces with a function parameter to the eigenvalue distribution of the compact operators including the integral operators with the logarithmic kernel. We shall show quantitatively how the singularity of the integral kernel effects the compactness of the integral operator.

For example, the integral operator in $L_2(\Omega)$ defined by

$$Tf(x)=\int_\Omega |x-y|^{-1}f(y)dy, \qquad \Omega=\{x\in \boldsymbol{R}^3\,;\,|x|<1\}$$

has the range $R(T)$ contained in the Sobolev space $H^2(\Omega)$. Since we know the s-numbers of the imbedding mapping from $H^2(\Omega)$ into $L_2(\Omega)$, we can evaluate the absolute values of the eigenvalues of T from above. We note that $H^2(\Omega)$ is the smallest Sobolev space which includes $R(T)$. On the other hand, when we consider the integral operator defined by

$$Sf(x)=\int_\Omega |x-y|^{-1}(\log|x-y|)^\mu f(y)dy,$$

where μ is a positive integer, there is not the smallest Sobolev space which includes the range $R(S)$, although we have $R(S)\subset H^\lambda(\Omega)$ for any $\lambda<2$. But appealing to the Fourier transform we can find the space which is the most suitable for the range $R(S)$. It is the interpolation space $(L_2(\Omega), H^m(\Omega))_{\varphi,2}$ with a function parameter. That's why we need the theory of interpolation spaces with a function parameter to obtain the estimates of the eigenvalues of the operators including S.

The plan of this paper is as follows. In Section 1 we review the theory of interpolation spaces with a function parameter. Our useful tools to estimate the eigenvalues of the compact operators are the mean space $S(2, \varphi_1, X_1; 2, \varphi_2, X_2)$, the abstract Besov space $D^{\varphi}(A)$, the Sobolev space with a function parameter $H^{\varphi}(\boldsymbol{R}^n)$, etc.

In Section 2 we consider the eigenvalues of the compact operator with the range contained in the interpolation space with a function parameter. Our result is the extension of the result on the operator with the range contained in the usual Sobolev space $H^m(\Omega)$.

In Section 3 we treat the integral operator with the kernel which has the singularity at the diagonal set, and whose Fourier transform has a certain property. We show that the range of the integral operator is contained in the interpolation space with a function parameter, and apply the result of Section 2 to obtain the estimate of the eigenvalues. We also gain the characterization of the Sobolev space with a function parameter $H^{\varphi}(\boldsymbol{R}^n)$ by the Fourier transform.

In Section 4 we give an example of the integral operator with the logarithmic kernel which satisfies the condition of Section 3.

1. Interpolation spaces.

In this section we will review the interpolation theory with a function parameter. The details can be found in Muramatu [10].

Weight functions on $\boldsymbol{R}_+$. Let φ be a *weight function on* $\boldsymbol{R}_+ = (0, \infty)$, that is, a measurable function from $\boldsymbol{R}_+$ onto $\boldsymbol{R}_+$. We define the *similarity ratio function* $\tilde{\varphi}$ of φ by

$$\tilde{\varphi}(t) = \operatorname{ess.}\sup_{s>0} \frac{\varphi(ts)}{\varphi(s)}.$$

Throughout this paper we assume that $\tilde{\varphi}(t)$ is finite for any $t>0$, and that the left and the right indexes of φ coincide:

$$\lim_{t\to 0} \frac{\log \tilde{\varphi}(t)}{\log t} = \lim_{t\to\infty} \frac{\log \tilde{\varphi}(t)}{\log t},$$

which we denote by $\operatorname{ind}\varphi$. The following properties of $\operatorname{ind}\varphi$ will be often used.

LEMMA 1.1. *Let φ be a weight function on $\boldsymbol{R}_+$.*
(i) *For any $s>0$ and $t>0$ we have*

$$\tilde{\varphi}(t^{-1})^{-1} \leq \frac{\varphi(ts)}{\varphi(s)} \leq \tilde{\varphi}(t).$$

(ii) *Let $\alpha = \operatorname{ind}\varphi$. For any $\varepsilon>0$ there exists $C>0$ such that*

$$C^{-1}t^{\alpha+\varepsilon} \leq \varphi(t) \leq Ct^{\alpha-\varepsilon} \qquad \text{when } 0<t\leq 1,$$
$$C^{-1}t^{\alpha-\varepsilon} \leq \varphi(t) \leq Ct^{\alpha+\varepsilon} \qquad \text{when } 1\leq t.$$

The above inequalities also hold when we replace φ with $\tilde{\varphi}$.

(iii) *Let $\lambda \neq 0$. Then $\varphi(t^\lambda)$ and $\varphi(t)^\lambda$ are also weight functions on $\boldsymbol{R}_+$, and we have*

$$\operatorname{ind}\varphi(t^\lambda) = \operatorname{ind}\varphi(t)^\lambda = \lambda \operatorname{ind}\varphi(t).$$

In particular we have

$$\operatorname{ind}\varphi(t^{-1})^{-1} = \operatorname{ind}\varphi(t).$$

Mean spaces. For a weight function φ, a Banach space X, and p with $1\leq p<\infty$, we denote by $L_p^*(X)$ the space of all X-valued measurable functions $f(t)$ on $\boldsymbol{R}_+$ such that

$$\|f\|_{L_p^*(X)} = \left(\int_0^\infty \|f(t)\|_X^p \frac{dt}{t}\right)^{1/p} \leq \infty.$$

Let $\{X_0, X_1\}$ be a compatible couple of Banach spaces. Let φ_0 and φ_1 be weight functions on $\boldsymbol{R}_+$ which satisfy $\operatorname{ind}\varphi_0>0>\operatorname{ind}\varphi_1$ or $\operatorname{ind}\varphi_0<0<\operatorname{ind}\varphi_1$. We define the *mean space with a function parameter* $S(2, \varphi_0, X_0; 2, \varphi_1, X_1)$ as the space of means

$$a = \int_0^\infty u(t)\frac{dt}{t}$$

with $\varphi_0(t)u(t) \in L_2^*(X_0)$ and $\varphi_1(t)u(t) \in L_2^*(X_1)$. The mean space $S(2, \varphi_0, X_0; 2, \varphi_1, X_1)$ is a Banach space with the norm

$$\|x\|_{S(2,\varphi_0,X_0;2,\varphi_1,X_1)} = \inf\ \max\{\|\varphi_0 u\|_{L_2^*(X_0)}, \|\varphi_1 u\|_{L_2^*(X_1)}\}$$

where the infinimum is taken over all $u(t)$ such that $a=\int_0^\infty u(t)dt/t$. In particular, when $\varphi_0(t)=\varphi(t)$ and $\varphi_1(t)=t\varphi(t)$ for some weight function φ

on $\boldsymbol{R}_+$ with $-1<\operatorname{ind}\varphi<0$, we denote $S(2, \varphi_0, X_0; 2, \varphi_1, X_1)$ simply by $(X_0, X_1)_{\varphi,2}$:

$$(X_0, X_1)_{\varphi,2}=S(2, \varphi(t), X_0; 2, t\varphi(t), X_1).$$

Moreover when $\varphi(t)=t^{-\theta}$ for some θ with $0<\theta<1$, we get the mean space $(X_0, X_1)_{\theta,2}$ which Lions and Peetre [7] originally defined:

$$(X_0, X_1)_{\theta,2}=(X_0, X_1)_{t^{-\theta},2}=S(2, t^{-\theta}, X_0; 2, t^{1-\theta}, X_1).$$

