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Intersection forms of 4-manifolds with a homology 3-sphere boundary

Dedicated to Professor Akio Hattori on his 60th birthday

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§ 1. Introduction.

In this paper we study the intersection forms of smooth 4-manifolds with a boundary. Through this paper the boundary will be a Z homology 3-sphere.

In [4], [5] and [6], S. K. Donaldson proved powerful three theorems on smooth closed 4-manifolds, Theorem A, B, C. Theorem A implies that any definite intersection form of smooth closed 4-manifold is diagonalizable over Z, and Theorem B (resp. C) implies that any even intersection form of 1-connected smooth closed 4-manifold with one positive part (resp. two positive parts) is equivalent over Z to the standard hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\left(\text{resp. } 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$.

On the other hand in [15], C. H. Taubes extended Theorem A to open definite end-periodic 4-manifolds. Now, in this paper we shall extend Theorem B and Theorem C for almost definite 4-manifolds with a boundary (i.e. almost definite open 4-manifolds with a product end). Essential difference between manifolds with boundary and closed manifolds is the influence of the fundamental group of the boundary. For example, Taubes pointed out in [15] that his theorem holds when the fundamental group of the boundary has the only trivial $SU(2)$ representation, but 'generally' false when it has non trivial $SU(2)$ representations. (e.g. Poincaré homology 3-sphere has two representations and bounds non diagonalizable definite $|E_8|$.) In this paper we use the same boundary condition as that of Taubes, which seems to be a strong condition. Roughly speaking, under the boundary condition the assertion is the same as in closed 4-manifolds cases.

In forthcoming paper [11], we will discuss the same problems under the weaker boundary condition in terms of some vanishing of Floer's instanton homology groups ([8]) or some modified instanton homology groups, which seem to be more natural in these problems.

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§ 2. Statement of the results.

In this paper, we shall show the following theorems.

THEOREM 1. *Let X be a compact, simply connected, oriented smooth spin 4-manifold with a Z homology 3-sphere boundary S. If the intersection form has one positive part and $\pi_1(S)$ has only the trivial representation into $SU(2)$, then the form is equivalent over Z to* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

THEOREM 2. *Let X be a compact, simply connected, oriented smooth spin 4-manifold with a Z homology 3-sphere boundary S. If the intersection form has two positive parts and $\pi_1(S)$ has only the trivial representation into $SU(2)$, then the form is equivalent over Z to* $2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

REMARK 1.1. (1) According to [15] we can generalize Theorem 1 and Theorem 2 to the case of an end-periodic 4-manifold. But we state our theorems as above for simplicity.

(2) By the argument of Donaldson in the closed manifold case [6], perturbing the anti-self-dual equations, we may replace the assumption of simply connectedness of X by the one that $H_1(X; Z)$ has no 2-torsion. But the details are omitted here.

As some applications of the theorems, we can obtain;

COROLLARY 1. *Any homotopy 3-sphere which bounds a simply connected spin 4-manifold such that its intersection form has one positive part or two positive parts has zero Rohlin invariant.*

A. Casson proved that any homotopy 3-sphere has zero Rohlin invariant by introducing his invariant (Casson's invariant) which is equal (mod 2) to Rohlin invariant ([1]). Corollary 1 implies that we can show that a certain homotopy 3-sphere has zero Rohlin invariant not by using Casson's invariant, but by definition of Rohlin invariant directly.

COROLLARY 2. *Let n be any negative integer and X_n be a closed simply connected TOP 4-manifold with the intersection form* $nE_8\oplus\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ *or* $nE_8\oplus 2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. *(Here E_8 is the positive definite Cartan matrix for the exceptional Lie algebra e_8.) Then X_n has no simplicial triangulation.*

REMARK 1.2. Freedman's classification theorem ([9]) implies that for each n such a TOP 4-manifold X_n exists uniquely in TOP category. A. Casson has proved (in his lecturers given at M.S.R.I., 1985 (unpublished, see [15])) that any closed TOP 4-manifold with non trivial Kirby-Siebenmann obstruction has no simplicial triangulation. On the other hand for any simply connected closed spin TOP 4-manifold, its Kirby-Siebenmann obstruction is equal to the signature/8 (mod 2) [14]. So in our case, Casson's theorem asserts that when n is odd, then the manifold in Corollary 2 has no simplicial triangulation. When n is even, Corollary 2 yields new examples of closed simply connected TOP 4-manifolds with no simplicial triangulations.

PROOF OF COROLLARY 2. If X_n were triangulated, it would be PL except at a finite number of vertices. We may assume these vertices are only single point p. The link L of p is a homotopy 3-sphere (fake 3-sphere). Now, since $X_n - p$ is PL, it is smoothable along the PL structure. Then the end of $X_n - p$ is a smooth product end $L\times \boldsymbol{R}$. This contradicts Theorem 1 or 2. (See [15].) ∎

We shall prove above the theorems by using gauge theory. The idea of the proof is the combination of the analysis on an end-periodic 4-manifold by Taubes [15] (and also on a product end 4-manifold by Lockhart and McOwen [13]) modified to the case of instanton number 2 or 3 and the method of Donaldson [5] [7] modified to the end-periodic manifold case. Compared with the closed case, the most different point is that the moduli space of the anti-self-dual (ASD) connections has "sliding off" ends. But the key is that cutting the moduli space by the zero sets of transversal sections of certain line bundles (see § 4) in a similar manner as Donaldson's, we can avoid such "sliding off" ends. Here we use the boundary condition $\mathrm{Hom}(\pi_1(S), SU(2))=\{1\}$ essentially.

(See § 5.)

We shall prove Theorem 1. Since Theorem 2 is got in a similar way, we shall only sketch the outline of the proof.

§ 3. Reduction and the gauge theory setting on a 4-manifold with a product end.

First, we reduce the claim of Theorem 1 to that of conformity with the gauge theory framework.

According to the Hasse-Minkowski classification, any even unimodular indefinite quadratic form over Z is equivalent to $nE_8 \oplus m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for some $n \in Z, m \geq 0$. Moreover for any even unimodular form of rank r $(\ ,\ )$: $Z^r \times Z^r \rightarrow Z$, we know the following equivalence;

$$\begin{aligned} r \leq 2 \Longleftrightarrow\ & Q_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ & := (\alpha_1, \alpha_2)(\alpha_3, \alpha_4) + (\alpha_1, \alpha_3)(\alpha_2, \alpha_4) + (\alpha_1, \alpha_4)(\alpha_2, \alpha_3) \\ & \equiv 0 \pmod 2 \\ & \text{for all } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in Z^r. \end{aligned}$$

In our case, since the boundary is a Z homology 3-sphere, the intersection form of X is unimodular by Alexander-Poincaré duality. Under the assumption that $H_1(X; Z)$ has no 2-torsion, X is a spin 4-manifold if and only if the intersection form is of even type. Therefore the form is equivalent to $nE_8 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for some $n \leq 0$ and it suffices to show that $Q_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \equiv 0 \pmod 2$.

Let Σ_i $(i=1,2,3,4)$ be a surface in general position in X, which represents the homology class $\alpha_i \in H_2(X, \partial X; Z) \cong H_2(X; Z)$ smoothly. We note that since the boundary of X is a Z homology 3-sphere, we may choose Σ_i as a closed surface (cf. [10] Appendix E). Then, geometrically $Q_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = Q_4([\Sigma_1], [\Sigma_2], [\Sigma_3], [\Sigma_4])$ is equal (mod 2) to the number of configurations of unordered couples $\{x_1, x_2\}$ of points in X such that each surface Σ_i contains a point x_j. Namely $x_1 \in \Sigma_i \cap \Sigma_j$, $x_2 \in \Sigma_k \cap \Sigma_l$, here (i, j, k, l) is some permutation of $(1, 2, 3, 4)$. The expression Q_4 will appear as the number of boundary components of 2-multi-instanton solutions.

