

**The restriction homomorphism $\text{Res}_H^G: \text{Wh}_G(X) \rightarrow \text{Wh}_H(X)$
 for G a compact Lie group**

Dedicated to Professor Akio Hattori on his sixtieth birthday

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Let G be a compact Lie group and H a closed subgroup of G . A main objective of this paper is to prove the following result.

THEOREM I. *Given a finite G -CW complex X there exist a finite H -CW complex $R_H X$ and an H -homotopy equivalence*

$$(1) \quad \eta: X \longrightarrow R_H X$$

such that this construction is unique up to simple H -homotopy type; i.e., if $\eta': X \rightarrow R'_H X$ is another choice then

$$(2) \quad \sigma_{\eta', \eta} = \eta' \circ \eta^-: R_H X \longrightarrow R'_H X$$

is a simple H -homotopy equivalence. Furthermore we have that

$$(3) \quad \dim(R_H X)^{H_1} = \dim X^{H_1}$$

for each closed subgroup H_1 of H , and the H -isotropy types occurring in $R_H X$ are exactly the same ones as in the H -space X .

Here η^- denotes an H -homotopy inverse of η and \dim denotes topological dimension. The notion of equivariant simple-homotopy equivalence is as defined in [3]. In order to put the above result into its right perspective one should recall the following two facts. First, when restricting the given action of G on X to the closed subgroup H , the H -space X does not in general inherit an induced structure of an H -CW complex, at least not in any natural way, see [8, Section 2]. This means that we cannot expect to find a finite H -CW complex $R_H X$ that is H -homeomorphic to the H -space X . Secondly, we recall that equivariant Whitehead torsion is not an equivariant topological invariant; i.e., there exist H -homeomorphisms between finite H -CW complexes, that are not

simple H -homotopy equivalences. In other words; we can have two finite H -CW complex structures Y_1 and Y_2 on the same H -space Y such that the identity map $\text{id}_Y: Y_1 \rightarrow Y_2$ is *not* a simple H -homotopy equivalence between the finite H -CW complexes Y_1 and Y_2 . In the light of these facts we see that Theorem I gives as precise information as one can expect.

We call such an H -homotopy equivalence $\eta: X \rightarrow R_H X$ as in Theorem I a *preferred H -reduction* of X . The existence of such a class of preferred H -reductions of X is proved in Section 6, where we also establish relative versions of Theorem I. In particular we show that if (V, X) is a finite G -CW pair and $\theta: X \rightarrow R_H X$ is a preferred H -reduction of X then θ can be extended to a preferred H -reduction $\eta: V \rightarrow R_H V$ of V , such that $R_H V$ contains $R_H X$ as an H -subcomplex. Furthermore, any G -map $f: X \rightarrow W$ between finite G -CW complexes induces an H -map $R_H f: R_H X \rightarrow R_H W$ between finite H -CW complexes, and $R_H f$ is an H -homotopy equivalence if f is a G -homotopy equivalence.

In Section 8 we define the H -equivariant Whitehead group $\text{Wh}_H(X)$ of a finite G -CW complex X . We also show that if $f: X \rightarrow W$ is a G -homotopy equivalence then the induced H -homotopy equivalence $R_H f: R_H X \rightarrow R_H W$ has a well-defined H -equivariant Whitehead torsion $\tau(R_H f) \in \text{Wh}_H(X)$, in the group $\text{Wh}_H(X)$. Then we go on to prove in Section 9 that the preferred H -reduction operation R_H respects equivariant simple-homotopy type in the sense that it takes formal G -deformations into formal H -deformations, see Corollary 9.2. This last mentioned fact leads directly to the existence of a well-defined restriction homomorphism

$$\text{Res}_H^G: \text{Wh}_G(X) \longrightarrow \text{Wh}_H(X).$$

In the last section of this paper, Section 10, we prove that if $f: X \rightarrow W$ is a G -homotopy equivalence between finite G -CW complexes and $R_H f: R_H X \rightarrow R_H W$ is the H -homotopy equivalence induced by f , then

$$\text{Res}_H^G(\tau(f)) = \tau(R_H f) \in \text{Wh}_H(X).$$

In particular we see that if $f: X \rightarrow W$ is a simple G -homotopy equivalence then $R_H f: R_H X \rightarrow R_H W$ is a simple H -homotopy equivalence.

The proof of the fact that the construction given in Theorem I is unique up to a simple H -homotopy equivalence easily becomes involved and messy. We have tried to make the argument clear by formalizing some parts of it. The notions k -equivalence and λ -map, and the technical results concerning these notions, established in Section 3, serve this

purpose. In Section 4 we prove three basic results concerning equivariant simple homotopy type of adjunction spaces, and combined into one theorem these results give Theorem 4.5. It is in fact essentially this result, Theorem 4.5, that lies behind the uniqueness up to a simple H -homotopy equivalence in Theorem I. In Section 5 we combine Theorem 4.5 with the technical results from Section 3 into one result, which is in a form that is convenient to use in the proof of the main result in Section 6.

The main result of this paper, Theorem 6.1, which proves the existence of preferred H -reductions, was announced in [7, Theorem C]. The corresponding restriction homomorphism $\text{Res}_H^G: \text{Wh}_G(X) \rightarrow \text{Wh}_H(X)$ is discussed in [7, Section 2]. (In [7] we denoted $R_H X$ by $\text{esh}_H(X)$.)

A different approach to the restriction homomorphism between equivariant Whitehead groups is given in W. Lück [9].

We shall in a later paper prove that if X is a finite G -CW complex and $K < H < G$, then the K -CW complexes $R_K(R_H X)$ and $R_K X$ have the same simple K -homotopy type. Curiously enough this transitivity property is a non-trivial fact.

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1. Preliminaries

Let G be a compact Lie group. We will consider the basic properties of G -CW complexes as well-known and use them without further reference. For example the equivariant skeletal approximation theorem, also in its relative form, and the fact that a G -CW pair (X, X_0) has the G -homotopy extension property, are facts that are used freely in this paper. The following result concerning different choices of characteristic G -maps for a G -cell c does perhaps not appear in the literature, so we give it here since we have explicit use of it in this paper.

LEMMA 1.1. *Suppose that $\xi: (D^n \times G/P, S^{n-1} \times G/P) \rightarrow (c, \dot{c}) \hookrightarrow (X^n, X^{n-1})$ and $\xi': (D^n \times G/P', S^{n-1} \times G/P') \rightarrow (c, \dot{c}) \hookrightarrow (X^n, X^{n-1})$ are two characteristic G -maps for some G -cell c of X . Then there exist an isometry $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$ and an element $g_0 \in G$ such that the maps $\xi, \xi' \circ (f \times g_0): (D^n \times G/P, S^{n-1} \times G/P) \rightarrow (X, X_0)$ are G -homotopic.*

In Lemma 1.1 one may always choose $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$ to be either the identity map or the isometry given by $f(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, -x_n)$. These two choices give a map from S^{n-1} to itself of degree 1 or of degree -1 , respectively. The map $g_0: G/P \rightarrow G/P'$, given by $gP \mapsto gg_0P'$ for every $gP \in G/P$, is a real analytic G -isomorphism between real analytic G -manifolds. The proof of Lemma 1.1 is easy and left to the reader.

For the notion of simple G -homotopy type and other facts from equivariant simple-homotopy theory we refer to [3]. In particular the basic notions of an elementary G -expansion and an elementary G -collapse, see [3, Definition II.1.1], are important. The notions G -expansion, G -collapse, formal G -deformation, and simple G -homotopy equivalence are defined in [3, Section II.1]. For the definition of the equivariant Whitehead group $\text{Wh}_G(X)$ of a finite G -CW complex X we refer to [3, Section II.2].

Let us now change the notation so that we let H denote an arbitrary compact Lie group. (In most cases H will be a closed subgroup of a given compact Lie group G .) In [10] Matumoto and Shiota show that one can associate to any compact smooth H -manifold a well-defined simple H -homotopy type. In this paper we only use a very special case of this result, namely the following one. Let H be a closed subgroup of G and consider the standard action of H , by multiplication from the left, on a homogeneous space G/P , where P is a closed subgroup of G .

The compact H -manifold G/P can be given a well-defined simple H -homotopy type in the following way. By a well-known theorem, due independently to Mostow [11] and Palais [12], there exist a linear representation space $\mathbf{R}^n(\rho)$ for G (where $\rho: G \rightarrow O(n)$) and a point $x \in \mathbf{R}^n(\rho)$ such that $G_x = P$. Then the G -orbit Gx through the point x is a real analytic G -submanifold of $\mathbf{R}^n(\rho)$ and Gx is G -isomorphic to G/P . We now consider $\mathbf{R}^n(\rho)$ as a linear representation space for H ; i.e., we consider the linear representation space $\mathbf{R}^n(\rho|H)$. By a well-known result the orbit space $\mathbf{R}^n(\rho|H)/H$ can be considered as a closed semi-algebraic subset of some euclidean space \mathbf{R}^k . Since the H -manifold G/P is a real analytic H -submanifold of $\mathbf{R}^n(\rho|H)$ it follows that the orbit space $(G/P)/H$, where $(G/P)/H \subset \mathbf{R}^n(\rho|H)/H \subset \mathbf{R}^k$, is a compact subanalytic subset of \mathbf{R}^k .

DEFINITION 1.2. A distinguished H -triangulation of the H -manifold G/P consists of a finite H -CW complex F such that the orbit space F/H is a finite simplicial complex and an H -homeomorphism

$$u: F \xrightarrow{\cong} G/P$$

such that the induced homeomorphism $\bar{u}: F/H \rightarrow (G/P)/H$ is subanalytic.

THEOREM 1.3. *Let G be a compact Lie group and let H and P be closed subgroups of G . Let H act on the homogeneous space G/P by multiplication from the left. Then there exists a distinguished H -triangulation $u: F \rightarrow G/P$ of the H -manifold G/P . If $u: F \rightarrow G/P$ and $u': F' \rightarrow G/P$ are distinguished H -triangulations of G/P the map $(u')^{-1} \circ u: F \rightarrow F'$ is a simple H -homotopy equivalence.*

PROOF. It follows from the results concerning existence of subanalytic triangulations of subanalytic sets, due to Hironaka [2] and Hardt [1], that there exists a subanalytic triangulation $t: L \rightarrow (G/P)/H$ of the subanalytic set $(G/P)/H$ such that each open simplex of the triangulation lies completely within one isotropy type. It now follows by the lifting procedure of Illman and Matsumoto, see [4], that there exists a finite H -CW complex F , with $F/H = L$, and an H -homeomorphism $u: F \rightarrow G/P$ which covers t .

The uniqueness part of Theorem 1.3 follows from the fact that two subanalytic triangulations of a subanalytic set have a common subanalytic subdivision, see Hironaka [2] and Hardt [1], and the fact that an arbitrary H -equivariant subdivision map of finite H -CW complexes is a simple H -homotopy equivalence, see Illman [6, Theorem 12.2]. \square

LEMMA 1.4. *Suppose that $u: F \rightarrow G/P$ is a distinguished H -triangulation of G/P . Let P' be a closed subgroup of G that is conjugate to P and let $g_0: G/P \rightarrow G/P'$ be the G -map given by $gP \mapsto gg_0P$ for all $gP \in G/P$. Then $g_0 \circ u: F \rightarrow G/P'$ is a distinguished H -triangulation of G/P' .*

PROOF. The map $g_0: G/P \rightarrow G/P'$ is a real analytic G -isomorphism between real analytic G -manifolds, and hence the induced map $\bar{g}_0: (G/P)/H \rightarrow (G/P')/H$, between the orbit spaces, is subanalytic. \square

We shall in this paper use the following terminology. An H -map

$$\sigma: (Y, Y_0) \longrightarrow (Z, Z_0)$$

between finite H -CW pairs is a simple H -homotopy equivalence if both $\sigma: Y \rightarrow Z$ and $\sigma|: Y_0 \rightarrow Z_0$ are simple H -homotopy equivalences. The following definition will be used throughout the paper.