THEOREM 1.2. *For any real number λ with $\lambda\neq 0$, the mean space $S(2, \varphi_0(t), X_0; 2, \varphi_1(t), X_1)$ remains the same even if we replace $\varphi_j(t)$ $(j=0, 1)$ with $\varphi_j(t^\lambda)$ $(j=0, 1)$. That is, we have*

$$S(2, \varphi_0(t^\lambda), X_0; 2, \varphi_1(t^\lambda), X_1)=S(2, \varphi_0(t), X_0; 2, \varphi_1(t), X_1),$$

with the equivalent norms.

THEOREM 1.3. *Let φ_0 and φ_1 be weight functions on $\boldsymbol{R}_+$ which satisfy* $\operatorname{ind}\varphi_0>0>\operatorname{ind}\varphi_1$ *or* $\operatorname{ind}\varphi_0<0<\operatorname{ind}\varphi_1$. *Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be compatible couples of Banach spaces. Let T be a compatible bounded linear operator from $\{X_0, X_1\}$ into $\{Y_0, Y_1\}$. Then T maps $S(2, \varphi_0, X_0; 2, \varphi_1, X_1)$ into $S(2, \varphi_0, Y_0; 2, \varphi_1, Y_1)$ and this mapping is continuous.*

Complex interpolation spaces. Let $0<\theta<1$. We denote by $[X_0, X_1]_\theta$ the *complex interpolation space* (Calderón [3]).

THEOREM 1.4. *Let A be a positive self-adjoint operator in a Hilbert space X. Then we have for any θ with $0<\theta<1$,*

$$[X, D(A)]_\theta=D(A^\theta).$$

Here we donote by $D(A)$ the domain of definition of A.

Abstract Besov spaces. A is called a *non-negative operator* in a Banach space X, if A is a closed linear operator in X with the resolvent set containing the negative real ray $(-\infty, 0)$ and $\{t(t+A)^{-1}; 0<t<\infty\}$ is bounded.

Let φ be a weight function on $\boldsymbol{R}_+$ with $0<\operatorname{ind}\varphi<m$ for some integer m. Let A be a non-negative operator in a Banach space X. We define the *abstract Besov space $D_2^\varphi(A)$* as the space of all $x\in X$ such that

$$\varphi(t)A^m(t+A)^{-m}x\in L_2^*(X)$$

with the norm

$$\|x\|_{D_2^\varphi(A)}=\|x\|_X+\|\varphi(t)A^m(t+A)^{-m}x\|_{L_2^*(X)}.$$

THEOREM 1.5. *Under the above situation we have*

$$D_2^\varphi(A)=S(2,\varphi(t),X;2,t^{-m}\varphi(t),D(A^m)).$$

In particular, for θ with $0<\theta<1$ we have

$$(X,D(A^m))_{\theta,2}=D_2^{m\theta}(A).$$

Sobolev spaces in R^n. Let φ be a weight function on R_+ with $0<\operatorname{ind}\varphi<m$ for some integer m. We define the *Sobolev space with a function parameter* $H^\varphi(R^n)$ as the space of all functions $f\in L_2(R^n)$ such that

$$|f|_{H^\varphi(R^n)}=\left(\iint_{R^n\times R^n}\frac{|\Delta_y^m f(x)|^2}{\varphi(|y|)^2|y|^n}dxdy\right)^{1/2}<\infty$$

with the norm

$$\|f\|_{H^\varphi(R^n)}=\|f\|_{L_2(R^n)}+|f|_{H^\varphi(R\ )}$$

where $\Delta_y^m f(x)$ is the difference of m-th order:

$$\Delta_y^m f(x)=\sum_{j=0}^m\binom{m}{j}(-1)^{m-j}f(x+jy).$$

In particular, when $\varphi(t)=t^\theta$ for $\theta>0$, we get the usual Sobolev space $H^\theta(R^n)$:

$$H^\theta(R^n)=H^{t^\theta}(R^n).$$

We have the following theorems with respect to the interpolation spaces of the Sobolev spaces.

THEOREM 1.6. *Let φ be a weight function with $0<\operatorname{ind}\varphi<m$ for some integer m. Put $\psi(t)=\varphi(t^{1/m})^{-1}$. Then we have*

$$(L_2(R^n),\ H^m(R^n))_{\psi,2}=H^\varphi(R^n).$$

In particular, for θ with $0<\theta<1$ we have

$$(L_2(R^n),\ H^m(R^n))_{\theta,2}=H^{m\theta}(R^n).$$

THEOREM 1.7. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ with the restricted cone property (cf. Agmon* [1]). *Let m be a positive integer, and let $0<\theta<1$. Then we have*

$$[L_2(\Omega),\ H^m(\Omega)]_\theta = H^{m\theta}(\Omega).$$

2. Compactness of compact operators.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with the restricted cone property. It is known that the bounded linear operator T from $L_2(\Omega)$ into $L_2(\Omega)$ with the range $R(T)\subset H^m(\Omega)$ for some positive integer m is compact, and that its eigenvalues $\{\lambda_j(T)\}_{j=1}^{\infty}$ are estimated for some $C>0$ by

$$|\lambda_j(T)| \leq Cj^{-m/n} \qquad (j=1, 2, \cdots). \tag{2.1}$$

We shall generalize this fact when the range $R(T)$ is contained in the interpolation space with a function parameter.

Here and in what follows we denote by $\lambda_j(T)$ $(j=1, 2, \cdots)$ the eigenvalues of a compact operator T which have been arranged in *decreasing* order with respect to the absolute values:

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq 0,$$

and we denote by $s_j(T)$ $(j=1, 2, \cdots)$ the s-numbers of T, that is, the eigenvalues of $(T^*T)^{1/2}$. Contrary we denote by $\lambda_j(A)$ $(j=1, 2, \cdots)$ the eigenvalues of an unbounded positive self-adjoint operator A with the compact resolvents which have been arranged in *increasing* order:

$$0<\lambda_1(A) \leq \lambda_2(A) \leq \cdots.$$

No confusion may occur.