In the rest of this section, we recall the gauge theory setting on a

4-manifold with a product end according to Taubes [15].

Let X be a 4-manifold with a boundary S as Theorem 1 or Theorem 2. We define a smooth 4-manifold M with a product end to be:

$$M = X \cup_S S \times [0, \infty).$$

Fix a Riemannian metric on S and choose and fix a metric on M which is the product metric of the given metric on S and the standard metric on $[0, \infty)$ over End M. Let $P = M \times SU(2) \to M$ be a trivial principal $SU(2)$ bundle, and Ad P be the $su(2)$ $(=\boldsymbol{R}^3)$ bundle associate with P by the adjoint representation (or the standard representation). Let

$$\mathcal{E}\mathcal{A}(P) := \{\text{connections on } P \text{ which are isomorphic over End } M \text{ to the trivial flat connection on End } M \times SU(2)\}.$$

For each $A \in \mathcal{E}\mathcal{A}(P)$, we set

$$p_1(A) = -\frac{1}{8\pi^2}\int_M \mathrm{tr}(F_A \wedge F_A), \tag{3.1}$$

where F_A is the curvature of A, which is Ad P-valued 2-form. Then $p_1(A)$ is an integer and the value depends on A (different from closed cases). (See [15], Lemma 7.1.)

We consider the following classes of connections and gauge transformations on P. For each $k \in \mathrm{Im}(p_1 : \mathcal{E}\mathcal{A}(P) \to \boldsymbol{Z})$, fix $A_0 \in \mathcal{E}\mathcal{A}(P)$ with $p_1(A_0) = k$. Fix a positive number δ, and let

$$\mathcal{A}_k(\delta) := \{A_0 + a \mid a \in L^2_{2,\mathrm{loc}}(\mathrm{Ad}\ P \otimes T^*M),\ \|a\|_\delta < \infty\}.$$

Here we fix a smooth map $\tau : M \to \boldsymbol{R}$ which is equal to the projection from $S \times [0, \infty)$ to $[0, \infty)$ over $S \times [0, \infty)$, and is zero on X. Then we define the weighted Sobolev norm $\| \ \|_\delta$ by

$$\|a\|^2_{A_0, \delta} = \int_M e^{\tau\delta}\{|\nabla^{(2)}_{A_0} a|^2 + |\nabla_{A_0} a|^2 + |a|^2\},$$

where ∇_{A_0} is a covariant derivative defined by A_0 and the Levi-Civita connection with respect to the given metric on M. The affine space $\mathcal{A}_k$ has a Banach manifold structure with respect to this norm.

With the given $A_0 \in \mathcal{E}\mathcal{A}(P) \cap p_1^{-1}(k)$, we define the gauge transformations on P by

$$\mathcal{G}_k := \{g \in L^2_{3,\mathrm{loc}}(\mathrm{Aut}\ P) \mid \|\nabla_{A_0} g\|_\delta < \infty\}.$$

The topology of $\mathcal{G}_k$ is defined by using $\|\nabla_{A_0} g\|_\delta$ and the 'limitting value'. (See [15] § 7.) Then the quotient space $\mathcal{B}_k=\mathcal{A}_k/\mathcal{G}_k$ with quotient topology is a C^∞-Banach manifold away from reducible orbits. We note that $\mathcal{B}_k$ does not depend on the choice of A_0 in $\mathcal{E}\mathcal{A}(P)\cap p_1^{-1}(k)$.

Now, let $*$ be the Hodge's star operator acting on Ad P-valued forms, and $F_+=1/2(F+*F):\mathcal{A}_k\to L^2_{1,\mathrm{loc}}(\mathrm{Ad}\,P\otimes\wedge^2 T^*M)$ be the self-dual part of curvature. The moduli space of the anti-self-dual connections $M_k(\delta)$ is defined to be $M_k(\delta)=F_+^{-1}(0)/\mathcal{G}_k$ with the quotient topology. Then by [15 ; § 8] we know some facts about the interior structure of $M_k(\delta)$. Namely there exists $\delta_1>0$ such that for all $\delta\in(0,\delta_1)$ and for generic metrics on M, $M_k(\delta)$ contains no orbits of reducible connections when M has an indefinite intersection forms (Proposition 8.3). Further there exists $\delta_2>0$ such that for all $\delta\in(0,\delta_2)$ and for generic metrics on M, $M_k(\delta)$ is a smooth manifold of dimension

$$\dim M_k(\delta)=8k-3(1-b_1(X)+b_2^+(X)), \tag{3.2}$$

here $b_1(X)$ means the first Betti number of X and $b_2^+(X)$ means the maximal dimension of the subspace of $H_2(X,\partial X;\boldsymbol{R})\cong H_2(X;\boldsymbol{R})$ on which the intersection form is positive definite. Note that we use here the Pontrjagin charge $p_1=k$ defined in (3.1) where $\mathrm{tr}(\cdot)$ is not the trace on the adjoint representation of $SU(2)$ like as [15], but on the standard representation. So k in [15] is just our k multiplied by 4. (See Proposition 8.2 and also [2]). Thus we fix δ and metric on M as above, and after this we omit the suffix δ.

§ 4. The method of Donaldson.

First, we prepare some notations. As in § 3, let $\Sigma_i\subset X$ be a closed surface which represents the homology class $\alpha_i\in H_2(X,\partial X;\boldsymbol{Z})$. Let

$\mathcal{A}^*_{\Sigma_i}:=$ {all irreducible connections on $P|_{\Sigma_i}=\Sigma_i\times SU(2)$ over Σ_i},

$\mathcal{G}_{\Sigma_i}:=$ the gauge group of automorphisms of $P|_{\Sigma_i}$,

$\mathcal{G}_{\Sigma_i,0}:=$ the subgroup of automorphisms which fix the fibre P_{x_0} over a base point x_0 in Σ_i.

We define,

$$\mathcal{B}^*_{\Sigma_i}:=\mathcal{A}^*_{\Sigma_i}/\mathcal{G}_{\Sigma_i},\qquad \widetilde{\mathcal{B}_{\Sigma_i}}^*:=\mathcal{A}^*_{\Sigma_i}/\mathcal{G}_{\Sigma_i,0}.$$

Then we have the following principal fibration,

$$SO(3)=\pm 1\backslash SU(2)\longrightarrow \widetilde{\mathcal{B}_{\Sigma_i}}^{*}\longrightarrow \mathcal{B}^{*}_{\Sigma_i}.$$

Now, following Donaldson [5], we can construct a line bundle over $\mathcal{B}^*_{\Sigma_i}$ as the determinant index line bundle of twisted Dirac operators with the vector bundle coefficients. Let $A\in\widetilde{\mathcal{B}_{\Sigma_i}}^{*}$ and let E be a complex vector bundle over Σ_i associated to the principal $SU(2)$ bundle $P|_{\Sigma_i}$. Fixing a spin structure of Σ_i, we denote a positive (negative) spinor bundle by V_i^+ (V_i^-). Then the Dirac operator $D_{\Sigma_i,A}$ twisted by A is

$$D_{\Sigma_i,A}:\Gamma(V_i^+\otimes_c E)\longrightarrow\Gamma(V_i^-\otimes_c E),$$

which has numerical index 0. Setting

$$\mathrm{Det}(\mathrm{ind}\,D_{\Sigma_i,A})=\left(\overset{\max}{\wedge}\mathrm{Ker}\,D_{\Sigma_i,A}\right)^{*}\otimes\left(\overset{\max}{\wedge}\mathrm{Coker}\,D_{\Sigma_i,A}\right),$$

the family parametrized by $A\in\widetilde{\mathcal{B}_{\Sigma_i}}^{*}$ defines the complex line bundle $\mathcal{L}_{\Sigma_i}$ over $\widetilde{\mathcal{B}_{\Sigma_i}}^{*}$;

$$\mathcal{L}_{\Sigma_i}=\mathrm{Det}(\mathrm{ind}\,D_{\Sigma_i})\in K(\widetilde{\mathcal{B}_{\Sigma_i}}^{*}).$$

Using the Atiyah-Singer index formula for families ([3], see also [5] Proposition 2.26), this line bundle $\mathcal{L}_{\Sigma_i}$ descends from $\widetilde{\mathcal{B}_{\Sigma_i}}^{*}$ to $\mathcal{B}^*_{\Sigma_i}$. Thus we obtain

$$\mathcal{L}_{\Sigma_i}\in K(\mathcal{B}^*_{\Sigma_i}).$$

We note that the isomorphism class of this line bundle is independent of spin structures of Σ_i but only depends on the homology class of Σ_i in X.

