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On polarized Calabi-Yau 3-folds

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Introduction.

In this paper, by a Calabi-Yau 3-fold, we mean a 3-dimensional non-singular projective variety whose canonical bundle and irregularity are both trivial. This is a 3-dimensional analogue of a $K3$ surface. In what follows, the ground field is assumed to be the complex number field $\boldsymbol{C}$. Recently, many mathematicians begin to study Calabi-Yau 3-folds from various viewpoints (cf. [W]). In this paper, we will study Calabi-Yau 3-folds as polarized manifolds. By definition, a pair (X, L) consisting of a non-singular projective variety X and an ample line bundle L on it is called a polarized manifold. A polarized Calabi-Yau 3-fold is in some sense a 3-dimensional analogue of a canonically polarized surface (S, K_S). For a given polarized manifold (X, L), the following questions naturally arise:

QUESTION (B). When is $\Phi_{|nL|}$ (:=the rational map defined by $|nL|$) birational?

QUESTION (F). When is nL free?

QUESTION (E). When is $\Phi_{|nL|}$ an embedding?

It is interesting to estimate n in question from below by an explicit value (cf. [Ka3], [Re1]). Such estimates were found for a polarized $K3$ surface ([SD]), a canonically polarized surface ([Bo]) and a canonically polarized 3-fold ([Be1], [Be2], [Ma]), but there are no such estimates for a polarized Calabi-Yau 3-fold (cf. the following table).

The first purpose of this paper is to give such estimates for a polarized Calabi-Yau 3-fold. The result is:

MAIN THEOREM I (cf. Th. (1.1), (2.1), Cor. (3.3), (3.5)). *Let (X, L) be a polarized Calabi-Yau 3-fold. Then,*
(1) $\Phi_{|nL|}$ *is birational for all* $n \geq 5$,
(2) nL *is free for all* $n \geq 20$,

(3)(a) *nL is very ample for all $n \geq 60$,*
(b) *nL is simply generated (in particular, very ample) for all $n \geq 2$ if L is free and $\Phi_{|L|}$ is birational.*
Moreover, the estimates in (1), (3)(b) *are best possible.*

Table.

Answer to	polarized $K3$ surface	canonically polarized surface	canonically polarized 3-fold
(B)	$*n \geq 3$ ([SD])	$*n \geq 5$ ([Bo])	$n \geq 7$ ([Ma], [Be2])
(F)	$*n \geq 2$ (*ibid.*)	$*n \geq 4$ (*ibid.*)	$n \geq 34$ ([Be1])
(E)	$*n \geq 3$ (*ibid.*)	$*n \geq 5$ (*ibid.*)	$n \geq 4 \cdot 34 \cdot 33$ (*ibid.*)

(Here, the symbol * means that the estimate is best possible.)

The author was very much inspired by the previous works [Be1, 2], [Ma] for proof. In (1.1) and (3.7), we construct examples which indicate that the estimates in (1) and (3)(b) are best possible. We notice that for a polarized $K3$ surface (X, L), nL is simply generated for all $n \geq 1$ under the assumption of (3)(b) (cf. [SD]).

In this paper, we put not a little stress on a weighted complete intersection (WCI) in a weighted projective space (WPS). This is fairly natural because any polarized manifold (X, L) is regarded as a polarized subvariety of a suitable WPS via the graded ring $R = \bigoplus H^0(X, nL)$ and WCI is a simple one among such subvarieties. In this viewpoint, we determine all the polarized Calabi-Yau 3-folds arising as general WCI (cf. §4) under the assumption $h^0(L) \geq 2$. Moreover, we consider some converse problems via the theory of Δ-genus ([Fu1]). This is the second purpose of this paper. The result is:

MAIN THEOREM II (cf. Th. (4.1), (5.1)). (1) *Any polarized Calabi-Yau 3-fold (X, L) with $h^0(L) \geq 2$ arising as a general WCI is one of the following*:

[1] $(10) \subset \boldsymbol{P}(1^3, 2, 5)$ [2] $(8) \subset \boldsymbol{P}(1^4, 4)$ [3] $(4, 6) \subset \boldsymbol{P}(1^3, 2^2, 3)$
[4] $(6) \subset \boldsymbol{P}(1^4, 2)$ [5] $(4, 4) \subset \boldsymbol{P}(1^4, 2^2)$ [6] $(3, 4) \subset \boldsymbol{P}(1^5, 2)$
[7] $(2, 6) \subset \boldsymbol{P}(1^5, 3)$ [8] $(2, 2, 2, 2) \subset \boldsymbol{P}^7$ [9] $(2, 2, 3) \subset \boldsymbol{P}^6$
[10] $(2, 4) \subset \boldsymbol{P}^5$ [11] $(3, 3) \subset \boldsymbol{P}^5$ [12] $(5) \subset \boldsymbol{P}^4$

[13] $(6, 6)\subset \boldsymbol{P}(1^2, 2^2, 3^2)$. (*As for notation, see* § 4.)
(2) *Any polarized Calabi-Yau* 3*-fold with* Δ*-genus* ≤ 2 *is essentially the same as one of the above. In particular, these are WCI of codimension* ≤ 2 *in WPS.* (*For more details, see* § 5.)

These 13 examples not only offer simple examples of graded ring structures of polarized Calabi-Yau 3-folds, but also play important roles as inevitable examples in the proof of Main Theorem I (cf. (0.13), (1.1)(3), (1.4), and (3.7)). The result (2) is a supplement to a series of splendid works of T. Fujita and E. Horikawa (cf. [Fu4, 5, 6, 8, 9], [Ho1, 2, 3]) from a viewpoint of graded ring structure. We notice that (X, L) is not necessarily WCI if $\Delta\geq 3$.

The outline of the proof is as follows. As for the birationality of $\Phi_{|nL|}$, we treat the following 3 cases separately: $\dim \Phi_{|L|}(X)\geq 2$, $\dim \Phi_{|L|}=1$, $\dim \Phi_{|L|}=0$. When $\dim \Phi_{|L|}(X)\geq 2$, we first take an embedded resolution of a general member of the movable part of $|L|$. Next, we reduce our problem to the birationality of an adjoint map of the resolved surface T. In this step, Lemma (1.3) in § 1, which is a clarification of the idea used in [Be2], [Ma], is essential. Finally, we apply to T Reider's criterion on the birationality of an adjoint map of a non-singular surface ([Rdr]). We cannot reach the sharpest bound in Main Theorem I (1) if we start from a resolution of indeterminacy of $|L|$ nor if we make use of more classical Bombieri's method in stead of Reider's one (cf. [Be2], [Ma]). When $\dim \Phi_{|L|}(X)\leq 1$, the method above does not work well if we take a resolution of the member of $|L|$. We first study a polarized Calabi-Yau 3-fold (X, L) such that $\dim \Phi_{|nL|}(X)=1$ for some n more closely. As a result, we see that $n=1$ and $|L|=(\dim |L|)\cdot|S|+B$, where any general member of $|S|$ is irreducible reduced (cf. (0.4)). By this, when $\dim \Phi_{|L|}(X)=1$ (resp. $\dim \Phi_{|L|}(X)=0$), we take an embedded resolution of a general member of $|S|$ (resp. $|2L|$) in stead of $|L|$ and apply the same method as above to the resolved surface. In the course of the proof, some polarized Calabi-Yau 3-folds of type WCI arise as exceptional cases (cf. (1.4)). The analysis of these are also important to complete the proof.

As for freeness of nL, we make use of the Benveniste's version ([Be1]) of Kawamata's technique ([Ka2]) directly. First, we take a resolution of indeterminacy of $|mL|$, say $\pi: Y\longrightarrow X$, and consider the two divisors: the fixed part of $|\pi^*mL|$ and K_Y. These are written as $\sum u_iE_i$, $\sum \rho_iE_i$ respectively, where $\sum E_i$ is the simple normal crossing divisor arising from the base locus of $|mL|$. The next is crucial for the proof.

KEY FORMULA (cf. Step 7 in §2). If $m\geq 10$, then $\frac{u_i}{\rho_i+1}<1$ for $\rho_i=0, 1$.

By this formula, any general member S_m of $|mL|$ ($m\geq 10$) is irreducible normal so that we can apply to S_m the freeness criterion of an adjoint linear system on a normal Gorenstein surface ([Be1, Th. 1-0]) and complete the proof (cf. Steps 8-10). We prove the key formula by selecting a single component E_i of maximal modified multiplicity $\frac{u_i+\varepsilon}{\rho_i+1}$ and making use of Benveniste's lemma (2.2) (cf. Steps 2, 7). The method of the proof here is almost identical to [Be1] for a canonically polarized 3-fold (X, K_X) except that a little more careful treatment of K_Y is needed (cf. the end of Step 2). The author does not have any good idea to sharpen the estimate.

We will explain very ampleness and simply generatedness of nL. We say that nL is *birationally very ample* if nL is free and $\Phi_{|nL|}$ is birational. Very ampleness is closely related to birationally very ampleness. In fact, we prove:

THEOREM III (cf. Th. (3.1)). *Let (X, L) be a polarized Calabi-Yau 3-fold such that mL is birationally very ample for all $m\geq f$. Then,* (1) *nfL is simply generated for all $n\geq 2$,* (2) *nL is very ample for all $n\geq 3f$.*

From this and Theorem I (1), (2), Theorem I (3) follows. The idea of the proof of Theorem III is as follows. First we take a curve C obtained as general twice cutting by $|fL|$ and examine the graded ring of $(C, L|_C)$ by making use of vanishing theorem and classical Castelnuovo's theorem (cf. (3.9), [ACGH], [Fu6]). Next we recover some information of the graded ring $R=\bigoplus R_m$ of (X, L) from the one of $(C, L|_C)$ by the ladder method. Theorem III is a refinement of the famous m-regularity theorem due to Mumford ([Mu]).

Main Theorem II is also proved by the same principle (the ladder method) as Theorem III. In addition to this principle, we make use of Reid's result [Rd2] on a half canonical divisor on an irreducible Gorenstein curve (cf. [Rd2], (4.3) in §4). The curve C obtained as general twice cutting by $|L|$ is actually irreducible Gorenstein and $L|_C$ is a half canonical bundle of C (i.e., $2L|_C=K_C$) in both cases. Thus we can apply [Rd2] to $(C, L|_C)$ so that the degree of both generators and relations of the graded ring R of (X, L) is bounded. By this, we can reduce our problem to some combinatorial calculation and get the results.

We will summarize the context of this paper briefly. Section 0 is a preliminary section. After recalling the Riemann-Roch theorem ((0.2)) and basic restriction maps ((0.3)), we study a polarized Calabi-Yau 3-fold (X, L) such that $\dim \Phi_{|nL|}(X)=1$ ((0.4)). We get $n=1$ as mentioned above. As for the dimension $N:=\dim|L|$ of the ambient projective space, two different phenomena arise: if $|L|$ has no fixed components, $N=1$; otherwise, N becomes arbitrarily large. In (0.13) and (0.14) we demonstrate examples of these two. Sections 1, 2, 3 are devoted to the proof of Main Theorem I. We prove Main Theorem II in Sections 4 and 5.

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§0. Preliminaries.

Throughout this paper, we assume that the ground field is the complex number field $\boldsymbol{C}$.

DEFINITION (0.1). (1) A non-singular projective 3-fold X is called *a Calabi-Yau* 3-*fold* if it's canonical bundle K_X is 0 and it's irregularity $q:=h^1(\mathcal{O}_X)$ is 0.
(2) A pair (X, L) consisting of a Calabi-Yau 3-fold X and an ample line bundle L on it is called *a polarized Calabi-Yau* 3-*fold*.

The following lemma is an easy consequence of Riemann-Roch, Serre duality, Kodaira vanishing, and the fact that $L.c_2(X)\geq 0$ ([Mi]).

