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On maximal versions of the Large Sieve

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1. In the present paper I consider maximal variants of the Large Sieve. The following new result is typical.

THEOREM 1. *Let $\varepsilon>0$. Then*

$$\sum_{\substack{qr\leq Q\\(q,r)=1}}\frac{q}{\phi(qr)}\sum_{\chi(\bmod q)}^{*}\max_{y\leq x}\left|\sum_{n\leq y}a_n\chi(n)c_r(n)\right|^2\ll\left(x+Q^2(\log x)^{2+\varepsilon}\right)\sum_{n\leq x}|a_n|^2$$

*where * denotes that χ runs through the primitive Dirichlet characters $(\bmod q)$, and $c_r(n)$ is the Ramanujan sum*

$$c_r(n)=\sum_{\substack{b=1\\(b,r)=1}}^{r}\exp(2\pi ibn/r).$$

The inequality is uniform in $Q\geq 1$, $x\geq 2$, and complex numbers a_n.

COROLLARY.

$$\sum_{q\leq Q}\sum_{\chi(\bmod q)}^{*}\max_{y\leq x}\left|\sum_{p\leq y}a_p\chi(p)\right|^2\ll\left(\frac{x}{\log x}+Q^{2+\varepsilon}\right)\sum_{p\leq x}|a_p|^2$$

where p denotes a prime number.

These appear to be the first inequalities of Large Sieve type in which the presence of the maxima under the character summation does not degrade the leading term in the upper bound by introducing an extraneous logarithmic factor, or increase the term Q^2 to Q^3. In certain applications to the theory of arithmetic functions such a refinement is important.

In two sections following the proof of Theorem 1 I discuss related results, their applications, and problems.

For convenience of exposition I shall formally denote the triple summation

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$$\sum_{\substack{qr\le Q\\(q,r)=1}}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)} \quad \text{by} \quad \widehat{\sum},$$

and

$$\sum_{n\le x}|a_n|^2 \quad \text{by} \quad |\boldsymbol{a}|^2.$$

LEMMA 1.

$$\widehat{\sum}\left|\sum_{n=M+1}^{M+N} a_n\chi(n)c_r(n)\right|^2 \ll (N+Q^2)\sum_{n=M+1}^{M+N}|a_n|^2.$$

PROOF. This now standard inequality may be found as Théorème 7A, p. 27 of Bombieri [1]. Lemma 1 is due to Selberg [20].

LEMMA 2. *If* $T\ge 1$

$$\widehat{\sum}\int_{-T}^{T}\left|\sum_{n\le x} a_n\chi(n)c_r(n)n^{it}\right|^2 dt \ll \sum_{n\le x}|a_n|^2(n+Q^2T).$$

PROOF. This result may be derived from Théorème 10, p. 30 of Bombieri [1], if the Théorème 7 that is applied there is replaced by Théorème 7A. Results in the style of Lemma 2 are obtained by the method of Gallagher [16].

For a more elaborate discussion of the Large Sieve and some of its ramifications in Analytic Number Theory see Motohashi [19].

LEMMA 3. *If* $s=\sigma+it$, $\sigma=\mathrm{Re}(s)$ *denotes a complex variable*

$$\int_{-1}^{1}\left|\frac{K^s-1}{s}\right|dt \le 4K^\sigma(1+\log\log K)$$

uniformly for $K\ge 2$, $\sigma\ge 0$. *The integrand is defined to be* $\log K$ *when* $\sigma=0=t$.

PROOF. On the one hand, for $s\ne 0$

$$K^s-1=\int_0^{s\log K} e^z dz,$$

the integral being taken along the straight line joining the origin to the point $s\log K$. Hence

$$|K^s-1|\le |s|\log K\exp(\sigma\log K).$$

On the other hand, for $t\neq 0$

$$|s^{-1}(K^{s}-1)|\leq |t|^{-1}(K^{\sigma}+1).$$

Applying these estimates for the ranges $|t|\leq(\log K)^{-1}$, $(\log K)^{-1}<|t|\leq 1$ respectively of the integral in the statement of Lemma 3, we readily obtain the asserted bound.

It is not difficult to show that for $\sigma=0$ the integral considered in Lemma 3 exceeds a multiple of $\log\log K$.

LEMMA 4. *Let* $\varepsilon>0$. *Then*

$$\widehat{\sum}\max_{0<y\leq x}\Big|\sum_{y/K<n\leq y} a_n\chi(n)c_r(n)\Big|^2 \ll (x+Q^2(\log x)^{2+\varepsilon})|\boldsymbol{a}|^2(1+\log\log K)^2$$

uniformly for $2\leq K\leq x$, $Q\geq 1$, $\boldsymbol{a}$.