Now, this line bundle $\mathcal{L}_{\Sigma_i}$ extends continuously to

$$\mathcal{B}^*_{\Sigma_i}\cup\{\text{degree zero reductions on }\Sigma_i\}.$$

Note that the trivial flat connection over Σ_i is included in {degree zero reductions on Σ_i}. We pull back to M_k^* this line bundle $\mathcal{L}_{\Sigma_i}$ over $\mathcal{B}^*_{\Sigma_i}\cup$ {degree zero reductions} by restriction map

$$r_{\Sigma_i}:M_k^*\longrightarrow\mathcal{B}^*_{\Sigma_i}\cup\{\text{degree zero reductions}\}$$

(if necessary, perturbing Σ_i in the homology class. See [5; § III (iii)]). For the section s of $\mathcal{L}_{\Sigma_i}$ over $\mathcal{B}^*_{\Sigma_i}\cup$ {degree zero reductions}, we define

$$V_{\Sigma_i} := (r^*_{\Sigma_i} s)^{-1}(0) \subset M^*_k.$$

Then, the transversality Lemma holds. ([5, Lemma 3.16], [7, Proposition 2.4])

LEMMA 4.1. (1) *For each i, there is a section s_i of $\mathcal{L}_{\Sigma_i} \to \mathcal{B}^*_{\Sigma_i} \cup$ {degree zero reductions} such that all the pull back sections of the $r^*_{\Sigma_i}(\mathcal{L}_{\Sigma_i})$ over M^*_k vanish transversally on codimension 2 submanifolds $V_{\Sigma_i} \cap M^*_k$.*

(2) *Above section s_i can be chosen so as not to vanish on a neighborhood of the trivial flat connection over Σ_i.*

(3) *We may choose sections s_i of the bundles $\mathcal{L}_{\Sigma_i}$ as in* (1) *such that all multiple intersections:*

$$V_{\Sigma_{i_1}} \cap \cdots \cap V_{\Sigma_{i_p}} \cap M^*_k$$

are transverse.

PROOF. The key of the proof of this lemma is Sard's theorem in the usual finite dimensional case applied to M^*_k. In our case, although the based manifold M is not closed, M^*_k is a finite dimensional manifold. So the proof is similar to that of Lemma 3.16 in [5]. ∎

Now, choosing the transversal sections s_j ($j=1,2,3,4$) in above the lemma, we define the main moduli space N_2 to be;

$$N_2 := M^*_2 \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_4}.$$

By the transversality and (3.2) we obtain

$$\dim N_2 = \dim M^*_2 - 2 \times 4 = 10 - 8 = 2.$$

§ 5. Ends of moduli and transversality argument.

In this section we study the ends of M_2 and which of them can be avoided as the ends of N_2 using the transversality argument (dimension counting and Lemma 4.1 (and also Uhlenbeck's L^p-boundness theorem [16])).

First, we study the bubble theorem in an end-periodic version ([15] § 10, [10] chap. 8). Essentially the bubble theorem is a local theorem, so it is valid even for non compact manifolds. Namely, we establish;

PROPOSITION 5.1 (Bubble Theorem [15; Lemma 10.2]). *Let M be an* *end-periodic 4-manifold with $\pi_1(M)=1$. Let $P=M\times SU(2)\to M$ be a trivial principal $SU(2)$ bundle. For a generic C^m-metric on M, the following is true.*

Suppose $\{[A_j]\}\subset M_k$, $(k\geq 0)$.

(1) *There exist an anti-self-dual connection A with*

$$\frac{1}{8\pi^2}\int_M |F_A|^2=l\leq k,$$

(here, l is an integer), a finite set of points $\{x_1,\cdots,x_\alpha\}\in M$, a sequence of the gauge transformations $\{h_j\}\in C^\infty$ $(\mathrm{Aut}\,P|_{M-\{x_1,\cdots,x_\alpha\}})$ and a subsequence $\{[A_j]\}$ (now relabeled) such that $\{h_j^*A_j\}$ converges on compact domains in $M-\{x_1,\cdots,x_\alpha\}$ in the C^m-topology.

(2) *Under* (1), *suppose* $\mathrm{Hom}(\pi_1(S), SU(2))=1$. *The sequence $\{h_j\}$ can be chosen ('the exponential gauge') so that $[A]\in M_l$.*

(3) *Suppose* $\mathrm{Hom}(\pi_1(S), SU(2))=1$ *and that $\{[A_j]\}$ satisfies*

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=0. \tag{5.2}$$

Then, there exists $\delta_1>0$ and $n<\infty$ such that

$$\lim_{j\to\infty}\int_{\tau\geq n} e^{\tau\delta}\sum_{p=0}^{m}|\nabla_A^{(p)}(A-h_j^*A_j)|^2=0$$

for all $\delta\in(0,\delta_1)$. Further, $l=k$ if and only if $\{x_\alpha\}=\emptyset$, whence $[h_j^*A_j]\to[A]$ in M_k.

The assertion (2) is stated by Taubes in [15] under the assumption that (5.2) holds. But in our case since we deal with the moduli space of instanton number 2 or 3, we need the exponential decay estimates of 'remaining connections' after 1 or 2-instantons slide off away. (That is, in the case when (5.2) does not hold. See Case (2)—a below.) Thus we prove only the assertion (2). The other is Lemma 10.2 in [15].

PROOF OF PROPOSITION 5.1 (2). We show that A has the exponential decay estimate. By the construction of A in the proof of Lemma 10.2 in [15], for given $\varepsilon>0$, there exists $n_0<\infty$ such that

$$\int_{\tau\geq n_0}|F_A|^2<\varepsilon$$

and $\{x_1, \cdots, x_\alpha\} \subset \{x \mid \tau(x) < n_0\}$.

Now to construct the exponential gauge the following lemma is crucial.

LEMMA 5.3 ([15; Lemma 10.5]). *Let M be as in Proposition* 5.1 (1) *and* (2) *and suppose* $\mathrm{Hom}(\pi_1(S), SU(2))=1$. *There exists $\rho<\infty$ and $\varepsilon>0$, with the following significance: Let A be an anti-self-dual connection on* End $M\times SU(2)$. *Suppose that a sufficient large $n<\infty$ exists such that*

$$\int_{\tau\geq n} |F_A|^2 < \varepsilon.$$

Then there exists an element $h \in C^{m+1}(\tau^{-1}[n, \infty); SU(2))$ such that

$$\sup_{j+1\leq\tau\leq j+2} \left\{\sum_{p=0}^{m} |\nabla_\Gamma^{(p)}(h^*A-\Gamma)|^2\right\} \leq \rho \int_{j\leq\tau\leq j+4} |F_A|^2 \tag{5.4}$$

for all $j\geq n$, where Γ is the product connection on $\tau^{-1}([n, \infty))\times SU(2)$.

REMARK 5.5. The condition $\mathrm{Hom}(\pi_1(S), SU(2))=1$ is too strong. In [12] Furuta proved the similar claim under the weaker boundary condition. (There, S is a rational homology 3-sphere. See Lemma 4.2 in [12].)