First we consider the case when the range $R(T)$ is contained in the domain of definition $D(A)$ of the self-adjoint operator A with the compact resolvents.

LEMMA 2.1 (Pham The Lai [11]). *Let X be a Hilbert space. Let A be a positive self-adjoint operator in X with the compact resolvents. Let I be the inclusion mapping from $D(A)$ into X. Then I is compact and we have*

$$s_j(I) = \lambda_j(A)^{-1} \qquad (j=1, 2, \cdots). \tag{2.2}$$

LEMMA 2.2. *Let X be a Hilbert space. Let Λ be a positive self-adjoint operator in X with the compact resolvents. Let T be a bounded linear operator in X with $R(T)\subset D(\Lambda)$. Then T is compact and it follows that*

$$s_j(T)\leq\|\tilde{T}\|\lambda_j(\Lambda)^{-1} \qquad (j=1,2,\cdots)$$

where $\tilde{T}$ is the operator from X into $D(\Lambda)$ such that $\tilde{T}x=Tx$ for $x\in X$, and $\|\tilde{T}\|$ is the operator norm of $\tilde{T}$.

PROOF. From the closed graph theorem it follows that $\tilde{T}$ is a bounded operator. Let I be the inclusion mapping from $D(\Lambda)$ into X. Then we have

$$T=I\tilde{T}.$$

Since I is compact, T is compact. From the well known property of the s-numbers and Lemma 2.1 it follows that

$$s_j(T)\leq\|\tilde{T}\|s_j(I)\leq\|\tilde{T}\|\lambda_j(\Lambda)^{-1}$$

which is the desired formula. q.e.d.

The next lemma is a generalization of Lemma 2.2.

LEMMA 2.3. *Let X be a Hilbert space. Let Λ be a positive self-adjoint operator in X with the compact resolvents. Let φ be a measurable function from $\boldsymbol{R}_+$ into $\boldsymbol{R}_+$ which satisfies $\varphi(t)\to\infty$ as $t\to\infty$. Let T be a bounded linear operator in X with $R(T)\subset D(\varphi(\Lambda))$. Here $\varphi(\Lambda)$ is the self-adjoint operator defined by using the spectral resolution of Λ. Then T is compact and there exists $C>0$ such that*

$$s_j(T)\leq C\lambda_j(\varphi(\Lambda))^{-1} \qquad (j=1,2,\cdots). \tag{2.3}$$

If φ is increasing in the interval $[c_0,\infty)$ for some $c_0>0$ in addition, we have

$$s_j(T)\leq C\varphi(\lambda_j(\Lambda))^{-1} \qquad (j=1,2,\cdots). \tag{2.4}$$

PROOF. The operator Λ admits the representation

$$\Lambda u=\sum_{j=1}^{\infty}\lambda_j(\Lambda)(u,e_j)e_j \qquad \text{for} \quad u\in D(\Lambda),$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal system of eigenvectors of Λ. Then $\varphi(\Lambda)$ has the bounded inverse $\varphi(\Lambda)^{-1}$ which admits the representation

$$\varphi(\Lambda)^{-1}u=\sum_{j=1}^{\infty}\varphi(\lambda_j(\Lambda))^{-1}(u,e_j)e_j \qquad \text{for} \quad u\in D(\varphi(\Lambda)).$$

Since $\varphi(\lambda_j(\Lambda))^{-1}\to 0$ as $j\to\infty$, $\varphi(\Lambda)^{-1}$ is a compact operator. Applying Lemma 2.2 for $\varphi(\Lambda)$, we conclude that T is compact and that (2.3) holds.

Next we shall show the latter part of the lemma. We note that the eigenvalues of $\varphi(\Lambda)$ are $\varphi(\lambda_j(\Lambda))$ $(j=1, 2, \cdots)$ but that the sequence $\{\varphi(\lambda_j(\Lambda))\}$ may not increase as j grows. If φ is increasing in $[c_0, \infty)$, there exists a positive integer j_0 such that

$$\lambda_j(\varphi(\Lambda))=\varphi(\lambda_j(\Lambda)) \qquad \text{for} \quad j\geq j_0.$$

Combining this equality and (2.3), we get (2.4) for $j\geq j_0$. If we take $C>0$ large enough, (2.4) holds for any $j\geq 1$. q.e.d.

Next we consider the case when $R(T)$ is contained in the interpolation space with a function parameter of $L_2(\Omega)$ and $H^m(\Omega)$. To do so, we prepare several lemmas.

LEMMA 2.4. *Let Λ be a positive self-adjoint operator in a Hilbert space X. Let φ be a weight function on $\boldsymbol{R}_+$ with* $\operatorname{ind}\varphi>0$. *Then we have*

$$D_2^{\varphi}(\Lambda)=D(\varphi(\Lambda))$$

with the equivalent norms.

PROOF. Let m be a positive integer such that $0<\operatorname{ind}\varphi<m$. Let $\{E_t\}$ be the spectral resolution of Λ:

$$\Lambda=\int_0^{\infty}t\,dE_t.$$

Changing the order of integration and changing the variable, we have for $x\in X$,

$$\int_0^{\infty}\|\varphi(t)\Lambda^m(t+\Lambda)^{-m}x\|_X^2\frac{dt}{t}$$
$$=\int_0^{\infty}d_s\|E_sx\|_X^2\int_0^{\infty}(\varphi(t)s^m(t+s)^{-m})^2\frac{dt}{t}$$

$$=\int_0^\infty \varphi(s)^2 d_s\|E_s x\|_X^2 \int_0^\infty \left(\frac{\varphi(ts)}{\varphi(s)}\right)^2 (t+1)^{-2m}\frac{dt}{t}.$$

From Lemma 1.1 it follows that

$$\int_0^\infty \bar{\varphi}(t^{-1})^{-2}(t+1)^{-2m}\frac{dt}{t} \leq \int_0^\infty \left(\frac{\varphi(ts)}{\varphi(s)}\right)^2 (t+1)^{-2m}\frac{dt}{t}$$
$$\leq \int_0^\infty \bar{\varphi}(t)^2(t+1)^{-2m}\frac{dt}{t}.$$

Both of the first and the third integrand of the above inequalities are integrable by virtue of $0<\operatorname{ind}\varphi<m$ and Lemma 1.1. Taking the positivity of Λ into account, we conclude that $x\in D_2^\varphi(\Lambda)$ is equivalent to $x\in D(\varphi(\Lambda))$ and that the norms are equivalent. q.e.d.