PROOF. Let w be half an odd positive integer, $\frac{1}{2}\leq w\leq 2x$, and n a positive integer not exceeding x. Then for $\sigma=(\log x)^{-1}$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{w}{n}\right)^{s}\frac{ds}{s}=\begin{cases}1 & \text{if } n\leq w,\\ 0 & \text{if } n>w.\end{cases}$$

If $T\geq 1$, then an integration by parts shows that the contribution to the integral arising from the range $|t|>T$ is $\ll(w/n)^{\sigma}|s\log w/n|^{-1}\ll x/T$, since

$$\left|\log\frac{w}{n}\right|\geq\min\left(\log\frac{\left(n+\frac{1}{2}\right)}{n},\left|\log\frac{\left(n-\frac{1}{2}\right)}{n}\right|\right)\geq\frac{1}{2n}.$$

Hence, if $y, y/K$ are half odd integers,

$$\sum_{y/K<n\leq y}a_n\chi(n)c_r(n)=\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\sum_{n\leq x}a_n\chi(n)c_r(n)n^{-s}\left(\frac{y^s-(y/K)^s}{s}\right)ds$$
$$+O\Big(\sum_{n\leq x}|a_n||\chi(n)c_r(n)|xT^{-1}\Big).$$

After several applications of the Cauchy-Schwarz inequality we see that the sum which we wish to estimate does not exceed a multiple of

$$E=\widehat{\sum}\left|\int_{\sigma-iT}^{\sigma+iT}A(\chi)\left(\frac{y^s-(y/K)^s}{s}\right)ds\right|^2+\frac{x}{T^2}\widehat{\sum}\sum_{n\leq x}|\chi(n)c_r(n)|^2|\boldsymbol{a}|^2,$$

where

$$A(\chi)=\sum_{n\le x}a_n\chi(n)c_r(n)n^{-s}.$$

For each integer n, an application of Lemma 1 shows that $\widehat{\sum}|\chi(n)c_r(n)|^2\ll Q^2$. With $T=x^2$ the second of the terms in E is therefore $\ll x^{-1}Q^2|\boldsymbol{a}|^2$, well within the asserted bound.

Since $|y^s-(y/K)^s|\le 2e$ uniformly for $\mathrm{Re}(s)=\sigma$, $0<y\le x$, an application of the Cauchy-Schwarz inequality gives

$$\left|\int_{1\le|t|\le T}A(\chi)\left(\frac{y^s-(y/K)^s}{s}\right)ds\right|^2$$
$$\le 4e^2\int_{1\le|t|\le T}\frac{dt}{|s|(\log(2+|s|))^{1+\varepsilon}}\int_{1\le|t|\le T}|A(\chi)|^2\frac{(\log(2+|s|))^{1+\varepsilon}}{|s|}dt.$$

The first of these two integrals is bounded above, uniformly for $T\ge 1$. We cover the range of the second by the intervals $2^\nu\le|t|\le 2^{\nu+1}$ with $0\le\nu\le\log T/\log 2$, and apply Lemma 2 to each of these intervals, with a_n replaced by $a_n n^{-\sigma}$. The corresponding contribution to the first of the terms in E is then

$$\ll\sum_\nu\frac{\nu^{1+\varepsilon}}{2^\nu}\int_{2^\nu<|t|\le 2^{\nu+1}}\widehat{\sum}|A(\chi)|^2dt$$
$$\ll\sum_\nu\frac{\nu^{1+\varepsilon}}{2^\nu}\left(\sum_{n\le x}|a_n|^2(n+Q^2 2^\nu)\right)$$
$$\ll\sum_{n\le x}|a_n|^2(n+Q^2(\log T)^{2+\varepsilon}),$$

which is also within the asserted upper bound.

For the range $|t|\le 1$ we apply the Cauchy-Schwarz inequality differently, taking one power of the kernel $|s^{-1}(y^s-(y/K)^s)|$ into each integral

$$\left|\int_{\sigma-i}^{\sigma+i}A(\chi)\left(\frac{y^s-(y/K)^s}{s}\right)ds\right|^2\le\int_{-1}^{1}\left|\frac{y^s-(y/K)^s}{s}\right|dt\int_{-1}^{1}|A(\chi)|^2\left|\frac{y^s-(y/K)^s}{s}\right|dt$$
$$\le e^2\int_{-1}^{1}\left|\frac{1-K^{-s}}{s}\right|dt\int_{-1}^{1}|A(\chi)|^2\left|\frac{1-K^{-s}}{s}\right|dt$$

this last step again since $|y^s|\le e$ uniformly for $0<y\le x$. The first of these two integrals is by Lemma $3\ll 1+\log\log K$. We apply Lemma 1 to estimate the sum involving $A(\chi)$, and complete the proof of the present lemma by a further appeal to Lemma 3.