Thus by Lemma 5.3 there is $h \in C^{m+1}(\tau^{-1}[n_0, \infty); SU(2))$ such that, if we set $a=h^*A-\Gamma$, then

$$\sup_{n+1\leq\tau\leq n+2} \left\{\sum_{p=0}^{m} |\nabla_\Gamma^{(p)}a|^2\right\} \leq \rho \int_{n\leq\tau\leq n+4} |F_A|^2$$

for all $n\geq n_0$. Therefore

$$\int_{\tau\geq n_0+1} e^{\tau\delta} \sum_{p=0}^{m} |\nabla_\Gamma^{(p)}a|^2 \leq C_1 \sum_{n=n_0}^{\infty} e^{(n+3)\delta} \|F_A\|^2_{L^2(n\leq\tau\leq n+4)}. \tag{5.6}$$

Now we claim the following.

LEMMA 5.7. *Let M and A be as in Proposition* 5.1 (2). *Then there exists $\gamma>0$ and $C>0$ depending only on S with the following significance: Suppose A is an anti-self-dual connection on P satisfying the condition of Lemma* 5.3, *then the following decay estimate holds.*

$$|F_A|^2 \leq Ce^{-\tau\gamma} \|F_A\|^2_{L^2}.$$

Admitting Lemma 5.7 for a moment, we continue the proof of Proposition 5.1 (2). Due to (5.6) and Lemma 5.7, we have

$$\begin{aligned}&\int_{\tau\geq n_0+1}e^{\tau\delta}\sum_{p=0}^{m}|\nabla_\Gamma^{(p)}a|^2\\ &\leq C_1\sum_{n=n_0}^{\infty}e^{(n+3)\delta}\frac{1}{8\pi^2}\int_{n\leq\tau\leq n+4}|F_A|^2\\ &\leq C_1\sum_{n=n_0}^{\infty}e^{(n+3)\delta}\frac{1}{8\pi^2}\int_{n\leq\tau\leq n+4}Ce^{-\tau\gamma}\|F_A\|^2\\ &\leq C_1\sum_{n=n_0}^{\infty}e^{(n+3)\delta}Ce^{-n\gamma}\,4\mathrm{vol}(S)\varepsilon\\ &=C_2e^{-n_0(\gamma-\delta)}\end{aligned}$$

where $C_2=4C_1C\varepsilon e^{3\delta}\mathrm{vol}(S)(1-e^{-(\gamma-\delta)})^{-1}$. We denote by $\|a\|^2_{\delta,[n_0+1,\infty)}$ the left hand side. Hence choosing $\delta<\gamma$, we have

$$\|a\|^2_{\delta,[n_0+1,\infty)}\leq C_2e^{-n_0(\gamma-\delta)}<\infty$$

which implies $A\in M_l$.

To show Lemma 5.7, we use Furuta's technique in [12 § 5].

CLAIM. *There exists $C_3>0$ depending only on S with the following significance. Suppose A is an anti-self-dual connection on P satisfying the condition of Lemma* 5.3. *Then we have*

$$\|F_A\|^2_{L^2(S\times[n_0,n_1])}<C_3(\|F_A\|^2_{L^2(S\times[n_0-1,n_0+2])}+\|F_A\|^2_{L^2(S\times[n_1-2,n_1+1])})$$

for any integers $n_0<n_1$.

Lemma 5.7 is an immediate consequence of the Claim together with following elementary lemma.

LEMMA 5.8 ([12, Lemma 5.4]). *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of non negative numbers satisfying $\sum_{k=1}^{\infty}a_k<\infty$. Suppose there is an integer N such that for any integers $k_0<k_1$,*

$$\sum_{k=k_0}^{k_1}a_k\leq N(a_{k_0}+a_{k_1}).$$

Then we obtain

$$a_k\leq 4Na_1 2^{-(k/2N)}.$$

So it suffices to show above Claim. By Lemma 5.3, there is h such that

$$(5.9)\qquad \sup_{n+1\leq\tau\leq n+2}(|a|^2+|\nabla_\Gamma a|^2)\leq C\|F_A\|^2_{L^2(S\times[n,n+3])}\leq C\varepsilon$$

for $a=h^*A-\Gamma$. We set $T(a):=\mathrm{tr}\left(da\wedge a+\frac{2}{3}a\wedge a\wedge a\right)$ so that $dT(a)=\mathrm{tr}(F_A\wedge F_A)$. Since F_A is anti-self-dual, $|F_A|^2=\mathrm{tr}(F_A\wedge F_A)$. Then, for integers $n_0<n_1$, we have

$$(5.10)\qquad \|F_A\|^2_{L^2(S\times[n_0,n_1])}=\int_{S\times n_1}T(a)-\int_{S\times n_0}T(a).$$

The claim follows from (5.9) and (5.10).

Therefore this completes the proof of Proposition 5.1 (2). ■

Note that by Proposition 5.1 (1) and (2), when $[A_j]\subset M_k$ and

$$\lim_{j\to\infty}\sup_{x\in M}|F_{A_j}|(x)<\infty,$$

then there exist h_j and $A\in M_l$ such that

$$\lim_{j\to\infty}\int_{\tau\leq n}\sum_{p=0}^{m}|\nabla_A^{(p)}(A-h_j^*A_j)|^2=0$$

for all $n<\infty$. Thus

$$\lim_{n\to\infty}\lim_{j\to\infty}\int_{\tau\leq n}e^{\tau\delta}\sum_{p=0}^{m}|\nabla_A^{(p)}(A-h_j^*A_j)|^2=0.$$

Next, we study "sliding off" ends. The boundary condition is essentially used here.

PROPOSITION 5.11 (Gap Theorem). *Suppose that M is as in Proposition 5.1 and* $\mathrm{Hom}(\pi_1(S), SU(2))=1$.

(1) *If* $\{[A_j]\}\subset M_2$ *obeys*

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2\geq\varepsilon>0,$$

then

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2\geq 1.$$

(2) *If* $\{[A_j]\}\subset M_2$ *obeys*

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2>1,$$

then

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2\geq 2.$$

PROOF OF PROPOSITION 5.11. We can prove this proposition by using the Chern-Weil formula and the boundary condition.

The assertion (1) follows from [15] Lemma 10.3 directly. So we prove the assertion (2), but the proof goes in a similar way. The crucial part is the following Uhlenbeck's L^p-boundness theorem ([16], [10; Theorem 8.8], [15]). Here we refer to [15].

LEMMA 5.12 ([15] Lemma 10.4). *Let U be an oriented open noncompact C^m-Riemannian 4-manifold ($m\gg 2$). Let $Q\subset U$ be a smooth submanifold with compact closure, $\bar{Q}\subset U$. Let $P\to U$ be a principal G-bundle. There exists $\varepsilon>0$ and $\zeta<\infty$ which depend on U, Q, P, and a C^m-neighborhood of the Riemannian metric on U with the following significance:*

Let A be an anti-self-dual connection on P with $\int_U|F_A|^2<\varepsilon$. Then $h\in C^{m+1}(\bar{Q}\,;G)$ exists such that

$$\sup_Q\left\{\sum_{p=0}^m|\nabla_\Gamma^{(p)}(h^*A-\Gamma)|^2\right\}\leq\zeta\int_U|F_A|^2,$$

where Γ is a flat connection on $P|_Q$.

Now by Proposition 5.1 (1), there exist an anti-self-dual connection A on M with

$$\frac{1}{8\pi^2}\int_M|F_A|^2<\infty, \tag{5.13}$$

a sequence of the gauge transformations $\{h_j\}$ and a subsequence of $\{[A_j]\}$ (now relabeled) such that $\{h_j^*A_j\}$ converges to A on compact domains of M—finite set of points in C^{m+1} (M—finite set of points) topology.