DEFINITION 1.5. Let (U, U_0) be an arbitrary H -pair and let $f_1: (U, U_0) \rightarrow (Y, Y_0)$ and $f_2: (U, U_0) \rightarrow (Z, Z_0)$ be H -maps, where (Y, Y_0) and (Z, Z_0) are finite H -CW pairs. We say that the H -maps f_1 and f_2 are s -equivalent if there exists a simple H -homotopy equivalence $\sigma: (Y, Y_0) \rightarrow (Z, Z_0)$ such that the maps $\sigma \circ f_1, f_2: (U, U_0) \rightarrow (Z, Z_0)$ are H -homotopic. If $Y_0 = Z_0$ and we in addition can choose σ such that $\sigma|_{Y_0} = \text{id}$ we say that f_1 and f_2 are s -equivalent rel Y_0 .

We will have use of the following result.

LEMMA 1.6. *Suppose $\sigma: Y \rightarrow Z$ is a simple H -homotopy equivalence between finite H -CW complexes and that $f: K \rightarrow L$ is a simple homotopy equivalence between ordinary CW complexes. Then $f \times \sigma: K \times Y \rightarrow L \times Z$ is a simple H -homotopy equivalence.*

PROOF. This is a direct consequence of the product formula for equivariant Whitehead torsion, see Illman [5, Section 3]. \square

Furthermore we will also use the following result.

LEMMA 1.7. *Suppose that (L, L_0) is a finite CW-pair such that L collapses to L_0 by a finite sequence of elementary collapses. Let F be a finite H -CW complex. Then $L \times F$ collapses to $L_0 \times F$ by a finite sequence of elementary H -collapses.*

PROOF. The proof is easy and left to the reader. \square

We will also use a slight generalization of the notion of a skeletal map, namely the following one.

DEFINITION 1.8. Let (C, D) be a finite H -CW pair and let Y be a finite H -CW complex. We say that an H -map $\varphi: D \rightarrow Y$ is (C, D) -skeletal if for every H -cell c , of say dimension m , in $C - D$ we have $\varphi(\dot{c}) \subset Y^{m-1}$.

Observe that if $\varphi: B \rightarrow Y$ is a (C, D) -skeletal H -map then the adjunction space $Y \cup_{\varphi} C$ is a finite H -CW complex.

If $\eta: X \rightarrow Y$ is an H -homotopy equivalence we let $\eta^{-}: Y \rightarrow X$ denote an H -homotopy inverse of η , and the same convention holds for H -homotopy equivalences between H -pairs. If $\eta_1, \eta_2: X \rightarrow Y$ are two H -maps we sometimes denote the fact that η_1 and η_2 are H -homotopic by $\eta_1 \simeq \eta_2$, and we also use this same notation in connection with H -maps between H -pairs. When we speak of a pair (C, D) we assume throughout the

paper that D is closed in C .

2. Background information on equivariant homotopy type of adjunction spaces

Let H denote an arbitrary compact Lie group. (In this section and in Section 3 the role of the transformation group H is completely formal, and hence H could as well be any locally compact group.) By X and Y we denote arbitrary H -spaces, and (A, B) and (C, D) denote H -pairs that have the H -homotopy extension property.

Let $\varphi_0, \varphi_1: B \rightarrow X$ be two H -maps that are H -homotopic. Then the adjunction spaces $X \cup_{\varphi_0} A$ and $X \cup_{\varphi_1} A$ have the same H -homotopy type. An H -homotopy equivalence from $X \cup_{\varphi_0} A$ to $X \cup_{\varphi_1} A$ can be constructed as follows. Let $\Phi: B \times I \rightarrow X$ be an H -homotopy from φ_0 to φ_1 . The fact that (A, B) has the H -homotopy extension property is equivalent to the fact that $A \times \{0\} \cup B \times I$ is a strong H -deformation retract of $A \times I$. Let $i_0: A \times \{0\} \cup B \times I \rightarrow A \times I$ denote the inclusion and let $r_1: A \times I \rightarrow A \times \{1\} \cup B \times I$ be an H -retraction. We now define $k(\Phi): X \cup_{\varphi_0} A \rightarrow X \cup_{\varphi_1} A$ to be the composite map

$$\begin{aligned} X \cup_{\varphi_0} A &\xrightarrow{\cong} X \cup_{\Phi} (A \times \{0\} \cup B \times I) \xrightarrow{\hat{i}_0} X \cup_{\Phi} (A \times I) \\ &\xrightarrow{\hat{r}_1} X \cup_{\Phi} (A \times \{1\} \cup B \times I) \xrightarrow{\cong} X \cup_{\varphi_1} A \end{aligned}$$

where the first map and the last map are natural H -homeomorphisms, which we shall use as identifications. Since $A \times \{0\} \cup B \times I$ and $A \times \{1\} \cup B \times I$ are strong H -deformation retracts of $A \times I$ it follows that both \hat{i}_0 and \hat{r}_1 are H -homotopy equivalences. Hence $k(\Phi)$ is an H -homotopy equivalence, and we also have $k(\Phi)|_X = \text{id}_X$. Different choices of the retraction $r_1: A \times I \rightarrow A \times \{1\} \cup B \times I$ give, strictly speaking, rise to different maps $k(\Phi)$, but $k(\Phi)$ is uniquely determined up to an H -homotopy rel X by the homotopy Φ . For the effect of different choices of H -homotopies Φ from φ_0 to φ_1 see Lemma 2.2 below. (A more detailed discussion of the H -homotopy equivalence $k(\Phi)$ can be found in [8, Section 3].)

We will in this paper need the following additional observation concerning the map $k(\Phi)$. Suppose that $B_0 \subset A$ is a closed subset of A such that $B_0 \cap B = \emptyset$ and such that $(A, B_0 \cup B)$ has the H -homotopy extension property. Then $A \times \{1\} \cup (B_0 \cup B) \times I$ is a strong H -deformation retract of $A \times I$, and hence there is an H -retraction $r_1: A \times I \rightarrow A \times \{1\} \cup B \times I$ such

that $r_1(b', t) = (b', 1)$ for all $(b', t) \in B_0 \times I$. With this choice of retraction we obtain that

$$k(\Phi)|_{B_0} = \text{id}_{B_0}$$

where we consider $B_0 \subset X \cup A$ and $B_0 \subset X \cup A$ in the obvious way.

All results in this section are well known and have easy proofs. We shall simply state the results here, and leave the proofs to the reader. (Proofs of Lemmas 2.4 and 2.5 are given in [8, Section 3].)

LEMMA 2.1. *Suppose that the H -maps $\varphi_0, \varphi_1 : B \rightarrow X$ are H -homotopic and that $\Phi : B \times I \rightarrow X$ is an H -homotopy from φ_0 to φ_1 . Then*

$$k(\Phi) : X \underset{\varphi_0}{\cup} A \longrightarrow X \underset{\varphi_1}{\cup} A$$

is an H -homotopy equivalence. Furthermore $k(\Phi)|_X = \text{id}_X$ and $k(\Phi^{-1})$ is an H -homotopy inverse of $k(\Phi)$ rel X . If in addition B_0 is a closed subset of A such that $B_0 \cap B = \emptyset$ and $(A, B_0 \dot{\cup} B)$ has the H -homotopy extension property, we may choose $k(\Phi)$ such that $k(\Phi)|_{B_0} = \text{id}_{B_0}$.

LEMMA 2.2. *Suppose that $\Phi, \Phi' : B \times I \rightarrow X$ are two H -homotopies from φ_0 to φ_1 such that Φ and Φ' are H -homotopic rel $B \times \dot{I}$. Then the two H -homotopy equivalences*

$$k(\Phi), k(\Phi') : X \underset{\varphi_0}{\cup} A \longrightarrow X \underset{\varphi_1}{\cup} A$$

are H -homotopic rel X . By abuse of notation we denote the conclusion of Lemma 2.2 simply by $k(\Phi) = k(\Phi')$.

If $\Phi_1 : B \times I \rightarrow X$ is an H -homotopy from φ_0 to φ_1 and $\Phi_2 : B \times I \rightarrow X$ is an H -homotopy from φ_1 to φ_2 the join $\Phi_1 * \Phi_2 : B \times I \rightarrow X$ is an H -homotopy from φ_0 to φ_2 .

LEMMA 2.3. *We have*

$$k(\Phi_1 * \Phi_2) = k(\Phi_2) \circ k(\Phi_1) : X \underset{\varphi_0}{\cup} A \longrightarrow X \underset{\varphi_2}{\cup} A.$$

If $f : X \rightarrow Y$ is an H -map we let

$$\bar{f} : X \underset{\varphi}{\cup} A \longrightarrow Y \underset{f\varphi}{\cup} A$$

be the H -map induced by f and by the identity map on A . We call

\bar{f} the canonical extension of f . We have $\bar{f}|_X = f$.

LEMMA 2.4. Suppose that the H -maps $f_0, f_1: X \rightarrow Y$ are H -homotopic and that $F: X \times I \rightarrow Y$ is an H -homotopy from f_0 to f_1 . Then the diagram

$$\begin{array}{ccc} X \cup A & \xrightarrow{\bar{f}_0} & Y \cup A \\ \varphi \downarrow & & \downarrow f_0 \varphi \\ & \searrow \bar{f}_1 & \downarrow k(\theta) \\ & & Y \cup A \\ & & \downarrow f_1 \varphi \end{array}$$

is H -homotopy commutative. Here $\theta = F \circ (\varphi \times \text{id}): B \times I \rightarrow Y$, and $k(\theta)$ denotes the corresponding H -homotopy equivalence given by Lemma 2.1.

LEMMA 2.5. If $f: X \rightarrow Y$ is an H -homotopy equivalence then so is its canonical extension $\bar{f}: X \cup A \rightarrow Y \cup A$.

LEMMA 2.6. Let $\Phi: B \times I \rightarrow X$ be an H -homotopy from φ_0 to φ_1 , and let $f: X \rightarrow Y$ be an H -map. Then the diagram

$$\begin{array}{ccc} X \cup A & \xrightarrow{\bar{f}} & Y \cup A \\ \varphi_0 \downarrow & & \downarrow f \varphi_0 \\ k(\Phi) \downarrow & & \downarrow k(f \circ \Phi) \\ X \cup A & \xrightarrow{\bar{f}} & Y \cup A \\ \varphi_1 \downarrow & & \downarrow f \varphi_1 \end{array}$$

is H -homotopy commutative.

If $\alpha: (A, B) \rightarrow (C, D)$ is an H -map we define

$$\hat{\alpha}: X \cup A \longrightarrow X \cup C$$

$\phi \alpha \downarrow \qquad \qquad \downarrow \phi$

to be the H -map induced by the identity map on X and the H -map α on A . Observe that $\hat{\alpha}|_X = \text{id}_X$. By ϕ we denote an H -map $\phi: D \rightarrow X$.

LEMMA 2.7. Suppose that the H -maps $\alpha_0, \alpha_1: (A, B) \rightarrow (C, D)$ are H -homotopic (as maps of pairs) and that $\Lambda: (A, B) \times I \rightarrow (C, D)$ is an H -homotopy from α_0 to α_1 . Then the diagram

$$\begin{array}{ccc}
 X \cup A & \xrightarrow{\hat{\alpha}_0} & X \cup C \\
 \downarrow k(\Gamma) & \nearrow \hat{\alpha}_1 & \downarrow \phi \\
 X \cup A & & X \cup C \\
 \phi\alpha_1| & & \phi
 \end{array}$$

is H -homotopy commutative. Here $\Gamma = \phi \circ (A|) : B \times I \rightarrow X$, and $k(\Gamma)$ denotes the corresponding H -homotopy equivalence as given by Lemma 2.1.

LEMMA 2.8. Suppose that $\alpha : (A, B) \rightarrow (C, D)$ is an H -homotopy equivalence of H -pairs. Then $\hat{\alpha} : X \cup A \rightarrow X \cup C$ is an H -homotopy equivalence.

LEMMA 2.9. Suppose that $\Psi : D \times I \rightarrow X$ is an H -homotopy from ϕ_0 to ϕ_1 and that $\alpha : (A, B) \rightarrow (C, D)$ is an H -map. Then the diagram

$$\begin{array}{ccc}
 X \cup A & \xrightarrow{\hat{\alpha}} & X \cup C \\
 \downarrow k' & & \downarrow k \\
 X \cup A & \xrightarrow{\hat{\alpha}} & X \cup C \\
 \phi_1\alpha| & & \phi_1
 \end{array}$$

is H -homotopy commutative. Here $k = k(\Psi)$ and $k' = k(\Psi \circ (\alpha| \times \text{id}))$.