LEMMA (0.2). *Let* (X, L) *be a polarized Calabi-Yau* 3-*fold. Then,*
(1) *for all* $n\geq 1$ *and* $i\geq 1$,

$$h^0(X, nL)=\frac{n^3L^3}{6}+\frac{nL.c_2(X)}{12}=\frac{n^3-n}{6}L^3+nh^0(X, L)\geq 1,\ h^i(X, nL)=0;$$

(2) *if* $i=1, 2$, *then* $h^i(X, mL)=0$ *for all* $m\in \boldsymbol{Z}$. □

LEMMA (0.3). *Let (X, L) be a polarized Calabi-Yau 3-fold. Let $S \in |nL|$ (resp. $C \in |nL|_S|$) be a surface (resp. a curve). Then, the natural restriction maps $r_S : H^0(X, mL) \longrightarrow H^0(S, mL|_S)$, $r_C : H^0(S, mL|_S) \longrightarrow H^0(C, mL|_C)$ are surjective for all $m \in \boldsymbol{Z}$.*

PROOF. By the exact sequence $0 \longrightarrow (m-n)L \longrightarrow mL \xrightarrow{r_S} mL|_S \longrightarrow 0$, and (0.2)(2), $r_S : H^0(X, mL) \longrightarrow H^0(S, mL|_S)$ is surjective and $H^1(S, mL|_S) = 0$ for all $m \in \boldsymbol{Z}$. Hence by the exact sequence $0 \longrightarrow (m-n)L|_S \longrightarrow mL|_S \xrightarrow{r_C} mL|_C \longrightarrow 0$, the restriction map r_C is also surjective. □

PROPOSITION (0.4). *Let (X, L) be a polarized Calabi-Yau 3-fold. Assume that $\dim \Phi_{|nL|}(X) = 1$ for some $n > 0$. Then,*
(1) $n = 1$;
(2) *$|L| = N|S| + B$ where B is the fixed part of $|L|$, $N = \dim|L|$, $\dim|S| = 1$ and any general member of $|S|$ is irreducible reduced;*
(3) *$\mathrm{Im}\,\Phi_{|L|}(X)$ is a rational normal curve, i.e., a non-singular rational curve embedded into $\boldsymbol{P}^N$ by the complete linear system $|\mathcal{O}_{P^1}(N)|$.*

PROOF. First we prove:

CLAIM (0.5). $|nL| = N|S| + B$ where B is the fixed part of $|nL|$, $N = \dim|nL|$, $\dim|S| = 1$ and any general member of $|S|$ is irreducible reduced. Moreover $\mathrm{Im}\,\Phi_{|nL|}(X)$ is a rational normal curve.

PROOF. Put $|nL| = |M| + B$ where B is the fixed part of $|nL|$. Let $\pi : \tilde{X} \longrightarrow X$ be a resolution of the indeterminacy of $\Phi := \Phi_{|nL|} = \Phi_{|M|}$ obtained by a successive blow up along a non-singular center in the base locus (of codim≥ 2):

$$\begin{array}{ccl} X & \overset{\Phi}{\dashrightarrow} & W := \mathrm{Im}\,\Phi_{|nL|}(X) \subset \boldsymbol{P}^N \\ \pi \uparrow & \overset{\Phi_{|F|}}{\nearrow} & \uparrow \rho \\ \tilde{X} & \xrightarrow[\varphi]{} & V := \text{the Stein factorization of } \Phi_{|F|}, \end{array}$$

where $|\pi^* nL| = |F| + \tilde{B}$, $|F|$ is free and $\tilde{B}$ is the fixed part of $|\pi^* nL|$. Since $\dim \mathrm{Alb}(\tilde{X}) = h^1(\mathcal{O}_{\tilde{X}}) = h^1(\mathcal{O}_X) = 0$, V is a smooth rational curve. Let T be a general fiber of φ, i.e., $T \in |\varphi^* \mathcal{O}_V(1)|$. Then $F = \mathcal{O}_{\tilde{X}}(abT)$ and $|F| = ab|T|$ where $a :=$ degree of W and $b :=$ degree of ρ. Then, by $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mT)) = h^0(V, \mathcal{O}_V(m)) = m+1$ for $m \geq 0$ and our choice of π, we get:

(0.6) $N=ab$;
(0.7) codim $W+1=N=ab\geq a=\deg W$;
(0.8) $|\pi^{*}nL|=N|T|+\tilde{B}$, i.e., $|nL|=N|S|+B$, where $S=\pi_{*}T$.

On the other hand, since codim $W+1\leq\deg W$ because W is not contained in any hyperplane in $\boldsymbol{P}^{N}$, the inequality in (0.7) must be an equality, i.e., $a=N$, $b=1$. Thus W is a rational normal curve. □

In order to complete our proof, it is enough to show:

CLAIM (0.9). $n=1$.

PROOF. By (0.8), $nL\sim NS+B$. We treat the following two cases separately: Case 1. $B=\emptyset$, Case 2. $B\neq\emptyset$.

Case 1. Since S is ample in this case, the following equality holds by (0.2)(1):

$$N+1=h^{0}(nL)=h^{0}(\mathcal{O}_{X}(NS))=\frac{N^{3}-N}{6}S^{3}+Nh^{0}(\mathcal{O}_{X}(S))\geq 2N.$$

Thus $N=1$. So by (0.2)(1) again, we get:
$2=h^{0}(nL)=\frac{n^{3}-n}{6}L^{3}+nh^{0}(L)\geq\frac{n^{3}-n}{6}+n=\frac{n^{3}+5n}{6}$. Hence $n=1$.

Case 2. Multiplying $nL\sim NS+B$ by L^{2}, we have:
$nL^{3}=NSL^{2}+BL^{2}\geq N+1=h^{0}(nL)=\frac{n^{3}-n}{6}L^{3}+nh^{0}(L)$, i.e., $6L^{3}\geq(n^{2}-1)L^{3}+6h^{0}(L)\geq(n^{2}-1)L^{3}$. Hence $n=1$ or 2. Assume that $n=2$. Then we have:

(0.10). $2L\sim NS+B$ and $N=h^{0}(2L)-1=d+2h-1$, where we put $d:=L^{3}$ and $h:=h^{0}(L)$.

CLAIM (0.11). $S.L^{2}=1$.

PROOF. Assume that $S.L^{2}\geq 2$. Then multiplying (0.10) by L^{2}, we get: $2d\geq 2N+BL^{2}>2(d+2h-1)\geq 2d$. But this is absurd. □

CLAIM (0.12). $S^{2}.L=0$, especially $|S|$ is free.

PROOF. By (0.10), we have:
$4d=(2L)^{2}.L=(NS+B)^{2}.L=N^{2}S^{2}.L+NS.B.L+2B.L^{2}$. Assume $S^{2}.L>0$. Note that $S.B.L>0$ because $NS+B\in|2L|$ is connected by the ampleness

of L. Hence we get the next from the above equality: $4d \geq N^2+N+2=(d+2h-1)^2+d+2h-1+2>4d$. But this is impossible. $\square$

By (0.12), any general member S of $|S|$ is smooth (irreducible) and $K_S \sim \mathcal{O}_S$. Then by Riemann-Roch and (0.11), $\chi(L|_S)-\chi(\mathcal{O}_S)=\frac{(L|_S)^2}{2}=\frac{1}{2}$, which is obviously absurd. Hence $n=1$. Q.E.D.

REMARK-EXAMPLE (0.13). Let (X, L) be same as in (0.4). As was seen in the proof of (0.4) (Case 1), if $|L|$ has no fixed components, then $N:=\dim|L|=1$, i.e., $\operatorname{Im}\Phi_{|L|}(X)$ is a rational curve of degree 1. Such a polarized Calabi-Yau 3-fold actually exists. In fact, a general weighted complete intersection $X=(6,6)\subset \boldsymbol{P}(1,1,2,2,3,3)$ with polarization $L=\mathcal{O}_X(1)$ is a polarized Calabi-Yau 3-fold (cf. §4) with $h^0(L)=2$. Note that $|L|$ has no fixed components because $\operatorname{Pic}(X)=\boldsymbol{Z}L$ (cf. (4.2)). But, in general, N becomes arbitrarily large under the condition (0.4) as will be seen in the next (0.14).

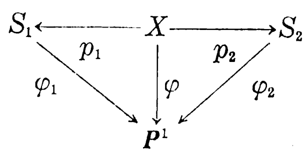
PROPOSITION-EXAMPLE (0.14). Let $\varphi_i: S_i \longrightarrow \boldsymbol{P}^1$ $(i=1,2)$ be a relatively minimal rational elliptic surface satisfying that:

(a) φ_i has no reducible fibers;

(b) for any $t \in \boldsymbol{P}^1$, either $\varphi_1^{-1}(t)$ or $\varphi_2^{-1}(t)$ is smooth;

(c) φ_i has a section C_i for each $i=1,2$.

Put $X:=S_1 \times_{\boldsymbol{P}^1} S_2$. Then X is a Calabi-Yau 3-fold (cf. [Sc]). Note the following natural diagram:



Put $F:=$ a general fiber of φ (an abelian surface), $S_1:=p_2^*C_2=S_1\times_{\boldsymbol{P}^1} C_2$, $S_2:=p_1^*C_1=C_1\times_{\boldsymbol{P}^1} S_2$. By this identification, we will consider $S_i \subset X$. Then,

(1) $L:=S_1+S_2+kF$ $(k\geq 3)$ is ample on X,

(2) $|L|=k|F|+(S_1+S_2)$ and $h^0(L)=k+1$. In particular, $\Phi_{|L|}(X)$ is a rational normal curve of degree k in $\boldsymbol{P}^k$.

PROOF. First note that $L=p_i^*(C_i+(k-2)F_i)+p_j^*(C_j+2F_j)$ $(\{i,j\}=\{1,2\})$, where F_i (resp. F_j) is a general fiber of φ_i (resp. φ_j). For any irreducible curve $C\subset X$, $C.L=[C:p_i(C)](p_i(C).(C_i+(k-2)F_i))+[C:p_j(C)]\times(p_j(C).(C_j+2F_j))$ by projection formula. Since $k-2\geq 1$, $C_i+(k-2)F_i$ (resp. C_j+2F_j) is nef on S_i (resp. ample on S_j) under the assumption (a). Thus $C.L>0$ because either $p_1(C)$ or $p_2(C)$ must be a curve. Hence (0.14)(1) holds by Kleiman's criterion.

Next we will prove (0.14)(2). First we will show the following

CLAIM (0.15). (1) $S_i^3=F^3=S_i^2.F=S_i.F^2=0$ for $i=1,2$.
(2) $S_1^2.S_2=S_1.S_2^2=-1$.
(3) $S_1.S_2.F=1$.
(4) $S_1.c_2(X)=S_2.c_2(X)=12$.
(5) $F.c_2(X)=0$.

PROOF. (1) is trivial because $S_1=p_2^*C_2$, $S_2=p_1^*C_1$, $F=p_i^*F_i$.
(2) Let us put $C=S_1\cap S_2$. Then C is a section of both φ_i. Hence $S_1^2.S_2=(C|_{S_2})^2=-1$. Similarly we have $S_1.S_2^2=-1$.
(3) $S_1.S_2.F=(S_2|_{S_1}).(F|_{S_1})=C.F_1=1$ because C is a section.
(4) By the exact sequence $0\longrightarrow T_{S_i}\longrightarrow T_X|_{S_i}\longrightarrow N_{X/S_i}\longrightarrow 0$, and (0.15)(1), we get:

$$\begin{aligned}c_2(X).S_i&=c_2(S_i)+c_1(S_i).c_1(N_{X/S_i})=12+(-K_{S_i}).(S_i|_{S_i})\\&=12+F_i.(S_i|_{S_i})=12+F.S_i^2=12.\end{aligned}$$

(5) Since F is a fiber and an abelian surface, $c_2(X).F=c_2(F)=0$. □

Since $L=S_1+S_2+kF$ $(k\geq 3)$ is ample,

$$h^0(L)\underset{(0.2)(1)}{=}\frac{(S_1+S_2+kF)^3}{6}+\frac{(S_1+S_2+kF).c_2(X)}{12}\underset{(0.15)}{=}k+1.$$

Hence (0.14)(2) holds because $|kF|=k|F|$ and $h^0(kF)=k+1$. Q.E.D.