Generally, for two sequences $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=1}^\infty$ we denote by

$$a_j \approx b_j \qquad \text{as} \quad j\to\infty,$$

if there exists $C>0$ and a positive integer j_0 such that

$$C^{-1}b_j \leq a_j \leq Cb_j \qquad (j\geq j_0).$$

LEMMA 2.5. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ with the restricted cone property. Let m be a positive integer. Then there exists a positive self-adjoint operator Λ_m in $L_2(\Omega)$ which satisfies*

$$D(\Lambda_m^j)=H^j(\Omega) \qquad (j=0,1,\cdots,m), \tag{2.5}$$

and

$$\lambda_j(\Lambda_m)\approx j^{1/n} \qquad \text{as} \quad j\to\infty. \tag{2.6}$$

PROOF. We consider the integro-differential sesquilinear form

$$a[u,v]=\int_\Omega \sum_{|\alpha|\leq m} D^\alpha u(x)\overline{D^\alpha v(x)}dx$$

on $H^m(\Omega)$. Let A be the self-adjoint operator associated with the variational triple $\{a, H^m(\Omega), L_2(\Omega)\}$. Then it is known that

$$D(A^{1/2})=H^m(\Omega) \tag{2.7}$$

(cf. Tanabe [12]) and that

$$\lambda_j(A) \approx j^{2m/n} \quad \text{as} \quad j \to \infty \tag{2.8}$$

(cf. Maruo and Tanabe [8]). Now let us put $\Lambda_m = A^{1/2m}$. Then (2.6) follows from (2.8). (2.5) is clear if $j=0$ or m. From Theorem 1.4, (2.7) and Theorem 1.7 it follows that for $j=1, 2, \cdots, m-1$,

$$\begin{aligned} D(\Lambda_m^j) = D(A^{j/2m}) &= [L_2(\Omega), D(A^{1/2})]_{j/m} \\ &= [L_2(\Omega), H^m(\Omega)]_{j/m} = H^j(\Omega) \end{aligned}$$

which is the desired formula. q.e.d.

LEMMA 2.6. *Let φ be a weight function on $\boldsymbol{R}_+$ with $0 < \operatorname{ind} \varphi < m$ for some integer m. Put $\phi(t) = \varphi(t^{1/m})^{-1}$. Let Λ_m be as defined in Lemma 2.5. Then we have*

$$(L_2(\Omega), H^m(\Omega))_{\phi,2} = D(\varphi(\Lambda_m^{-1})^{-1}).$$

PROOF. We have only to combine the results which have been stated. Let us put $\Lambda = \Lambda_m$ for simplicity. It follows that

$$\begin{aligned} &(L_2(\Omega), H^m(\Omega))_{\phi,2} = (L_2(\Omega), D(\Lambda^m))_{\phi,2} \qquad \text{(Lemma 2.5)} \\ &= S(2, \phi(t), L_2(\Omega); 2, t\phi(t), D(\Lambda^m)) \\ &= S(2, \phi(t^{-m}), L_2(\Omega); 2, t^{-m}\phi(t^{-m}), D(\Lambda^m)) \qquad \text{(Theorem 1.2)} \\ &= S(2, \varphi(t^{-1})^{-1}, L_2(\Omega); 2, t^{-m}\varphi(t^{-1})^{-1}, D(\Lambda^m)) \\ &= D_2^{\varphi(t^{-1})^{-1}}(\Lambda) \qquad \text{(Theorem 1.5)} \\ &= D(\varphi(\Lambda^{-1})^{-1}) \qquad \text{(Lemma 2.4)} \end{aligned}$$

which is the desired formula. q.e.d.

Now we are ready to give the main result of this section concerning the operator whose range is contained in the interpolation space with a function parameter of $L_2(\Omega)$ and $H^m(\Omega)$.

THEOREM 2.7. *Let φ be a weight function on $\boldsymbol{R}_+$ with $0 < \operatorname{ind} \varphi < m$ for some integer m. Put $\phi(t) = \varphi(t^{1/m})^{-1}$. Let T be a bounded linear operator in $L_2(\Omega)$ with the range $R(T) \subset (L_2(\Omega), H^m(\Omega))_{\phi,2}$. Let Λ_m be as defined in Lemma 2.5. Then T is compact and we have for some $C>0$,*

$$s_j(T) \leq C\lambda_j(\varphi(\Lambda_m^{-1})^{-1})^{-1} \qquad (j=1, 2, \cdots). \tag{2.9}$$

If φ is increasing in the interval $(0, c_0]$ in addition, we have

(2.10) $$s_j(T)\leq C\varphi(j^{-1/n}) \qquad (j=1,2,\cdots).$$

PROOF. From Lemma 2.6 we have

$$R(T)\subset D(\varphi(\Lambda_m^{-1})^{-1}).$$

Put $\alpha=\operatorname{ind}\varphi>0$. From Lemma 1.1 we have

$$\varphi(t^{-1})^{-1}\geq\varphi(1)^{-1}\tilde{\varphi}(t^{-1})^{-1}\geq C^{-1}\varphi(1)^{-1}t^{\alpha/2} \qquad \text{for} \quad t\geq 1,$$

from which it follows that $\varphi(t^{-1})^{-1}\to\infty$ as $t\to\infty$. Applying Lemma 2.3 for $\varphi(t^{-1})^{-1}$, we conclude that T is compact and that (2.9) holds.

If φ is increasing in $(0, c_0]$, Lemma 2.3 gives

(2.11) $$s_j(T)\leq C\varphi(\lambda_j(\Lambda_m)^{-1}).$$

From Lemma 2.5 there exists $c_1>0$ such that

(2.12) $$c_1^{-1}j^{1/n}\leq\lambda_j(\Lambda_m)\leq c_1 j^{1/n} \qquad (j=1,2,\cdots).$$

Let us fix a positive integer j_0 satisfying $c_1 j_0^{-1/n}\leq c_0$. From (2.11), (2.12) and Lemma 1.1 it follows that for $j\geq j_0$,

$$s_j(T)\leq C\varphi(\lambda_j(\Lambda_m)^{-1})\leq C\varphi(c_1 j^{-1/n})\leq C\tilde{\varphi}(c_1)\varphi(j^{-1/n}),$$

which implies (2.10) for $j\geq j_0$. If we take $C>0$ large enough, (2.10) is valid for any $j\geq 1$. This completes the proof. q.e.d.

REMARK. In Lemma 2.2, Lemma 2.3 and Theorem 2.7 if T is a self-adjoint operator in addition, $s_j(T)$ can be replaced with $|\lambda_j(T)|$.