(5.13) implies that, given $\varepsilon>0$, there exists $n<\infty$ such that

$$\int_{\tau\geq n}|F_A|^2<\varepsilon.$$

We can choose n so that the finite set above does not lie in $\tau^{-1}[n, n+3]$. We denote the segment $\{x\in M|\,\tau(x)\in[n, n+1]\}$ by W_n. Lemma 5.12 provides $h\in C^{m+1}(W_{n+1}\,;SU(2))$ such that

(5.14) $$\sup_{W_{n+1}}\left(\sum_{p=0}^{m}|\nabla_{\Gamma}^{(p)}(h^*A-\Gamma)|^2\right)<\zeta\varepsilon$$

for some ζ. Here, because $\mathrm{Hom}(\pi_1(W_{n+1}), SU(2))=1$, Γ is the trivial flat connection on $W_{n+1}\times SU(2)$.

For all j sufficiently large, (5.14) implies that

$$\sup_{W_{n+1}}\left(\sum_{p=0}^{m}|\nabla_{\Gamma}^{(p)}(h^*h_j^*A_j-\Gamma)|^2\right)<\zeta\varepsilon.$$

On the other hand since $A_j\in\mathcal{A}_k$ and A_j is anti-self-dual, given $\varepsilon>0$ and $j<\infty$, there exists $Q=Q(j)<\infty$ such that

$$\int_{\tau\geq Q+n} e^{\tau\delta}|F_{A_j}|^2<\varepsilon.$$

Therefore applying Lemma 5.12 to A_j and W_{Q+n+1}, we get $q_j\in C^{m+1}(W_{Q+n+1}, SU(2))$ such that

$$\sup_{W_{Q+n+1}}\left(\sum_{p=0}^{m}|\nabla_{\Gamma}^{(p)}(q_j^*A_j-\Gamma)|^2\right)<\zeta\varepsilon.$$

Here again Γ is the trivial flat connection on $W_{Q+n+1}\times SU(2)$ (by definition of $\mathcal{A}_k$).

Now, let $\beta_n\in C^\infty(M)$ obey $\beta_n\equiv1$ if $\tau\geq n+1$, $\beta_n\equiv0$ if $\tau\leq n$, and $|d\beta_n|<10$. For large j, define a connection (and principal $SU(2)$ bundle) on $B_\varepsilon^j:=W_n\cup W_{n+1}\cup\cdots\cup W_{n+Q+1}\cup W_{n+Q+2}$, by specifying

$$\bar{A}_j=\begin{cases}\beta_{n+1}(hh_j)^*A_j & \text{on } \tau^{-1}[n, n+2]\\ A_j & \text{on } \tau^{-1}[n+2, n+Q+1]\\ (1-\beta_{n+Q+1})q_j^*A_j & \text{on } \tau^{-1}[n+Q+1, n+Q+3]\end{cases}$$

which satisfies $\bar{A}_j=\Gamma$ (the trivial flat connection) on W_n and W_{Q+n+2}. So furl up B_ε^j to obtain the closed manifold Y_ε^j to be

$$Y_\varepsilon^j:=B_\varepsilon^j/\sim,\quad \tau^{-1}[n, n+1/2]\sim\tau^{-1}[n+Q+2+1/2, n+Q+3].$$

The connection $\bar{A}_j$ descends to Y_ε^j, say $\tilde{A}_j$. (The bundle $\tilde{P}$ on Y_ε^j is twisted by the isotropy group of the trivial flat connection on S ($\cong SU(2)$).)

Then, for sufficient large j, we have the following estimates

(5.15) $$\frac{1}{8\pi^2}\int_{Y_\varepsilon^j}|P_+F_{\tilde{A}_j}|^2\leq\zeta\varepsilon$$

$$0 \leq \frac{1}{8\pi^2}\int_{\tau \geq n} |F_{A_j}|^2 - \frac{1}{8\pi^2}\int_{Y^j_\varepsilon} |F_{\tilde{A}_j}|^2 \leq \zeta\varepsilon. \tag{5.16}$$

Here P_+ (P_-) means the projection to the self-dual (anti-self-dual) part. According to the Chern-Weil formula, we have

$$\begin{aligned} \frac{1}{8\pi^2}\int_{Y^j_\varepsilon} |F_{\tilde{A}_j}|^2 &= \frac{1}{8\pi^2}\int_{Y^j_\varepsilon} (|P_+F_{\tilde{A}_j}|^2 + |P_-F_{\tilde{A}_j}|^2) \\ &= \frac{2}{8\pi^2}\int_{Y^j_\varepsilon} |P_+F_{\tilde{A}_j}|^2 + c_2(\tilde{P}) \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \frac{1}{8\pi^2}\int_{Y^j_\varepsilon} |F_{\tilde{A}_j}|^2 &\geq \frac{1}{8\pi^2}\int_{Y^j_\varepsilon} (|P_-F_{\tilde{A}_j}|^2 - |P_+F_{\tilde{A}_j}|^2) \\ &= c_2(\tilde{P}). \end{aligned} \tag{5.18}$$

Therefore (5.17) and (5.18) implies that

$$\begin{aligned} c_2(\tilde{P}) &\leq \frac{1}{8\pi^2}\int_{Y^j_\varepsilon} |F_{\tilde{A}_j}|^2 \\ &= \frac{2}{8\pi^2}\int_{Y^j_\varepsilon} |P_+F_{\tilde{A}_j}|^2 + c_2(\tilde{P}). \end{aligned}$$

From (5.15), we have

$$c_2(\tilde{P}) \leq \frac{1}{8\pi^2}\int_{Y^j_\varepsilon} |F_{\tilde{A}_j}|^2 \leq 2\zeta\varepsilon + c_2(\tilde{P}). \tag{5.19}$$

By (5.16) plus (5.19), we obtain

$$c_2(\tilde{P}) \leq \frac{1}{8\pi^2}\int_{\tau \geq n} |F_{A_j}|^2 \leq 3\zeta\varepsilon + c_2(\tilde{P}). \tag{5.20}$$

Hence noticing that $c_2(\tilde{P})$ is an integer, we see that if

$$\lim_{n\to\infty} \lim_{j\to\infty} \frac{1}{8\pi^2}\int_{\tau \geq n} |F_{A_j}|^2 > 1,$$

then $c_2(\tilde{P}) = 2$. This implies the assertion (2) of Proposition 5.11. ■

By this proof, we obtain more generally the following

PROPOSITION 5.11. (3) *Let l be an integer with $0 \leq l < k$. If* $\{[A_j]\} \subset$

M_k obeys

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2>l$$

then

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2\geq l+1.$$

REMARK 5.21. When $\mathrm{Hom}(\pi_1(S), SU(2))\neq 1$, then the above 'integral' Gap Theorem does not hold. Non trivial flat connections on S appear on the "feet" of the curvature density. In that case, the Chern-Simons functional of the flat connection affects the value of the gap. This argument will be related to that of instanton homology groups. Details will be argued in forthcoming paper [11].

We note that

$$\frac{1}{8\pi^2}\int_M |F_{A_j}|^2=2$$

for $\{[A_j]\}\subset M_2$.

Then due to Proposition 5.11 we obtain

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=0 \quad \text{or } 1 \text{ or } 2,$$

as follows.

Case (1). When $1<\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2\leq 2$, then

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=2.$$

Case (2). When $0<\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2\leq 1$, then

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=1.$$

Case (3). The other cases,

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=0.$$

Combining the bubble theorem (Proposition 5.1) with the gap theorem (Proposition 5.11), we classify the type of ends of M_2 according to the following cases.

Suppose $\{[A_j]\}\subset M_2$.

Case (1). $\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=2.$

Case (2). $\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=1.$

Case (2)—a. $\lim_{j\to\infty}\sup_{\tau\leq n}|F_{A_j}|<\infty$ for all $n<\infty$.

Case (2)—b. $\lim_{j\to\infty}\sup_{\tau\leq n}|F_{A_j}|=\infty$ for some $n<\infty$.

Case (3). $\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\geq n}|F_{A_j}|^2=0.$

Case (3)—a. $\lim_{j\to\infty}\sup_{\tau\leq n}|F_{A_j}|<\infty$ for all $n<\infty$.