§1. Birationality of $\Phi_{|nL|}$.

The purpose of this section is to prove:

THEOREM (1.1). *Let (X, L) be a polarized Calabi-Yau 3-fold. Let $\Phi_{|nL|}$ be the rational map associated to the complete linear system $|nL|$ $(n\geq 1)$. Then,*

(1) $\Phi_{|nL|}$ *is birational (onto it's image) for* $n\geq 5$;
(2) $\Phi_{|4L|}$ *is birational except for the following two cases*:
    (a) $X=(10)\subset \boldsymbol{P}(1,1,1,2,5)$ *with* $L=\mathcal{O}_X(1)$,
    (b) $h^0(X, L)=1$;
(3) *for any polarized Calabi-Yau 3-fold* (X, L) *in* (2)(a), $\Phi_{|4L|}$ *is not birational. In particular, the estimate in* (1) *is best possible.*

First we prepare some lemmas.

LEMMA (1.2) ([Rdr, Cor. 2.]). *Let* T *be a non-singular projective surface and* P *be a nef divisor on* T. *Then the rational map* $\Phi_{|P+K_T|}$ *is birational if the following two conditions are satisfied;*
(i) $P^2\geq 10$, (ii) $P.Z\neq 1, 2$ *for any effective divisor* Z *on* T. □

The next lemma which is implicitly mentioned in [Ma], [Be2] is useful.

LEMMA (1.3). *Let* Y *be a non-singular complex projective variety. Assume that a linear system* $|F|$ *with* $\dim|F|\geq 1$ *and an effective divisor* D *on* Y *are given. Then,* $\Phi_{|F+D|}$ *is birational if so is the restriction* $\Phi_{|F+D|}|_T$ *for general* $T\in|F|$.

PROOF. Assume contrary. Since the characteristic of the ground field is zero, there is a non-empty Zariski open set U on Y such that:
(1) $U\cap(D\cup Bs|F|)=\emptyset$,
(2) $\Phi:=\Phi_{|F+D|}$ is a morphism on U,
(3) for any point $x\in U$, there is another point $y\in U$ satisfying that $y\neq x$ and $\Phi(y)=\Phi(x)$.
Note that under the condition (1), the last condition in (3) implies that:
(4) there is a non-zero constant a satisfying that $\varphi(y)=a\varphi(x)$ for any $\varphi\in H^0(Y, F)$.
Take general $T\in|F|$. Then $T\cap U\neq\emptyset$ since $\dim|F|\geq 1$. Thus for any point $x\in T\cap U$, the point y taken in (3) would be a point on $T\cap U$ by (4). But this contradicts our assumption that $\Phi|_T$ is birational for general $T\in|F|$. □

LEMMA (1.4). *Let* (X, L) *be a polarized Calabi-Yau 3-fold. Assume that* $|L|=m|S|+B$ *where* $m\geq 1$, $\dim|S|\geq 1$ *and any general member of* $|S|$ *is irreducible reduced.*
If $L^2.S=1$, *then* $|L|=|S|$, $L^3=1$ *and* (X, L) *is one of the following two*:
(i) $X=(10)\subset \boldsymbol{P}(1,1,1,2,5)$, $L=\mathcal{O}_X(1)$;
(ii) $X=(6,6)\subset \boldsymbol{P}(1,1,2,2,3,3)$, $L=\mathcal{O}_X(1)$.

PROOF. First remark that any member of $|L|$ is connected because L is ample. Then, by $1=L^2.S=L.(mS+B).S=mL.S^2+L.S.B$, either (a) $m=1$, $L.S^2=1$, $L.S.B=0$ or (b) $L.S^2=0$, $L.S.B=1$ holds. We treat these two cases separately.

Case (a). Since $B=0$ by the remark above, we have $|L|=|S|$. Then $L^3=1$. Since the delta genus $\Delta(X, L):=L^3+3-h^0(L)\geq 1$, we see that $h^0(L)=2$ or 3. (Note that $h^0(L)\geq 2$ by our assumption.) Thus we get the desired assertion by [Fu8] (See also §5 in this paper.) in Case (a).

Case (b). Note that $S\cap S'=\emptyset$ for general $S, S'\in|S|$ because $L.S^2=0$. In particular, $|S|$ is free and $\mathcal{O}_S(S)=\mathcal{O}_S$. Thus any general member S of $|S|$ is smooth with $K_S=0$. Since $S^2=0$, we have $1=L.S.B=(mS+B).S.B$ $=B^2.S=(B|_S)^2$. Thus by Riemann-Roch, $\chi(B|_S)-\chi(\mathcal{O}_S)=\dfrac{B|_S(B|_S-K_S)}{2}$ $=\dfrac{B^2.S}{2}=\dfrac{1}{2}$. But this is absurd. Hence Case (b) does not occur. □

PROOF OF (1.1) (1), (2). Let us put $W_n:=\operatorname{Im}\Phi_{|nL|}$ and $W:=W_1$. We will treat the following 3 cases separately:
Case 1. $\dim W\geq 2$; Case 2. $\dim W=1$; Case 3. $\dim W=0$.

Case 1. Put $|L|=|M|+B$, where B is the fixed part of $|L|$. Note that any general member of $|M|$ is irreducible reduced because $\dim W\geq 2$. Let us denote an embedded resolution of the singularities of any general member S of $|M|$ by $\pi:\tilde{X}\longrightarrow X$. Note that $\operatorname{Sing} S\subset Bs|M|$ by Bertini's theorem. Since π can be obtained by a successive blow up only along a non-singular center at which general S is singular (in each step), $|\pi^*M|$ is written as $|\pi^*M|=|T|+\sum a_iE_i$, where T is a proper transform of S, E_i is the total transform of the center of the i-th blow up, and a_i is an integer. Note that $a_i\geq 2$ by our choice of π. Since $K_{\tilde{X}}=\sum\alpha_iE_i$ with $\alpha_i=1$ or 2, we get:
$\pi^*L=\pi^*M+\pi^*B=T+K_{\tilde{X}}+\pi^*B+\sum(a_i-\alpha_i)E_i$. In particular,

(1.5) $\pi^*L=T+K_{\tilde{X}}+$(some effective divisor).

CLAIM (1.6). Put $R:=\pi^*L|_T$. For a positive integer k, $\Phi_{|(k+1)L|}$ is birational if so is $\Phi_{|K_T+kR|}$ for general $T\in|T|$.

PROOF. First note that if $\Phi_{|T+K_{\tilde{X}}+k\pi^*L|}$ is birational then so is $\Phi_{|(k+1)L|}$ by (1.5). Moreover, if $\Phi_{|T+K_{\tilde{X}}+k\pi^*L||_T}$ is birational for general $T\in|T|$ then so is $\Phi_{|T+K_{\tilde{X}}+kL|}$ by (1.3). On the other hand, from the exact sequence $0\longrightarrow K_{\tilde{X}}+k\pi^*L\xrightarrow{\otimes t}T+K_{\tilde{X}}+k\pi^*L\xrightarrow{r_T}K_T+kR\longrightarrow 0$, we get $\Phi_{|T+K_{\tilde{X}}+k\pi^*L|}|_T=\Phi_{|K_T+kR|}$ because $H^1(\tilde{X}, K_{\tilde{X}}+k\pi^*L)=0$ by Kawamata-Viehweg vanishing theorem. Hence (1.6) is proved. □

Now in order to prove (1.1)(1)(2) it is enough to show the following claim.

CLAIM (1.7). For general $T\in|T|$, $\Phi_{|K_T+kR|}$ is birational in the following two cases:
(1) $k\geq 4$;
(2) $k\geq 3$ and (X, L) is different from the one in (i) in (1.4).

PROOF. Note that any general member of $|T|$ is irreducible smooth. Then it is enough to check the condition (i), (ii) in (1.2) for $P=kR$. Since the condition (ii) is obviously satisfied because $k\geq 3$. Let us check the condition (i). Since $P^2=k^2R^2=k^2(\pi^*L)^2.T=k^2L^2.S$ and $L^2.S\geq 1$, we have $P^2\geq 10$ if either $k\geq 3$ and $L^2.S\neq 1$, or $k\geq 4$ holds. Since $|L|=|S|+B$ and $h^0(L)\geq 3$ by our assumption, we can see at once from (1.4) that $L^2.S\neq 1$ unless (X, L) is same as the one in (i) in (1.4). □

Case 2. In this case $|L|=N|S|+B$, where $\dim|S|=1$ and any general member S of $|S|$ is irreducible reduced by (0.4). Then, by considering an embedded resolution of S, we can see that $\Phi_{|nL|}$ is birational for $n\geq 4$ unless (X, L) is same as the one in (ii) in (1.4) by the same manner as in Case 1. So, in order to complete our proof, it is enough to show the following:

CLAIM (1.8). Let (X, L) be a polarized Calabi-Yau 3-fold such that $X=(6, 6)\subset P:=\boldsymbol{P}(1, 1, 2, 2, 3, 3)$ and $L=\mathcal{O}_X(1)$. Then $\Phi_{|nL|}$ is birational for $n\geq 3$.

PROOF. Since $H^0(nL)$ $(n\geq 3)$ contains monomials $x_2x_1^{n-1}$, $x_3x_1^{n-2}$, $x_4x_1^{n-2}$, $x_5x_1^{n-3}$, $x_6x_1^{n-3}$, where x_i are the weighted homogeneous coordinates of P in order, $\Phi_{|nL|}$ is one to one on the non-empty Zariski open set $(x_1\neq 0)\cap X$. □

Case 3. Since $|L|$ consists of the single member, say S_0, we start from $|2L|$ instead of $|L|$ in order to make use of (1.3). Let us put $|2L|=|M|+B$. Since $\dim W_2\geq 2$ by (0.4), any general member S of $|M|$ is irreducible reduced. By considering an embedded resolution $\pi:\tilde{X}\longrightarrow X$

of S as before, we can see that $\Phi_{|(k+2)L|}$ is birational if so is $\Phi_{|K_T+kR|}$ $(R:=\pi^*L|_T)$ for a proper transform T of general S. Again by the same manner as in case 1, we can also see that $\Phi_{|K_T+kR|}$ is birational for $k\geq 3$ unless $L^2.M=1$. Thus in order to complete the proof, it is sufficient to show:

CLAIM (1.9). $L^2.M\neq 1$.

PROOF. Assume contrary. Then every member of $|M|$ is irreducible and reduced. In particular, some irreducible component Q of S_0 must be contained in $|M|$ since $2S_0\in|2L|$. But this is absurd because S_0 (in particular, Q) cannot move. □

Finally we will show (1.1)(3).

Let us denote the weighted homogeneous coordinates of $\boldsymbol{P}(1,1,1,2,5)$ by x_1, x_2, x_3, x_4, x_5 in order. By changing the coordinates if necessary, we may assume that the defining equation of X is written as follows: $x_5^2+g(x_1,x_2,x_3,x_4)=0$ where $g\neq 0$. Remark that no elements of $H^0(4L)$ contain the variable x_5 in their polynomial expression of x_i's. Take a non-empty Zariski open set $U:=X\cap(x_1\neq 0)\cap(g\neq 0)$ of X. Consider $\Phi_{|4L|}$ on U. By definition of U we may put $x_1(x)=1$ for all $x\in U$. Then $\Phi_{|4L|}|_U$ is given by the following diagram:

$$\begin{array}{ccl} X & \dashrightarrow & W:=\operatorname{Im}\Phi_{|4L|}\subset \boldsymbol{P}^{\dim|4L|} \\ \cup & & \cup \\ U & \longrightarrow & V:=\operatorname{Im}\Phi_{|4L|}|_U,\ x\longmapsto(1,x_2(x),x_3(x),x_4(x),\cdots). \end{array}$$

Take $w=(1,a,b,c,\cdots)\in V$. Then $\Phi_{|4L|}(x)=w$ for $x\in U$ if and only if $x_2(x)=a$, $x_3(x)=b$, $x_4(x)=c$ and $x_5^2(x)+g(1,a,b,c)=0$ by the above remark. Thus the inverse image of $w\in V$ consists of two points. Hence $\Phi_{|4L|}$ is not birational. Q.E.D.

