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Generalized ground states for quasilinear elliptic equations

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Abstract. We discuss the existence of generalized ground states. This notion was introduced in the recent work of Atkinson, Peletier and Serrin [2], where they considered the prescribed mean curvature problem and observed its multi-valued nature. We extend their results for more general quasilinear elliptic equations. In the course of the proof we exploit the Pohozaev type identity for u-variable.

1. In this note we shall discuss the existence of positive radial solutions of the problem

$$\text{(I)}\qquad \begin{cases} \operatorname{div}[A(|Du|)Du]+f(u)=0 & \text{in } \boldsymbol{R}^n, \\ u(x)\longrightarrow 0 & \text{as } x\to\infty. \end{cases} \qquad (1)$$

where Du denotes the gradient of u. The functions $A(p)$, $f(u)$ will be assumed to satisfy the following hypotheses:

(A1) $A\in C^1[0,\infty)$,
(A2) $A'(p)\leq 0, \qquad E(p)\equiv A'(p)p+A(p)>0$,
(A3) $\lim_{p\to\infty} p^3E(p)=1$,
(F1) $f\in C^1[0,\infty)$,
(F2) $f(0)=0, \qquad f(u)>0 \qquad \text{for } u>0$,
(F3) $uf'(u)\geq \frac{n+2}{n-2}f(u) \qquad \text{for } u>0$.

Normalization to "1" in (A3) does not involve a loss of generality. (A1)-(A3) are satisfied, for example, in the case

$$A(p)=m^{-1}p^{m-1}(1+p^2)^{-m/2}$$

if $m\geq 1$.

Our study was motivated by recent work of F. V. Atkinson, L. A. Peletier and J. Serrin [2]. They considered the question of existence and nonexistence of ground states for the prescribed mean curvature equation in $\boldsymbol{R}^n$, that is, the problem (I) with $A(p)=(1+p^2)^{-1/2}$. The typical feature for this problem is the occurrence of a *vertical point* in the graph of solution; for large values of $u_0 \equiv u(0)$ there exists a finite R such that a unique C^2 solution $u(r)$ exists over the interval $0 \leq r < R$ and has the property

$$|u'(r)| \longrightarrow \infty \qquad \text{as } r \to R.$$

They sought ways of continuing the solution beyond a vertical point. Regarding u as an independent variable, they proved that r is a well-defined function of u with a sequence of maxima and minima, and that under the hypotheses (F1)-(F3) there exists a solution in this generalized sense, called *the generalized ground states*, for every $u_0 > 0$. This solution, which is possibly *multi-valued* as a function of r, is a natural one from the viewpoint of the prescribed mean curvature problem; the problem may be seen as seeking for a C^2-embedded surface in $\boldsymbol{R}^n \times \boldsymbol{R}^+$ whose mean curvature H at each point (x, u) is given by $H(x, u) = -n^{-1} f(u)$, and which is asymptotic to the hyperplane $u = 0$ as $x \to \infty$.

Our aim is to extend these procedures to more general quasilinear elliptic equations. Solutions of our problem generally have vertical points for u_0 large (see, Lemma 1, below). Although there seems to be no clear geometrical meaning in the general case, we intend to continue the solution beyond the vertical point. Following [2] we regard u as an independent variable and rewrite the problem as follows:

$$\text{(II)} \quad \begin{cases} E\left(\dfrac{1}{q}\right) q^{-3} r'' - \dfrac{n-1}{r} A\left(\dfrac{1}{q}\right) \dfrac{1}{q} + f(u) = 0 & \text{for } 0 < u < u_0, \qquad (2) \\ r(u) > 0 \qquad \text{for } 0 < u < u_0, & \\ r(u_0) = 0, \qquad r'(u) \longrightarrow -\infty \qquad \text{as } u \to u_0, & \\ r(u) \longrightarrow \infty \qquad \text{as } u \to 0, & \end{cases}$$

where $q = |r'(u)|$. Note that if $u' < 0$ (1) is equivalent to (2) and that (A3) ensures the continuability of solutions of (II) beyond each local maxima and minima, where $q = 0$.