3. Integral operators.

In order to apply Theorem 2.7 to the integral operator we first give the characterization of the Sobolev space $H^{\varphi}(\boldsymbol{R}^n)$ by the Fourier transform. This characterization is the extension of that of the usual Sobolev space. Let $\mathcal{S}=\mathcal{S}(\boldsymbol{R}^n)$ be the space of rapidly decreasing functions and let $\mathcal{S}'$ be the dual space of $\mathcal{S}$. The Fourier transform is defined by the formula

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=\int_{\boldsymbol{R}^n}e^{-ix\xi}f(x)dx \qquad \text{for} \quad f\in\mathcal{S}(\boldsymbol{R}^n).$$

THEOREM 3.1. *Let φ be a weight function on $\boldsymbol{R}_+$ with* $\operatorname{ind}\varphi>0$.

Then we have

$$H^{\varphi}(\boldsymbol{R}^n)=\{f\in\mathcal{S}'(\boldsymbol{R}^n);\ (1+\varphi(|\xi|^{-1})^{-1})\hat{f}(\xi)\in L_2(\boldsymbol{R}^n)\}.$$

PROOF. Let m be a positive integer such that $0<\operatorname{ind}\varphi<m$. Let $f\in L_2(\boldsymbol{R}^n)$. By Plancherel's formula we have

$$|f|^2_{H^{\varphi}(\boldsymbol{R}^n)}=(2\pi)^{-n}\int_{\boldsymbol{R}^n}w_{\varphi}(\xi)|\hat{f}(\xi)|^2d\xi,$$

where

$$w_{\varphi}(\xi)=\int_{\boldsymbol{R}^n}\frac{|e^{-iy\xi}-1|^{2m}}{\varphi(|y|)^2|y|^n}dy. \tag{3.1}$$

Let us divide the integral of (3.1) into two parts:

$$w_{\varphi}(\xi)=\int_{|y|>e^{-1}}+\int_{|y|<e^{-1}}=I_1(\xi)+I_2(\xi). \tag{3.2}$$

From Lemma 1.1 (ii) and $0<\operatorname{ind}\varphi$ it follows that

$$I_1(\xi)\leq 2^{2m}\int_{|y|>e^{-1}}\frac{dy}{\varphi(|y|)^2|y|^n}<\infty. \tag{3.3}$$

Changing the variable and denoting by y_1 the first component of y, we have

$$I_2(\xi)=\varphi(|\xi|^{-1})^{-2}\int_{|y|<e^{-1}|\xi|}\left(\frac{\varphi(|\xi|^{-1})}{\varphi(|\xi|^{-1}|y|)}\right)^2\frac{|e^{-iy_1}-1|^{2m}}{|y|^n}dy.$$

From Lemma 1.1 and $0<\operatorname{ind}\varphi<m$ it follows that

$$\int_{|y|<e^{-1}}\tilde{\varphi}(|y|)^{-2}\frac{|e^{-iy_1}-1|^{2m}}{|y|^n}dy\leq\varphi(|\xi|^{-1})^2I_2(\xi)\qquad\text{if}\quad |\xi|\geq 1, \tag{3.4}$$

and

$$\varphi(|\xi|^{-1})^2I_2(\xi)\leq\int_{\boldsymbol{R}^n}\tilde{\varphi}(|y|^{-1})^2\frac{|e^{-iy_1}-1|^{2m}}{|y|^n}dy<\infty\qquad\text{for any}\ \xi\in\boldsymbol{R}^n. \tag{3.5}$$

Noting $\varphi(|\xi|^{-1})^{-1}\to 0$ as $|\xi|\to 0$, and combining (3.2)-(3.5), we conclude that there exists $C>0$ such that for any $\xi\in\boldsymbol{R}^n$

$$C^{-1}(1+\varphi(|\xi|^{-1})^{-2})\leq 1+w_{\varphi}(\xi)\leq C(1+\varphi(|\xi|^{-1})^{-2}),$$

from which the theorem is easily verified. q.e.d.

We note that Theorem 3.1 shows that $H^{\varphi}(\boldsymbol{R}^n)$ is determined by the behavior of $\varphi(t)$ in the neighbourhood of $t=0$.

Now we consider the integral operator of the form

$$Tf(x)=\int_{\Omega} K(x-y)f(y)dy \tag{3.6}$$

where $K(x)$ is a locally integrable function on $\boldsymbol{R}^n$, and Ω is a bounded domain in $\boldsymbol{R}^n$. It is easily seen that the integral operator defined above is a bounded operator from $L_2(\Omega)$ into $L_2(\Omega)$. We investigate the range of T.

Generally we denote by

$$f(x)\approx g(x) \qquad \text{as} \quad |x|\to\infty$$

for f, g which are defined in $\{x;\, |x|>R\}$ for some $R>0$, if there exists $C>0$ and $R_0>R$ such that

$$C^{-1}g(x)\leq f(x)\leq Cg(x) \qquad \text{when} \quad |x|\geq R_0.$$

LEMMA 3.2. *Let φ be a weight function on $\boldsymbol{R}_+$ with $0<\operatorname{ind}\varphi<m$ for some positive integer m. Let $K(x)$ satisfy the following conditions.*

(i) $K\in L_{1,loc}(\boldsymbol{R}^n)\cap\mathcal{S}'$,

(ii) $\hat{K}(\xi)$ *is a measurable function in* $\{\xi;\, |\xi|>R\}$ *for some* $R>0$, *and* $\hat{K}(\xi)\approx\varphi(|\xi|^{-1})$ *as* $|\xi|\to\infty$.

Take a function $a\in C_0^{\infty}(\boldsymbol{R}^n)$ such that $a(x)\equiv 1$ on $\{x;\, |x|\leq\operatorname{diam}(\Omega)\}$, where $\operatorname{diam}(\Omega)$ *denotes the diameter of Ω. We put*

$$\tilde{T}f(x)=\int_{\Omega} a(x-y)K(x-y)f(y)dy$$

which is considered as an operator from $L_2(\Omega)$ into $L_2(\boldsymbol{R}^n)$. Then we have

$$\tilde{T}f(x)=Tf(x) \quad \text{for} \quad f\in L_2(\Omega) \quad \text{and} \quad x\in\Omega, \tag{3.7}$$

and

$$R(\tilde{T})\subset H^{\varphi}(\boldsymbol{R}^n). \tag{3.8}$$

PROOF. (3.7) is clear. Let $f\in C_0^{\infty}(\boldsymbol{R}^n)$. Using the formula

$$\mathcal{F}[g*h]=\hat{g}\hat{h} \qquad \text{for } g\in\mathcal{S} \text{ and } h\in\mathcal{S}'$$