The above argument assumed that the $y, y/K$ which appear under the maxima have only half odd integer values. If they do not, we may change them to a nearest such value at the expense of introducing an error $O(|\boldsymbol{a}|^2\widehat{\sum}1)$. Since every character to a modulus not exceeding Q is induced by a primitive character to a modulus not exceeding Q

$$\widehat{\sum}1 \leq \sum_{q\leq Q}\ \sum_{\chi(\mathrm{mod}\ q)} 1 \leq Q^2.$$

The extra error lies within the asserted bound.

REMARK. This same argument also allows us to replace the maxima over $y/K<n\leq y$ by maxima over $y/K\leq n\leq y$.

PROOF OF THEOREM 1. Define the integer k by $2^k\leq(\log x)^\varepsilon<2^{k+1}$. If

$$T(y)=\sum_{n\leq y}{}' a_n\chi(n)c_r(n)$$

then we decompose the range $1\leq n\leq y$ into intervals, to give

$$T(y)=T(2^{-k-1}y)+\sum_{\nu=0}^{k}\ \sum_{y2^{-\nu-1}<n\leq y2^{-\nu}} a_n\chi(n)c_r(n).$$

Since the series $\sum 2^{-\nu/2}$ converges, applications of the Cauchy-Schwarz inequality show that

$$\max_{y\leq x}|T(y)|^2 \ll \max_{w\leq x(\log x)^{-\varepsilon}}|T(w)|^2+\sum_{\nu=0}^{k}2^{\nu/2}\max_{w\leq x2^{-\nu}}\left|\sum_{w/2<n\leq w}a_n\chi(n)c_r(n)\right|^2.$$

We estimate

$$L=\widehat{\sum}\max_{w\leq x(\log x)^{-\varepsilon}}|T(w)|^2$$

by Lemma 4, bearing in mind the remark made at the end of its proof, replacing x and K there by $x(\log x)^{-\varepsilon}$. Hence

$$L\ll(x(\log x)^{-\varepsilon}+Q^2(\log x)^{2+\varepsilon/2})|\boldsymbol{a}|^2(1+\log\log x)^2,$$

which is within the asserted bound in the statement of Theorem 1.

For each sum corresponding to a value of ν, $0\leq\nu\leq k$, we apply Lemma 4 with $K=2$, x replaced by $x2^{-\nu}$. Then

$$\widehat{\sum}\sum_{\nu}2^{\nu/2}\max_{w\leq x2^{-\nu}}\left|\sum_{w/2<n\leq w}a_n\chi(n)c_r(n)\right|^2\ll\sum_{\nu}2^{\nu/2}(x2^{-\nu}+Q^2(\log x)^{2+\varepsilon/2})|\boldsymbol{a}|^2,$$

and since

$$\sum_{\nu=0}^{k} 2^{\nu/2} \ll 2^{k/2} \ll (\log x)^{\varepsilon/2},$$

the proof of Theorem 1 is complete.

PROOF OF THE COROLLARY. We may assume that $0<\varepsilon\leq 1/6$. If $1\leq r\leq x^{\varepsilon}$, then the primes p in the interval $x^{\varepsilon}<p\leq x$ do not divide r. For such primes $c_r(p)=\mu(r)$. Replacing Q in Theorem 1 by Qx^{ε} we see that

$$\sum_{q\leq Q} \frac{q}{\phi(q)} \sum_{\substack{r\leq x^{\varepsilon} \\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)} \sum_{\chi (\mathrm{mod}\, q)}^{*} \max_{y\leq x} \Big| \sum_{x^{\varepsilon}<p\leq y} a_p \chi(p) \Big|^2 \ll (x+Q^2x^{2\varepsilon}(\log x)^{2+\varepsilon})|\boldsymbol{a}|^2.$$

Bearing in mind that

$$\sum_{\substack{r\leq w \\ (r,q)=1}} \frac{\mu^2(r)}{\phi(r)} > \frac{\phi(q)\log w}{q}$$

uniformly for $w\geq 1$ (see, for example, Halberstam and Richert [17], Lemma (3.1), p. 102), we see that

$$\sum_{q\leq Q} \sum_{\chi (\mathrm{mod}\, q)}^{*} \max_{y\leq x} \Big| \sum_{x^{\varepsilon}<p\leq y} a_p \chi(p) \Big|^2 \ll \Big(\frac{x}{\log x} + Q^2 x^{2\varepsilon} (\log x)^{1+\varepsilon} \Big) |\boldsymbol{a}|^2.$$

The condition $x^{\varepsilon}<p$ may be removed by a further appeal to Theorem 1, and the proof of the corollary may be completed by considering separately the cases $Q\leq x^{1/6}$, $Q>x^{1/6}$.

2. Further results.

The following result widens the uniformity in Theorem 1 at the expense of replacing $Q^2(\log x)^{2+\varepsilon}$ by $Q^3(\log Q)^{1+\varepsilon}$.