§2. Freeness of nL.

In this section we will prove the following theorem by making use of the Benveniste's method ([Be1]) directly.

THEOREM (2.1). *Let (X,L) be a polarized Calabi-Yau 3-fold. Then,*
(1) *$|nL|$ has no fixed components if $n\geq 5$;*
(2) *any general member of $|nL|$ is normal if $n\geq 10$;*
(3) *nL is free if $n\geq 20$.*

The following lemma due to Benveniste is essential for proof.

Lemma (2.2) ([Be1, Prop. 2-1]). *Let Y be a non-singular projective 3-fold. Let I be a finite index set including 0. Assume that there are integral divisors P, $\{E_i\}_{i\in I}$ and G on Y such that:*
(a) $P\in \mathrm{Div}(Y)$ *is nef and big,*
(b) $\sum_{i\in I} E_i$ *is a simple normal crossing divisor,*
(c) $G=\lceil A\rceil$ (:=*the round up of* A)$=A+\sum_{i\neq 0} c_iE_i$ *for some ample* Q-divisor A.
Set $D:=mP+\sum_{i\neq 0} m_iE_i$, *where* m *is a positive integer and* m_i ($i\in I\backslash\{0\}$) *are non-negative integers.*
Then E_0 *is not contained in the fixed part of* $|D|$ *if* $K_Y+P+E_0+G\in|D|$.
□

Proof of (2.1). In what follows, we assume that $|mL|$ is not free for some $m\geq 2$.

Step 1. There exist a resolution $\pi: Y\longrightarrow X$, a divisor with simple normal crossings $\sum_{i\in I} E_i$ on Y, and non-negative constants u_i, $u_i(2)$, $u_i(3)$, ρ_i, α_i, β_i such that
(i) $|mP|=|F|+\sum_{i\in I} u_iE_i$, $|2P|=|F_2|+\sum_{i\in I} u_i(2)E_i$, $|3P|=|F_3|+\sum_{i\in I} u_i(3)E_i$ for $P:=\pi^*L$, where $|F|$, $|F_2|$ and $|F_3|$ are free, u_i, $u_i(2)$, $u_i(3)$ are non-negative integers, and at least one u_i is positive;
(ii) $K_Y=\sum_{i\in I}\rho_iE_i$, where ρ_i are non-negative integers and ρ_i is positive if and only if E_i is exceptional for π;
(iii) $Q:=P-\sum_{i\in I} t_iE_i\in \mathrm{Div}(Y)\otimes Q$ is ample for any $t_i\in Q$ with $\alpha_i<t_i<\beta_i$.

Proof. This is well-known. □

Step 2. $\dfrac{u_i}{\rho_i+1}\leq\dfrac{m}{m-1}$ for every $i\in I$.

Proof. Set $a(r):=\mathrm{Max}\{(u_i+rt_i)/(\rho_i+1)|\, i\in I\}$ and $\tilde{a}(r):=\{i\in I|\,(u_i+rt_i)/(\rho_i+1)=a(r)\}$ for $r\in Q$, $r>0$. Assume contrary that $\dfrac{u_\alpha}{\rho_\alpha+1}>\dfrac{m}{m-1}$ for some $\alpha\in I$. Since $a(r)>\dfrac{n_\alpha}{\rho_\alpha+1}>\dfrac{m}{m-1}$ and the middle term is independent on r, by changing r smaller if necessary, we may assume that $a(r)\geq\frac{m+r}{m-1}$, i.e., $m-1-\frac{m+r}{a(r)}\geq 0$, and $u_i\neq 0$ for every $i\in\tilde{a}(r)$. Moreover, by changing t_i's slightly if necessary, we may also assume that $\tilde{a}(r)$ consists of only one element, say, 0. Then, we have:

(2.3) $a(r)=\frac{u_0+rt_0}{\rho_0+1}$, i.e., $\rho_0+1=\frac{u_0}{a(r)}+\frac{rt_0}{a(r)}$,

(2.4) $a(r)>\frac{u_i+rt_i}{\rho_i+1}$, i.e., $\rho_i-\frac{u_i+rt_i}{a(r)}>-1$ for $i\neq 0$.

Set $m_i:=\left\lceil\left(\rho_i-\frac{u_i+rt_i}{a(r)}\right)\right\rceil$ for $i\neq 0$. These are non-negative integers by (2.4). Put $D:=mP+\sum_{i\neq 0}m_iE_i$.

Then we get,

$$\begin{aligned}D-K_Y-E_0-P&=(m-1)P+\sum_{i\neq 0}(m_i-\rho_i)E_i-(\rho_0+1)E_0\\&=(m-1)P+\sum_{i\neq 0}(m_i-\rho_i)E_i-\left(\frac{u_0}{a(r)}+\frac{rt_0}{a(r)}\right)E_0\\&\qquad\text{(by eliminating } E_0)\\&=\frac{r}{a(r)}\Big(P-\sum_{i\in I}t_iE_i\Big)+\frac{1}{a(r)}\Big(mP-\sum_{i\in I}u_iE_i\Big)\\&\quad+\Big(m-1-\frac{m+r}{a(r)}\Big)P+\sum_{i\neq 0}\Big\{m_i-\Big(\rho_i-\frac{u_i+rt_i}{a(r)}\Big)\Big\}E_i.\end{aligned}$$

Set $A:=\frac{r}{a(r)}\Big(P-\sum_{i\in I}t_iE_i\Big)+\frac{1}{a(r)}\Big(mP-\sum_{i\in I}u_iE_i\Big)+\Big(m-1-\frac{m+r}{a(r)}\Big)P$. Then, A is ample by the choice of r and step 1;
$\lceil A\rceil=A+\sum_{i\neq 0}\Big\{m_i-\Big(\rho_i-\frac{u_i+rt_i}{a(r)}\Big)\Big\}E_i$ by the choice of m_i; $D=K_Y+P+E_0+\lceil A\rceil$. Hence E_0 is not contained in the fixed part of $|D|$ by (2.2). On the other hand, since $u_0>0$ and $\rho_i\geq m_i$ $(i\neq 0)$ by our choice of u_0, m_i, the following inequality holds:
$E_0\leq mP\leq D=mP+\sum_{i\neq 0}m_iE_i\leq mP+K_Y$. Note that $h^0(mP-E_0)=h^0(mP)$ because E_0 is a fixed component of mP. Hence $h^0(mP)=h^0(mP-E_0)$ $\leq h^0(mP+K_Y-E_0)\leq h^0(mP+K_Y)$ because K_Y and E_0 are effective. But, since $mP=\pi^*mL$ and $\pi_*K_Y=K_X=\mathcal{O}_X$, we have $h^0(mP)=h^0(mL)$ $=h^0(mP+K_Y)$ by projection formula. Thus $h^0(mP+K_Y-E_0)=h^0(mP+K_Y)$, i.e., E_0 is a fixed component of $mP+K_Y$. Then E_0 must be contained in the fixed part of $|D|$. But this contradicts the above. Therefore the

inequality $\frac{u_i}{\rho_i+1} \leq \frac{m}{m-1}$ must hold for every $i \in I$. □

Step 3. If $u_i \geq 1$, then either $u_i(2) \geq 1$ or $u_i(3) \geq 1$ holds.

PROOF. Since $m \geq 2$, there are non-negative integers a, b such that $m=2a+3b$. Thus $u_i \leq au_i(2)+bu_i(3)$ so that our assertion holds. □

Step 4. Assume that $m \geq 4$. Then $\frac{u_i}{\rho_i+1} \leq 1$ for every i such that $\rho_i \in \{0, 1\}$.

PROOF. By step 2 and $m \geq 4$, $\frac{u_i}{\rho_i+1} \leq \frac{m}{m-1} \leq 1+\frac{1}{3}$. Hence $\frac{u_i}{\rho_i+1} \leq 1$ if $\rho_i \in \{0, 1\}$. □

Let us put

$$c_n(s) := \mathrm{Max}\left\{c_i(s) := \frac{u_i+su_i(n)}{\rho_i+1} \,\middle|\, i \in I\right\},$$

$$b_n(r, s) := \mathrm{Max}\left\{\frac{u_i+rt_i+su_i(n)}{\rho_i+1} = c_i(s)+\frac{rt_i}{\rho_i+1} \,\middle|\, i \in I\right\},$$

where $n \in \{2, 3\}$, $r>0$, $s>0$, and $r, s \in \boldsymbol{Q}$.

Step 5. Assume that $m \geq 10$. Moreover assume that there is $\alpha \in I$ such that $\frac{u_\alpha}{\rho_\alpha+1}=1$ and $\rho_\alpha \in \{0, 1\}$.

Then, for suitable $r>0$, $s>0$, and $n \in \{2, 3\}$, the following (i) and (ii) hold:

(i) $m-1-\frac{m+r+ns}{b_n(r, s)}>0$;

(ii) there is an element $0 \in I$ such that

$$b_n(r, s)=\frac{u_0+rt_0+su_0(n)}{\rho_0+1}, \qquad u_0 \neq 0,$$

$$b_n(r, s)>\frac{u_i+rt_i+su_i(n)}{\rho_i+1} \qquad \text{for} \quad i \neq 0.$$

PROOF. By step 4, either $u_\alpha(2) \geq 1$ or $u_\alpha(3) \geq 1$ holds. Take n such that $u_\alpha(n)=\mathrm{Max}\{u_\alpha(2), u_\alpha(3)\}$. Since $b_n(r, s)>c_n(s) \geq \frac{u_\alpha+su_\alpha(n)}{\rho_\alpha+1}>1$, in order to prove our claim, it is sufficient to show the existence of $s>0$ satisfying that:

(1) $m-1-\dfrac{m+ns}{b_n(r,s)}>0$ and

(2) if $c_n(s)=\dfrac{u_i+su_i(n)}{\rho_i+1}$, then $u_i\neq 0$.

For this, by definition of $b_n(r,s)$ and $c_n(s)$, it is enough to prove the existence of s which satisfies the following three inequalities:

(3) $s>0$,

(4) $m-1-\dfrac{(\rho_\alpha+1)(m+ns)}{u_\alpha+su_\alpha(n)}\geq 0$,

(5) $\dfrac{u_\alpha+su_\alpha(n)}{\rho_\alpha+1}>\dfrac{su_i(n)}{\rho_i+1}$ for all $i\in I$.

Noting that $\rho_\alpha+1=u_\alpha$ by assumption, we can readily see that (4) (resp. (5)) is equivalent to the following (6) (resp. (7)) under (3):

(6) $\left\{(m-1)\dfrac{u_\alpha(n)}{\rho_\alpha+1}-n\right\}s\geq 1$, (7) $1>\left\{\dfrac{u_i(n)}{\rho_i+1}-\dfrac{u_\alpha(n)}{\rho_\alpha+1}\right\}s$.

Hence there exists s satisfying (3), (4), (5) if the following (8) and (9) hold: (8) $(m-1)\dfrac{u_\alpha(n)}{\rho_\alpha+1}-n>0$,

(9) $(m-1)\dfrac{u_\alpha(n)}{\rho_\alpha+1}-n>\dfrac{u_i(n)}{\rho_i+1}-\dfrac{u_\alpha(n)}{\rho_\alpha+1}$, i.e., $m\dfrac{u_\alpha(n)}{\rho_\alpha+1}-n>\dfrac{u_i(n)}{\rho_i+1}$.

We will find m which satisfies (8) and (9).

(8): Since $(m-1)\dfrac{u_\alpha(n)}{\rho_\alpha+1}-n\geq(m-1)\dfrac{1}{2}-3$ (because $\rho_\alpha\in\{0,1\}$), (8) is satisfied if $m\geq 8$.