LEMMA 2.10. Let $f : X \rightarrow Y$ and $\alpha : (A, B) \rightarrow (C, D)$ be H -maps. Then the diagram

$$\begin{array}{ccc}
 X \cup A & \xrightarrow{\bar{f}} & Y \cup A \\
 \downarrow \hat{\alpha} & & \downarrow \hat{\alpha} \\
 X \cup C & \xrightarrow{\bar{f}} & Y \cup C \\
 \phi & & f\phi
 \end{array}$$

commutes.

3. k -equivalence and λ -maps

In this section H denotes a compact Lie group. (In fact H could

also in this section, as in Section 2, be an arbitrary locally compact group.) By X, Y and Z we denote H -spaces and $(A, B), (C, D)$ and (E, F) denote H -pairs that have the H -homotopy extension property.

Assume that $\alpha: (A, B) \rightarrow (C, D)$ and $\eta: X \rightarrow Y$ are H -homotopy equivalences, and that $\varphi: B \rightarrow X$ and $\mu: D \rightarrow Y$ are arbitrary H -maps such that the lower square in the diagram

$$(S) \quad \begin{array}{ccc} A & \xrightarrow{\alpha} & C \\ \cup & & \cup \\ B & \xrightarrow{\alpha|} & D \\ \downarrow \varphi & & \downarrow \mu \\ X & \xrightarrow{\eta} & Y \end{array}$$

is H -homotopy commutative. Let $\Omega: B \times I \rightarrow Y$ be an H -homotopy from $\eta \circ \varphi$ to $\mu \circ \alpha|$. We form the composite map

$$(1) \quad \lambda(S; \Omega): X \cup_{\varphi} A \xrightarrow{\bar{\eta}} Y \cup_{\eta \varphi} A \xrightarrow{k(\Omega)} Y \cup_{\mu \alpha|} A \xrightarrow{\hat{\alpha}} Y \cup_{\mu} C.$$

Then $\lambda(S; \Omega)$ is an H -homotopy equivalence since $\bar{\eta}$, $k(\Omega)$ and $\hat{\alpha}$ are H -homotopy equivalences by Lemma 2.5, 2.1 and 2.8, respectively. Furthermore $\bar{\eta}|X = \eta$, $k(\Omega)|Y = \text{id}_Y$ and $\hat{\alpha}|Y = \text{id}_Y$, and hence $\lambda(S; \Omega)|Y = \eta$. Also recall from Section 2 that if B_0 is a closed subset of A , disjoint from B , and $(A, B_0 \cup B)$ has the H -homotopy extension property then we obtain $\lambda(S; \Omega)|B_0 = \alpha|B_0$. Furthermore the homotopy Ω determines the map $k(\Omega)$ uniquely up to H -homotopy rel X . We shall in the rest of the paper usually denote the map $\lambda(S; \Omega)$ by $\lambda(\eta, \alpha; \Omega)$. In using the notation

$$\lambda(\eta, \alpha; \Omega): X \cup_{\varphi} A \longrightarrow Y \cup_{\mu} C$$

one should keep in mind that $\varphi: B \rightarrow X$ denotes an arbitrary H -map and that $\mu: D \rightarrow Y$ is an H -map such that $\eta \circ \varphi$ is H -homotopic to $\mu \circ (\alpha|)$. By a slight abuse of terminology we call $\lambda(\eta, \alpha; \Omega)$ the λ -map induced by η, α and Ω .

We show in Corollary 3.6 that different choices of H -homotopies, say Ω and Ω' , from $\eta \circ \varphi$ to $\mu \circ \alpha|$ give rise to H -homotopy equivalences $\lambda(\eta, \alpha; \Omega)$ and $\lambda(\eta, \alpha; \Omega')$ that are k -equivalent in the sense defined below. In the following U denotes an arbitrary H -space, and in Definition 3.1

below $\mu_i : D \rightarrow Y, i=1, 2$, denote H -maps.

DEFINITION 3.1. We say that two H -maps $f_1 : U \rightarrow Y \cup C$ and $f_2 : U \rightarrow Y \cup C$ are k -equivalent if there exists an H -homotopy $\Psi : D \times I \rightarrow Y$ from μ_1 to μ_2 such that the diagram

$$\begin{array}{ccc}
 & Y \cup C & \\
 f_1 \nearrow & \mu_1 \downarrow k(\Psi) & \\
 U & & \\
 f_2 \searrow & \mu_2 \downarrow & \\
 & Y \cup C &
 \end{array}$$

is H -homotopy commutative.

In case $\Psi : D \times I \rightarrow Y$ is the constant homotopy from μ to μ , the map $k(\Psi) : Y \cup C \rightarrow Y \cup C$ is H -homotopic (in fact rel Y) to the identity map, see Section 2. Therefore, if $f, f' : U \rightarrow Y \cup C$ are H -homotopic H -maps then f is k -equivalent to f' . In particular any H -map $f : U \rightarrow Y \cup C$ is k -equivalent to itself. Furthermore we have

LEMMA 3.2. *The k -equivalence relation is both symmetric and transitive.*

PROOF. This follows immediately from Lemma 2.1 and Lemma 2.3. \square

Thus k -equivalence is an equivalence relation, a fact that we from now on will use freely. The notion of k -equivalence allows us to reformulate Lemma 2.4 in the following convenient form.

LEMMA 3.3. *If $f_i : X \rightarrow Y, i=0, 1$, are H -homotopic H -maps, then the canonical extensions $\tilde{f}_i : X \cup A \rightarrow Y \cup A, i=0, 1$, are k -equivalent.*

PROOF. This follows directly from Lemma 2.4 and Definition 3.1. \square

An analogous reformation of Lemma 2.7 says the following : Suppose that the H -homotopy equivalences $\alpha_i : (A, B) \rightarrow (C, D), i=0, 1$, are H -homotopic. Then the H -homotopy equivalences $(\hat{\alpha}_i)^- : X \cup C \rightarrow X \cup A, i=0, 1$, are k -equivalent. This fact follows directly from Lemmas 2.7 and 2.8, and Definition 3.1.

The following lemma will be very useful.

LEMMA 3.4. Let $\beta = \alpha^\leftarrow : (C, D) \rightarrow (A, B)$ be an H -homotopy inverse of α . Then the H -homotopy equivalences

$$\hat{\alpha} : Y \underset{\mu\alpha|}{\cup} A \longrightarrow Y \underset{\mu}{\cup} C$$

and

$$(\hat{\beta})^\leftarrow : Y \underset{\mu\alpha|}{\cup} A \longrightarrow Y \underset{\mu\alpha|\beta|}{\cup} C$$

are k -equivalent.

PROOF. Let $A : (C, D) \times I \rightarrow (C, D)$ be an H -homotopy from $\alpha \circ \beta$ to the identity map. Then it follows by Lemma 2.7 that the map $\hat{\alpha} \circ \hat{\beta} = \widehat{\alpha \circ \beta}$ is H -homotopic to $k(A) : Y \underset{\mu}{\cup} C \rightarrow Y \underset{\mu}{\cup} C$. Since $\hat{\beta} : Y \underset{\mu\alpha|\beta|}{\cup} C \rightarrow Y \underset{\mu\alpha|}{\cup} A$ is an H -homotopy equivalence by Lemma 2.8 it now follows that $\hat{\alpha}$ is H -homotopic to $k(A) \circ (\hat{\beta})^\leftarrow$; i.e., $\hat{\alpha}$ is k -equivalent to $(\hat{\beta})^\leftarrow$. \square

In Lemma 3.5 we let $\phi_i : B \rightarrow Y, i=1, 2$, denote H -maps.

LEMMA 3.5. Suppose that the H -maps $f_1 : U \rightarrow Y \underset{\phi_1}{\cup} A$ and $f_2 : U \rightarrow Y \underset{\phi_2}{\cup} A$ are k -equivalent. Then :

- (a) The H -maps $\bar{\theta} \circ f_1 : U \rightarrow Z \underset{\theta\phi_1}{\cup} A$ and $\bar{\theta} \circ f_2 : U \rightarrow Z \underset{\theta\phi_2}{\cup} A$ are k -equivalent for every H -map $\theta : Y \rightarrow Z$.
- (b) The H -maps $(\hat{\beta})^\leftarrow \circ f_1 : U \rightarrow Y \underset{\phi_1\beta|}{\cup} C$ and $(\hat{\beta})^\leftarrow \circ f_2 : U \rightarrow Y \underset{\phi_2\beta|}{\cup} C$ are k -equivalent for every H -homotopy equivalence $\beta : (C, D) \rightarrow (A, B)$.

PROOF. (a) is an immediate consequence of Lemma 2.6, and (b) follows from Lemma 2.8 and Lemma 2.9. \square

LEMMA 3.6. Assume that $\alpha : (A, B) \rightarrow (C, D)$ is an H -homotopy equivalence and let $\beta = \alpha^\leftarrow : (C, D) \rightarrow (A, B)$ be an H -homotopy inverse of α . By $\phi : B \rightarrow Y$ and $\mu : D \rightarrow Y$ we denote arbitrary H -maps. Suppose that the H -maps $f_1 : U \rightarrow Y \underset{\phi}{\cup} A$ and $f_2 : U \rightarrow Y \underset{\mu\alpha|}{\cup} A$ are k -equivalent. Then the H -maps $(\hat{\beta})^\leftarrow \circ f_1 : U \rightarrow Y \underset{\phi\beta|}{\cup} C$ and $\hat{\alpha} \circ f_2 : U \rightarrow Y \underset{\mu}{\cup} C$ are k -equivalent.

PROOF. First we note that by part b of Lemma 3.5 the maps $(\hat{\beta})^\leftarrow \circ f_1 : U \rightarrow Y \underset{\phi\beta|}{\cup} C$ and $(\hat{\beta})^\leftarrow \circ f_2 : U \rightarrow Y \underset{\mu\alpha|\beta|}{\cup} C$ are k -equivalent. It follows by Lemma 3.4 that $(\hat{\beta})^\leftarrow : Y \underset{\mu\alpha|}{\cup} A \rightarrow Y \underset{\mu\alpha|\beta|}{\cup} C$ is k -equivalent to $\hat{\alpha} : Y \underset{\mu\alpha|}{\cup} A \rightarrow Y \underset{\mu}{\cup} C$, and hence $(\hat{\beta})^\leftarrow \circ f_2$ is k -equivalent to $\hat{\alpha} \circ f_2$. Therefore $(\hat{\beta})^\leftarrow \circ f_1$ is k -equiv-

alent to $\hat{\alpha} \circ f_2$. \square

LEMMA 3.7. *Let the notation and assumptions be as in diagram (S) at the beginning of this section, and let $\beta : (C, D) \rightarrow (A, B)$ be an H -homotopy inverse of α . Then the H -homotopy equivalences $\lambda(\eta, \alpha; \Omega) : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu}{\cup} C$ and $(\hat{\beta})^\circ \circ \bar{\eta} : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\eta\varphi\beta|}{\cup} C$ are k -equivalent.*

PROOF. The H -maps $\bar{\eta} : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\eta\varphi}{\cup} A$ and $k(\Omega) \circ \bar{\eta} : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu}{\cup} A$ are k -equivalent by definition. Hence Lemma 3.6 implies that $(\hat{\beta})^\circ \circ \bar{\eta} : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu}{\cup} C$ and $\lambda(\eta, \alpha; \Omega) = \hat{\alpha} \circ k(\Omega) \circ \bar{\eta} : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu}{\cup} C$ are k -equivalent. \square

COROLLARY 3.8. *Suppose that $\Omega, \Omega' : B \times I \rightarrow Y$ are two H -homotopies from $\eta \circ \varphi$ to $\mu \circ \alpha|$. Then the H -homotopy equivalences $\lambda(\eta, \alpha; \Omega) : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu}{\cup} C$ and $\lambda(\eta, \alpha; \Omega') : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu}{\cup} C$ are k -equivalent.*

PROOF. This follows directly from Lemma 3.7. \square

Thus we see that the map $\lambda(\eta, \alpha; \Omega) : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu}{\cup} C$, induced by the diagram (S) at the beginning of this section, is up to k -equivalence independent of the choice of the connecting H -homotopy Ω from $\eta \circ \varphi$ to $\mu \circ \alpha|$. Hence we will denote such a map simply by

$$\lambda(\eta, \alpha) : X \underset{\varphi}{\cup} A \longrightarrow Y \underset{\mu}{\cup} C.$$

One should keep in mind that $\lambda(\eta, \alpha)$ is an H -homotopy equivalence that is uniquely determined up to k -equivalence by the diagram (S). By abuse of terminology we call $\lambda(\eta, \alpha)$ the λ -map induced by the diagram (S).