THEOREM 2. *If* $\varepsilon>0$, *then*

$$\sum_{\substack{qr\leq Q \\ (q,r)=1}} \frac{q}{\phi(qr)} \sum_{\chi (\mathrm{mod}\, q)}^{*} \max_{v-u\leq H} \Big| \sum_{u<n\leq v} a_n c_r(n) \chi(n) \Big|^2 \ll (H+Q^3(\log Q)^{1+\varepsilon}) \sum_{n=-\infty}^{\infty} |a_n|^2,$$

uniformly for $H\geq Q\geq 2$, *and all square summable sequences of complex numbers* a_n.

The logarithmic factor may be reduced a little.

COROLLARY.

$$\sum_{D\le Q}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, D)}\ \max_{v-u\le H}\Big|\sum_{u<p\le v} a_p\chi(p)\Big|^2 \ll \Big(\frac{H}{\log H}+Q^{3+\varepsilon}\Big)\sum_{p\ge 2}|a_p|^2$$

uniformly for $H\ge 2$, $Q\ge 2$.

The proof of Theorem 2 is an exercise in duality. It depends upon the following result.

LEMMA 5. (i) *If the positive integers* r_j *are prime to* D, $j=1,2$, *and* χ *is a non-principal character* (mod D), *then*

$$\sum_{n\le y} c_{r_1}(n)c_{r_2}(n)\chi(n) \ll D^{1/2}\log D\frac{(r_1r_2)^2}{\phi(r_1)\phi(r_2)}$$

uniformly in y.

(ii) *With the same restrictions upon the* r_j, *and* $r_1\ne r_2$,

$$\sum_{\substack{n\le y\\ (n,D)=1}} c_{r_1}(n)c_{r_2}(n) \ll 2^{\omega(D)}\frac{(r_1r_2)^2}{\phi(r_1)\phi(r_2)}$$

uniformly in y, *where* $\omega(D)$ *denotes the number of distinct prime divisors of* D.

(iii) *For positive integers* r *and* D, *with* $(r,D)=1$,

$$\sum_{n\le y}|c_r(n)|^2 = y\phi(r)\frac{\phi(D)}{D}+O(2^{\omega(D)}\phi(r)^2)$$

uniformly in y.

PROOF. These results are readily obtained by employing the representation

$$c_r(n)=\sum_{d|r,\,d|n}\mu\Big(\frac{r}{d}\Big)d,$$

and applying the Pólya-Vinogradov inequality.

PROOF OF THEOREM 2. Define

$$\phi_j(n)=\frac{c_r(n)}{\phi(r)^{1/2}}\Big(\frac{q}{\phi(q)}\Big)^{1/2}\chi(n),$$

one function ψ_j for each pair (r,χ), with χ a primitive character $(\bmod\, q)$, and r, q coprime integers satisfying $qr \le Q$. There are therefore at most Q^2 such functions. It follows directly from Lemma 5 that for $j \ne k$

$$\sum_{n\le y} \psi_j(n)\bar{\psi}_k(n) \ll Q(\log Q)^{1+\varepsilon} \tag{1}$$

uniformly in y. Moreover

$$\sum_{n\le y} |\psi_j(n)|^2 = y + O(Q). \tag{2}$$

Choosing a pair of reals u_j, v_j, for each j, to satisfy $v_j - u_j \le H$, we see that the inequality of the theorem asserts that

$$\sum_j \Big| \sum_{u_j<n\le v_j} a_n \psi_j(n) \Big|^2 \ll (H + Q^3(\log Q)^{1+\varepsilon}) \sum_{n=-\infty}^{\infty} |a_n|^2.$$

It will be enough to establish the dual inequality

$$\sum_n \Big| \sum_{u_j<n\le v_j} c_j \psi_j(n) \Big|^2 \ll (H + Q^3(\log Q)^{1+\varepsilon}) \sum_j |c_j|^2$$

for all complex c_j. The sum to be here estimated is an Hermitian form in the variables c_j, and does not exceed $\lambda \sum |c_j|^2$ where λ is the largest eigenvalue of the associated matrix. In particular λ lies in one of the Gershgorin discs

$$\Big| \lambda - \sum_{u_k<n\le v_k} |\psi_k(n)|^2 \Big| \le \sum_{j\ne k} \Big| \sum_{\substack{u_j<n\le v_j \\ u_k<n\le v_k}} \psi_j(n)\bar{\psi}_k(n) \Big|.$$

Appeal to the inequalities (2) and (1) completes the proof.