(9): Assume $n=2$. In this case, since $m\cdot\dfrac{u_\alpha(2)}{\rho_\alpha+1}-n\geq\dfrac{1}{2}m-2$ and $\dfrac{u_i(2)}{\rho_i+1}\leq\dfrac{2}{2-1}=2$ by step 2, (9) is satisfied if $m\geq 9$. Assume $n=3$. In this case, since $m\cdot\dfrac{u_\alpha(3)}{\rho_\alpha+1}-n\geq\dfrac{1}{2}m-3$ and $\dfrac{u_i(n)}{\rho_i+1}\leq\dfrac{3}{3-1}=\dfrac{3}{2}$ by step 2, (9) is satisfied if $m\geq 10$. This completes the proof. □

Step 6. Assume that $m\geq 5$. Moreover assume that there is $\alpha\in I$ such that $\dfrac{u_\alpha}{\rho_\alpha+1}=1$ and $\rho_\alpha=0$, i.e., $u_\alpha=1$ and $\rho_\alpha=0$. Then, for suitable

$r>0$, $s>0$, and $n\in\{2,3\}$, the following (i) and (ii) hold:

(i) $m-1-\dfrac{m+r+ns}{b_n(r,s)}>0$,

(ii) there is an element $0\in I$ such that

$$b_n(r,s)=\frac{u_0+rt_0+su_0(n)}{\rho_0+1}, \qquad u_0\neq 0,$$

$$b_n(r,s)>\frac{u_i+rt_i+su_i(n)}{\rho_i+1} \qquad \text{for} \quad i\neq 0.$$

PROOF. By the same argument as in step 5, it is enough to show that the following (8)′, (9)′ hold for $m\geq 5$ instead of (8) and (9) in step 5: (8)′ $(m-1)u_\alpha(n)-n>0$, (9)′ $mu_\alpha(n)-n>u_i(n)$. Then again by the same argument as in step 5, we complete the proof. □

Step 7. (i) Assume that $m\geq 10$. Then $\dfrac{u_i}{\rho_i+1}<1$ if $\rho_i\in\{0,1\}$.

(ii) Assume that $m\geq 5$. Then $\dfrac{u_i}{\rho_i+1}<1$ if $\rho_i=0$.

PROOF OF (i). Assume contrary. Then there is $\alpha\in I$ such that $\dfrac{u_\alpha}{\rho_\alpha+1}=1$ and $\rho_\alpha\in\{0,1\}$ by step 4. So we can take n, r, s found in step 5. Put $b:=b_n(r,s)$ for these r, s, n. Consider the divisor $D=mP+\sum_{i\neq 0} n_iE_i$, where $n_i:=\left\lceil \rho_i-\dfrac{u_i+rt_i+su_i(n)}{b}\right\rceil$ $(i\neq 0)$. Note that n_i is a non-negative integer by step 5 (ii). The divisor $D-K_Y-P-E_0$ can be calculated as follows:

$$\begin{aligned} D-K_Y-E_0-P&=(m-1)P+\sum_{i\neq 0}(n_i-\rho_i)E_i-(\rho_0+1)E_0 \qquad \text{(step 5 (i))}\\ &=(m-1)P+\sum_{i\neq 0}(n_i-\rho_i)E_i-\left(\frac{u_0}{b}+\frac{rt_0}{b}+\frac{su_0(n)}{b}\right)E_0\\ &\qquad\qquad \text{(by eliminating } E_0)\\ &=\frac{r}{b}\left(P-\sum_{i\in I}t_iE_i\right)+\frac{1}{b}\left(mP-\sum_{i\in I}u_iE_i\right)\\ &\quad+\frac{s}{b}\left(nP-\sum_{i\in I}u_i(n)E_i\right)+\left(m-1-\frac{m+r+ns}{b}\right)P \end{aligned}$$

$$+\sum_{i\neq 0}\left\{n_i-\left(\rho_i-\frac{u_i+rt_i+su_i(n)}{b}\right)\right\}E_i$$
$$=\frac{r}{b}Q+\frac{1}{b}F+\frac{s}{b}F(n)+\left(m-1-\frac{m+r+ns}{b}\right)P$$
$$+\sum_{i\neq 0}\left\{n_i-\left(\rho_i-\frac{u_i+rt_i+su_i(n)}{b}\right)\right\}E_i.$$

Since $m-1-\dfrac{m+r+ns}{b}>0$ by step 5 (i), the divisor $\dfrac{r}{b}Q+\dfrac{1}{b}F+\dfrac{s}{b}F(n)+\left(m-1-\dfrac{m+r+ns}{b}\right)P$ is ample. Then by the same argument as in step 2, E_0 is not contained in the fixed part of $|D|$. On the other hand, since $u_0\neq 0$ by step 5 (ii) and $n_i\leq\rho_i$ by our choice of n_i, again by the same argument as in the last part of step 2, E_0 must be contained in the fixed part of $|D|$. But this is absurd. Hence (i) is proved.

PROOF OF (ii). Use step 6 instead of step 5. □

Step 8. (i) $|mL|$ has no fixed components for $m\geq 5$.
(ii) General member of $|mL|$ is irreducible normal for $m\geq 10$.

PROOF. (i) Assume contrary. Then, corresponding to a fixed component, there exists $i\in I$ such that $u_i\geq 1$ and $\rho_i=0$. Then $\dfrac{u_i}{\rho_i+1}\geq 1$. But this contradicts step 7 (ii).
(ii) Since $\dim\Phi_{|mL|}(X)\geq 2$ by (0.4) and $|mL|$ is free from the fixed part by (i), any general member of $|mL|$ is irreducible and reduced. In order to prove normality, it is enough to show that any general member of $|mL|$ has only isolated singularities by the Serre's criterion. For this, it is sufficient to prove that $|mL|$ has no fixed curves of multiplicity ≥ 2 by Bertini's theorem. Assume contrary. Then corresponding to the first blow up of a fixed curve of multiplicity ≥ 2, there exists $i\in I$ such that $\rho_i=1$ and $u_i\geq 2$. Then $\dfrac{u_i}{\rho_i+1}\geq 1$, which contradicts step 7(i). Thus (ii) has been proved. □

Step 9. Assume that $m\geq 10$. Let S be a general member of $|mL|$. (Thus S is a normal Gorenstein surface.) Put $R=L|_S$. Then $|(a+m)R|$ is free on S for all $a\geq 3$.

PROOF. Since $h^0((a+m)R)=h^0((a+m)L)-h^0((aL))\geq 4$, step 9 follows from [Be1, Th. 1-0]. □

Step 10. $|nL|$ is free for every $n\geq 20$.

PROOF. First we assume that n is even. Put $n=2m$. Let $S:=\mathrm{div}(s)$ be any general member of $|mL|$ $(s\in H^0(mL))$. Put $R=L|_S$. Note that $K_S=mR$. From the exact sequence $0\longrightarrow \mathcal{O}_X(mL)\xrightarrow{\otimes s}\mathcal{O}_X(2mL)\longrightarrow \mathcal{O}_S(2mR)\longrightarrow 0$, we get $0\longrightarrow H^0(\mathcal{O}_X(mL))\xrightarrow{\cdot s}H^0(\mathcal{O}_X(2mL))\longrightarrow H^0(S,\mathcal{O}_S(2mR))\longrightarrow 0$. Hence $Bs|2mL|=Bs|2mR|$. But the right hand side is empty by step 9. Next we assume that n is odd. Put $n=2m+1$. Let S_m (resp. S_{m+1}) be any general member of $|mL|$ (resp. $|(m+1)L|$) corresponding to $s_m\in H^0(mL)$ (resp. $s_{m+1}\in H^0((m+1)L)$). Put $R_m=L|_{S_m}$ (resp. $R_{m+1}=L|_{S_{m+1}}$). Note that $K_{S_m}=mR_m$ and $K_{S_{m+1}}=(m+1)R_{m+1}$. From two exact sequences $0\longrightarrow \mathcal{O}_X(mL)\xrightarrow{\otimes s_{m+1}}\mathcal{O}_X((2m+1)L)\longrightarrow \mathcal{O}_{S_{m+1}}((2m+1)R_{m+1})\longrightarrow 0$, and $0\longrightarrow \mathcal{O}_X((m+1)L)\xrightarrow{\otimes s_m}\mathcal{O}_X((2m+1)L)\longrightarrow \mathcal{O}_{S_m}((2m+1)R_m)\longrightarrow 0$, we can see by the same manner as in the previous case that:
$Bs|(2m+1)L|\subset Bs|(2m+1)R_{m+1}|\cup Bs|(2m+1)R_m|$. But the right hand side is empty by step 9. Thus $Bs|(2m+1)L|=\emptyset$. □

Now (2.1) has been proved. Q.E.D.

§3. Very ampleness and simply generatedness of nL.

Throughout this section (X, L) is a polarized Calabi-Yau 3-fold. Put $R_m:=H^0(X, mL)$, $R:=\bigoplus_{m\geq 0}R_m$. We say that fL $(f>0)$ is *birationally very ample* if fL is free and $\Phi_{|fL|}$ is birational. The purpose of this section is to prove:

THEOREM (3.1). *Assume that mL is birationally very ample for all $m\geq f$. Then,*
(1) *the graded ring R is generated by some elements of degree $<3f$;*
(2) *nfL is simply generated (in particular, very ample) for all $n\geq 2$;*
(3) *nL is very ample for all $n\geq 3f$.*

REMARK (3.2). A famous m-regularity theorem [Mu, Th. 3] says that nfL is simply generated for all $n\geq 4$. Our theorem is a refinement of this. □

The next two corollaries are immediate consequences of (3.1), (2.1) and (1.1).

COROLLARY (3.3). (1) *The graded ring R is generated by some elements of degree* <60;
(2) $40L$ *is simply generated*;
(3) nL *is very ample for all* $n\geq 60$. □

REMARK (3.4). T. Fujita conjectured in [Fu7] that nL is free for all $n\geq 4$. If this conjecture is true, then except for the exceptional cases in (1.1)(2),
(1) R is generated by some element of degree <12;
(2) $8L$ is simply generated;
(3) nL is very ample for all $n\geq 12$. □

COROLLARY (3.5). *Assume that L is birationally very ample. Then,*
(1) R *is generated by some elements of degree* ≤ 2;
(2) nL *is simply generated for all* $n\geq 2$. □

REMARK (3.6). As was seen by [SD], if (S, L) is a polarized $K3$ surface such that L is birationally very ample, then nL is simply generated for all $n\geq 1$. But in Calabi-Yau case this is not true as will be seen in the next (3.7). In this sense, the estimate in (3.5) is best possible.

PROPOSITION-EXAMPLE (3.7). Let $X:=(3,4)\subset P:=\boldsymbol{P}(1,1,1,1,1,2)$ be a general weighted complete intersection. Put $L:=\mathcal{O}_X(1)$. Then (X, L) is a polarized Calabi-Yau 3-fold (cf. §4). This (X, L) satisfies the assumption of (3.5) but L is not very ample (in particular, L is not simply generated).

PROOF. Let $x_1, x_2, \cdots, x_5, x_6$ be the weighted homogeneous coordinates of P in order. Then, the defining equation of X is written as follows (after suitable change of coordinate):

$$F:=x_6 f_1(x_1, \cdots, x_5)+f_3(x_1, \cdots, x_5)=0$$

$$G:=x_6^2+f_4(x_1, \cdots, x_5)=0,$$

where f_i $(i=1,3,4)$ are not zero polynomials of degree i. Note that $H^0(X, L)=\boldsymbol{C}\langle x_1, \cdots, x_5\rangle$ by definition. Then $\bigcap_{i=1}^{5}(x_i=0)\cap(G=0)=\emptyset$. In particular, L is free. Moreover, we can readily see that $\Phi_{|L|}$ is birational by the shape of F. But $\Phi_{|L|}:X\longrightarrow W:=\operatorname{Im}\Phi_{|L|}(X)\subset \boldsymbol{P}^4$ cannot be an isomorphism because $\deg W=L^3=6$. □

PROOF OF (3.1). Put $M:=fL$. Let $s, t, u\in H^0(X, M)$ be general elements. Put $S:=\mathrm{div}(s)$, $C:=\mathrm{div}(t|_S)$, $P:=\mathrm{div}(u|_C)$. Note that S (resp. C, resp. P) is a smooth surface (resp. curve, resp. 0-dimensional scheme of degree M^3). Put $P=P_1+\cdots+P_{M^3}$.