where $g*h$ stands for the convolution (cf. [6]), we have

$$\mathcal{F}[\tilde{T}f](\xi)=(\hat{a}*\hat{K})(\xi)\hat{f}(\xi).$$

Take a function $b(\xi)\in C_0^\infty(\boldsymbol{R}^n)$ such that $b(\xi)\equiv 1$ in $\{|\xi|\le R\}$. We shall evaluate

$$\hat{a}*\hat{K}=\hat{a}*(b\hat{K})+\hat{a}*\{(1-b)\hat{K}\}.$$

Since $\hat{a}\in\mathcal{S}$ and $b\hat{K}\in\mathcal{E}'$ (the set of distributions of compact support), it follows that

$$\hat{a}*(b\hat{K})\in\mathcal{S}. \tag{3.9}$$

We put $L(\xi)=(1-b(\xi))\hat{K}(\xi)$ and note that

$$L(\xi)\approx\varphi(|\xi|^{-1}) \quad \text{as} \quad |\xi|\to\infty, \tag{3.10}$$

and

$$L(\xi)\equiv 0 \quad \text{in} \quad \{\xi;\ |\xi|\le R\}. \tag{3.11}$$

We have

$$\hat{a}*L(\xi)=L(\xi)\int\hat{a}(\eta)\frac{L(\xi-\eta)}{L(\xi)}d\eta.$$

If $|\xi-\eta|\ge R$, it follows from Lemma 1.1 and (3.10) that

$$\begin{aligned}\left|\frac{L(\xi-\eta)}{L(\xi)}\right| &\le C\frac{\varphi(|\xi-\eta|^{-1})}{\varphi(|\xi|^{-1})}\le C\tilde{\varphi}\left(\frac{|\xi|}{|\xi-\eta|}\right)\\ &\le C\left\{1+\left(\frac{|\xi|}{|\xi-\eta|}\right)^m\right\}\le C\left\{1+\left(1+\frac{|\eta|}{R}\right)^m\right\}.\end{aligned}$$

If $|\xi-\eta|\le R$, (3.11) gives

$$\left|\frac{L(\xi-\eta)}{L(\eta)}\right|=0.$$

Hence we have

$$\sup_\xi\left|\int\hat{a}(\eta)\frac{L(\xi-\eta)}{L(\xi)}d\eta\right|<\infty,$$

from which we get

$$|\hat{a}*L(\xi)|\leq C|L(\xi)|\leq C\varphi(|\xi|^{-1}). \tag{3.12}$$

By (3.9), (3.11) and (3.12) we obtain

$$|\hat{a}*\hat{K}(\xi)|\leq C\min\{1, \varphi(|\xi|^{-1})\},$$

from which it follows that

$$(1+\varphi(|\xi|^{-1})^{-1})|\mathcal{F}[\tilde{T}f](\xi)|\leq C|\hat{f}(\xi)|. \tag{3.13}$$

Noting that C in (3.13) is independent of $f\in C_0^\infty(\Omega)$ and that $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$, and applying Theorem 3.1 we get (3.8). q.e.d.

REMARK. It is not expected that $R(\tilde{T})$ is contained in the space smaller than $H^\varphi(\boldsymbol{R}^n)$. In fact, if we take the function $a(x)$ with $\hat{a}(\xi)\geq 0$, it follows that there exist $C>0$ and $R'>0$ such that

$$|\mathcal{F}[\tilde{T}f](\xi)|\geq C^{-1}\varphi(|\xi|^{-1})|\hat{f}(\xi)| \quad \text{when} \quad |\xi|>R', \tag{3.14}$$

which is proved as follows. From Fatou's lemma, (3.10) and Lemma 1.1 we have

$$\begin{aligned}\varliminf_{|\xi|\to\infty}\frac{\hat{a}*L(\xi)}{L(\xi)} &\geq\int \hat{a}(\eta)\varliminf_{|\xi|\to\infty}\frac{L(\xi-\eta)}{L(\xi)}d\eta \\ &\geq C^{-1}\int \hat{a}(\eta)\varliminf_{|\xi|\to\infty}\frac{\varphi(|\xi-\eta|^{-1})}{\varphi(|\xi|^{-1})}d\eta \\ &\geq C^{-1}\Big(\varlimsup_{t\to 1}\tilde{\varphi}(t)\Big)^{-1}\int\hat{a}(\eta)d\eta.\end{aligned} \tag{3.15}$$

It is known that $\varlimsup_{t\to 1}\tilde{\varphi}(t)<\infty$ (cf. [10]). Therefore (3.14) follows from (3.9) and (3.15).

Now we are ready to give the estimate of the eigenvalues of the integral operator.

THEOREM 3.3. *Let Ω be a bounded domain with the restricted cone property in $\boldsymbol{R}^n$. Let φ be a weight function on $\boldsymbol{R}_+$ with $0<\operatorname{ind}\varphi<m$ for some positive integer m. Let $K(x)$ satisfy the following conditions.*

(i) $K\in L_{1,loc}(\boldsymbol{R}^n)\cap\mathcal{S}'$,

(ii) *$\hat{K}(\xi)$ is a measurable function in $\{\xi; |\xi|>R\}$ for some $R>0$, and $\hat{K}(\xi)\approx\varphi(|\xi|^{-1})$ as $|\xi|\to\infty$.*

Then the integral operator

$$Tf(x)=\int_\Omega K(x-y)f(y)dy$$

is a compact operator in $L_2(\Omega)$ *and we have*

$$s_j(T)\leq C\lambda_j(\varphi(\Lambda_m^{-1})^{-1})^{-1} \qquad (j=1,2,3,\cdots),$$

where Λ_m *is as defined in Lemma* 2.5. *In particular if* φ *is increasing in the interval* $(0,c_0]$ *for some* c_0 *in addition, we have*

$$s_j(T)\leq C\varphi(j^{-1/n}) \qquad (j=1,2,3,\cdots).$$

PROOF. Denoting by R the restriction from $L_2(\boldsymbol{R}^n)$ into $L_2(\Omega)$, we can write

$$T=R\tilde{T}$$

where $\tilde{T}$ is the operator defined in Lemma 3.2. From Lemma 3.2, the closed graph theorem and Theorem 1.6, the operator $\tilde{T}$ can be considered as a bounded linear operator from $L_2(\Omega)$ into $H^{\varphi}(\boldsymbol{R}^n)=(L_2(\boldsymbol{R}^n), H^m(\boldsymbol{R}^n))_{\phi,2}$ where $\phi(t)=\varphi(t^{1/m})^{-1}$. R is a bounded linear operator from $L_2(\boldsymbol{R}^n)$ into $L_2(\Omega)$ and also a bounded linear operator from $H^m(\boldsymbol{R}^n)$ into $H^m(\Omega)$. Hence Theorem 1.3 shows that R is a bounded linear operator from $H^{\varphi}(\boldsymbol{R}^n)=(L_2(\boldsymbol{R}^n), H^m(\boldsymbol{R}^n))_{\phi,2}$ into $(L_2(\Omega), H^m(\Omega))_{\phi,2}$. Therefore it follows that T is considered as a bounded linear operator from $L_2(\Omega)$ into $(L_2(\Omega), H^m(\Omega))_{\phi,2}$. Applying Theorem 2.7 we get the theorem. q.e.d.

