J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
37 (1990), 659-702.

On fiber bundles over S^1 having small Seifert manifolds as fibers

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0. Introduction

P. Orlik divided Seifert manifolds into small and large ones in his paper [11]. Our purpose in this paper is to classify fiber bundles over S^1 having small Seifert manifolds as fibers in terms of

(A) diffeomorphism,
(B) bundle isomorphism.

In dividing small Seifert manifolds into aspherical ones and non-aspherical ones, we have the following table of classification up to diffeomorphism (the notation of Seifert invariant conforms to that of [11]).

aspherical ones

$$\begin{array}{ll} \{b:(o_1,1)\} & (b\in Z,\quad b\geq 0) \\ \{b:(o_2,1)\} & (b\in Z_2) \\ \{b:(n_2,2)\} & (b\in Z,\quad b\geq 0) \\ \{b:(n_3,2)\} & (b\in Z_2) \end{array}$$

non-aspherical ones

Lens spaces, $S^2\times S^1$, $S^2\tilde{\times}S^1$ (nontrivial S^2 bundle over S^1), $RP^2\times S^1$, $RP^3\#RP^3$, prism manifolds and the following

$$\begin{array}{ll} \{b:(o_1,0):(2,1),(3,1),(3,1)\} & (b\in Z) \\ \{b:(o_1,0):(2,1),(3,1),(3,2)\} & (b\in Z, b\geq -1) \\ \{b:(o_1,0):(2,1),(3,1),(4,\beta)\} & (b\in Z, \beta=1 \text{ or } 3) \\ \{b:(o_1,0):(2,1),(3,1),(5,\beta)\} & (b\in Z, \beta=1,2,3 \text{ or } 4). \end{array}$$

From §2 to §4 we shall consider the aspherical cases and non-aspherical cases will be treated in §5. Theorem 2 in §4 is an answer to problem (A) and this gives an algorithm whether given fiber bundles are diffeomorphic or not. Theorem 1 is the answer to problem (B) and being combined with Theorem 2, it gives the algorithm to judge whether there exists bundle isomorphism or not.

THEOREM 1. (1) *If* $b_1(E)=b_1(E')=1$, *or* $F\cong F'$ *and* $b=0,1,2,3$, *or* $F\cong F'\cong\{4:(o_1,1)\}$ *or* $\{6:(o_1,1)\}$, *then the following three assertions are equivalent*

$$\text{(a)}\quad \Pi_1(E)\cong\Pi_1(E') \qquad \text{(b)}\quad E\cong E' \qquad \text{(c)}\quad E\cong_b E.$$

(2) *If* $F\cong F'=\{b\neq 0:(o_1,2)\}$, *then the following two assertions are equivalent*

(a) $E(F:N:p,q)\cong_b E(F':N':p',q')$

(b) *there exist some* $A\in GL(2,Z)$ *such that*

$$\text{(i)}\quad A^{-1}NA=N' \quad \text{or} \quad A^{-1}NA=N'^{-1}$$

$$\text{(ii)}\quad (p,q)-(p',q')^tA\in\left\langle\begin{pmatrix}b&0\\0&b\end{pmatrix},\ {}^tN-(\det N)\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\rangle.$$

(3) *If* $F\cong F'=\{b\neq 0:(n_2,2)\}$ *and* $b_1(E)=b_1(E')=2$, *the bundle isomorphism is reduced to the following three types.*

(a) *If there exists* $\varepsilon\in\{\pm 1\}$ *s.t.*

$$n'\equiv n \mod 2b$$

$$(p',q')-(p,q)\begin{pmatrix}1&\frac{1-\varepsilon}{2}(n+b)\\0&1\end{pmatrix}\in\left\langle\begin{pmatrix}2&b\\0&2\end{pmatrix}\right\rangle$$

then $E(n,1,p,q)\cong_b E(n',1,p',q')$.

(b) *If* $0\leq n\leq b$, *then* $E(n,1,1,1)\cong_b E(n,1,1,0)$.