LEMMA 3.9. *Assume that the H -homotopy equivalences $\alpha_i : (A, B) \rightarrow (C, D)$, $i=0, 1$, are H -homotopic and that the H -homotopy equivalences $\eta_i : X \rightarrow Y$, $i=0, 1$, are H -homotopic. Let $\varphi : B \rightarrow X$ be an H -map and let $\mu_i : D \rightarrow Y$, be H -maps such that μ_i is H -homotopic to $\eta_i \circ \varphi \circ (\alpha_i|)^\circ$, $i=0, 1$. Then the H -homotopy equivalences $\lambda(\eta_0, \alpha_0) : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu_0}{\cup} C$ and $\lambda(\eta_1, \alpha_1) : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\mu_1}{\cup} C$ are k -equivalent.*

PROOF. Let $\beta : (C, D) \rightarrow (A, B)$ be an H -homotopy inverse of α_0 . Then β is also an H -homotopy inverse of α_1 . Hence we know by Lemma 3.7 that $\lambda(\eta_0, \alpha_0)$ is k -equivalent to $(\hat{\beta})^\circ \circ \bar{\eta}_0 : X \underset{\varphi}{\cup} A \rightarrow Y \underset{\eta_0\varphi\beta|}{\cup} C$, and that $\lambda(\eta_1, \alpha_1)$ is

k -equivalent to $(\hat{\beta})^{\leftarrow} \circ \bar{\eta}_1: X \cup_{\varphi} A \rightarrow Y \cup_{\eta_1 \circ \varphi} C$. Now the H -maps $\bar{\eta}_i: X \cup_{\varphi} A \rightarrow Y \cup_{\eta_i \circ \varphi} C$, $i=0,1$, are k -equivalent by Lemma 3.3, and therefore part b of Lemma 3.5 implies that the H -maps $(\hat{\beta})^{\leftarrow} \circ \bar{\eta}_i: X \cup_{\varphi} A \rightarrow Y \cup_{\eta_i \circ \varphi} C$, $i=0,1$, are k -equivalent. It now follows that the H -maps $\lambda(\eta_0, \alpha_0)$ and $\lambda(\eta_1, \alpha_1)$ are k -equivalent. \square

Assume that $\alpha: (A, B) \rightarrow (C, D)$, $\gamma: (C, D) \rightarrow (E, F)$, $\eta: X \rightarrow Y$ and $\theta: Y \rightarrow Z$ are H -homotopy equivalences. Let $\varphi: B \rightarrow X$ be an H -map and let $\mu: D \rightarrow Y$ an H -map which is H -homotopic to $\eta \circ \varphi \circ (\alpha|)^{-}$, and let $\omega: F \rightarrow Z$ be an H -map which is H -homotopic to $\theta \circ \mu \circ (\gamma|)^{-}$. In this situation we have

LEMMA 3.10. *The maps $\lambda(\theta \circ \eta, \gamma \circ \alpha)$ and $\lambda(\theta, \gamma) \circ \lambda(\eta, \alpha): X \cup_{\varphi} A \rightarrow Z \cup_{\omega} E$ are k -equivalent.*

PROOF. Let $\beta: (C, D) \rightarrow (A, B)$ and $\delta: (E, F) \rightarrow (C, D)$ be H -homotopy inverses of α and γ , respectively. It follows by Lemma 3.7 that $(\hat{\beta})^{\leftarrow} \circ \bar{\eta}: X \cup_{\varphi} A \rightarrow Y \cup_{\eta \circ \varphi} C$ is k -equivalent to $\lambda(\eta, \alpha): X \cup_{\varphi} A \rightarrow Y \cup C$. By part a of Lemma 3.5 we have that $\bar{\theta} \circ (\hat{\beta})^{\leftarrow} \circ \bar{\eta}$ is k -equivalent to $k(\Omega_2) \circ \bar{\theta} \circ \lambda(\eta, \alpha)$, where $\Omega_2: D \times I \rightarrow Z$ is an H -homotopy from $\theta \circ \mu$ to $\omega \circ \gamma|$. Therefore we obtain by applying Lemma 3.6 that $(\hat{\delta})^{\leftarrow} \circ \bar{\theta} \circ (\hat{\beta})^{\leftarrow} \circ \bar{\eta}$ is k -equivalent to $\hat{\gamma} \circ k(\Omega_2) \circ \bar{\theta} \circ \lambda(\eta, \alpha) = \lambda(\theta, \gamma) \circ \lambda(\eta, \alpha)$. But $\bar{\theta} \circ \hat{\beta} = \hat{\beta} \circ \bar{\theta}$ (see Lemma 2.10) and therefore $\bar{\theta} \circ (\hat{\beta})^{\leftarrow}$ is H -homotopic to $(\hat{\beta})^{\leftarrow} \circ \bar{\theta}$. Thus $(\hat{\delta})^{\leftarrow} \circ \bar{\theta} \circ (\hat{\beta})^{\leftarrow} \circ \bar{\eta} = (\hat{\delta})^{\leftarrow} \circ (\hat{\beta})^{\leftarrow} \circ \bar{\theta} \circ \bar{\eta} = (\widehat{\beta \circ \delta})^{\leftarrow} \circ (\overline{\theta \circ \eta})$, and by Lemma 3.7 the map $(\widehat{\beta \circ \delta})^{\leftarrow} \circ (\overline{\theta \circ \eta})$ is k -equivalent to $\lambda(\theta \circ \eta, \gamma \circ \alpha)$. \square

4. Equivariant simple-homotopy type of adjunction spaces

PROPOSITION 4.1. *Let Y be a finite H -CW complex and let (C, D) be a finite H -CW pair. Suppose that $\mu_0, \mu_1: D \rightarrow Y$ are skeletal H -maps that are H -homotopic and that $\Psi: D \times I \rightarrow Y$ is an H -homotopy from μ_0 to μ_1 . Then*

$$k = k(\Psi): Y \cup_{\mu_0} C \longrightarrow Y \cup_{\mu_1} C$$

is a simple H -homotopy equivalence.

PROOF. By the relative equivariant skeletal approximation theorem Ψ is H -homotopic rel $D \times I$ to a skeletal H -map $\tilde{\Psi}: D \times I \rightarrow Y$. Then $\tilde{\Psi}$ is a skeletal H -homotopy from μ_0 to μ_1 . By Lemma 2.2 we know that $k(\Psi)$ is H -homotopic (in fact rel Y) to $k(\tilde{\Psi})$, and hence it is enough to

prove that $k(\tilde{\Psi})$ is a simple H -homotopy equivalence. The adjunction space $Y \cup_{\tilde{\Psi}} (C \times I)$ is a finite H -CW complex. Since $C \times I$ H -collapses to $C \times \{0\} \cup D \times I$ it follows that $Y \cup_{\tilde{\Psi}} (C \times I)$ H -collapses to $Y \cup_{\tilde{\Psi}} (C \times \{0\} \cup D \times I) = Y \cup_{\mu_0} C$, see [3, Corollary II.1.10 and Lemma II.1.6]. In the same way we see that $Y \cup_{\tilde{\Psi}} (C \times I)$ collapses to $Y \cup_{\mu_1} C$ by a finite sequence of elementary H -collapses. Hence, the map

$$k(\tilde{\Psi}) : Y \cup_{\mu_0} C \xrightarrow{\hat{i}_0} Y \cup_{\tilde{\Psi}} (C \times I) \xrightarrow{\hat{r}_1} Y \cup_{\mu_1} C$$

is a formal H -deformation and in particular $k(\tilde{\Psi})$ is a simple H -homotopy equivalence. \square

COROLLARY 4.2. *Let Y and (C, D) and $\mu_0, \mu_1 : D \rightarrow Y$ be as in Proposition 4.1. Suppose that $f_0 : U \rightarrow Y \cup_{\mu_0} C$ and $f_1 : U \rightarrow Y \cup_{\mu_1} C$ are k -equivalent H -maps. Then f_0 and f_1 are s -equivalent, in fact $f_0^{\mu_0}$ and $f_1^{\mu_1}$ are s -equivalent rel Y . \square*

Propositions 4.3 and 4.4 given below have proofs that are very similar to each other. Of these the proof of Proposition 4.4 is the more enlightening one, and also the somewhat more complicated one. Therefore we give the proof of Proposition 4.4 in detail and leave the proof of Proposition 4.3 to the reader.

PROPOSITION 4.3. *Let Y and Z be finite H -CW complexes and let (C, D) be a finite H -CW pair, and let $\mu : D \rightarrow Y$ be a skeletal H -map. Suppose that $\sigma : Y \rightarrow Z$ is a skeletal simple H -homotopy equivalence. Then its canonical extension $\bar{\sigma} : Y \cup_{\mu} C \rightarrow Z \cup_{c\mu} C$ is a simple H -homotopy equivalence.*

PROPOSITION 4.4. *Let Y be a finite H -CW complex, and let (C, D) and (E, F) be finite H -CW pairs, and let $\nu : F \rightarrow Y$ be a skeletal H -map. Suppose that $\gamma : (C, D) \rightarrow (E, F)$ is a skeletal simple H -homotopy equivalence. Then $\bar{\gamma} : Y \cup_{\nu} C \rightarrow Y \cup_{\nu} E$ is a simple H -homotopy equivalence.*

PROOF OF PROPOSITION 4.4. We consider the diagram

$$\begin{array}{ccc}
Y \cup (C \times \{0\} \cup D \times I) & \xrightarrow{\gamma \times \{0\} \cup \gamma| \times I} & Y \cup (E \times \{0\} \cup F \times I) \\
\downarrow \nu_{\gamma| \{1\}} & & \downarrow \nu_{\{1\}} \\
i \downarrow & & \downarrow i' \\
Y \cup (C \times I) & \xrightarrow{\gamma \times I} & Y \cup (E \times I) \\
\downarrow \nu_{\gamma| \{1\}} & & \downarrow \nu_{\{1\}} \\
r \downarrow & & \downarrow r' \\
Y \cup C & \xrightarrow{\hat{\gamma}} & Y \cup E \\
\downarrow \nu_{\gamma|} & & \downarrow \nu
\end{array}$$

Here $\nu_{\{1\}}: F \times \{1\} \rightarrow Y$, denotes the map given by $\nu_{\{1\}}(x, 1) = \nu(x)$ for all $x \in F$ and $\nu_{\gamma| \{1\}}: D \times \{1\} \rightarrow Y$ is defined similarly.

The inclusions $C \times \{0\} \cup D \times I \hookrightarrow C \times I$ and $E \times \{0\} \cup F \times I \hookrightarrow E \times I$ are H -expansions, see [3, Corollary II. 1.10]. Hence it follows by [3, Lemma II. 1.6] that the inclusions i and i' in the above diagram are H -expansions. The upper square of the above diagram clearly commutes.

In the lower square of the above diagram r and r' denote the retractions induced by the standard projections $C \times I \rightarrow C \times \{1\} = C$ and $E \times I \rightarrow E \times \{1\} = E$, respectively. With this choice of retractions r and r' the lower square clearly commutes. Since $C \times I$ H -collapses to $C \times \{1\}$, see [3, Corollary II. 1.10], it follows by [3, Lemma II. 1.6] that $Y \cup (C \times I)$ H -collapses to $Y \cup C$. Therefore any H -retraction from $Y \cup (C \times I)$ onto $Y \cup C$ is a simple H -homotopy equivalence. In particular r is a simple H -homotopy equivalence, and the same argument shows that r' is a simple H -homotopy equivalence.

Thus we have shown that the maps $r \circ i$ and $r' \circ i'$ are simple H -homotopy equivalences. Therefore, in order to prove that $\hat{\gamma}$ is a simple H -homotopy equivalence, it is enough to show that the map $\tilde{\gamma} = \gamma \times \{0\} \cup \gamma| \times I$, at the top of the diagram, is a simple H -homotopy equivalence. This we do in the following way.