LEMMA (3.8). *$R_m=R_f\cdot R_{m-f}$ for all $m\geq 4f+1$.*

PROOF. Let k be a positive integer. By the exact sequence $0\longrightarrow(3f+k)L\xrightarrow{\otimes s}(4f+k)L\xrightarrow{r_S}(4f+k)L|_S\longrightarrow 0$ and (0.3), we get the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} & & H^0(M)\otimes H^0((3f+k)L) & \xrightarrow[r_S]{} & H^0(M|_S)\otimes H^0((3f+k)L|_S) & \longrightarrow & 0\\ & & \downarrow m & & \downarrow m_S & & \\ 0\longrightarrow H^0((3f+k)L) & \xrightarrow[\cdot s]{} & H^0((4f+k)L) & \xrightarrow[r_S]{} & H^0((4f+k)L|_S) & \longrightarrow & 0.\end{array}$$

Hence the multiplication map m is surjective if so is m_S. By the exact sequence $0\longrightarrow(3f+k)L|_S\xrightarrow{\otimes t}(4f+k)L|_S\xrightarrow{r_C}(4f+k)L|_C\longrightarrow 0$ and (0.3), we get similarly the commutative diagram of exact sequences:

$$\begin{array}{ccccccc} & & H^0(M|_S)\otimes H^0((3f+k)L|_S) & \xrightarrow[r_C]{} & H^0(M|_C)\otimes H^0((3f+k)L|_C) & \longrightarrow & 0\\ & & \downarrow m_S & & \downarrow m_C & & \\ 0\longrightarrow H^0((3f+k)L|_S) & \xrightarrow[\cdot t]{} & H^0((4f+k)L|_S) & \xrightarrow[r_C]{} & H^0((4f+k)L|_C) & \longrightarrow & 0.\end{array}$$

Hence the multiplication map m_S is surjective if so is m_C. Note that $2fL|_C=K_C$ so that $h^1((2f+m)L|_C)=h^0(-mL|_C)=0$ for all $m>0$. Then, by the exact sequences

$$0\longrightarrow(3f+k)L|_C\xrightarrow{\otimes u}(4f+k)L|_C\xrightarrow{r_P}(4f+k)L|_P=\bigoplus_{i=1}^{M^3}C_{P_i}\longrightarrow 0$$

and

$$0\longrightarrow(2f+k)L|_C\xrightarrow{\otimes u}(3f+k)L|_C\xrightarrow{r_P}(3f+k)L|_P=\bigoplus_{i=1}^{M^3}C_{P_i}\longrightarrow 0,$$

we get again the commutative diagram of exact sequences:

$$\begin{array}{ccccccc} & & H^0(M|_C)\otimes H^0((3f+k)L|_C) & \xrightarrow[r_P]{} & H^0(M|_C)\otimes(\bigoplus C_{P_i}) & \longrightarrow & 0\\ & & \downarrow m_C & & \downarrow m_P & & \\ 0\longrightarrow H^0((3f+k)L|_C) & \xrightarrow[\cdot u]{} & H^0((4f+k)L|_C) & \xrightarrow[r_P]{} & \bigoplus C_{P_i} & \longrightarrow & 0.\end{array}$$

But m_P is surjective because $M|_C$ is free. Thus m_C is also surjective. Hence the assertion holds. □

LEMMA (3.9). *$H^0((2f+k)L|_C)=H^0(2M|_C)\cdot H^0(kL|_C)$ if $k\geq f$.*

PROOF. Since $2M|_C=K_C$ and $kL|_C$ is birationally very ample, the equality holds by the Castelnuvo's theorem (cf. [ACGH, P. 151], [Fu6, Th. A7]). □

LEMMA (3.10). *$R_m=R_f\cdot R_{m-f}+R_{2f}\cdot R_{m-2f}$ if $m\geq 3f$.*

PROOF. Let k be an integer such that $k\geq f$. By (3.9) and (0.3), we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & H^0(2M|_S)\otimes H^0((kL|_S) & \xrightarrow{r_C} & H^0(2M|_C)\otimes H^0(kL|_C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle m_S} & & \downarrow{\scriptstyle m_C} & & \\
0\longrightarrow H^0((f+k)L|_S) & \xrightarrow{\cdot t} & H^0((2f+k)L|_S) & \xrightarrow{r_C} & H^0((2f+k)L|_C) & \longrightarrow & 0. \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Thus $H^0((2f+k)L|_S)=H^0(2M|_S)\cdot H^0(kL|_S)+H^0(M|_S)\cdot H^0((f+k)L|_S)$.
Hence we get the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & \begin{array}{c} H^0(M)\otimes H^0((f+k)L) \\ \oplus H^0(2M)\otimes H^0(kL) \end{array} & \xrightarrow{r_S} & \begin{array}{c} H^0(M|_S)\otimes H^0((f+k)L|_S) \\ \oplus H^0(2M|_S)\otimes H^0(kL|_S) \end{array} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m_S} & & \\
0\longrightarrow H^0((f+k)L) & \xrightarrow{\cdot s} & H^0((2f+k)L) & \xrightarrow{r_S} & H^0((2f+k)L|_S) & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Thus $H^0((2f+k)L)=H^0(M)\cdot H^0((f+k)L)+H^0(2M)\cdot H^0(kL)$, i.e., $R_m=R_f\cdot R_{m-f}+R_{2f}\cdot R_{m-2f}$ if $m\geq 3f$. □

By (3.10), (3.1)(1) holds.

LEMMA (3.11). *$R_{knf}=(R_{nf})^k$ if $n\geq 2$ and $k\geq 1$.*

PROOF. We will treat the following 3 cases separately:

Case 1. $n=2$; Case 2. $n=3$; Case 3. $n\geq 4$.

Case 1. By (3.10),

$$\begin{aligned}R_{2(k+1)f} &= R_f \cdot R_{(2k+1)f} + R_{2f} \cdot R_{2kf} \\ &= R_f \cdot (R_f \cdot R_{2kf} + R_{2f} \cdot R_{(2k-1)f}) + R_{2f} \cdot R_{2kf} \\ &= R_f^2 \cdot R_{2kf} + R_{2f} \cdot R_f \cdot R_{(2k-1)f} + R_{2f} \cdot R_{2kf} \\ &= R_{2f} \cdot R_{2kf} \qquad \text{for} \quad k \geq 1.\end{aligned}$$

Thus $R_{2kf} = (R_{2f})^k$ by induction.

Case 2. By successive use of (3.8), we get:

$$R_{3(k+1)f} = R_f \cdot R_{(3k+2)f} = \cdots = (R_f)^{3k-2} \cdot R_{5f} = (R_f)^{3k-1} \cdot R_{4f}.$$

Note that $R_{4f} = (R_{2f})^2$ by case 1 and $R_{3f} = R_f \cdot R_{2f}$ by (3.10). Thus $R_{3(k+1)f} = (R_f)^{3(k-1)} \cdot (R_f)^2 \cdot (R_{2f})^2 = (R_f)^{3(k-1)} \cdot (R_{3f})^2$. Since $(R_f)^{3(k-1)} \cdot R_{3f} \subset R_{3kf}$, we get $R_{3(k+1)f} = R_{3kf} \cdot R_{3f}$. Hence $R_{3kf} = (R_{3f})^k$ by induction.

Case 3 (cf. [Mu]). Since $nf+1 \geq 4f+1$ in this case, we get $R_{(k+1)nf} = (R_f)^{kn} \cdot R_{nf}$ by (3.8). Since $(R_f)^{kn} \subset R_{knf}$, we have $R_{(k+1)nf} = R_{knf} \cdot R_{nf}$. Thus $R_{knf} = (R_{nf})^k$. □

This completes the proof of (3.1)(2). Now (3.1)(3) is clear because $2fL$ is very ample and mL is free for all $m \geq f$. Q.E.D.

§4. Polarized Calabi-Yau 3-folds arising as general weighted complete intersections.

Recall that any polarized manifold (X, L) is isomorphic to $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R}(1))$ where $R = \bigoplus_{n \geq 0} H^0(X, nL)$ and R is a quotient of some weighted polynomial ring, say $\boldsymbol{C}[x_1, \cdots, x_m]$ with $a_i := \deg x_i$, by some homogeneous prime ideal I. Geometrically saying, (X, L) is a polarized subvariety of the weighted projective space $P := \boldsymbol{P}(a_1, \cdots, a_m)$ with the equations I and polarization $L = \mathcal{O}_X(1)$. We call (X, L) a weighted complete intersection (abbreviated by WCI) in P if I is generated by some $r := \operatorname{codim} X$ homogeneous elements $(f_1, \cdots, f_r)$. For simplicity, in this case we denote (X, L) by $(b_1, \cdots, b_r) \subset P$ where $b_i := \deg f_i$. We call a WCI $(b_1, \cdots, b_r) \subset P$ *general* if each generator f_i is chosen generally in $H^0(\mathcal{O}_P(b_i)) = \{$homogeneous polynomials of degree $b_i\}$. We refer the reader to [Mo], [Fu3], [Do] for basic properties of WCI. Here we only mention that our definition is equivalent to the one of [Mo] (cf. [Mo], [Fu3]). In particular, X does not meet the singular spectrum $\mathcal{S} := \bigcup_{k \in \boldsymbol{Z}} \{x_i = 0\,; k$ does not divide $a_i\}$.

In this section, we determine every polarized Calabi-Yau 3-fold (X, L) arising as a general WCI under the assumption that $h^0(L)\geq 2$. We call such (X, L) of type GWCI. The result is:

THEOREM (4.1). *Any polarized Calabi-Yau 3-fold (X, L) of type GWCI is one of the following*:

[1]	$(10)\subset \boldsymbol{P}(1^3, 2, 5)$	[2]	$(8)\subset \boldsymbol{P}(1^4, 4)$	[3]	$(4, 6)\subset \boldsymbol{P}(1^3, 2^2, 3)$
[4]	$(6)\subset \boldsymbol{P}(1^4, 2)$	[5]	$(4, 4)\subset \boldsymbol{P}(1^4, 2^2)$	[6]	$(3, 4)\subset \boldsymbol{P}(1^5, 2)$
[7]	$(2, 6)\subset \boldsymbol{P}(1^5, 3)$	[8]	$(2, 2, 2, 2)\subset \boldsymbol{P}^7$	[9]	$(2, 2, 3)\subset \boldsymbol{P}^6$
[10]	$(2, 4)\subset \boldsymbol{P}^5$	[11]	$(3, 3)\subset \boldsymbol{P}^5$	[12]	$(5)\subset \boldsymbol{P}^4$
[13]	$(6, 6)\subset \boldsymbol{P}(1^2, 2^2, 3^2)$.				

REMARK (4.2). These 13 types are actually polarized Calabi-Yau 3-folds. In fact, we can easily check the smoothness of X for generically chosen generators (by writing down the equations explicitely (cf. [Fl])) and the equalities $K_X=0$, $q=0$ (by the general theory of weighted complete intersections). Moreover, in each case, Pic $X=\boldsymbol{Z}L$ and $(h^0(L), L^3)$ is calculated as follows.