(c) *If* $0\leq n\leq b$ *and* n *is odd, then* $E(n,1,0,1)\cong_b E(n,1,0,0)$.

THEOREM 2. $E\cong E'$ *if and only if* $\Pi_1(E)\cong\Pi_1(E')$ *and more precisely,*

(1) *If* $\mathrm{Ker}(\Pi_1(E)\to H_1(E)/torsion)\cong Z$, *then the diffeomorphism type of* E *is determined uniquely by* $H_*(E)$.

(2) *If* $\mathrm{Ker}(\Pi_1(E)\to H_1(E)/torsion)\cong\Pi_1(K)$, *then the diffeomorphism type of* E *is determined by* $H_*(E)$, *center of* $\Pi_1(E)$, $\mathrm{Abel}(\Pi_1(E)/center)$, *and* $s(\Pi_1(E))$.

(3) *If* $\mathrm{Ker}(\Pi_1(E)\to H_1(E)/torsion)\cong Z^2$, *then the diffeomorphism type of* E *is determined by the three procedure below*

(i) *to give the representation as the bundle over* S^1.

(ii) *to transform the representation into that of* T^3*-bundle over* S^1 *or fiber bundle over* S^1 *with fiber* $\{0:(n_2,2)\}$.

(iii) *to apply Propositions* 2.4, 2.6, 2.7.

(4) *If* $\mathrm{Ker}(\Pi_1(E)\to H_1(E)/torsion)$ *is trivial, then* $E\cong T^4$.

The main result in non-aspherical case is Theorem 3 in §5.

THEOREM 3. *Except for* $E_{f_2} \cong E_{\mathrm{id}_{S^2 \tilde{\times} S^1}}$ *and* $E_{f_1 f_2} \cong E_{g_1}$, *the following three assertions hold.*

(1) $E \cong E'$ *if and only if* $E \cong_b E'$.

(2) *When* F *is not lens space,* $E \cong_{(b)} E'$ *if and only if* $E \simeq E'$.

(3) *When* $F \cong F'$, $E \cong_{(b)} E'$ *if and only if* $E \simeq E'$.

The complex analytic surfaces of type VII_0 with zero second Betti number b_2, i.e. three kinds of Inoue surfaces, all have the fiber bundle structure over S^1 with small Seifert manifold as fiber [7]. (The case when $b_2 \neq 0$ is studied in [9]). Fiber bundles over torus with torus as fiber (we call it T^2-bundle over T^2 for short), Klein bottle bundles over T^2 and Klein bottle bundles over klein bottle have also the fiber bundle structure over S^1 having small Seifert manifolds as fibers. In Corollary 1 and Corollary 2 in §4, the classification of these manifolds up to diffeomorphism is established.

In considering fiber bundles over S^1, attentions shall be paid to diffeotopy of fibers. There exist a natural isomorphism between the group of outer automorphisms of the fundamental group and diffeotopy of the fiber if the fiber is aspherical small Seifert manifold. The diffeotopy of non-aspherical small Seifert manifolds are determined recently ([1], [2], [3], [4], [10], [12]). The classification of fiber bundles over S^1 with small Seifert manifolds as fiber can be made based upon these results.

Finally I would like to express my heartfelt gratitude to Prof. Yukio Matsumoto, who advised the author on many subjects, Dr. Masaaki Ue and others, who gave precious advices.

1. Notations

We summarize the special notations that will be used in this paper.

$\cong$: diffeomorphism between manifolds or isomorphism between groups.

$\cong_b$: bundle isomorphism between fiber bundles.

$\cong_{(b)}$: bundle isomorphism in the case when diffeomorphism and bundle isomorphism is known to be equivalent.

$\simeq$: homotopy equivalent.

T^n : n-dimensional torus.

K : Klein bottle.

$b_i(X)$: i-th Betti number of X.