We now state our

MAIN RESULT. *We assume* (A1)-(A3) *and* (F1)-(F3). *For every* $u_0 > 0$ *there exists a solution* $r(u)$ *of* (II) *such that the inverse function* $u(r)$ *satisfies either*

$$(r^{n-1} A(|u'|) u')' + r^{n-1} f(u) = 0$$

or

$$(r^{n-1}A(|u'|)u')' - r^{n-1}f(u) = 0$$

according to whether $r'<0$ *or* $r'>0$, *respectively.*

We remark here that in our main result the change of sign for the equation (1) is natural. To see this we consider the prescribed mean curvature equation $A(p)=(1+p^2)^{-1/2}$ and its geometrical meaning. In order to define the sign of the mean curvature we must attach a continuous normal direction to the surface of solution. If this direction points downward we have

$$\mathrm{div}\Big(\frac{Du}{(1+|Du|^2)^{1/2}}\Big) - f(u) = 0.$$

Although the notion of normal direction is ambiguous in our general case, we legitimately define the sign so that it is consistent with the prescribed mean curvature problem.

To prove our main result we exploit the following Pohozaev type identity for u-variable. Many authors develop the ordinary r-variable Pohozaev type identity (see, for instance, [6, 7, 8]). But it seems to be new that the explicit formulation for general u-variable identity is given.

PROPOSITION. *Let* $r(u)$ *be a solution to* (II). *We define the functional*

$$H(u) = r^n \int_{u_0}^{u} E\Big(\frac{1}{q}\Big) q^{-3} r''(u)\,du - \frac{n-2}{2} r^{n-1} u A\Big(\frac{1}{q}\Big)\frac{1}{q} + \frac{n-2}{2n} r^n u f(u),$$

where we denote $q = |r'(u)|$.

Then there holds the identity:

$$H'(u) = n r^{n-1}\Big\{(r' + |r'|)\int_0^\infty E(p)p\,dp - q\int_0^{1/q} E(p)p\,dp + \frac{1}{2}A\Big(\frac{1}{q}\Big)\frac{1}{q}\Big\} + \frac{n-2}{2n} r^n \Big\{ u f'(u) - \frac{n+2}{n-2} f(u)\Big\}.$$

The proof goes on an easy calculation. So we omit the details.

We end this section with a few comments. First, the ordinary ground state problem for quasilinear equations has been considered by several authors under various assumptions (see, for instance, [5, 6, 7, 8, 9], and references therein). They all discuss the r-variable single-valued function, however, and do not mention the multi-valued case. Second, the multi-valued nature of the problem was first observed in the case of capillary. For more physical backgrounds and more informations about this problem, see the remarkable paper of P. Concus and R. Finn [3] or the recent monograph of R. Finn [4]. Finally, the occurrence of a vertical point for various equations was investigated extensively in the recent paper of F. V. Atkinson and L. A. Peletier [1]. Their assumptions are less stringent than ours, of course. But they do not discuss the continuability of solutions beyond vertical points.

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2. We want to prove our main result. Our proof proceeds along the same line as in [2]. Let

$$M \equiv \lim_{p\to\infty} A(p)p,$$
$$N \equiv \int_0^\infty pE(p)dp.$$

We see from (A2) (A3) that M, N are well-defined and finite. Note that (A2) and N finite suffice to produce a vertical point (see [1]).

First we consider the initial value problem

$$\begin{cases} (r^{n-1}A(|u'|)u')' + r^{n-1}f(u) = 0 & \text{for } r>0, \\ u(0)=u_0, \quad u'(0)=0. \end{cases} \tag{3}$$

For every $u_0>0$ this problem has a unique C^2 solution $u(r)=u(r;u_0)$ in a neighbourhood of $r=0$, and

$$u(r)<u_0, \qquad u'(r)<0$$

as long as it exists.

We put

$$r_1 = \sup\{r \mid C^2 \text{ solution } u(s;u_0) \text{ exists for } 0<s<r\},$$
$$u_1 = u(r_1).$$

If $r_1<\infty$, we call r_1 the *first vertical point*.