Case (3)—b. $\lim_{j\to\infty}\sup_{\tau\leq n}|F_{A_j}|=\infty$ for some $n<\infty$.

Case (3)—b.1. In Proposition 5.1 $\alpha=1$. (one bubble)

Case (3)—b.2. In Proposition 5.1 $\alpha=2$. (two bubbles)

First of all, in the Case (3)—a (i.e. $\{x_\alpha\}=\emptyset$ in Proposition 5.1), $[A_j]$ converges to $[A]$ (after gauge transformations) in M_2 by Proposition 5.1 (3). Hence this is not an end.

Now, among above cases, which can be the ends of N_2? By the transversality argument we assert the followings.

LEMMA 5.22. *There exist sections s_j $(j=1,2,3,4)$, of the line bundles $\mathcal{L}_{\Sigma_j}$ over $\mathcal{B}^*_{\Sigma_j}\cup\{$degree zero reductions$\}$ so that N_2 can avoid ends of types Case* (1), *Case* (2)—a, *Case* (2)—b *and Case* (3)—b.1. *That is, only the ends of type Case* (3)—b.2 *can appear in those of N_2.*

LEMMA 5.23. *Suppose sections s_j are chosen as in Lemma* 5.22. *Then in the Case* (3)—b.2, *concentrate points x_1, x_2 must be $x_1\in\Sigma_i\cap\Sigma_j$, $x_2\in\Sigma_k\cap\Sigma_l$. (Here (i,j,k,l) is some permutation of* $(1,2,3,4)$.)

PROOF OF LEMMA 5.22. Case (1). Obviously

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\leq n}|F_{A_j}|^2=0,$$

hence for all $n<\infty$

$$\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\le n}|F_{A_j}|^2=0. \tag{5.24}$$

Now applying Lemma 5.12 to the case of $A=A_j$ and $Q=X$ ($\tau\le 0$), we have

$$\begin{aligned}\sup_X\sum_{p=0}^m|\nabla_\Gamma^{(p)}(h_j^*A_j-\Gamma)|^2&\le\zeta\int_X|F_{A_j}|^2\\&\le\zeta\int_{\tau\le n}|F_{A_j}|^2.\end{aligned}$$

Here ζ does not depend on connections (i.e. on j), but only depends on the bundle and X, and since $\pi_1(X)=1$, Γ is the trivial flat connection on X. Thus by (5.24)

$$\sup_X\sum_{p=0}^m|\nabla_\Gamma^{(p)}(h_j^*A_j-\Gamma)|^2\longrightarrow 0\qquad\text{as } j\to\infty.$$

Since the restriction to Σ_i of the trivial flat connection on X is also trivial, $\{[A_j]\}$ (after gauge transformations) converges to the trivial flat connection on each surface Σ_i. By the transversality Lemma 4.1 (2), this is not included in V_{Σ_i}. Hence choosing the sections s_j as Lemma 4.1, these ends do not appear in $N_2=M_2\cap V_{\Sigma_1}\cap V_{\Sigma_2}\cap V_{\Sigma_3}\cap V_{\Sigma_4}$.

Case (3)—b.1. Of course, in the case (3) the sliding off ends do not appear. By Proposition 5.1 (3), these ends correspond to $([A], x)\in M_1\times M$. Thus an argument similar to [5, § III (iv)] allows N_2 to avoid these ends. By the transversality,

$$\dim(M_1\cap V_{\Sigma_j}\cap V_{\Sigma_j})=2-2\times 2=-2<0\quad(i\ne j),$$

thus $M_1\cap V_{\Sigma_i}\cap V_{\Sigma_j}$ is empty. Since x lies on at most two surfaces, this means that no point $([A], x)\in M_1\times M$ can be in the closure of N_2.

Case (2)—a. By Proposition 5.1 (1) and (2), there exist h_j and $A\in M_1$ such that

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{8\pi^2}\int_{\tau\le n}e^{\tau\delta}\sum_{p=0}^m|\nabla_A^{(p)}(A-h_j^*A_j)|^2=0.$$

(We say that "after 1-instantons slide off away, $A\in M_1$ (with the exponential decay estimate) remains".) This is also similar to Case (3)—b.1. But the slight difference from Case (3)—b.1 is that there are no concentrate points on surfaces. So in this case it is enough to count the following dimension,

$$\dim(M_1 \cap V_{\Sigma_1} \cap V_{\Sigma_2} \cap V_{\Sigma_3} \cap V_{\Sigma_4}) = 2 - 2 \times 4 = -6 < 0.$$

Thus $M_1 \cap V_{\Sigma_1} \cap V_{\Sigma_2} \cap V_{\Sigma_3} \cap V_{\Sigma_4}$ is empty.

Case (2)—b. By the assumption and

$$2 = \frac{1}{8\pi^2}\int_M |F_{A_j}|^2 = \frac{1}{8\pi^2}\int_{\tau \geq n} |F_{A_j}|^2 + \frac{1}{8\pi^2}\int_{\tau \leq n} |F_{A_j}|^2,$$

we have

$$\lim_{n\to\infty} \lim_{j\to\infty} \frac{1}{8\pi^2}\int_{\tau \leq n} |F_{A_j}|^2 = 1.$$

Moreover since there exists $N < \infty$ such that

$$\lim_{j\to\infty} \sup_{\tau \leq N} F_{A_j} = \infty,$$

there is only one point x with $\tau(x) \leq N$ so that the curvature of $\{[A_j]\}$ concentrates at x and outside of a neighborhood of x $\{[A_j]\}$ converges to a flat connection on X (in Proposition 5.1 (1), $l=0$). Because $\pi_1(X)=1$, this flat connection is trivial. Therefore on at least two surfaces, say Σ_k, Σ_l, (in general position in X) $\{[A_j]\}$ converges to the trivial flat connection. Then by Lemma 4.1 and the argument of Case (1), these ends do not intersect V_{Σ_k}, V_{Σ_l} and neither N_2. ■

Proof of Lemma 5.23. When $x_i \notin \Sigma_k$ for some $i=1,2$, $k=1,2,3,4$, by the argument of the proof of Lemma 5.22, Case (1) and Case (2)—b, $\{[A_j]\}$ converges to the trivial flat connection on Σ_k. Thus this is not included in V_{Σ_k}, so it does not appear in ends of N_2. Hence each surface Σ_k $(k=1,2,3,4)$ must contain a point x_i $(i=1,2)$, that is $x_1 \in \Sigma_i \cap \Sigma_j$, and $x_2 \in \Sigma_k \cap \Sigma_l$. (Here (i,j,k,l) is some permutation of $(1,2,3,4)$.) ■

Eventually, we have obtained, like in closed cases, $N_2 = M_2 \cap V_{\Sigma_1} \cap V_{\Sigma_2} \cap V_{\Sigma_3} \cap V_{\Sigma_4}$ such that, choosing suitably transversal sections s_j, the ends of N_2 are covered by $N_{(x_1,x_2)} = N_2 \cap W_{(x_1,x_2)}$, where $x_1 \in \Sigma_i \cap \Sigma_j$, $x_2 \in \Sigma_k \cap \Sigma_l$ and $W_{(x_1,x_2)}$ is a neighborhood of $([\Gamma]; x_1, x_2) \in M_0 \times S^2(X)$ in $\overline{M}_2$, (S^l means l-power symmetric product (cf. [5] § III (iii))). So the number of end components of N_2 is equal (mod 2) to $Q_4([\Sigma_1], [\Sigma_2], [\Sigma_3], [\Sigma_4])$.

§ 6. The proofs of Theorem 1 and Theorem 2.

By § 2 and the conclusion of the previous section, it is enough to show that the number of end components of N_2 is even. According to [5 ; § II and § III], we define the first Stiefel-Whitney class $u_1 \in H^1(N_2 ; \boldsymbol{Z}_2)$ to show this. Recall that our moduli space N_2 has no sliding off ends. Then by the similar argument of Taubes [15 ; § 11], N_2 can be embedded in a moduli space of connections on a certain closed manifold as below.