In order to simplify the notation we denote

$$K = Y \cup (C \times \{0\} \cup D \times I)$$

and

$$L = Y \cup \underset{\nu_{\{1\}}}{(E \times \{0\} \cup F \times I)}.$$

We shall prove that

$$\tilde{\gamma} = \gamma \times \{0\} \cup \gamma| \times I : K \longrightarrow L$$

is a simple H -homotopy equivalence. Let K^* denote the H -equivariant subdivision of K obtained by subdividing the unit interval $I=[0, 1]$ at the point $1/2$. Then

$$K_1^* = C \times \{0\} \cup D \times [0, 1/2]$$

and

$$K_2^* = Y \cup \underset{\nu\gamma|_{\{1\}}}{(D \times [1/2, 1])}$$

are H -subcomplexes of K^* . Furthermore we have that

$$K_1^* \cup K_2^* = K^*$$

and

$$K_1^* \cap K_2^* = D \times \{1/2\}.$$

In complete analogy with the above we define an H -equivariant subdivision L^* of L , and obtain H -subcomplexes L_1^* and L_2^* of L^* such that

$$L_1^* \cup L_2^* = L^*$$

and

$$L_1^* \cap L_2^* = F \times \{1/2\}.$$

We claim that $\tilde{\gamma}| : K_1^* \rightarrow L_1^*$ is a simple H -homotopy equivalence. In order to see this we consider the commutative diagram

$$\begin{array}{ccc} K_1^* & \xrightarrow{\tilde{\gamma}|} & L_1^* \\ r_0 \downarrow & & \downarrow r'_0 \\ C \times \{0\} & \xrightarrow{\gamma \times \{0\}} & E \times \{0\} \end{array}$$

where r_0 denotes the retraction induced by the standard projection $D \times [0, 1/2] \rightarrow D \times \{0\}$ and r'_0 is defined similarly. Then r_0 and r'_0 are simple H -homotopy equivalences, and since $\gamma : C \rightarrow E$ is a simple H -homotopy

equivalence, by assumption, the claim follows.

The map $\tilde{\gamma}|: K_2^* \rightarrow L_2^*$ is also a simple H -homotopy equivalence. This follows from the fact that in the commutative diagram

$$\begin{array}{ccc} K_2^* & \xrightarrow{\tilde{\gamma}|} & L_2^* \\ r_1 \downarrow & & \downarrow r'_1 \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

the retractions r_1 and r'_1 , induced by the standard projections $D \times [1/2, 1] \rightarrow D \times \{1\} = D$ and $F \times [1/2, 1] \rightarrow F \times \{1\} = F$, are simple H -homotopy equivalences.

Furthermore we have that $\tilde{\gamma}|: K_1^* \cap K_2^* \rightarrow L_1^* \cap L_2^*$ is a simple H -homotopy equivalence since it equals $\gamma| \times \{1/2\}: D \times \{1/2\} \rightarrow F \times \{1/2\}$, which is a simple H -homotopy equivalence by assumption.

By the sum theorem for equivariant Whitehead torsion, see [3, Theorem II. 3.12], it now follows that $\tilde{\gamma}: K^* \rightarrow L^*$ is a simple H -homotopy equivalence. Furthermore, the G -equivariant subdivision maps $j: K^* \rightarrow K$ and $j': L^* \rightarrow L$ are simple G -homotopy equivalences by [6, Theorem 12.2]. Hence $\tilde{\gamma}: K \rightarrow L$ is a simple H -homotopy equivalence. \square

We shall now combine the results of this section into one theorem. In Theorem 4.5 below Y and Z denote finite H -CW complexes, and (C, D) and (E, F) denote finite H -CW pairs.

THEOREM 4.5. *Suppose that $\sigma: Y \rightarrow Z$ and $\gamma: (C, D) \rightarrow (E, F)$ are simple H -homotopy equivalences. Let $\mu: D \rightarrow Y$ be a skeletal H -map and $\nu: F \rightarrow Z$ a skeletal H -approximation of $\sigma \circ \mu \circ (\gamma|)^+$. Then*

$$\lambda(\sigma, \gamma): Y \cup_{\mu} C \longrightarrow Z \cup_{\nu} E$$

is a simple H -homotopy equivalence.

PROOF. It follows by the equivariant skeletal approximation theorem, Corollary 3.7 and Corollary 4.2 that it is enough to consider the case when γ and σ are skeletal H -maps. In this case $\sigma \circ \mu: D \rightarrow Z$ and $\nu \circ \gamma|: D \rightarrow Z$ are skeletal H -maps. Since $\sigma \circ \mu \circ (\gamma|)^+$ is H -homotopic to ν it follows that $\sigma \circ \mu$ is H -homotopic to $\nu \circ (\gamma|)$. Let $\Omega: D \times I \rightarrow Z$ be an H -homotopy from $\sigma \circ \mu$ to $\nu \circ (\gamma|)$. Then the composite map

$$\lambda(\sigma, \gamma; \Omega) : Y \underset{\mu}{\cup} C \xrightarrow{\bar{\sigma}} Z \underset{\sigma\mu}{\cup} C \xrightarrow{k(\Omega)} Z \underset{\nu\gamma_1}{\cup} C \xrightarrow{\hat{\gamma}} Z \underset{\nu}{\cup} E$$

is a simple H -homotopy equivalence since $\bar{\sigma}$, $k(\Omega)$ and $\hat{\gamma}$ are simple H -homotopy equivalences by Propositions 4.3, 4.1 and 4.4, respectively. \square

5. A key result

In Theorem 5.1 below X denotes an H -space, (A, B) an H -pair that has the H -homotopy extension property, and $\varphi : B \rightarrow X$ an arbitrary H -map. Furthermore, Y_i denote finite H -CW complexes, (C_i, D_i) finite H -CW pairs and $\mu_i : D_i \rightarrow Y_i$ skeletal H -maps, $i=1, 2$.

THEOREM 5.1. *Suppose that the H -homotopy equivalences $\alpha_1 : (A, B) \rightarrow (C_1, D_1)$ and $\alpha_2 : (A, B) \rightarrow (C_2, D_2)$ are s -equivalent as maps of pairs and that the H -homotopy equivalences $\theta_1 : X \rightarrow Y_1$ and $\theta_2 : X \rightarrow Y_2$ are s -equivalent. Then the H -homotopy equivalences $\lambda(\theta_1, \alpha_1) : X \cup A \rightarrow Y_1 \cup C_1$ and $\lambda(\theta_2, \alpha_2) : X \cup A \rightarrow Y_2 \cup C_2$ are s -equivalent. In fact, if $\sigma : Y_1 \xrightarrow{\varphi} Y_2$ is a simple H -homotopy equivalence such that $\sigma \circ \theta_1$ is H -homotopic to θ_2 , then, σ can be extended to a simple H -homotopy equivalence $\Sigma : Y_1 \underset{\mu_1}{\cup} C_1 \rightarrow Y_2 \underset{\mu_2}{\cup} C_2$ such that $\Sigma \circ \lambda(\theta_1, \alpha_1)$ is H -homotopic to $\lambda(\theta_2, \alpha_2)$.*

PROOF. Let $\gamma : (C_1, D_1) \rightarrow (C_2, D_2)$ be a simple H -homotopy equivalence such that $\gamma \circ \alpha_1 \simeq \alpha_2$, and let $\sigma : Y_1 \rightarrow Y_2$ be a simple H -homotopy equivalence such that $\sigma \circ \theta_1 \simeq \theta_2$. Since $\theta_1 \circ \varphi \simeq \mu_1 \circ \alpha_1|$ and $\theta_2 \circ \varphi \simeq \mu_2 \circ \alpha_2|$ it now follows that $\sigma \circ \mu_1 \simeq \mu_2 \circ \gamma$. Thus we can form the λ -map $\lambda(\sigma, \gamma) : Y_1 \underset{\mu_1}{\cup} C_1 \rightarrow Y_2 \underset{\mu_2}{\cup} C_2$, and by Theorem 4.5 $\lambda(\sigma, \gamma)$ is a simple H -homotopy equivalence, and we also have that $\lambda(\sigma, \gamma)|_{Y_1} = \sigma$. By Lemma 3.10 the composite map $\lambda(\sigma, \gamma) \circ \lambda(\theta_1, \alpha_1)$ is k -equivalent to $\lambda(\sigma \circ \theta_1, \gamma \circ \alpha_1)$, and by Lemma 3.9 we know that $\lambda(\sigma \circ \theta_1, \gamma \circ \alpha_1)$ is k -equivalent to $\lambda(\theta_2, \alpha_2)$. Therefore $\lambda(\sigma, \gamma) \circ \lambda(\theta_1, \alpha_1)$ is k -equivalent to $\lambda(\theta_2, \alpha_2)$; i.e., there exists an H -homotopy $\Lambda : D_2 \times I \rightarrow D_2$ from μ_2 to μ_2 such that setting $k = k(\Lambda) : Y_2 \underset{\mu_2}{\cup} C_2 \rightarrow Y_2 \underset{\mu_2}{\cup} C_2$ we have that $k \circ \lambda(\sigma, \gamma) \circ \lambda(\theta_1, \alpha_1)$ is H -homotopic to $\lambda(\theta_2, \alpha_2)$. By Corollary 4.2 we have that k is a simple H -homotopy equivalence and $k|_{Y_2} = \text{id}$. Hence $\Sigma = k \circ \lambda(\sigma, \gamma) : Y_1 \underset{\mu_1}{\cup} C_1 \rightarrow Y_2 \underset{\mu_2}{\cup} C_2$ is a simple H -homotopy equivalence which extends σ and $\Sigma \circ \lambda(\theta_1, \alpha_1)$ is H -homotopic to $\lambda(\theta_2, \alpha_2)$. \square

6. Preferred H -reductions

THEOREM 6.1. *Let G be a compact Lie group and H a closed subgroup of G . Then, given a finite G -CW complex X , there exist a finite H -CW complex $R_H X$ and an H -homotopy equivalence*

$$\eta: X \longrightarrow R_H X$$

such that this construction is unique up to a simple H -homotopy equivalence.

The last statement in Theorem 6.1 means that if one by some other choices in the construction arrives at the finite H -CW complex $R'_H X$ and the H -homotopy equivalence $\eta': X \rightarrow R'_H X$, then

$$\eta' \circ \eta^{-1}: R_H X \longrightarrow R'_H X$$

is a simple H -homotopy equivalence; i.e., η and η' are s -equivalent.

PROOF. The proof is by induction on the number of G -cells in X . First assume that X consists of one 0-dimensional G -cell, say $X = G/P$, where P is a closed subgroup of G . By Theorem 1.3 there exists a distinguished H -triangulation $u: F \xrightarrow{\cong} G/P$ of the H -manifold G/P . We define $R_H(G/P) = F$ and $\eta = u^{-1}$. Then $R_H(G/P)$ is a finite H -CW complex and $\eta: G/P \rightarrow R_H(G/P)$ is an H -homeomorphism and hence in particular an H -homotopy equivalence. If we choose another distinguished H -triangulation $u': F' \rightarrow G/P$ of G/P , and define $R'_H(G/P) = F'$ and $\eta' = (u')^{-1}$, we have by Theorem 1.3 that $\eta' \circ \eta^{-1} = (u')^{-1} \circ u: F \rightarrow F'$ is a simple H -homotopy equivalence.

Let $m \geq 2$ and assume inductively that we have proved Theorem 6.1 for all finite G -CW complexes with at most $m-1$ G -cells. Let X be a finite G -CW complex with m G -cells. Choose a G -subcomplex X_0 of X such that X is obtained from X_0 by adjoining one G -cell. By the inductive assumption there exists a construction which gives a finite H -CW complex $R_H X_0$ and an H -homotopy equivalence

$$(1) \quad \theta: X_0 \longrightarrow R_H X_0$$

and the construction is unique up to a simple H -homotopy equivalence. Suppose that $X = X_0 \cup c$, where c is a G -cell, of say dimension n and type G/P . Let

$$(2) \quad \xi: (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (c, \dot{c}) \hookrightarrow (X, X_0)$$

be a characteristic G -map for c and let

$$(3) \quad \phi = \xi| : S^{n-1} \times G/P \longrightarrow X_0$$

denote the corresponding attaching map. We then have the G -homeomorphism

$$(4) \quad \hat{\xi} : X_0 \cup_{\phi} (D^n \times G/P) \xrightarrow{\cong} X$$

induced by ξ .