It has been known for some time that from an inequality of Large Sieve type, say,

$$\sum_{D\le Q} \sideset{}{^*}\sum_{\chi (\bmod D)} \Big| \sum_{n\le \chi} a_n \chi(n) \Big|^2 \ll (x + Q^2) \sum_{n\le x} |a_n|^2$$

one may derive a maximal variant

$$\sum_{D\le Q} \sideset{}{^*}\sum_{\chi (\bmod D)} \max_{y\le x} \Big| \sum_{n\le y} a_n \chi(n) \Big|^2 \ll (x + Q^2) \sum_{n\le x} |a_n|^2 (\log(2+x))^2.$$

An example is stated in Montgomery [18] p. 807. An earlier example for quadratic character sums, using Fourier analysis in the complex plane, may be found in Elliott [3]. Indeed, for Fourier analysis on finite fields results of this type go back at least to I. M. Vinogradov's work on power residues.

Improving on a result of Uchiyama [21], Montgomery [18] proved that for points x_j (mod 1) which satisfy $\|x_i - x_j\| \geq \delta > 0$ whenever $i \neq j$

$$\sum_j \max_{k\leq N} \left| \sum_{n=M}^{M+k} a_n \exp(2\pi i x_j n) \right|^2 \ll (N+\delta^{-1}) \sum_{n=M}^{M+N} |a_n|^2, \tag{3}$$

and deduced the corollary

$$\sum_{D\geq Q} \max_{k\leq N} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ D)} \left| \sum_{n=M}^{M+k} a_n \chi(n) \right|^2 \ll (N+Q^2) \sum_{n=M}^{M+N} |a_n|^2. \tag{4}$$

These inequalities may be compared with

$$\sum_j \max_{v-u\leq H} \left| \sum_{u<n\leq v} a_n \exp(2\pi i x_j n) \right|^2 \leq (H+\delta^{-1}\log \delta^{-1}) \sum_{n=-\infty}^{\infty} |a_n|^2 \tag{5}$$

and

$$\sum_{D\leq Q} \max_{v-u\leq H} \sideset{}{^*}\sum_{\chi (\mathrm{mod}\ D)} \left| \sum_{u<n\leq v} a_n \chi(n) \right|^2 \leq (H+6Q^2\log Q) \sum_{n=-\infty}^{\infty} |a_n|^2 \tag{6}$$

of Elliott [8], valid for all square-summable sequences of complex numbers a_n. In applications to the study of arithmetic functions the uniformity of (5) over all intervals $(u, v]$ plays an important rôle. I afterwards realized that a result of this type may be deduced from Montgomery's inequality (3). Each pair of reals u, v with $v-u\leq H$ must belong to one of the intervals $(rH, (r+2)H]$ where $r=0, \pm 1, +2, \cdots$. The sum in (5) does not exceed

$$4 \sum_{r=-\infty}^{\infty} \sum_j \max_{k\leq 2H} \left| \sum_{rH<n\leq rH+k} a_n \exp(2\pi i x_j n) \right|^2 \ll (H+\delta^{-1}) \sum_{n=-\infty}^{\infty} |a_n|^2,$$

since

$$\sum_{r=-\infty}^{\infty} \sum_{rH\leq n\leq (r+2)H} |a_n|^2 = 2 \sum_{n=-\infty}^{\infty} |a_n|^2.$$

The upper bound factor $H+\delta^{-1}\log \delta^{-1}$ in (5) has been replaced by $O(H+\delta^{-1})$. This improvement has been bought at a rather high price, however. The proof which I gave of (5) is quite elementary. The mainspring of Montgomery's proof of the maximal inequality (3) is Hunt's quantitative development

$$\int_0^1 \max_{k\leq N}\left|\sum_{r=1}^{k} a_n \exp(2\pi i\alpha n)\right|^2 d\alpha \ll \sum_{n=1}^{N}|a_n|^2$$

from Carlson's proof of the almost sure convergence of the Fourier series of functions of the class $L^2(0,1)$. Similar remarks can be made concerning inequalities (4) and (6).

In applications to the study of character sums over primes, the extra factor $\log Q$ in (6) is not particularly significant.

The inequality (6) suggests that the term $Q^3(\log Q)^{1+\varepsilon}$ in Theorem 2 should be replaced by something more near to Q^2. Improvements of this kind in the study of the Large Sieve are generally effected by moving to additive characters. With the maxima under the summation over the primitive characters to a given modulus that procedure seems no longer viable.