First, we prepare some notations. Let

$$L_0 := M - X = S \times [0, \infty)$$
$$L_n := \{x \in M \mid \tau(x) \geq n\} \qquad \text{for } n > 0$$

and $T_n : L_0 \to L_n$ be a natural translation diffeomorphism. Let $P \to M$ be a trivial principal $SU(2)$ bundle and

$$\mathcal{A}(L_0) := \{a \in L^2_{2,\delta}\Omega^1(\mathrm{Ad}P|_{L_0})\}$$
$$\mathcal{G}(L_0) := \left\{ g \in L^2_{3,\mathrm{loc}}(L_0, SU(2)) \,\middle|\, \int_{L_0} e^{\tau\delta} \sum_{p=0}^{3} |\nabla^{(p)} g|^2 < \infty \right\}.$$

Fix $x \in L_0$ and let P_x be the fibre $\{x\} \times SU(2)$ of P. We set

$$\widetilde{\mathcal{B}}(L_0) := (\mathcal{A}(L_0) \times P_x)/\mathcal{G}(L_0)$$

where $\mathcal{G}(L_0)$ acts on P_x by $(g(\cdot), p) \mapsto g(x)p$. $\widetilde{\mathcal{B}}(L_0)$ admits a smooth $SO(3)$ action with fixed point $[\Gamma, 1]$ $(1 \in SU(2)$, and Γ is the trivial flat connection on $L_0 \times SU(2))$.

Now back on M, let $x_n = T_n(x) \in L_n$ and N_2^n denote the full inverse image of the projection

$$(\mathcal{A}_2 \times P_{x_n})/\mathcal{G}_2 \longrightarrow \mathcal{A}_2/\mathcal{G}_2.$$

(Here $\mathcal{A}_2$ and $\mathcal{G}_2$ are defined in § 3.) Away from the reducible orbits, $SO(3) \to N_2^n \to N_2$ is a principal $SO(3)$ bundle.

By restriction to L_n and via pull back by T_n, we obtain a smooth $SO(3)$ equivariant map

$$j_n : N_2^n \longrightarrow \widetilde{\mathcal{B}}(L_0).$$

Then we have

LEMMA 6.1. *Given a neighborhood* $\mathcal{D}$ *of* $[\Gamma, 1] \in \widetilde{\mathcal{B}}(L_0)$, *there exists* $m < \infty$ *such that for all* $n \geq m$ $j_n(N_2^n) \hookrightarrow \mathcal{D}$.

PROOF OF LEMMA 6.1. The key is that N_2 has no sliding off ends, and each $A \in N_2$ is asymptotic the trivial flat connection on End M. So the proof is a direct translation of Lemma 11.2 in [15]. ■

Using this lemma and parallel argument as in the proof of Lemma 11.3 in [15], we obtain the following Lemma.

LEMMA 6.2. *There exists a smooth homotopy*

$$h_t : N_2 \longrightarrow \mathcal{A}_2/\mathcal{G}_2 =: \mathcal{B}_2, \qquad 0 \leq t \leq 1$$

and $n < \infty$ *with the following properties.*

(1) h_0 *is identity of* N_2.

(2) *If* $[A] \in N_2$, *there is a lift of* $h([A])$ *to a path* $h : [0, 1] \times N_2 \to \mathcal{A}_2$ *which is the constant path when restricted to* $\tau^{-1}([0, n-1])$.

(3) $h_1(A)$ *is gauge equivalent to the trivial flat connection on* $\tau^{-1}([n, \infty)) \times SU(2)$.

(4) *For each* $t \in [0, 1]$, $h_t(N_2) \cap \mathcal{B}_2$ *is diffeomorphic to* N_2, *and* $h_t([A]) \in \mathcal{B}_2$ *is an irreducible orbit if and only if* $[A] \in N_2$ *is an irreducible orbit.*

With h in Lemma 6.2, let $\widetilde{N}_2 := h_1(N_2)$ diffeomorphic to N_2. For $m > n+2$ with n as given in Lemma 6.2, we denote by Q_m a closed manifold $Q_m := U_m \cup_S (-U_m)$ where $U_m := M - L_m$. Then following [15], the family $\widetilde{N}_2$ defines a family of orbits of connections on a principal $SU(2)$ bundle $P' \to Q_m$ with $c_2(P') = 2$. Namely, let $\mathcal{A}_2(Q_m), \mathcal{G}_2(Q_m)$ be the space of L_2^2-connections and L_2^3-gauge transformations on P'. Then we have a one to one map

$$\Psi : \widetilde{N}_2 \hookrightarrow \mathcal{A}_2(Q_m)/\mathcal{G}_2(Q_m) = \mathcal{B}_2(Q_m)$$

which is an embedding away from reducible orbits. In particular, since for a generic metric N_2 admits no reducible orbits (by indefiniteness of M and Freed-Uhlenbeck's generosity theorem ([10; Theorem 3.17 and Corollary 3.21] and [15; Proposition 8.3])), we have by Lemma 6.2 (4)

$$\Psi \circ h_1 ; N_2 \xrightarrow{\sim} \widetilde{N}_2 \hookrightarrow \mathcal{B}_2^*(Q_m). \tag{6.3}$$

Using the fact that $\Psi \circ h_1(N_2)$ is diffeomorphic to N_2, we shall show the number of end components of $\Psi \circ h_1(N_2)$ is even. To show this we define the first Stiefel-Whitney class $u_1 \in H^1(\mathcal{B}_2^*(Q_m); \boldsymbol{Z}_2)$. Since U_m is a spin manifold and S is a $\boldsymbol{Z}$ homology 3-sphere, $Q_m = U_m \cup_S (-U_m)$ is spin, too. Fix a spin structure of Q_m. For a closed spin manifold X and a principal $SU(2)$ bundle $P \to X$ with $c_2(P) = l$ even, Donaldson defined in [5; Definition 2.25] the Stiefel-Whitney class u_1, using the real determinant index line bundle over $\mathcal{B}^*(X)$ (descending from $\widetilde{\mathcal{B}^*}(X)$) of twisted Dirac operators $D_{X,A}$ $(A \in \mathcal{B}_2^*(X))$,

$$u_1 = w_1(\mathrm{Det\ ind} D_X) \in H^1(\mathcal{B}_2^*(X); \boldsymbol{Z}_2). \tag{6.4}$$

Then Donaldson asserts that the sum of connections $\Gamma \sharp I \sharp I \sharp \cdots \sharp I$, (here Γ is the trivial flat connection on X and I is a basic BPST-instanton on S^4) determined by a bundle gluing map ρ $(\pm\rho \in SU(2)/\pm 1 \cong SO(3))$ forms a multi-instanton family of based connections, flat away from $x_1, \cdots, x_l$, $(x_i \in X)$ parametrized by

$$\underbrace{SO(3) \times \cdots \times SO(3)}_{l} \hookrightarrow \widetilde{\mathcal{B}^*}(X).$$

Forgetting the base point, the fibration $SO(3) \to \widetilde{\mathcal{B}^*}(X) \to \mathcal{B}^*(X)$ is represented by the left action of $SO(3)$ on $SO(3) \times \cdots \times SO(3)$. So the quotient may be represented by the first $(l-1)$-$SO(3)$ factors.

$$\underbrace{SO(3) \times \cdots \times SO(3)}_{l-1} \hookrightarrow \mathcal{B}^*(X). \tag{6.5}$$

Denote by t_i $(i = 1, \cdots, l-1) \in H^1(SO(3); \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2$ the mod 2 class corresponding to the cohomology generator in the i-th factor. Lemma 3.25 in [5] implies that in this representation the class u_1 defined in (6.4) pulls back via (6.5) so that $u_1 = t_1 + t_2 + \cdots + t_{l-1}$.