Lemma 1. *Suppose for all sufficiently large values of u there holds $4n^2Nf'(u) \le f(u)^2$. Then for u_0 sufficiently large the first vertical point*

(r_1, u_1) exists on the graph of solution $u(r)$, with

$$u_0-\frac{2n}{f(u_0)}N<u_1<u_0-\frac{n}{f(u_0)}N,$$

$$\frac{n}{f(u_0)}M<r_1<\frac{2n}{f(u_0)}M.$$

The inverse function $r(u)$ satisfies $r'(u_1)=0$, $r''(u_1)<-\frac{f(u_0)}{2n}$.

PROOF. Although the proof of Lemma 1 is a slight modification of that of Theorem 2.1 in [2], we do it for completeness.

We introduce the inverse function Q of $A(p)p$, which is possible by (A2). We also set

$$G(r)=r^{1-n}\int_0^r s^{n-1}f(u(s))ds.$$

Then we obtain

$$u'(r)=-Q(G(r)) \tag{4}$$

on an integration of (3).

We first prove by contradiction the lower bound for u for u_0 sufficiently large. Suppose there exists a point $\bar{r}$ in the interval $(0, r_1)$ satisfying $u(\bar{r})=u_0-\frac{2n}{f(u_0)}N$, and choose $\bar{r}$ to be the least of such values. Then we have

$$\begin{aligned}G'(r)&=f(u(r))-(n-1)r^{-n}\int_0^r s^{n-1}f(u(s))ds\\&>f\left(u_0-\frac{2n}{f(u_0)}N\right)-\frac{n-1}{n}f(u_0)\\&\geq\frac{1}{n}f(u_0)-\frac{2n}{f(u_0)}N\cdot\frac{1}{4n^2N}f(u_0)^2\\&=\frac{1}{2n}f(u_0).\end{aligned}$$

Here we have used the assumption $4n^2N\cdot f'(u)\leq f(u_0)^2$ over the interval $(0, \bar{r})$. Hence, integrating (4), we obtain a contradiction:

$$\frac{2n}{f(u_0)}N=\int_0^{\bar{r}} Q(G(s))ds$$

$$=\int_0^{G(r)} Q(t)\frac{dt}{G'(s)}$$
$$<\frac{2n}{f(u_0)}\int_0^M Q(t)dt$$
$$=\frac{2n}{f(u_0)}N.$$

With this lower bound, we get the upper bound for r immediately, namely,

$$M>G(r)=\int_0^r G'(s)ds>\frac{f(u_0)}{2n}r.$$

In other words, a vertical point (r_1, u_1) must appear with

$$u_1>u_0-\frac{2n}{f(u_0)}N,\quad r_1<\frac{2n}{f(u_0)}M.$$

For the remaining bounds we use

$$G'(r)=\frac{1}{n}f(u(r))+\frac{n-1}{n}r^{-n}\int_0^r s^{-n}f'(u(s))u'(s)ds$$
$$<\frac{1}{n}f(u_0),$$

which follows on an integration by parts. This yields

$$u_0-u_1=\int_0^M Q(t)\frac{dt}{G'(s)}$$
$$>\frac{n}{f(u_0)}\int_0^M Q(t)dt=\frac{n}{f(u_0)}N,$$

by integration of (4), and

$$\frac{f(u_0)}{n}r_1>\int_0^{r_1} G'(r)dr=G(r_1)=M.$$

Since it is clear that $r'(u_1)=0$, we only have to estimate $r''(u_1)$. This follows from (2) in view of the lower bound for r_1 and u_1. That is,

$$r''(u_1)=\frac{n-1}{r_1}M-f(u_1)$$

$$
\begin{aligned}
&<\frac{n-1}{n}f(u_0)-f(u_1)\\
&<-\frac{1}{n}f(u_0)+(u_0-u_1)\frac{1}{4n^2N}f(u_0)^2\\
&=-\frac{1}{2n}f(u_0)\\
&<0.
\end{aligned}
$$

This completes the proof. ■

We want to continue the trajectory beyond the point (r_1, u_1). To do that, we consider the inverse function $r(u)$, which is a solution of the initial value problem (II). Note that local existence and uniqueness of a solution $r(u)$ is assured by that of $u(r)$. We then have

LEMMA 2. *The solution $r(u)$ of* (II) *exists over the interval* $0<u\leq u_0$.