Disregarding this difficulty, in order to lower $Q^{3+\varepsilon}$ in the Corollary to Theorem 2 to $Q^{2+\varepsilon}$ a natural approach would be to investigate the analogue of the inequality (3) when the innermost variable n is restricted to prime values. Suppose that for some $c>1$

$$\sideset{}{'}\sum_{q\leq Q}\sum_{\substack{b=1\\(b,q)=1}}^{q}\left|\sum_{p\leq x} a_p \exp(2\pi i p b q^{-1}\right|^2 \ll \left(\frac{x}{\log x}+Q^c\right)\sum_{p\leq x}|a_p|^2 \tag{7}$$

where $'$ indicates that q runs over a selection of the positive integers not exceeding Q. The innersums over the b in (7) have the alternative expression

$$q\sum_{h=1}^{q}\left(\sum_{d|q}\frac{\mu(d)}{d}\sum_{\substack{p\leq x\\ p\equiv h \pmod{qd^{-1}}}} a_p\right)^2.$$

Since no prime in the range $Q<p\leq x$ will satisfy $p\equiv q \pmod{qd^{-1}}$ with $d<q\leq Q$,

$$\sideset{}{'}\sum_{q\leq Q}\frac{\mu^2(q)}{q}\left|\sum_{Q<p\leq x} a_p\right|^2 \ll \left(\frac{x}{\log x}+Q^c\right)\sum_{p\leq x}|a_p|^2.$$

The choice $a_p=1$ shows that for $Q\ll(x/\log x)^{1/c}$, the sum $\sum'\mu^2(q)q^{-1}$ is bounded uniformly in x. This is a severe restriction upon the moduli q. An improvement of the following result would therefore seem to require a severe reformulation.

THEOREM 3. *Let $\varepsilon>0$. Let $x_j, j=1,\cdots,J$ be real numbers* (*mod* 1) *that satisfy $\|m(x_i-x_j)\|\geq\delta>0$ for $i\neq j$ and $1\leq m\leq M$. Then*

$$\sum_{j=1}^{J} \max_{v-u\leq H}\Big|\sum_{\substack{u<n\leq v\\(n,P)=1}} a_n \exp(2\pi i n x_j)\Big|^2 \ll \Big(H\prod_{\substack{p|P\\p\leq M}}\Big(1-\frac{1}{p}\Big)+M^{\varepsilon}\delta^{-1}\log\delta^{-1}\Big)\sum_{\substack{n=-\infty\\(n,P)=1}}^{\infty}|a_n|^2$$

uniformly for $H\geq 1$, $M\geq 1$, *integers* P, *and square-summable sequences of complex numbers* a_n.

PROOF. For each j choose real numbers u_j, v_j, with $v_j-u_j\leq H$, and define

$$t_j(n)=\begin{cases}\exp(2\pi i x_j n) & \text{if} \quad u_n<n\leq v_j,\\ 0 & \text{otherwise.}\end{cases}$$

Consider the dual inequality

$$S=\sum_{\substack{n=-\infty\\(n,P)=1}}^{\infty}\Big|\sum_{j=1}^{J} t_j(n)c_j\Big|^2\leq \rho\sum_{j=1}^{J}|c_j|^2, \tag{8}$$

required to be valid for all complex c_j. If $\sigma(n)$ is real, ≥ 1 whenever $(n,P)=1$, and is non-negative otherwise, then

$$\begin{aligned} S&\leq \sum_{n=-\infty}^{\infty}\sigma(n)\Big|\sum_{j=1}^{J} t_j(n)c_j\Big|^2\\ &=\sum_{j,k=1}^{J} c_j\bar{c}_k\sum_{n=-\infty}^{\infty}\sigma(n)t_j(n)\overline{t_k(n)}. \end{aligned} \tag{9}$$

Since $|c_j\bar{c}_k|\leq(|c_j|^2+|c_k|^2)/2$ we see that a permissible value for ρ is

$$\max_{1\leq k\leq J}\Big(\sum_{n=-\infty}^{\infty}\sigma(n)|t_k(n)|^2+\sum_{\substack{j=1\\j\neq k}}^{J}\Big|\sum_{n=-\infty}^{\infty}\sigma(n)t_j(n)\overline{t_k(n)}\Big|\Big). \tag{10}$$

Adopting the sieve device of Selberg, we replace $\sigma(n)$ by

$$\Big(\sum_{d|n}\lambda_d\Big)^2$$

where the real numbers λ_d vanish unless $d\leq z$ and d is made up only of powers of the primes which divide P. Then for $j\neq k$ a typical innermost sum at (10) has the form

$$\sum_{d_1,d_2\leq z}\lambda_{d_1}\lambda_{d_2}\sideset{}{'}\sum_{n\equiv 0 (\mathrm{mod}\, d_i),i=1,2.}\exp(2\pi i(x_j-x_k)n)$$

where the innersum is taken over the intersection of the intervals $(u_j, v_j]$ and $(u_k, v_k]$. Since the terms of this sum are in a geometric progression, it does not exceed $(\operatorname{Sin}\pi\|(x_j-x_k)[d_1, d_2]\|)^{-1}\leq(2\|(x_j-x_k)[d_1, d_2]\|)^{-1}$ in absolute value. The final (multiple sum) in (10) is

$$\ll\sum_{d_1,d_2\leq z}|\lambda_{d_1}\lambda_{d_2}|\sum_{\substack{j=1\\ j\neq k}}^{J}\|(x_j-x_k)[d_1, d_2]\|^{-1}.$$