Next we choose a distinguished H -triangulation $u : F \rightarrow G/P$ of G/P , which exists by Theorem 1.3, and consider the H -homeomorphism

$$(5) \quad \alpha = \text{id} \times u^{-1} : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (D^n \times F, S^{n-1} \times F).$$

In order to simplify the notation we shall in the following denote

$$(C, D) = (D^n \times F, S^{n-1} \times F).$$

(Here D^n and S^{n-1} have standard CW structures and $(D^n \times F, S^{n-1} \times F)$ is then a finite H - CW pair.)

We form the λ -map

$$(6) \quad \lambda(\theta, \alpha) : X_0 \cup_{\phi} (D^n \times G/P) \longrightarrow R_H X_0 \cup_{\mu} C,$$

where $\mu : D \rightarrow R_H X_0$ denotes a skeletal H -approximation of $\theta \circ \phi \circ (\alpha|)^{-1}$. Then $\lambda(\theta, \alpha)$ is an H -homotopy equivalence and $\lambda(\theta, \alpha)$ extends θ , see Section 3. We define $\eta : X \rightarrow R_H X$ to be the composite map

$$(7) \quad \eta = \lambda(\theta, \alpha) \circ (\hat{\xi})^{-1} : X \longrightarrow R_H X_0 \cup_{\mu} C = R_H X.$$

Then $\eta : X \rightarrow R_H X$ is an H -homotopy equivalence and we have $\eta|_{X_0} = \theta : X_0 \rightarrow R_H X_0$.

If we at the inductive level choose $\theta' : X_0 \rightarrow R'_H X_0$, instead of θ in (1), we have by the inductive assumption that θ and θ' are s -equivalent. Let $\sigma : R_H X_0 \rightarrow R'_H X_0$ be a simple H -homotopy equivalence such that $\sigma \circ \theta$ is H -homotopic to θ' . Suppose that we also choose another distinguished H -triangulation $u' : F' \rightarrow G/P$ of G/P , and set $\alpha' = \text{id} \times (u')^{-1} : (D^n \times G/P, S^{n-1} \times G/P) \rightarrow (C', D')$, where $(C', D') = (D^n \times F', S^{n-1} \times F')$. It then follows by Theorem 1.3 and Lemma 1.6 that $\alpha' \circ \alpha^{-1} = \text{id} \times ((u')^{-1} \circ u) : (C, D) \rightarrow (C', D')$ is a simple H -homotopy equivalence; i.e., the H -maps α and α' are s -equivalent, as maps of pairs. Hence Theorem 5.1 implies that

$$(8) \quad \lambda(\theta', \alpha') : X_0 \cup_{\phi} (D^n \times G/P) \longrightarrow R'_H X_0 \cup_{\mu'} C' := R'_H X$$

is s -equivalent to $\lambda(\theta, \alpha)$ in (6). In fact we have by Theorem 5.1 that there exists a simple H -homotopy equivalence $\Sigma : R_H X \rightarrow R'_H X$ that extends σ such that $\Sigma \circ \lambda(\theta, \alpha)$ is H -homotopic to $\lambda(\theta', \alpha')$. Hence $\Sigma \circ \eta$, where η is as in (7), is H -homotopic to

$$\eta' = \lambda(\theta', \alpha') \circ (\hat{\xi})^{-1} : X \longrightarrow R'_H X,$$

and thus η is s -equivalent to η' rel $R_H X_0$. We have now shown that the map $\eta : X \rightarrow R_H X$ in (7) is up to s -equivalence rel $R_H X_0$ independent of the choice of the map θ in (1) and also of the choice of distinguished H -triangulation of G/P and the corresponding map α in (5).

Suppose that

$$\xi' : (D^n \times G/P', S^{n-1} \times G/P') \longrightarrow (c, \dot{c}) \hookrightarrow (X, X_0)$$

is another characteristic G -map for the G -cell c , and let $\phi' = \xi'| : S^{n-1} \times G/P' \rightarrow X_0$ denote the corresponding attaching map for c , and let

$$\hat{\xi}' : X_0 \cup_{\phi'} (D^n \times G/P') \xrightarrow{\cong} X$$

be the G -homeomorphism induced by ξ' . By Lemma 1.1 there exist an isometry f of R^n and an analytic G -isomorphism $g_0 : G/P \rightarrow G/P'$ such that the maps

$$\xi' \circ (f \times g_0), \quad \xi : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (X, X_0)$$

are G -homotopic. We denote

$$\xi_1 = \xi' \circ (f \times g_0) \quad \text{and} \quad \phi_1 = \phi' \circ (f \times g_0).$$

If $u : F \rightarrow G/P$ is a distinguished H -triangulation of G/P then $g_0 \circ u : F \rightarrow G/P'$ is a distinguished H -triangulation of G/P' by Lemma 1.4. We showed above that our construction is, up to a simple H -homotopy equivalence rel $R_H X_0$, independent of the choice of distinguished H -triangulation, and hence we may use $g_0 \circ u : F \rightarrow G/P'$. Moreover we note that the H -maps

$$\text{id} \times (g_0 \circ u)^{-1}, \quad (\text{id} \times u^{-1}) \circ (f \times g_0)^{-1} : (D^n \times G/P', S^{n-1} \times G/P') \rightarrow (D^n \times F, S^{n-1} \times F)$$

are s -equivalent, since $f \times \text{id} : (D^n \times F, S^{n-1} \times F) \rightarrow (D^n \times F, S^{n-1} \times F)$ is a simple H -homotopy equivalence by Lemma 1.6. Hence it follows by

Theorem 5.1 that the λ -maps

$$\lambda(\theta, \alpha_0) = \lambda(\theta, \text{id} \times (g_0 \circ u)^{-1}) : X \bigcup_{\phi'} (D^n \times G/P') \longrightarrow R_H X_0 \bigcup_{\mu'} C$$

and

$$\lambda(\theta, \alpha \circ (f \times g_0)^{-1}) : X \bigcup_{\phi'} (D^n \times G/P') \longrightarrow R_H X_0 \bigcup_{\mu} C$$

are s -equivalent $\text{rel } R_H X_0$. Here $\mu' : D \rightarrow R_H X_0$ is a skeletal H -approximation of $\theta \circ \phi' \circ (\alpha_0|)^{-1}$. Furthermore it is immediately seen that the map $\lambda(\theta, \alpha \circ (f \times g_0)^{-1}) \circ (\widehat{f \times g_0})$ and the map $\lambda(\theta, \alpha)$, as in (6) but with ϕ_1 in place of ϕ , are k -equivalent (in fact H -homotopic maps for a suitable choice of connecting homotopies). Hence the maps $\lambda(\theta, \alpha \circ (f \times g_0)^{-1}) \circ (\hat{\xi}')^{-1}$ and $\lambda(\theta, \alpha) \circ (\hat{\xi}_1)^{-1}$ are k -equivalent, since $\hat{\xi}' \circ (\widehat{f \times g_0}) = \hat{\xi}_1$. Since ξ_1 is H -homotopic to ξ it follows that the map $\lambda(\theta, \alpha) \circ (\hat{\xi}_1)^{-1}$ is k -equivalent to $\lambda(\theta, \alpha) \circ \hat{\xi}^{-1}$. All in all we have now shown that the map

$$\lambda(\theta, \alpha_0) \circ (\hat{\xi}')^{-1} : X \longrightarrow R_H X_0 \bigcup_{\mu'} C$$

is s -equivalent $\text{rel } R_H X_0$ to

$$\lambda(\theta, \alpha) \circ (\hat{\xi})^{-1} : X \longrightarrow R_H X_0 \bigcup_{\mu} C,$$

where $\lambda(\theta, \alpha)$ is as in (6). This shows that the map $\eta : X \rightarrow R_H X$ in (7) is up to s -equivalence $\text{rel } R_H X_0$ independent of the choice of characteristic map ξ in (2).

The choice of a G -subcomplex X_0 of X such that X is obtained from X_0 by adjoining one G -cell corresponds to the choice of one specific filtration of X by an increasing sequence of G -subcomplexes each obtained from the preceding one by adjoining one G -cell. It is easy to see, again using an inductive argument, that the construction of $\eta : X \rightarrow R_H X$ is independent up to simple H -homotopy type of this choice of filtration. This completes the proof of Theorem 6.1. \square

It follows from the construction of the finite H -CW complex $R_H X$, given in the above proof of Theorem 6.1, that $R_H X$ also satisfies the following conditions

- (i) $\dim(R_H X)^{H_1} = \dim X^{H_1}$, for each $H_1 < H$.
- (ii) The H -isotropy types occurring in $R_H X$ and in X are exactly the same.

This is seen exactly as in the proof of Theorem A in [8].

DEFINITION 6.2. Let X be a finite G -CW complex. A preferred H -reduction of X consists of a finite H -CW complex Y and an H -homotopy equivalence

$$\theta: X \longrightarrow Y$$

such that θ is s -equivalent to an H -homotopy equivalence $\eta: X \rightarrow R_H X$ constructed in Theorem 6.1 and Y satisfies (i) and (ii) above.

We will in the rest of this paper not pay any specific attention to conditions (i) and (ii). All constructions used are such that conditions (i) and (ii) hold and hence we concentrate our attention only to questions concerning simple H -homotopy type.

The proof of Theorem 6.1 also establishes the following.

THEOREM 6.3. *Let G be a compact Lie group and H a closed subgroup of G . Suppose that (X, X_0) is a finite G -CW pair and that $\eta_0: X_0 \rightarrow R_H X_0$ is a preferred H -reduction of X_0 . Then there exists a preferred H -reduction $\eta: X \rightarrow R_H X$ of X , which extends η_0 , and this construction is unique up to a simple H -homotopy equivalence rel $R_H X_0$.*

The last statement in Theorem 6.3 means that if one by some other choices in the construction arrives at the preferred H -reduction $\eta': X \rightarrow R'_H X$ of X , which also extends η_0 , then there exists a simple H -homotopy equivalence $\sigma: R_H X \rightarrow R'_H X$, which extends the identity on $R_H X_0$, such that $\sigma \circ \eta$ is H -homotopic to η' .

In fact the proof of Theorem 6.1 shows that if $\eta: (X, X_0) \rightarrow (R_H X, R_H X_0)$ and $\eta': (X, X_0) \rightarrow (R'_H X, R'_H X_0)$ are two preferred H -reductions of the pair (X, X_0) then there exists a simple H -homotopy equivalence $\sigma: (R_H X, R_H X_0) \rightarrow (R'_H X, R'_H X_0)$ such that $\sigma \circ \eta$ is H -homotopic to η' .

Suppose that $f: X \rightarrow W$ is a G -map between finite G -CW complexes. Let $\theta: X \rightarrow R_H X$ and $\eta: W \rightarrow R_H W$ be preferred H -reductions of X and W , respectively. Then we obtain an induced H -map

$$R_H f: R_H X \longrightarrow R_H W$$

by defining $R_H f = \eta \circ f \circ \theta^-$. If f is a G -homotopy equivalence then $R_H f$ is an H -homotopy equivalence.

The following Lemma is easy to prove, and we leave the details to the reader.

LEMMA 6.4. *Let $\theta: X \rightarrow R_H X$ be a preferred H -reduction of the finite*

G -CW complex, and let K be an ordinary finite CW complex. Then $\text{id} \times \theta : K \times X \rightarrow K \times_{R_H} X$ is a preferred H -reduction of the finite G -CW complex $K \times X$.

7. An important property of preferred H -reductions

In Proposition 7.1 below X denotes a finite G -CW complex and (K, L) denotes a finite G -CW pair, and $\varphi : L \rightarrow X$ is a (K, L) -skeletal G -map, (see Definition 1.8). The adjunction space $X \cup_{\varphi} K$ is then a finite G -CW complex.

PROPOSITION 7.1. *Let $\eta : X \rightarrow R_H X$ and $\theta : (K, L) \rightarrow (R_H K, R_H L)$ be preferred H -reductions of X and (K, L) , respectively. Then*

$$\lambda(\eta, \theta) : X \cup_{\varphi} K \longrightarrow R_H X \cup_{\mu} R_H K$$

is a preferred H -reduction of $X \cup_{\varphi} K$. Here $\mu : R_H L \rightarrow R_H X$ denotes a skeletal H -approximation of the H -map $\eta \circ \varphi \circ (\theta|)^{\leftarrow}$.