In our case $l = 2$. Therefore we have

$$u_1 = w_1(\mathrm{Det\ ind} D_{Q_m}) \in H^1(\mathcal{B}_2^*(Q_m); \boldsymbol{Z}_2)$$

which pulls back so that $u_1 = t_1$.

Now returning to N_2, by the parallel translation of [5; Proposition 3.21, § IV and § V] to Taubes' Fredholm theory framework, we obtain the following local description of a neighborhood of pure bubble ends (i.e. no sliding off ends)

$$([A_0], x_1, \cdots, x_l) \in M_{k-l} \times S^l(M) \subset \overline{M}_k.$$

Let $x_1, \cdots, x_l \in M$ and $\Omega_i \subset M$ be a disjoint small open neighborhood of x_i. Let L_i be the parameter space, a copy of $\boldsymbol{R}^4$, combining the bundle gluing map and scales. (See [5 ; § V].)

Taubes asserts in [15 ; Proposition 5.1] that the Atiyah-Hitchin-Singer deformation complex on an end-periodic 4-manifold

$$(6.6) \quad 0 \longrightarrow L^p_{k+2,\delta}\Omega^0(\mathrm{Ad}P) \xrightarrow{d_A} L^p_{k+1,\delta}\Omega^1(\mathrm{Ad}P) \xrightarrow{d_A^+} L^p_{k,\delta}\Omega^2_+(\mathrm{Ad}P) \longrightarrow 0$$

is Fredholm for all but a discrete set of $\delta \in \boldsymbol{R}_+$, and when A is the trivial flat connection on M, $\dim H^2_A = b_2^+(M) = b_2^+(X)$.

Now let $B_{r/2}(0) \subset H^1_{A_0}$ be a small $r/2$-ball at 0 and Γ_{A_0} be the isotropy subgroup of A_0. Then we establish by an argument parallel to [5] together with Kuranishi method the following

LEMMA 6.7. *A neighborhood of the point* $([A_0], x_1, \cdots, x_l) \in M_{k-l} \times S^l(M)$ *in* $\overline{M}_k$ *is modelled on the quotient by* $\Gamma_{A_0} \times \{\pm 1\} \times \cdots \times \{\pm 1\}$ *of the zero set of a map*

$$\Phi : \overline{N} = \Omega_1 \times \cdots \times \Omega_l \times L_1 \times \cdots \times L_l \times B_{r/2}(0) \longrightarrow H^2_{A_0}$$

where $H^2_{A_0}$ *is the second cohomology of* (6.6) *with weighted Sobolev-norm. Moreover, for* $\omega \in H^2_{A_0}$, *we have a 'suitable approximation' of* $\langle \Phi, \omega \rangle$. *(See* [5 ; *Theorem* 5.5, *Proposition* 4.11].)

The key is slice lemma ([15 ; Lemma 7.3]) and a local description of M_k by Kuranishi method. ([15; § 8, p. 400-p. 402]). Then by Lemma 6.7 and the argument of [5 ; § V (ii)] we have immediately the following conclusion. That is, each of the ends $N_{(x,y)}$ has the form $\boldsymbol{R}_+ \times L_{(x,y)}$, where the link $L_{(x,y)}$ is a circle. Moreover $u_1(L_{(x,y)}) = t_1(L_{(x,y)}) = 1$. Here $L_{(x,y)}$ is identified with $\Psi \circ h_1(L_{(x,y)}) \subset \Psi \circ h_1(N_2) \cong N_2$ by the diffeomorphism $\Psi \circ h_1$. Hence let $\widehat{N_2}$ be the obvious truncation of N_2, then

$$\begin{aligned} 0 = u_1(\partial(\Psi \circ h_1(\widehat{N_2}))) &= \sum_{\substack{x \in \Sigma_i \cap \Sigma_j \\ y \in \Sigma_k \cap \Sigma_l}} u_1(\Psi \circ h_1(L_{(x,y)})) \\ &= \sum_{\substack{x \in \Sigma_i \cap \Sigma_j \\ y \in \Sigma_k \cap \Sigma_l}} t_1(L_{(x,y)}) = \sum_{\substack{x \in \Sigma_i \cap \Sigma_j \\ y \in \Sigma_k \cap \Sigma_l}} 1 \\ &\equiv Q_4([\Sigma_1], [\Sigma_2], [\Sigma_3], [\Sigma_4]) \pmod 2. \end{aligned}$$

This completes the proof of Theorem 1.

We can prove Theorem 2 in a similar way. We sketch the outline of the proof in the rest of this paper. First let Σ_i $(i=1,2,3,4,5,6)$ be a closed surface in general position in X, which represents the homology class $\alpha_i \in H_2(X, \partial X; \boldsymbol{Z})$ smoothly. As in § 2, it suffices to show that

$$\begin{aligned} &Q_6([\Sigma_1],[\Sigma_2],\cdots,[\Sigma_6]) \\ :=&\sum([\Sigma_{i_1}],[\Sigma_{i_2}])([\Sigma_{i_3}],[\Sigma_{i_4}])([\Sigma_{i_5}],[\Sigma_{i_6}]) \\ \equiv&0 \pmod 2. \end{aligned}$$

Now we consider the moduli space N_3 to be

$$N_3 := M_3 \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_6}$$

with dimension 3. By the transversality argument as in § 5, we obtain that there exist sections s_i $(i=1,2,\cdots,6)$ of the line bundles $\mathcal{L}_{\Sigma_i}$ so that only the three-points bubble ends can be appear in those of N_3. That is, the ends of N_3 are covered by $N_{(x,y,z)}=W_{(x,y,z)} \cap N_3$, where $x \in \Sigma_{i_1} \cap \Sigma_{i_2}$, $y \in \Sigma_{i_3} \cap \Sigma_{i_4}$ and $z \in \Sigma_{i_5} \cap \Sigma_{i_6}$ ($(i_1,\cdots,i_6)$ is some permutation of $(1,2,\cdots,6)$), and $W_{(x,y,z)}$ is a neighborhood of $([\Gamma]; x, y, z) \in M_0 \times S^3(X)$ in $\bar{M}_3$. Then N_3 has also no sliding off ends. Therefore as in § 6, N_3 can be embedded in the moduli space $\mathcal{B}_3^*(Q_m)$

$$\Psi : N_3 \hookrightarrow \mathcal{B}_3^*(Q_m)$$

of irreducible connections on a certain closed manifold Q_m. Here the bundle $P \to Q_m$ has $c_2(P)=3$.

To count the number of end components of $\Psi(N_3)$, we must define the second Stiefel-Whitney class $u_2 \in H^2(\mathcal{B}_3^*(Q_m); \boldsymbol{Z}_2)$. Now for a closed spin 4-manifold Q_m, by [5] when $c_2(P)$ is odd we have $u_2 \in H^2(\mathcal{B}_3^*(Q_m); \boldsymbol{Z}_2)$ by using the universal bundle over $\mathcal{B}_3^*(Q_m)$. (See [5, § II (ii)]). Moreover by the local description of pure bubble ends Lemma 6.7, we conclude that each of the ends $N_{(x,y,z)}$ has the form $\boldsymbol{R}_+ \times L_{(x,y,z)}$, where the link $L_{(x,y,z)}$ is a 2-torus and $u_2(L_{(x,y,z)})=1$. Hence we obtain

$$\begin{aligned} 0=u_2(\partial(\widehat{\Psi(N_3)}))=&\sum_{\substack{x \in \Sigma_{i_1} \cap \Sigma_{i_2} \\ y \in \Sigma_{i_3} \cap \Sigma_{i_4} \\ z \in \Sigma_{i_5} \cap \Sigma_{i_6}}} u_2(L_{(x,y,z)}) \\ \equiv& Q_6([\Sigma_1],\cdots,[\Sigma_6]) \pmod 2. \end{aligned}$$

This is Theorem 2.

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