PROOF. We use the idea of [2].

Let $r(u)$ be the solution of (II) as long as it exists. We introduce the functional

$$F(u)=r^{n-1}A\left(\frac{1}{q}\right)\frac{1}{q}-\frac{1}{n}f(u)r^n.$$

By (2) we have

$$F'(u)=-\frac{1}{n}f'(u)r^n<0,$$

and hence

$$
\begin{aligned}
Mr^{n-1}&\geq r^{n-1}A\left(\frac{1}{q}\right)\frac{1}{q}\\
&=F(u)+\frac{1}{n}f(u)r^n\\
&>F(u_0)+\frac{1}{n}f(u)r^n\\
&=\frac{1}{n}f(u)r^n.
\end{aligned}
$$

This implies

$$\frac{nM}{f(u)} > r(u),$$

from which we conclude that $r(u)$ is bounded on any interval $[\bar{u}, u_0]$ with $0<\bar{u}<u_0$. Moreover, since $F(u)>F(u_0)$ we have

$$A\left(\frac{1}{q}\right)\frac{1}{q} > \frac{1}{n} f(u) r > 0$$

in the right neighbourhood of $\bar{u}$. This implies that $|r'(u)|$ is bounded above on the same neighbourhood. Finally we note that there exists no u^* in the interval $(0, u_0)$ such that

$$\lim_{u \downarrow u^*} r(u) = 0.$$

This can be seen once we observe that in such case we have $0=F(u^*)>F(u_0)=0$, a contradiction. Hence the result. ■

Next we want to prove that $\lim_{u\to 0} r(u)$ exists, which is possibly infinite, even in the case that $r(u)$ has vertical points. Let $r(u)$ be such a solution. We define inductively the sequence of minima and maxima of $r(u)$ as follows:

$$u_{2m} = \inf\{0<u<u_{2m-1} \mid r' \geq 0 \text{ on the interval } (u, u_{2m-1})\},$$
$$u_{2m+1} = \inf\{0<u<u_{2m} \mid r' \leq 0 \text{ on the interval } (u, u_{2m})\} \qquad m=1, 2, \cdots$$
$$r_k = r(u_k) \qquad k=1, 2, 3, \cdots$$

Then we have

LEMMA 3. (i) $r_{2m} \leq \dfrac{1}{f(u_{2m})}(n-1)M, \qquad r_{2m-1} \geq \dfrac{1}{f(u_{2m-1})}(n-1)M,$

(ii) $r_{2m+2} > r_{2m}, \qquad r_{2m+1} > r_{2m-1}.$

PROOF. (i) is a straight consequence of (2). To prove (ii) we consider two successive maxima r_{2m-1} and r_{2m+1}. We set $a=u_{2m-1}$, $b=u_{2m}$, $c=u_{2m+1}$, for simplicity. On intervals (c, b) and (b, a), we can define inverses $u^-(r)$ and $u^+(r)$, respectively. Then we find, integrating (2) over the interval $(u^-(r), u^+(r))$, that

$$r^{n-1}\left\{A\left(\frac{1}{q^+}\right)\frac{1}{q^+} - A\left(\frac{1}{q^-}\right)\frac{1}{q^-}\right\} = \int_{u^-}^{u^+} r^{n-1}(u) f(u) r'(u) du$$

$$=\int_r^b \{f(u^+(t))-f(u^-(t))\}t^{n-1}dt$$

$$>0, \tag{5}$$

for r satisfying $r(b)\leq r\leq \min\{r(a), r(b)\}$. Here we put $q^+=|r'(u^+)|$ and $q^-=|r'(u^-)|$. Therefore

$$|r'(u^-(r))|>|r'(u^+(r))|,$$

from which we conclude by increasing r that

$$r(a)\leq r(c).$$

If $r(a)=r(c)$ then (5) implies a contradiction. This is the desired result.
The case of minima proceeds in a similar fashion. ■

REMARK. We can also prove the fact:
1. $rf(u)<nM$.
2. r_k is bounded above by a constant independent of u_0.