PROOF. First of all we note the following. If $\theta : (K, L) \rightarrow (R_H K, R_H L)$ and $\theta' : (K, L) \rightarrow (R'_H K, R'_H L)$ are two preferred H -reductions of (K, L) , then, θ and θ' are s -equivalent as maps of pairs. Hence Theorem 5.1 implies that the H -homotopy equivalences $\lambda(\eta, \theta)$ and $\lambda(\eta, \theta')$ are s -equivalent, and hence if one of them is a preferred H -reduction of $X \cup_{\varphi} K$ then so is the other one. Thus it is enough to exhibit one specific preferred H -reduction $\theta : (K, L) \rightarrow (R_H K, R_H L)$ of (K, L) and prove that for this specific choice the λ -map $\lambda(\eta, \theta) : X \cup_{\varphi} K \rightarrow R_H X \cup_{\mu} R_H K$ is a preferred H -reduction of $X \cup_{\varphi} K$.

The proof of Proposition 7.1 is now by induction on the number of G -cells in $K - L$. Let K_0 be a G -subcomplex of K such that $L \subset K_0$ and $K - K_0$ consists of one G -cell c , of say dimension n and type G/P . Let

$$\xi : (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (c, \dot{c}) \hookrightarrow (K, K_0)$$

be a characteristic G -map for c , and let

$$\phi = \xi| : S^{n-1} \times G/P \longrightarrow K_0$$

be the corresponding attaching G -map for c . Then

$$\hat{\xi} : K_0 \cup_{\phi} (D^n \times G/P) \xrightarrow{\cong} K$$

is a G -homeomorphism and $\hat{\xi}|_{K_0} = \text{id}_{K_0}$.

We now construct a specific preferred H -reduction of (K, L) with which it will be convenient to work. Let

$$\theta_0: (K_0, L) \longrightarrow (R_H K_0, R_H L)$$

be a preferred H -reduction of the pair (K_0, L) . Let $u: F \rightarrow G/P$ be a distinguished H -triangulation of G/P , which exists by Theorem 3.1, and consider the H -homeomorphism

$$\alpha = \text{id} \times u^{-1}: (D^n \times G/P, S^{n-1} \times G/P) \longrightarrow (D^n \times F, S^{n-1} \times F).$$

In order to shorten the notation we denote the finite G -CW pair $(D^n \times G/P, S^{n-1} \times G/P)$ by (A, B) and the finite H -CW pair $(D^n \times F, S^{n-1} \times F)$ by (C, D) .

We now claim that the composite map

$$(1) \quad \begin{aligned} \theta: K &\xrightarrow{(\hat{\xi})^{-1}} K_0 \cup_{\phi} A \xrightarrow{\bar{\theta}_0} R_H K_0 \cup_{\theta_0 \phi} A \xrightarrow{\hat{\alpha}} R_H K_0 \cup_{\omega} C \\ &\xrightarrow{k'} R_H K_0 \cup_{\omega} C := R_H K \end{aligned}$$

is a preferred H -reduction of K . (Here $\omega = \theta_0 \circ \phi \circ (\alpha|)^{-1}$ and $\bar{\omega}: D \rightarrow R_H K_0$ is a skeletal H -approximation of ω .) This is seen as follows. It follows by Lemma 3.7 that the composite $\hat{\alpha} \circ \bar{\theta}_0$ is k -equivalent to $\lambda(\theta_0, \alpha)$. Hence $\theta = k' \circ \hat{\alpha} \circ \bar{\theta}_0 \circ (\hat{\xi})^{-1}$ is k -equivalent to $\lambda(\theta_0, \alpha) \circ (\hat{\xi})^{-1}$. By the very construction of preferred H -reductions (see the proof of Theorem 6.1) the map $\lambda(\theta_0, \alpha) \circ (\hat{\xi})^{-1}$ is a preferred H -reduction of K . Therefore θ is a preferred H -reduction of K by Corollary 4.2.

By the inductive assumption the map

$$(2) \quad \zeta_0 = \lambda(\eta, \theta_0): X \cup_{\phi} K_0 \longrightarrow R_H X \cup_{\mu} R_H K_0$$

is a preferred H -reduction of $X \cup_{\phi} K_0$. Let $j: K_0 \rightarrow X \cup_{\phi} K_0$ denote the obvious map. Observe that $X \cup_{\phi} K$ is obtained from $X \cup_{\phi} K_0$ by adjoining the G -cell $A = D^n \times G/P$ by the attaching map $j \circ \phi: B = S^{n-1} \times G/P \rightarrow X \cup_{\phi} K_0$. We claim that the composite map

$$(3) \quad \begin{aligned} \zeta: X \cup_{\phi} K &\xrightarrow{(j \circ \hat{\xi})^{-1}} (X \cup_{\phi} K_0) \cup_{j \phi} A \xrightarrow{\bar{\zeta}_0} (R_H X \cup_{\mu} R_H K_0) \cup_{\zeta_0 j \phi} A \\ &\xrightarrow{\hat{\alpha}} (R_H X \cup_{\mu} R_H K_0) \cup_{\nu} C \xrightarrow{k''} (R_H X \cup_{\mu} R_H K_0) \cup_{\bar{\nu}} C \end{aligned}$$

is a preferred H -reduction of $X \cup_{\varphi} K$. Here $\nu = \zeta_0 \circ j \circ \phi \circ (\alpha|)^{-1}$ and $\tilde{\nu} : D \rightarrow R_H X \cup_{\mu} R_H K_0$ is a skeletal H -approximation of ν . This is seen in exactly the same way as we saw that θ in (1) is a preferred H -reduction of K .

Now the map ζ_0 in (2) is by definition a composite of three maps and by inserting this expression for ζ_0 in the formula (3) for ζ we obtain ζ expressed as a composite of six maps. Now we compare ζ with the map

$$(4) \quad \lambda(\eta, \theta) : X \cup_{\varphi} K \longrightarrow R_H X \cup_{\mu} R_H K.$$

The map $\lambda(\eta, \theta)$ in (4) is by definition a composite of three maps and if we in the defining expression for the λ -map $\lambda(\eta, \theta)$ insert the expression (1) for θ we obtain $\lambda(\eta, \theta)$ expressed as a composite of six maps. By comparing these two composites of six maps with each other we see that the maps ζ and $\lambda(\eta, \theta)$ are s -equivalent. Since we already showed that ζ is a preferred H -reduction of $X \cup_{\varphi} K$ it now follows that $\lambda(\eta, \theta)$ in (4) is a preferred H -reduction of $X \cup_{\varphi} K$. \square

8. The H -equivariant Whitehead group $\text{Wh}_H(X)$ of a finite G -CW complex X

Let as before G denote a compact Lie group and H a fixed closed subgroup of G . Suppose that X is a finite G -CW complex. We shall define the H -equivariant Whitehead group $\text{Wh}_H(X)$ of X . Let $\theta : X \rightarrow R_H X$ be a preferred H -reduction of X . Then $R_H X$ is a finite H -CW complex and we have the H -equivariant Whitehead group $\text{Wh}_H(R_H X)$ of $R_H X$. If $\theta' : X \rightarrow R'_H X$ is another preferred H -reduction of X , then $\sigma_{\theta', \theta} = \theta' \circ \theta^{-1} : R_H X \rightarrow R'_H X$ is a simple H -homotopy equivalence. In particular $\sigma_{\theta', \theta}$ is an H -homotopy equivalence and hence

$$(1) \quad (\sigma_{\theta', \theta})_* : \text{Wh}_H(R_H X) \xrightarrow{\cong} \text{Wh}_H(R'_H X)$$

is an isomorphism. Thus any two preferred H -reductions $\theta : X \rightarrow R_H X$ and $\theta' : X \rightarrow R'_H X$ of X give rise to a canonical isomorphism (1) between $\text{Wh}_H(R_H X)$ and $\text{Wh}_H(R'_H X)$. This defines for us a group $\text{Wh}_H(X)$, and for each preferred H -reduction $\theta : X \rightarrow R_H X$ of X there is a canonical isomorphism

$$(2) \quad i_{\theta} : \text{Wh}_H(R_H X) \xrightarrow{\cong} \text{Wh}_H(X).$$

These canonical isomorphisms are characterized by the fact that

$$(3) \quad (i_{\theta'})^{-1} \circ i_{\theta} = (\sigma_{\theta', \theta})_* : \text{Wh}_H(R_H X) \xrightarrow{\cong} \text{Wh}_H(R'_H X)$$

for any two preferred H -reductions $\theta: X \rightarrow R_H X$ and $\theta': X \rightarrow R'_H X$ of X .

Suppose that $f: X \rightarrow W$ is a G -homotopy equivalence between finite G -CW complexes. Let $\theta: X \rightarrow R_H X$ and $\eta: W \rightarrow R_H W$ be preferred H -reductions of X and W , respectively. By $R_H f: R_H X \rightarrow R_H W$ we denote the H -map induced by f . Recall that $R_H f$ by definition is an H -map which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & W \\ \theta \downarrow & & \downarrow \eta \\ R_H X & \xrightarrow{R_H f} & R_H W \end{array}$$

H -homotopy commutative; i.e., $R_H f = \eta \circ f \circ \theta^-$. Since f is a G -homotopy equivalence it follows that $R_H f$ is an H -homotopy equivalence between finite H -CW complexes, and hence its H -equivariant Whitehead torsion $\tau(R_H f) \in \text{Wh}_H(R_H X)$ is defined.

We now define the H -equivariant Whitehead torsion of $R_H f$, as an element of $\text{Wh}_H(X)$, by

$$(4) \quad \tau(R_H f) = i_{\theta}(\tau(R_H f)) \in \text{Wh}_H(X).$$

We claim that this H -equivariant Whitehead torsion of $R_H f$ is well-defined; i.e., it is independent of the choice of preferred H -reductions of X and W . Let $\theta: X \rightarrow R_H X$ and $\theta': X \rightarrow R'_H X$ be preferred H -reductions of X and let $\eta: W \rightarrow R_H W$ and $\eta': W \rightarrow R'_H W$ be preferred H -reductions of W . By $R_H f: R_H X \rightarrow R_H W$ and $R'_H f: R'_H X \rightarrow R'_H W$ we denote the corresponding H -maps induced by $f: X \rightarrow W$. We know that $\sigma_{\theta', \theta} = \theta' \circ \theta^-: R_H X \rightarrow R'_H X$ and $\sigma_{\eta', \eta} = \eta' \circ \eta^-: R_H W \rightarrow R'_H W$ are simple H -homotopy equivalences. Furthermore $R'_H f \circ \sigma_{\theta', \theta}$ is H -homotopic to $\sigma_{\eta', \eta} \circ R_H f$, and hence we obtain by the formula for the equivariant Whitehead torsion of a composite map, see [3, Proposition II. 3.8], that

$$\tau(\sigma_{\theta', \theta}) + (\sigma_{\theta', \theta})_*^{-1}(\tau(R'_H f)) = \tau(R_H f) + (R_H f)_*^{-1}(\tau(\sigma_{\eta', \eta})).$$

Since $\sigma_{\theta', \theta}$ and $\sigma_{\eta', \eta}$ are simple H -homotopy equivalences we have $\tau(\sigma_{\theta', \theta}) = 0$ and $\tau(\sigma_{\eta', \eta}) = 0$, and hence

$$\tau(R'_H f) = (\sigma_{\theta', \theta})_* (\tau(R_H f)) \in \text{Wh}_H(R'_H X)$$

which shows that

$$(5) \quad i_{\theta'}(\tau(R'_H f)) = i_{\theta}(\tau(R_H f)) \in \text{Wh}_H(X).$$

This proves our claim that (4) is well-defined.