Type	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]
$h^0(L)$	3	4	3	4	4	5	5	8	7	6	6	5	2
L^3	1	2	2	3	4	6	4	16	12	8	9	5	1 □

PROOF OF (4.1). Put $h:=h^0(L)$, $d:=L^3$. Let C be a general twice cut of X by elements of $H^0(\mathcal{O}_P(1))$. Then C is also general weighted complete intersection in P such that $C\cap \mathcal{S}=\emptyset$ because $X\cap \mathcal{S}=\emptyset$. Hence we can apply the adjunction formula, so that C is an irreducible Gorenstein curve with a half-canonical bundle $L|_C=\mathcal{O}_C(1)$. Recall the following remarkable result due to M. Reid:

THEOREM (4.3) ([Rd2, Th. 3.4]). *Let C be an irreducible Gorenstein curve with arithemetic genus $g(C)\geq 2$. Let D be a half-canonical divisor on C, i.e., a Cartier divisor satisfying that $2D\sim K_C$.*
Then the graded ring $R(C, D)$ is generated by some elements of degree ≤ 3 and their relations are generated by some weighted polynomials of degree ≤ 6, except for the following 4 cases;
(1) *C is a hyperelliptic curve of $g(C)\neq 2$ and $h^0(D)=0$;*
(*In this case $R(C, D)$ is generated by degree ≤ 4 and relations generated*

by degree ≤ 8.)
(2) $g(C)=2$, $D=P$, *where* P *is a Weierstrass point*;
(*In this case* $C=(10)\subset \boldsymbol{P}(1,2,5)$.)
(3) $g(C)=3$, $D=g_2^1$ (*hence, hyperelliptic*);
(*In this case* $C=(8)\subset \boldsymbol{P}(1,1,4)$.)
(4) C *is non-hyperelliptic with* $g(C)=3$ *and* $h^0(D)=0$.
(*In this case* $R(C,D)$ *has one more relation of degree* $=8$.) □

Note that $h^0(L|_C)=h-2$. Thus, (X,L) is one of as follows:

In the case of $h\geq 3$.
$(10)\subset \boldsymbol{P}(1^3,2,5)$, i.e., [1] in (4.1),
$(8)\subset \boldsymbol{P}(1^4,4)$, i.e., [2] in (4.1),
[I] $(4^a,5^b,6^c)\subset \boldsymbol{P}(1^h,2^m,3^n)$, [II] $(3^f,4^a,5^b,6^c)\subset \boldsymbol{P}(1^h,2^m)$,
[III] $(2^e,4^a,5^b,6^c)\subset \boldsymbol{P}(1^h,3^n)$, [IV] $(2^e,3^f,4^a,5^b,6^c)\subset \boldsymbol{P}^{h-1}$;

In the case of $h=2$.
[i] $(5^A,6^B,7^C,8^D)\subset \boldsymbol{P}(1^2,2^p,3^q,4^r)$,
[ii] $(3^F,5^A,6^B,7^C,8^D)\subset \boldsymbol{P}(1^2,2^p,4^r)$,
[iii] $(2^E,5^A,6^B,7^C,8^D)\subset \boldsymbol{P}(1^2,3^q,4^r)$,
[iv] $(4^G,5^A,6^B,7^C,8^D)\subset \boldsymbol{P}(1^2,2^p,3^q)$,
[v] $(2^E,3^F,5^A,6^B,7^C,8^D)\subset \boldsymbol{P}(1^2,4^r)$,
[vi] $(3^F,4^G,5^A,6^B,7^C,8^D)\subset \boldsymbol{P}(1^2,2^p)$,
[vii] $(2^E,4^G,5^A,6^B,7^C,8^D)\subset \boldsymbol{P}(1^2,3^q)$;

where $h\geq 3$, m,n,p,q,r are positive integers and the other letters are non-negative integers.

We treat these cases separately.

CLAIM (4.4). If (X,L) is of type [I], then (X,L) is $(4,6)\subset \boldsymbol{P}(1^3,2^2,3)$, i.e., of type [3] in (4.1).

PROOF. We can easily show by the following formulas:

(4.5) $a+b+c=h+m+n-4$ (because $\dim X=3$),

(4.6) $4a+5b+6c=h+2m+3n$ (because $K_X=0$),

(4.7) $d(:=L^3)=\dfrac{4^a\cdot 5^b\cdot 6^c}{2^m\cdot 3^n}$,

(4.8) $c\geq n$, $2a+c\geq m$ (by (4.7)).

In fact, by eliminating a from (4.5) and (4.6), we get;
$3h+3n+2m+b+2(c-n)=16$. Thus $h=3$ and $n=1$.
By eliminating h from (4.5) and (4.6), we have;
$(2a+c-m)+2(c-n)+a+4b+2c=4$. Thus $b=0$ and $c=1$ because $c\geq n=1$.
Hence $m=2$ and $a=1$. □

By a similar elementary calculation based on the formulas of $\dim X=3$, $K_X=0$, and $d=L^3$, we get:

CLAIM (4.9). (X, L) of types [II], $\cdots$, [vii] is one of as follows:
$(3, 4^2)\subset \boldsymbol{P}(1^3, 2^4)$ or of types [4], [5], [6] in (4.1) if (X, L) is of type [II];
of type [7] in (4.1) if (X, L) is of type [III];
of types [8], [9], [10], [11], [12] in (4.1) if (X, L) is of type [IV];
$(8)\subset \boldsymbol{P}(1^2, 2^3)$, $(4^3)\subset \boldsymbol{P}(1^2, 2^5)$, or of type [13] in (4.1) if $h^0(L)=2$. □

Thus in order to complete the proof of (4.1), it is enough to show the following:

CLAIM (4.10). (X, L) is not isomorphic to any one of below:
$(3, 4^2)\subset \boldsymbol{P}(1^3, 2^4)$, $(8)\subset \boldsymbol{P}(1^2, 2^3)$, $(4^3)\subset \boldsymbol{P}(1^2, 2^5)$.

PROOF. Assume that (X, L) is isomorphic to $(3, 4^2)\subset \boldsymbol{P}(1^3, 2^4)=: P$. Then, $h^0(L)=3$ and $L^3=\dfrac{3\cdot 4^2}{1^3\cdot 2^4}=3$. Hence $h^0(2L)=L^3+2h^0(L)=9$ by (0.2). On the other hand, by counting monomials of degree 2, we have $h^0(2L)=\binom{3+1}{2}+4=10$. This is absurd. By the same reason, (X, L) is not also isomorphic to the others in (4.10). □

This completes the proof of (4.1). Q.E.D.

§5. Graded rings of polarized Calabi-Yau 3-folds with delta genus ≤ 2.

Let (X, L) be a polarized Calabi-Yau 3-fold. As before, put $h:=h^0(L)$, $d:=L^3$. As will be seen in (5.5), if the delta genus $\Delta:=\Delta(X, L):=d+3-h\leq 2$, then $(h, d)=(3, 1)$, $(4, 2)$, $(2, 1)$, $(3, 2)$, $(4, 3)$, $(5, 4)$. On the other hand, by (4.2), a polarized Calabi-Yau 3-fold of type [1], [2], [13], [3], [4], [7] in (4.1) takes the above value (h, d) respectively.

In this section, we will prove the converse:

THEOREM (5.1). *Let (X, L) be a polarized Calabi-Yau 3-fold with delta*

genus ≤ 2. *Then* (X, L) *is a weighted complete intersection of codimension* ≤ 2 *in a suitable weighted projective space. More precisely,* (X, L) *is described as follows.*
(I). *Assume that* $\Delta(X, L)=1$. *Then* $(h, d)=(3,1)$ *or* $(4,2)$, *and,*
(I–1). *if* $(h, d)=(3,1)$, *then* $X=(10)\subset \boldsymbol{P}(1^3, 2, 5)$;
(I–2). *if* $(h, d)=(4,2)$, *then* $X=(8)\subset \boldsymbol{P}(1^4, 4)$.
(II). *Assume that* $\Delta(X, L)=2$. *Then* $(h, d)=(2,1)$, $(3,2)$, $(4,3)$ *or* $(5,4)$, *and,*
(II–1). *if* $(h, d)=(2,1)$, *then* $X=(6,6)\subset \boldsymbol{P}(1^2, 2^2, 3^2)$;
(II–2). *if* $(h, d)=(3,2)$, *then* $X=(4,6)\subset \boldsymbol{P}(1^3, 2^2, 3)$;
(II–3). *if* $(h, d)=(4,3)$, *then* X *is either*
(a) $(6)\subset \boldsymbol{P}(1^4, 2)$ *or* (b) $(3,6)\subset \boldsymbol{P}(1^4, 2, 3)$ *where the relation of degree* 3 *is a polynomial not containing the variable* x_6 *of degree* 3;
(II–4). *if* $(h, d)=(5,4)$, *then* $X=(2,6)\subset \boldsymbol{P}(1^5, 3)$.

COROLLARY (5.2). *Let* (X, L) *be a polarized* 3*-fold with* $\Delta(X, L)\leq 2$. *Then the deformation type of* (X, L) *are uniquely determined by the pair* (h, d). □

REMARK (5.3). (I–1), (II–1) are already known by T. Fujita [Fu8] in the course of his study on polarized manifolds of sectional genus two, and (I–2), (II–4) are also known by T. Fujita [Fu6] by his theory of hyperelliptic polarized manifolds. For the remaining two cases the graded ring structures (II–2) and (II–3) are new (cf. [Fu9, (10.10), (10.11)]).

The proof of us is based on the Reid's analysis (4.3) and the following theorem due to T. Fujita.

THEOREM (5.4) ([Fu4], [Fu5], [Fu9]). *Let* (X, L) *be a polarized manifold of dimension* $n\geq 3$. *Assume that* $\Delta\leq 2$ *and* $d:=L^n\geq 2$. *Then* $\dim Bs|L|\leq 1$ *and any general member of* $|L|$ *is smooth.* □

PROOF OF (5.1).

LEMMA (5.5). $(h, d)=(3,1)$, $(4,2)$ *if* $\Delta=1$ *and* $(h, d)=(2,1)$, $(3,2)$, $(4,3)$, $(5,4)$ *if* $\Delta=2$.

PROOF. Assume $d\geq 2$. By (5.4), general member of $|L|$, say S, is smooth with ample $K_S=L|_S$. Hence S is a minimal surface of general type. Note that $p_g(S)=h-1$. Hence $d=K_S^2\geq 2p_g(S)-4=2h-6$ by the Noether's inequality. Combining this with $\Delta=d+3-h$, we get the desired result. □

Next we will determine the graded ring for each (h, d). We only demonstrate the proof for the new cases:
Case 1. $(h, d)=(3, 2)$; Case 2. $(h, d)=(4, 3)$, while the verification here can be also applied to the known cases.

Case 1. $(h, d)=(3, 2)$. Let us take a smooth general $S\in|L|$.

CLAIM (5.6). (1) $\dim Bs|L|_S|=0$.
(2) any general $C\in|L|_S|$ is a smooth irreducible curve of genus$=3$ with $2L|_C=K_C$.

PROOF. (1) Assume contrary that $Bs|L|_S|$ contains a fixed curve B. Put $L_S:=L|_S$ and $|L_S|=|M|+B$. Then $2=L^3=L_S^2=M.L_S+B.L_S$. Hence $M.L_S=B.L_S=1$ because L_S is ample. Moreover, $1=M.L_S=M^2+M.B$. Since $M.B>0$ (by the ampleness of L_S) and $M^2\geq 0$ (because $|M|$ has no fixed curves), we get $M^2=0$. Then $2p_a(M)-2=M.(M+K_S)=M^2+M.K_S$ $=M^2+M.L_S=1$. But this is absurd.
(2) Since $\dim Bs|L_S|=0$ by (1), the assertion follows from the inequality $L_S^2=2<4$ and $g(C)=L^3+1$. □

Let us take a general $C\in|L|_S|$. By (0.3), we can apply the ladder method so that in order to complete the proof, it is enough to show the next claim.

CLAIM (5.7). $R(C, L|_C)$ is written as follows:
$R(C, L|_C)=\boldsymbol{C}[x_3, x_4, x_5, x_6]/(f, g)$ where $\deg x_3=1$, $\deg x_4=\deg x_5=2$, $\deg x_6$ $=3$, $\deg f=4$, and $\deg g=6$.