After these preliminaries we can prove

LEMMA 4. $\lim_{u\to 0} r(u)$ *exists (finite or infinite).*

PROOF. The only case we have to discuss is that there is a decreasing sequence $\{u_k\}$ of critical points of $r(u)$ convergent to 0.

An integration of (2) from u to u_k yields

$$-\int_u^{u_k} E\left(\frac{1}{q}\right)q^{-3}r''du=\operatorname{sgn}(r')\int_{1/q}^{\infty} E(p)pdp$$

$$=-(n-1)\int_u^{u_k}\frac{1}{r}A\left(\frac{1}{q}\right)\frac{1}{q}du+\int_u^{u_k}f(s)ds. \tag{6}$$

By Lemma 3 (ii) we see that $r(u)$ is bounded away from zero for $0<u<u_k$ uniformly in k. Thus (6) implies

$$\int_{1/q}^{\infty} E(p)pdp\leq C(u_k-u),$$

where constant C does not depend on k. Letting $k\to\infty$ this yields

$$\lim_{u\to 0} r'(u)=0,$$

from which we conclude that $\lim_{u\to 0} r(u)$ exists. ■

REMARK. We can actually prove that $r'(u)<0$ for all u sufficiently near zero. This can be seen as in Lemma 2.7 in [2].

In order to complete the proof of our main result we only have to show that

$$r(u)\longrightarrow\infty \qquad \text{as } u\to 0.$$

To prove this we use the identity formulated in the proposition: Let $r(u)$ be a solution of (II) and let us define the functional

$$H(u)=r^n\int_{u_0}^{u} E\Big(\frac{1}{q}\Big)q^{-3}r''(u)du-\frac{n-2}{2}r^{n-1}uA\Big(\frac{1}{q}\Big)\frac{1}{q}+\frac{n-2}{2n}r^n uf(u)$$

where $q=|r'(u)|$. Then there holds the identity

$$H'(u)=nr^{n-1}\Big\{(r'+|r'|)\int_0^{\infty} E(p)pdp-q\int_0^{1/q} E(p)pdp+\frac{1}{2}A\Big(\frac{1}{q}\Big)\frac{1}{q}\Big\}$$
$$+\frac{n-2}{2n}r^n\Big\{uf'(u)-\frac{n+2}{n-2}f(u)\Big\}.$$

In view of $A'(p)\leq 0$ we have $\int_0^p E(p)pdp\leq\frac{1}{2}A(p)p^2$ and

$$(r'+|r'|)\int_0^{\infty} E(p)pdp-q\int_0^{1/q} E(p)pdp+\frac{1}{2}A\Big(\frac{1}{q}\Big)\frac{1}{q}$$
$$\geq(r'+|r'|)N-\frac{1}{2}A\Big(\frac{1}{q}\Big)\frac{1}{q}+\frac{1}{2}A\Big(\frac{1}{q}\Big)\frac{1}{q}$$
$$>0.$$

It then follows by (F3) that $H'(u)>0$, and since $H(u_0)=0$ we obtain

$$\lim_{u\to 0} H(u)<0.$$

Suppose on the contrary that $r_\infty=\lim_{u\to 0}r(u)<\infty$. Because

$$\int_{u_0}^{u} E\Big(\frac{1}{q}\Big)q^{-3}r''du=\int_0^{\infty} E(p)pdp+\operatorname{sgn}(r')\int_{1/q}^{\infty} E(p)pdp$$
$$>0,$$

we find

$$\lim_{u\to 0} H(u)\geq\lim_{u\to 0} u\Big\{-\frac{n-2}{2}r^{n-1}A\Big(\frac{1}{q}\Big)\frac{1}{q}+\frac{n-2}{2n}r^n f(u)\Big\}$$
$$=0.$$

This is a contradiction, and hence the main result is proved. ■

REMARK. Under the same assumptions but with (F2) replaced by

(F2)′ $f(0)=0$, and there exists a number $a>0$ such that
$f(u)<0$ for $0<u<a$, $f(u)>0$ for $u>a$,

we can prove as in [2] that there exists no solution of (II).

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