In particular we note the following case. Let (V, X) be a finite G -CW pair where the inclusion $j: X \rightarrow V$ is a G -homotopy equivalence. Let $\theta: X \rightarrow R_H X$ be a preferred H -reduction of X and let $\eta: V \rightarrow R_H V$ be a preferred H -reduction of V which extends θ . Then $(R_H V, R_H X)$ is a finite H -CW pair such that the inclusion $R_H j: R_H X \rightarrow R_H V$ is an H -homotopy equivalence. Let $\theta': X \rightarrow R'_H X$ be another preferred H -reduction of X and let $\eta': V \rightarrow R'_H V$ be a preferred H -reduction of V which extends θ' , and let $R'_H j: R'_H X \rightarrow R'_H V$ denote the corresponding inclusion map, which is an H -homotopy equivalence. Then we have, by what we showed above, that

$$i_{\theta}(\tau(R_H j)) = i_{\theta'}(\tau(R'_H j)) \in \text{Wh}_H(X).$$

This can also be written in the form

$$(6) \quad i_{\theta}(s_H(R_H V, R_H X)) = i_{\theta'}(s_H(R'_H V, R'_H X)) \in \text{Wh}_H(X),$$

see [3, Lemma II. 3.11].

By taking $\theta = \theta'$ we conclude that if $\eta: V \rightarrow R_H V$ and $\eta': V \rightarrow R'_H V$ are two preferred H -reductions of V , which both extend $\theta: X \rightarrow R_H X$, then

$$s_H(R_H V, R_H X) = s_H(R'_H V, R_H X) \in \text{Wh}_H(R_H X).$$

This proves the following lemma.

LEMMA 8.1. *Let (V, X) be a finite G -CW pair, and let $\theta: X \rightarrow R_H X$ be a preferred H -reduction of X . Suppose that $\eta: V \rightarrow R_H V$ and $\eta': V \rightarrow R'_H V$ are two preferred H -reductions of V which both extend θ . Then there exists a formal H -deformation from $R_H V$ to $R'_H V$ rel $R_H X$. \square*

9. Construction of the restriction homomorphism $\text{Res}_H^G: \text{Wh}_G(X) \rightarrow \text{Wh}_H(X)$

LEMMA 9.1. *Suppose that X collapses to X_0 by an elementary G -collapse and let $\theta: X_0 \rightarrow R_H X_0$ be a preferred H -reduction of X_0 . Then there exists a preferred H -reduction $\eta: X \rightarrow R_H X$ of X , which extends θ , such*

that $R_H X$ H -collapses to $R_H X_0$.

PROOF. The assumption that X_0 is an elementary G -collapse of X means that X can be expressed as an adjunction space of the form

$$X = X_0 \cup_{\varphi} (I^n \times G/P)$$

where $P < G$ and $\varphi: J^{n-1} \times G/P \rightarrow X_0$ is a G -map such that $\varphi(J^{n-1} \times G/P) \subset X_0^{n-1}$ and $\varphi(\dot{I}^n \times G/P) \subset X_0^{n-2}$, see [3, p. 13]. In the terminology of Definition 1.8 the map $\varphi: J^{n-1} \times G/P \rightarrow X_0$ is an $(I^n \times G/P, J^{n-1} \times G/P)$ -skeletal G -map.

Let $u: F \rightarrow G/P$ be a distinguished H -triangulation of the H -manifold G/P , which exists by Theorem 1.3. Then $u^{-1}: G/P \rightarrow F$ is a preferred H -reduction of G/P , and by Lemma 6.4

$$\alpha = \text{id} \times u^{-1}: (I^n \times G/P, J^{n-1} \times G/P) \longrightarrow (I^n \times F, J^{n-1} \times F)$$

is a preferred H -reduction of the finite G -CW pair $(I^n \times G/P, J^{n-1} \times G/P)$. Hence we have by Proposition 7.1 that

$$\lambda(\theta, \alpha): X_0 \cup_{\varphi} (I^n \times G/P) \longrightarrow R_H X_0 \cup_{\mu} (I^n \times F)$$

is a preferred H -reduction of $X = X_0 \cup_{\varphi} (I^n \times G/P)$, and $\lambda(\theta, \alpha)$ extends the given preferred H -reduction $\theta: X_0 \rightarrow R_H X_0$ of X_0 . Here $\mu: J^{n-1} \times F \rightarrow R_H X_0$ denotes a skeletal H -approximation of the H -map $\theta \circ \varphi \circ (\alpha|)^{-1}$.

Since I^n collapses to J^{n-1} , in fact by one elementary collapse, it follows by Lemma 1.5 that $I^n \times F$ H -collapses to $J^{n-1} \times F$. Hence it follows, see [3, Lemma II. 1.6], that $R_H X = R_H X_0 \cup_{\mu} (I^n \times F)$ H -collapses to $R_H X_0 \cup_{\mu} (J^{n-1} \times F) = R_H X_0$. \square

COROLLARY 9.2. *Let (V, X) and (W, X) be finite G -CW pairs such that there exists a formal G -deformation from V to W rel X , and let $\theta: X \rightarrow R_H X$ be a preferred H -reduction of X . Suppose that $\eta_1: V \rightarrow R_H V$ and $\eta_2: W \rightarrow R_H W$ are preferred H -reductions of V and W , respectively, which extend θ . Then there exists a formal H -deformation from $R_H V$ to $R_H W$ rel $R_H X$.*

PROOF. Let

$$V = V_0 \xrightarrow{k_1} V_1 \xrightarrow{k_2} \dots \xrightarrow{k_{m+1}} V_{m+1} = W$$

be a formal G -deformation from V to W rel X ; i.e., each V_i , $0 \leq i \leq m+1$,

is a finite G -CW complex that contains X as a G -subcomplex and each k_i , $1 \leq i \leq m+1$, is either an elementary G -expansion or an elementary G -collapse. For each $i=1, \dots, m$ we choose a preferred H -reduction $\theta_i: V_i \rightarrow R_H V_i$ of V_i , such that θ_i extends $\theta: X \rightarrow R_H X$. We also denote the given preferred H -reductions η_1 and η_2 by $\theta_0: V_0 \rightarrow R_H V_0$ and $\theta_{m+1}: V_{m+1} \rightarrow R_H V_{m+1}$, respectively.

Clearly it is enough to show that for each $j=0, \dots, m$ there exists a formal H -deformation from $R_H V_j$ to $R_H V_{j+1}$ rel $R_H X$. Assume for example that $k_j: V_j \rightarrow V_{j+1}$ is an elementary G -expansion. By Lemma 9.1 there exists a preferred H -reduction $\bar{\theta}_{j+1}: V_{j+1} \rightarrow \bar{R}_H V_{j+1}$ of V_{j+1} , which extends $\theta_j: V_j \rightarrow R_H V_j$, such that $\bar{R}_H V_{j+1}$ collapses to $R_H V_j$ by a finite sequence of elementary H -collapses. Thus there exists a formal H -deformation from $R_H V_j$ to $\bar{R}_H V_{j+1}$ rel $R_H V_j$ and hence in particular rel $R_H X$.

Since $\bar{\theta}_{j+1}: V_{j+1} \rightarrow \bar{R}_H V_{j+1}$ and $\theta_{j+1}: V_{j+1} \rightarrow R_H V_{j+1}$ are preferred H -reductions of V_{j+1} , which both extend $\theta: X \rightarrow R_H X$, it follows by Lemma 8.1 that there exists a formal H -deformation from $\bar{R}_H V_{j+1}$ to $R_H V_{j+1}$ rel $R_H X$. Thus we have shown that there is a formal H -deformation from $R_H V_j$ to $R_H V_{j+1}$ rel $R_H X$ for each $j=0, \dots, m$. \square

We are now able to define the restriction homomorphism

$$\text{Res}_H^G: \text{Wh}_G(X) \longrightarrow \text{Wh}_H(X).$$

Choose a preferred H -reduction $\theta: X \rightarrow R_H X$ of X , and define

$$\text{Res}_H^G(s_G(V, X)) = i_\theta(s_H(R_H V, R_H X)) \in \text{Wh}_H(X)$$

for every $s_G(V, X) \in \text{Wh}_G(X)$. The fact that this gives a well-defined map is an immediate consequence of Corollary 9.2, and (6) in Section 8.

Lemma 9.3. *The map $\text{Res}_H^G: \text{Wh}_G(X) \rightarrow \text{Wh}_H(X)$ is a homomorphism.*

PROOF. Let $\theta: X \rightarrow R_H X$ be a preferred H -reduction of X . Clearly it is enough to prove that the restriction map $\text{Res}_H^G: \text{Wh}_H(X) \rightarrow \text{Wh}_H(R_H X)$ is a homomorphism. Let $s_G(V_1, X) \in \text{Wh}_G(X)$ and $s_G(V_2, X) \in \text{Wh}_G(X)$. Then

$$s_G(V_1, X) + s_G(V_2, X) = s_G(V_1 \cup_X V_2, X).$$

Let $\eta_1: V_1 \rightarrow R_H V_1$ and $\eta_2: V_2 \rightarrow R_H V_2$ be preferred H -reductions of V_1 and V_2 , respectively, which extend the given H -reduction $\theta: X \rightarrow R_H X$ of X . The space $V_1 \cup_X V_2$; i.e., the union of V_1 and V_2 along X , can also

be considered as the adjunction space obtained by adjoining V_2 to V_1 by the inclusion map $i: X \rightarrow V_1$. Hence it follows by Proposition 7.1 that

$$\eta: V_1 \bigcup_X V_2 \longrightarrow R_H V_1 \bigcup_{R_H X} R_H V_2$$

where $\eta = \eta_1 \bigcup_X \eta_2$, is a preferred H -reduction of $V_1 \bigcup_X V_2$. Hence

$$\begin{aligned} \text{Res}_H^G(s_G(V_1, X) + s_G(V_2, X)) &= \text{Res}_H^G(s_G(V_1 \bigcup_X V_2, X)) \\ &= s_H(R_H V_1 \bigcup_{R_H X} R_H V_2, R_H X) \\ &= s_H(R_H V_1, R_H X) + s_H(R_H V_2, R_H X) \\ &= \text{Res}_H^G(s_G(V_1, X)) + \text{Res}_H^G(s_G(V_2, X)) \in \text{Wh}_H(R_H X). \quad \square \end{aligned}$$

10. Behaviour of equivariant Whitehead torsion under the restriction homomorphism Res_H^G

PROPOSITION 10.1. Suppose that $f: X \rightarrow W$ is a G -homotopy equivalence between finite G -CW complexes and let $R_H f: R_H X \rightarrow R_H W$ be the H -map induced by f . Then

$$\text{Res}_H^G(\tau(f)) = \tau(R_H f) \in \text{Wh}_H(X).$$

PROOF. Recall that in defining the G -equivariant Whitehead torsion of a G -homotopy equivalence $f: X \rightarrow W$ one may assume that f is skeletal and then

$$\tau(f) = s_G(M_f, X) \in \text{Wh}_G(X),$$

where M_f denotes the mapping cylinder of f , see [3, page 23 and Proposition II.3.5]. We let $\theta: X \rightarrow R_H X$ and $\eta: W \rightarrow R_H W$ be preferred H -reductions of X and W respectively, and in order to simplify the notation we denote $\hat{f} = R_H f$. We may assume that \hat{f} is skeletal and then the H -equivariant Whitehead torsion of \hat{f} is given by

$$\tau(\hat{f}) = i_\theta(s_H(M_f, R_H X)) \in \text{Wh}_H(X).$$

By Lemma 6.3 $\theta \times \text{id}: X \times I \rightarrow R_H X \times I$ is a preferred H -reduction of $X \times I$. It now follows by Proposition 7.1 that

$$\zeta = \lambda(\eta; \theta \times \text{id}): W \bigcup_f (X \times I) \longrightarrow R_H W \bigcup_{\hat{f}} (R_H X \times I)$$

is a preferred H -reduction of $M_f = W \bigcup_f (X \times I)$. Here f and \hat{f} denote maps $f: X \times \{1\} \rightarrow W$ and $\hat{f}: R_H X \times \{1\} \rightarrow R_H W$, respectively. Since $X = X \times \{0\}$

is a G -subcomplex of $X \times I$ disjoint from $X \times \{1\}$ we may choose ζ so that ζ extends $\theta : X \times \{0\} \rightarrow R_H X \times \{0\}$. Since $M_f = R_H W \bigcup_f (R_H X \times I)$, we have shown that

$$\zeta : M_f \longrightarrow M_f$$

is a preferred H -reduction of M_f that extends $\theta : X \rightarrow R_H X$. Hence

$$\begin{aligned} \text{Res}_H^G(\tau(f)) &= \text{Res}_H^G(s_G(M_f, X)) \\ &= i_\theta(s_H(M_f, R_H X)) = \tau(\hat{f}) \in \text{Wh}_H(X). \quad \square \end{aligned}$$

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