PROOF. As is easily seen (e.g. by Riemann-Roch), we know that $h^0(L|_C)=1$, $g(C)=h^0(2L|_C)=3$, and $h^0(tL|_C)=2t-2$ for $t\geq 3$. In particular, $(C, L|_C)$ is not in the exceptional cases of (4.3). Let x_3 be a basis of $H^0(L|_C)$. Then there are x_4, $x_5\in H^0(2L|_C)$ such that $\{x_3^2, x_4, x_5\}$ is a basis of $H^0(2L|_C)$. Since $x_3^3, x_3x_4, x_3x_5\in H^0(3L|_C)$ are linearly independent, there is $x_6\in H^0(3L|_C)$ such that $\{x_3^3, x_3x_4, x_3x_5, x_6\}$ is a basis of $H^0(3L|_C)$. Then $R(C, L|_C)$ is generated by four elements x_3, x_4, x_5, x_6 by (4.3). We will find the relations among them. Again by virture of (4.3), it is enough to find the relations of x_i's of degree$=4, 5, 6$.
degree$=4$. There are 7 monomials of x_i's in $H^0(4L|_C)$. Then there is one relation of degree 4, say f, because $h^0(4L|_C)=6$.
degree$=5$. There are 9 monomials of x_i's in $H^0(5L|_C)$ while $h^0(5L|_C)=8$. Then there is one relation among them. But this is nothing but x_3f.

degree$=6$. There are 14 monomials of x_i's in $H^0(6L|_C)$ while $h^0(6L|_C)=10$. Then there are four linearly independent relations among them. But there are exactly three linearly independent relations coming from f, say $x_3^2 f$, $x_4 f$, $x_5 f$. Hence there is one more relation among them. Hence (5.7) has been proved. □

Case 2. $(h, d)=(4, 3)$. As before, let us take a smooth $S\in|L|$.

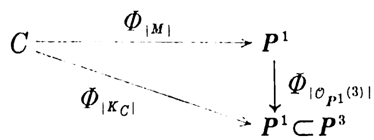
CLAIM (5.8). (1) $\dim Bs|L|_S|=0$.
(2) any general $C\in|L|_S|$ is a smooth irreducible curve with $2L|_C=K_C$, $h^0(L|_C)=2$, $h^0(2L|_C)=h^0(K_C)=4$, $h^0(tL|_C)=3t-3$ for $t\geq 3$. In particular, this $(C, L|_C)$ is not in the exceptional cases of (4.3).

PROOF. Similar to the one of (5.6). □

Let us take a general $C\in|L|_S|$.

CLAIM (5.9). (1) $Bs|L|_C|=\emptyset$ or $\{P\}$ (one point set).
(2) If $Bs|L|_C|=\{P\}$, then $|L|_C|=|2P|+P$.

PROOF. Let us put $|L|_C|=|M|+B$. Since $h^0(L|_C)=h-2=2$ and $\deg L|_C=L^3=3$, we have $\deg M=3$ or 2. If $\deg M=3$, then $B=0$. Assume that $\deg M=2$. Then $B=P$ (a single point). Put $P=\mathrm{div}(z)$ where $z\in H^0(B)$. Moreover, since $\Phi_{|M|}$ is a morphism of degree 2 onto $\boldsymbol{P}^1$ by $\deg M=h^0(M)=2$, C must be a hyperelliptic curve (of genus 4). In particular, we have the following commutative diagram:



Let $\{\varphi_0, \varphi_1\}$ be a basis of $H^0(M)=H^0(\mathcal{O}_{P^1}(1))$. Note that φ_0, φ_1 are algebraically independent over $\boldsymbol{C}$. Moreover, from the above diagram, $\{\varphi_0^3, \varphi_0^2\varphi_1, \varphi_0\varphi_1^2, \varphi_1^3\}$ is a basis of $H^0(K_C)$. On the other hand, $H^0(K_C)=H^0(2L|_C)$ contains $\varphi_0^2z^2$ and $\varphi_1^2z^2$ because $\varphi_0 z$, $\varphi_1 z\in H^0(L|_C)$. Hence $\varphi_0^2z^2$, $\varphi_1^2z^2$ are written as follows: $\varphi_0^2z^2=A\varphi_0^3+B\varphi_0^2\varphi_1+C\varphi_0\varphi_1^2+D\varphi_1^3$, $\varphi_1^2z^2=E\varphi_0^3+F\varphi_0^2\varphi_1+G\varphi_0\varphi_1^2+H\varphi_1^3$, where $A, \cdots, H$ are some constants. By eliminating z from these two equalities, we get:

$$(A\varphi_0^3+B\varphi_0^2\varphi_1+C\varphi_0\varphi_1^2+D\varphi_1^3)\varphi_1^2=(E\varphi_0^3+F\varphi_0^2\varphi_1+G\varphi_0\varphi_1^2+H\varphi_1^3)\varphi_0^2.$$

From this equality, we obtain the equality $C=D=0$ because φ_0 and φ_1 were algebraically independent over $\boldsymbol{C}$. Hence $\varphi_0^2 z^2 = A\varphi_0^3 + B\varphi_0^2\varphi_1$, i.e., $z^2 = A\varphi_0 + B\varphi_1$. This means $z^2 \in |M|$, i.e., $|M| = |2P|$. □

As before, we will calculate $R(C, L|_C)$ dividing into the two cases: Case (i) $Bs|L|_C| = \emptyset$; Case (ii) $Bs|L|_C| = \{P\}$.

Case (i).

CLAIM (5.10). $R(C, L|_C)$ is written as follows: $R(C, L|_C) = \boldsymbol{C}[x_3, x_4, x_5]/(f)$ where $\deg x_3 = \deg x_4 = 1$, $\deg x_5 = 2$, and $\deg f = 6$.

PROOF. Let $\{x_3, x_4\}$ be a basis of $H^0(L|_C) = H^0(\mathcal{O}_{P^1}(1))$. Since x_3, x_4 are algebraically independent over $\boldsymbol{C}$, there is $x_5 \in H^0(2L|_C)$ such that $\{x_3^2, x_3x_4, x_4^2, x_5\}$ is a basis of $H^0(2L|_C)$. Then the monomials of x_i $(i=3, 4, 5)$ contained in $H^0(3L|_C)$ are the following six:

$$x_3^3,\ x_3^2x_4,\ x_3x_4^2,\ x_4^3,\ x_3x_5,\ x_4x_5.$$

SUB-CLAIM (5.11). These six monomials are linearly independent in $H^0(3L|_C)$. In particular, these are the basis of $H^0(3L|_C)$.

PROOF. Assume contrary. Then we have the following equality: $(ax_3+bx_4)x_5 = \alpha x_3^3 + \beta x_3^2x_4 + \gamma x_3x_4^2 + \delta x_4^3$ where $(a, b) \neq (0, 0)$, $\alpha, \cdots, \delta$ are some constants. From this we have: $\varphi_1 x_5 = \varphi_2\varphi_3\varphi_4$ where φ_i are some non-zero elements of $H^0(L|_C)$. Put $Q \leq \operatorname{div}(\varphi_1)$. Then $Q \leq \operatorname{div}(\varphi_i)$ for some $i \in \{2, 3, 4\}$, say $i=2$. If $\operatorname{div}(\varphi_1) \neq \operatorname{div}(\varphi_2)$, then $\{\varphi_1, \varphi_2\}$ becomes a basis of $H^0(L|_C)$. Thus $Q \in Bs|L|_C|$. But this contradicts our assumption. Hence $\operatorname{div}(\varphi_1) = \operatorname{div}(\varphi_2)$. Thus $x_5 = c\varphi_3\varphi_4$ for some constant c. In particular, $x_5 \in \boldsymbol{C}\langle x_3^2, x_3x_4, x_4^2\rangle$. But this contradicts our choice of x_5. Hence the assertion holds. □

By Sub-Claim (5.11) and (4.3), $R(C, L|_C)$ is generated by three elements x_3, x_4, x_5. Moreover, all relations among them are generated by one polynomial of degree 6 by (4.3) and the following table (5.12).

Table (5.12).

	dimension	# (monomials of x_3, x_4, x_5)
$H^0(4L\|_C)$	9	9
$H^0(5L\|_C)$	12	12
$H^0(6L\|_C)$	15	16

This completes the proof of (5.10). □

Case (ii). Let us take basis of $H^0(\mathcal{O}_C(P))$ and $H^0(\mathcal{O}_C(2P))$ as in the following table (5.13).

Table (5.13).

	dimension	basis
$H^0(\mathcal{O}_C(P))$	1	z
$H^0(\mathcal{O}_C(2P))$	2	$y_1:=z^2$, y_2 (These two are algebraically independent over $\boldsymbol{C}$.)

We can easily see that basis of $H^0(L|_C)=H^0(\mathcal{O}_C(3P))$ and $H^0(2L|_C)$ are given by the next table (5.14).

Table (5.14).

	dimension	basis
$H^0(L\|_C)$	2	$x_3:=z^3=zy_1$, $x_4:=zy_2$
$H^0(2L\|_C)$	4	$x_3^2=z^6$, $x_3x_4=z^2y_2$, $x_4^2=z^2y_2^2$, $x_5:=y_2^2$

We consider $H^0(3L|_C)$. Note that there are exactly 6 monomials of x_3, x_4, x_5 in $H^0(3L|_C)$: x_3^3, $x_3^2x_4$, $x_3x_4^2$, x_4^3, x_3x_5, x_4x_5. By rewriting these by the variables z, y_2, we get the next:

CLAIM (5.15). These 6 monomials are related by $f:=x_4^3-x_3x_5=0$ and span the 5-dimensional subspace of $H^0(3L|_C)$. Hence there is one new generator x_6. □

Thus $R(C, L|_C)$ is generated by x_3, x_4, x_5 and x_6. On the other hand, we can at once see the following table.

Table (5.16).

	dimension	number of the monomials of $x_3, \cdots, x_6$	number of the (independent) relations coming from f
$H^0(4L\|_C)$	9	11	2 $(=h^0(L\|_C))$
$H^0(5L\|_C)$	12	15	3 $(=h^0(2L\|_C))$
$H^0(6L\|_C)$	15	22	6 $(=h^0(3L\|_C))$

Thus there is a new relation g of degree 6 and, again by (4.3), all the relations among $x_3, \cdots, x_6$ are generated by f and g. In order to complete the proof, it is enough to show the next

CLAIM (5.17). Let $W := C[w_3, w_4, w_5]$ be a weighted polynomial ring with $\deg w_3 = \deg w_4 = 1$ and $\deg w_5 = 2$. Let $h \in W$ be a general homogeneous element of degree 3. Then h is written as follows after a suitable change of homogeneous coordinates $w_i : h = w_4^3 - w_3 w_5$.

PROOF. First of all, any homogeneous element h of degree 3 is written as $h = a w_4^3 + b w_4^2 w_3 + c w_4 w_3^2 + d w_3^3 + (e w_4 + f w_3) w_5$, where a, b, c, d, e, f are constants. By generality, we may assume that $f \neq 0$. Replace w_3 by $(e w_4 + f w_3)$. Then, we may assume that h is written as $h = a w_4^3 + b w_4^2 w_3 + c w_4 w_3^2 + d w_3^3 + w_3 w_5$. By generality again, we may assume that $a \neq 0$. Then, by taking suitable constants $\alpha \neq 0$, β and replacing w_4 by $\alpha(w_4 + \beta w_3)$, we see that h is transformed into the following form:

$$h = w_4^3 + \gamma w_4 w_3^2 + \delta w_3^3 + w_3 w_5 = w_4^3 + w_3(\gamma w_4 w_3 + \delta w_3^2 + w_5).$$

Thus we get the result by putting $w_5 := -(\gamma w_4 w_3 + \delta w_3^2 + w_5)$. $\square$

This completes the proof of (5.1). Q.E.D.

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