Further, we consider the integral kernel which is a little more complicated than that of Theorem 3.3.

THEOREM 3.4. *Let* Ω, φ, m *and* $K(x)$ *be as in Theorem* 3.3. *In addition, let* $K(x)$ *satisfy*

(iii) *there exists a positive constant* $N>0$ *such that* $x^{\alpha}K(x)\in C^m(\boldsymbol{R}^n)$ *for* α *with* $|\alpha|=N$.

Let $L(x,y)\in C^{\infty}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$. *We define the integral operator* T *by*

$$Tf(x)=\int_{\Omega}K(x-y)L(x,y)f(y)dy.$$

Then we obtain the same conclusion as in Theorem 3.3.

PROOF. We use Taylor's expansion formula:

$$\begin{aligned} L(x,y)= &\sum_{|\alpha|<N}\frac{(x-y)^{\alpha}}{\alpha!}L^{(\alpha,0)}(y,y) \\ &+N\sum_{|\alpha|=N}\frac{(x-y)^{\alpha}}{\alpha!}\int_0^1(1-\theta)^{N-1}L^{(\alpha,0)}(y+\theta(x-y),y)d\theta \end{aligned}$$

where $L^{(\alpha,0)}(y,y)=\partial_x^\alpha L(x,y)|_{x=y}$. We can write

$$K(x-y)L(x,y)=\sum_{\alpha,\beta} C_{\alpha\beta}x^\alpha K(x-y)y^\beta L_{\alpha\beta}(y)+\sum_{|\alpha|=N} C_\alpha(x-y)^\alpha K(x-y)L_\alpha(x,y)$$

where $C_{\alpha\beta}$ and C_α are constants, and $L_{\alpha\beta}(y)$ and $L_\alpha(x,y)$ are C^∞-functions. The mapping $f(x)\mapsto x^\beta L_{\alpha\beta}(x)f(x)$ is a bounded operator in $L_2(\Omega)$. From the proof of Theorem 3.3 the mapping $f(x)\mapsto\int_\Omega K(x-y)f(y)dy$ is a bounded operator from $L_2(\Omega)$ into $(L_2(\Omega),H^m(\Omega))_{\phi,2}$ where $\phi(t)=\varphi(t^{1/m})^{-1}$. From Theorem 1.3 it is easily seen that the mapping $f(x)\mapsto x^\alpha f(x)$ is a bounded operator in $(L_2(\Omega),H^m(\Omega))_{\phi,2}$. Hence the integral operator with the kernel $x^\alpha K(x-y)y^\beta L_{\alpha\beta}(y)$ has the range contained in $(L_2(\Omega),H^m(\Omega))_{\phi,2}$.

From condition (iii) on $K(x)$ it follows that the range of the integral operator with the kernel $(x-y)^\alpha K(x-y)L_\alpha(x,y)$ is contained in $H^m(\Omega)$. Therefore we have

$$R(T)\subset(L_2(\Omega),H^m(\Omega))_{\phi,2}.$$

Then the theorem follows from Theorem 2.7. q.e.d.

4. An example.

In this section we shall give an example. Let us consider the integral operator T defined in Theorem 3.3 or Theorem 3.4 with the function $K(x)$ given by

$$K(x)=|x|^\lambda(\log|x|)^\mu$$

where $\lambda>-n$ and μ is a non-negative integer.

It is easily seen that $K\in L_{1,loc}(\boldsymbol{R}^n)\cap\mathcal{S}'$. To find the Fourier transform of $r^\lambda(\log r)^\mu$ $(r=|x|)$, we begin with the property of the function r^λ. For $\varphi\in\mathcal{S}(\boldsymbol{R}^n)$ we denote by $S_\varphi(r)$ the mean of the values $\varphi(x)$ on the sphere with radius r:

$$S_\varphi(r)=\frac{1}{\sigma_n r^{n-1}}\int_{|x|=r}\varphi(x)dS_x=\frac{1}{\sigma_n}\int_{|\omega|=1}\varphi(r\omega)d\omega$$

where $\sigma_n=\pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit sphere in $\boldsymbol{R}^n$. r^λ is considered as a $\mathcal{S}'(\boldsymbol{R}^n)$-valued holomorphic function of λ in the half plane $\operatorname{Re}\lambda>-n$ by the formula

$$\langle r^\lambda, \varphi\rangle = \int_{R^n} r^\lambda \varphi(x)dx = \sigma_n \int_0^\infty r^{\lambda+n-1} S_\varphi(r)dr \qquad \text{if} \quad \operatorname{Re}\lambda > -n,$$

where $\langle\,,\rangle$ stands for the duality between $\mathcal{S}$ and $\mathcal{S}'$. Moreover r^λ can be extended to the meromorphic function in the whole plane $\boldsymbol{C}$ by the formula

$$\begin{aligned}\langle r^\lambda, \varphi\rangle &= \sigma_n \int_1^\infty r^{\lambda+n-1} S_\varphi(r)dr \\ &+ \sigma_n \int_0^1 r^{\lambda+n-1}\Big\{S_\varphi(r) - \sum_{k=0}^{m-1} \frac{1}{k!} S_\varphi^{(k)}(0) r^k\Big\}dr \\ &+ \sigma_n \sum_{k=1}^{m} \frac{S_\varphi^{(k-1)}(0)}{(k-1)!(\lambda+n+k-1)}\end{aligned} \tag{4.1}$$

if $\operatorname{Re}\lambda > -n-m$ where m is any positive integer. Simple calculation shows that

$$S_\varphi^{(k)}(0) = 0 \qquad \text{if } k \text{ is odd,}$$

from which it follows that r^λ has the poles of order one only at $\lambda = -n-2k$ $(k=0, 1, 2, \cdots)$.