If $z\leq M^{1/2}$ the hypothesis concerning the points x_j ensures that the innersum here is at most

$$\sum_{s\leq\delta^{-1}}(s\delta)^{-1}<3\delta^{-1}\log\delta^{-1}.$$

The sum involving $|t_k(n)|^2$ arises from a diagonal term in the natural matrix representation of the operator underlying the inequality (9). It does not exceed

$$\sum_{u_k<n\leq u_k+H}\sigma(n)=\sum_{d_1,d_2\leq z}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{u_k<n\leq u_k+H\\ n\equiv 0(\text{mod } d_i),i=1,2}}1.$$

An elementary argument gives for the innersum an estimate $H[d_1, d_2]^{-1}+O(1)$. Thus

$$\rho=H\sum_{d_1,d_2\leq z}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]}+O\Big(\delta^{-1}\log\delta^{-1}\sum_{d_1,d_2\leq z}|\lambda_{d_1}\lambda_{d_2}|\Big)$$

is possible.

The first bilinear form in the λ_d arises when the method of Selberg is applied to estimate the number of integers, in a given interval, which have no prime factor in common with P. An extensive account of the theory of the Selberg sieve may be found in Halberstam and Richert [17], and Motohashi [19]. Here it will suffice to employ the short account given in Elliott [5], Chapter 2. We set $z=M^{\varepsilon/2}$ and choose for the λ_d those values which minimize the bilinear form when P is replaced by the product of those primes q which satisfy $2\leq q\leq M^{\varepsilon^2}, q|P$. In the notation of [5] we have $r=M^{\varepsilon^2}$, and it will be enough to know that $|\lambda_d|\leq 1$ for all d, and that if ε is fixed at a sufficiently small value, then

$$\sum_{d_1,d_2\leq z}\frac{\lambda_{d_1}, \lambda_{d_2}}{[d_1, d_2]}\leq 3\prod_{p\leq M^{\varepsilon^2}}\Big(1-\frac{1}{p}\Big).$$

Since

$$\prod_{M^{\varepsilon^2}<p\leq M}\Big(1-\frac{1}{p}\Big)=\exp\Big(-\sum_{M^{\varepsilon^2}<p\leq M}\frac{1}{p}+O(M^{-\varepsilon^2})\Big)=\exp(-2\log\varepsilon+O(1))$$

we see that the inequality (8) holds for some

$$\rho \ll H \prod_{\substack{p \mid P \\ p \leq M}} \left(1-\frac{1}{p}\right) + M^{\varepsilon}\delta^{-1}\log\delta^{-1}.$$

The assertion of the theorem now follows by duality.

As an application of Theorem 3, let p, q denote primes. If $\varepsilon > 0$ then

$$\sum_{Q^{\varepsilon} < q \leq Q} \sum_{b=1}^{q-1} \max_{v-u \leq H} \left| \sum_{u < p \leq v} a_p \exp(2\pi i p b q^{-1}) \right|^2 \ll \left(\frac{H}{\log H} + Q^{2+\varepsilon}\right) \sum_{p \geq 2} |a_p|^2.$$

For any modulus D

$$\frac{D}{\phi(D)} \sum_{\chi (\mathrm{mod}\, D)}^{*} \left| \sum_{n \leq x} a_n \chi(n) \right|^2 \leq \sum_{\substack{b=1 \\ (b, D)=1}}^{D} \left| \sum_{n \leq x} a_n \exp(2\pi i n b D^{-1}) \right|^2$$

so that

$$\sum_{\substack{Q^{\varepsilon} < q \leq Q \\ q \text{ prime}}} \max_{v-u \leq H} \sum_{\chi (\mathrm{mod}\, q)}^{*} \left| \sum_{u < p \leq v} a_p \chi(p) \right|^2 \ll \left(\frac{H}{\log H} + Q^{2+\varepsilon}\right) \sum_{p \geq 2} |a_p|^2.$$

Combining this result with the corollary to Theorem 2 gives

THEOREM 4. *Let $\varepsilon > 0$. Then*

$$\sum_{\substack{q \leq Q \\ q \text{ prime}}} \max_{v-u \leq H} \sum_{\chi (\mathrm{mod}\, q)}^{*} \left| \sum_{u < p \leq v} a_p \chi(p) \right|^2 \ll \left(\frac{H}{\log H} + Q^{2+\varepsilon}\right) \sum_{p \geq 2} |a_p|^2$$

uniformly in $Q \geq 1$, $H \geq 2$ and complex numbers a_p, one for each prime p.