It is known that

$$\mathcal{F}[r^\lambda] = c(\lambda) r^{-\lambda-n} \qquad \text{if} \quad \operatorname{Re}\lambda \neq -n-2k \qquad (k=0, 1, 2, \cdots), \tag{4.2}$$

where

$$c(\lambda) = \frac{\pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}.$$

We note that $c(\lambda)$ has the poles of order one at $\lambda = -n-2k$ $(k = 0, 1, 2, \cdots)$ and that $c(\lambda)$ has the zeroes of order one at $\lambda = 2k$ $(k = 0, 1, 2, \cdots)$. Although $r^{-\lambda-n}$ has the singularities at $\lambda = 2k$ $(k=0, 1, 2, \cdots)$, $c(\lambda) r^{-n-\lambda}$ is holomorphic in $\operatorname{Re}\lambda > -n$. From (4.1) and (4.2) it follows that if the support of φ is contained in $\boldsymbol{R}^n \backslash \{0\}$, we have

$$\begin{aligned}\langle \mathcal{F}[r^\lambda], \varphi\rangle &= \langle c(\lambda) r^{-\lambda-n}, \varphi\rangle \\ &= \int_{R^n} c(\lambda) r^{-\lambda-n} \varphi(x)dx.\end{aligned} \tag{4.3}$$

Differentiating both sides of (4.3) μ times, we get

$$\langle \mathcal{F}[r^{\lambda}(\log r)^{\mu}], \varphi\rangle = \int_{R^n} \sum_{k=0}^{\mu} \binom{\mu}{k} (\log r^{-1})^{\mu-k} r^{-\lambda-n} c^{(k)}(\lambda)\varphi(x)dx$$

for λ with $\operatorname{Re}\lambda > -n$. Noting the zeroes of $c(\lambda)$, we have as $r\to\infty$

(4.4) $$\mathcal{F}[r^{\lambda}(\log r)^{\mu}] \approx c(\lambda) r^{-\lambda-n}(\log r^{-1})^{\mu}$$
$$\text{when } \lambda > -n,\ \lambda \neq 2k\ (k=0,1,2,\cdots),\ \mu=0,1,2,\cdots,$$

(4.5) $$\mathcal{F}[r^{\lambda}(\log r)^{\mu}] \approx \mu c'(\lambda) r^{-\lambda-n}(\log r^{-1})^{\mu-1}$$
$$\text{when } \lambda=2k\ (k=0,1,2,\cdots),\ \mu=1,2,3,\cdots.$$

Now we put

$$\varphi(t)=\varphi(t;\lambda,\mu)=\begin{cases} t^{\lambda}(\log t^{-1})^{\mu} & (0<t\leq e^{-1}) \\ t^{\lambda} & (e^{-1}\leq t), \end{cases}$$

which is an increasing function in $\boldsymbol{R}_+$. Simple calculation shows that

$$\tilde{\varphi}(t)=\begin{cases} t^{\lambda}(1+\log t^{-1})^{\mu} & (0<t\leq 1) \\ t^{\lambda} & (1\leq t), \end{cases}$$

from which we get

$$\operatorname{ind}\tilde{\varphi}=\lambda.$$

From (4.4) and (4.5) we have

$$|\mathcal{F}[r^{\lambda}(\log r)^{\mu}]| \approx \varphi(|\xi|^{-1};\lambda+n,\mu) \quad \text{as} \quad |\xi|\to\infty$$
$$\text{when } \lambda>-n,\ \lambda\neq 2k,\ \mu=0,1,2,\cdots,$$

$$|\mathcal{F}[r^{\lambda}(\log r)^{\mu}]| \approx \varphi(|\xi|^{-1};\lambda+n,\mu-1) \quad \text{as} \quad |\xi|\to\infty$$
$$\text{when } \lambda=2k\ (k=0,1,2,\cdots),\ \mu=1,2,3,\cdots.$$

Applying Theorem 3.3 or Theorem 3.4 with the weight function $\varphi(t;\lambda+n,\mu)$ or $\varphi(t;\lambda+n,\mu-1)$, we get the following estimates.
(i) When $\lambda>-n$, $\lambda\neq 2k$ $(k=0,1,2,\cdots)$ and $\mu=0,1,2,\cdots$, we have

$$|\lambda_j(T)| \leq Cj^{-(\lambda+n)/n}(\log j)^{\mu}.$$

(ii) When $\lambda=2k$ $(k=0,1,2,\cdots)$ and $\mu=1,2,3,\cdots$, we have

$$|\lambda_j(T)| \leq Cj^{-(\lambda+n)/n}(\log j)^{\mu-1}.$$

For sake of completeness we mention the case of $\lambda=2k$ $(k=$

$0, 1, 2, \cdots)$ and $\mu=0$. In this case it is clear that T is an operator of finite rank. Hence T has only finite eigenvalues different from zero.

REMARK. If we replace $\varphi(t; \lambda, \mu)$ with $t^{\lambda}(|\log t|+1)^{\mu}$, we get the same result.

References

[1] Agmon, S., Lectures on Elliptic Boundary Value Problems, Van Nostrand, Princeton, 1965.

[2] Bergh, J. and J. Löfström, Interpolation Spaces, An Introduction, Grundlehren Math. Wiss. vol. 223, Springer, Berlin-Heidelberg-New York, 1976.

[3] Calderón, A. P., Intermediate spaces and interpolation, the complex method, Studia Math. **24** (1964), 113–190.

[4] Gohberg, I. C. and M. G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. Math. Monographs, Amer. Math. Soc., Providence, R. I., 1969.

[5] Kac, M., Distribution of eigenvalues of certain integral operators, Michigan Math. J. **3** (1955–1956), 141–148.

[6] Kumano-go, H., Pseudo-differential Operators, MIT, Massachusetts, 1980.

[7] Lions, J. L. and J. Peetre, Sur une classe d'espaces d'interpolation, Inst. Hautes Études Sci. Publ. Math. **19** (1964), 5–68.

[8] Maruo, K. and H. Tanabe, On the asymptotic distribution of eigenvalues of operators associated with strongly elliptic sesquilinear forms, Osaka J. Math. **8** (1971), 323–345.

[9] Merucci, C., Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces, Proc. of Interpolation Spaces and Allied Topics in Analysis, Lund 1983, (eds. M. Cwikel and J. Peetre), Lecture Notes in Math. vol. 1070, Springer, New York-Berlin, 1984, 183–201.

[10] Muramatu, T., Theory of Interpolation Spaces and Linear Operators, (in Japanese), Kinokuniya Shoten, Tokyo, 1985.

[11] Pham The Lai, Noyaux d'Agmon, Seminaire Jean Leray, College de France, Paris, 1973.

[12] Tanabe, H., Equations of Evolution, Monographs Stud. Math. vol. 6, Pitman, London, 1979.

[13] Triebel, H., Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.

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