For values of Q large in comparison with N, Wolke [22] showed that for prime moduli the term Q^2 in Lemma 1 may be decreased to $Q^2(\log Q)^{-1+\varepsilon}$.

3. Applications to arithmetic functions.

An inequality of the form

$$(11) \qquad \sum_{q \leq Q} \phi(q) \sum_{r=1}^{q-1} \left| \sum_{\substack{p \equiv r (\mathrm{mod}\, q) \\ p \leq x}} a_p - \frac{1}{\phi(q)} \sum_{p \leq x} a_p \right|^2 \ll \left(\frac{x}{\log x} + D\right) \sum_{p \leq x} |a_p|^2$$

where q runs through prime moduli and D is an unspecified function of Q may be found in Elliott [4], where it plays an essential rôle in the characterisation of those real additive functions f for which the second moments

$$[x]^{-1}\sum_{n\le x}|f(n+1)-f(n)|^2$$

are uniformly bounded. Most important in that application is that the leading term in the upper bound factor be $x/\log x$, and not x.

In a subsequent paper [6] I gave a method which shows that (11) certainly holds with Q^5 in place of D. Because it employs reasonably good estimates for the number of zeros of Dirichlet L-functions in the critical strip, this method cannot be regarded as elementary. However it will give also the uniform result

$$\sum_{d\le Q}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ d)}\max_{y\le x}\left|\sum_{p\le y}a_p\chi(p)\right|^2\ll\left(\frac{x}{\log x}+Q^{6+\varepsilon}\right)\sum_{p\le x}|a_p|^2.$$

Without the maxima the results of Bombieri and Davenport [2] (see for example Elliott [7], Lemma (6.3)) enable $Q^{6+\varepsilon}$ to be replaced by $Q^{2+\varepsilon}$. These partial results for prime variables are now superseded by the corollary to Theorem 1.

Maximal interval versions of the large sieve play an important rôle in the derivation of the inequality

$$(12)\qquad \sum_{q\le x^{1/2-\varepsilon}}\phi(q)\max_{(r,q)=1}\max_{y\le x}\left|\sum_{\substack{n\le y\\ n\equiv r(\mathrm{mod}\ q)}}f(n)-\frac{1}{\phi(q)}\sum_{\substack{n\le y\\ (n,q)=1}}f(n)\right|^2$$
$$\ll\frac{x}{\log x}(\log\log x)^4\sum_{q\le x}\frac{|f(q)|^2}{q}$$

for additive functions f, with prime-power moduli q, Elliott [8]. Inequalities of this last type may be applied to the study of the differences of additive functions f_j, $j=1,2$,

$$x^{-1}\sum_{n\le x}|f_1(an+b)-f_2(An+B)|^\alpha,\qquad \alpha>1,\ aB\ne Ab.$$

Maximal large sieves play an equally important rôle in the derivation of analogues of the inequality (12) for multiplicative functions g, especially those which are restricted only by $|g(n)|\le 1$ for all positive n, Elliott [10], [11], [12], [13]. Here

$$(13)\qquad \sum_{(\log x)^4<p\le x^{1/2-\varepsilon}}p\max_{(r,p)=1}\max_{y\le x}\left|\sum_{\substack{n\le y\\ n\equiv r(\mathrm{mod}\ p)}}g(n)-\frac{1}{\phi(p)}\sum_{\substack{n\le y\\ (n,p)=1}}g(n)\right|^2$$

$$\ll \frac{x^2}{(\log x)^2}(\log\log x)^4$$

is typical. If in Elliott [10] we apply the corollary to the present Theorem 1, rather than the Lemma 4 used there, then we may decrease the factor $(\log\log x)^4$ in this upper bound to $(\log\log x)^2$.

The inequality (13) may be applied to the study of the correlations of multiplicative functions

$$x^{-1}\sum_{n\le x} g_1(an+b)g_2(An+B), \qquad aB\ne Ab,$$

Elliott [9], [14], and these in turn to questions in Probabilistic Number Theory concerning the characterisation of renormalized additive functions

$$\sum_{j=1}^{2} \beta_j(x)^{-1}(f_j(a_jn+b_j)-\alpha_j(x)), \qquad a_1b_2\ne a_2b_1,\ \beta_j(x)>0,$$

with asymptotically measurable value distribution over long intervals $1\le n\le x$ of integers n, Elliott [9], [15].

It should be emphasized that for these applications a very important feature of the inequalities (12) and (13) is their wide generality. Investigations of the spectral radius of the underlying operator shows that for (12) a factor $x/\log x$ would be best possible. It seems likely that for (13) an upper bound $\ll x^2(\log x)^{-2}$ might be best possible. In particular, since only one or two powers of $\log x$ are being saved over trivial bounds, it is essential that any maximal Large Sieve inequalities applied during the proofs do not have the traditional logarithmic factors.

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