$\langle (a_1, b_1), \cdots, (a_n, b_n) \rangle$:
the subgroup of Z^2 generated by (a_i, b_i) $(i=1, 2, \cdots, n)$.

$\langle A_1, \cdots, A_n \rangle$, $A_i \in GL(2, Z)$:
the subgroup of Z^2 generated by column vectors of A_i $(i=1, 2, \cdots, n)$.

$\mathrm{Aut}(G)$: the group consisting of all automorphisms of group G.

$\mathrm{Out}(G)$: the group consisting of all outer-automorphisms of group G.

$\mathrm{Abel}(G)$: the abelianization of group G.

id_X : identity map of X.

We shall call X-bundle over Y for short instead of fiber bundle over Y with fiber X.

2. T^3-bundles and $\{0 : (n_2, 2)\}$-bundles

In the beginning we shall show the following proposition.

PROPOSITION 2.1. *F, F': aspherical small Seifert manifolds.*
E (resp. E'): fiber bundle over S^1 with fiber F (resp. F').
When $b_1(E)=b_1(E')=1$, following assertions are equivalent.

(1) $$\Pi_1(E) \cong \Pi_1(E')$$
(2) $$E \cong E'$$
(3) $$E \cong_b E'.$$

PROOF. First we consider the exact sequence associated to E;

$$1 \longrightarrow \Pi_1(F) \longrightarrow \Pi_1(E) \xrightarrow{\pi_*} \Pi_1(S^1) \longrightarrow 1.$$

The mapping $\pi_* : \Pi_1(E) \to \Pi_1(S^1)$ is determined uniquely because $b_1(E)=1$.

If we set $j : \Pi_1(E) \xrightarrow{\cong} \Pi_1(E')$, then the following commutative diagram is obtained.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_1(F) & \longrightarrow & \Pi_1(E) & \xrightarrow{\pi_*} & \Pi_1(S^1) & \longrightarrow & 1 \\
 & & \cong \downarrow i & & \cong \downarrow j & & \cong \downarrow k & & \\
1 & \longrightarrow & \Pi_1(F') & \longrightarrow & \Pi_1(E') & \xrightarrow{\pi'_*} & \Pi_1(S^1) & \longrightarrow & 1
\end{array}$$

There exist a diffeomorphism $\tilde{i}: F \to F'$ that induces i, and let $x \in \Pi_1(E)$ be an element such that $\pi_*(x)$ is the generator of $\Pi_1(S^1)$. Then

$$f_x : \Pi_1(F) \longrightarrow \Pi_1(F), \quad f_x(y) = xyx^{-1} \qquad (y \in \Pi_1(F))$$

gives the monodromy of E, here we regard $\Pi_1(F)$ as a subgroup of $\Pi_1(E)$.

On the other hand $\pi'_*(j(x))$ is also the generator of $\Pi_1(S^1)$, and

$$f_{j(x)} : \Pi_1(F') \longrightarrow \Pi_1(F'), \quad f_{j(x)}(z) = j(x)zj(x)^{-1}$$

gives the monodromy of E', too.

Regarding F and F' to be equivalent by means of $\tilde{i}$, we can conclude that $E \cong_b E'$. $\square$

According to Proposition 2.1, we have only to treat the cases when the 1st Betti number of the total space is larger than 1 for aspherical small Seifert manifolds.

Next we will observe T^3-bundles.

PROPOSITION 2.2. *Let E and E' be T^3-bundles over S^1, then the following three assertions are equivalent.*

(1) $\Pi_1(E) \cong \Pi_1(E')$

(2) $E \cong E'$

(3) $E \cong_b E'$.

Before proving the proposition, we remark the theorem due to A. Inagaki.

THEOREM (A. Inagaki [6]). *Let E and E' are T^3-bundles over S^1 of which monodromies are periodic, then the following three assertions are equivalent.*

(1) $\Pi_1(E) \cong \Pi_1(E')$

(2) $E \cong E'$

(3) $E \cong_b E'$.

It is possible to judge from the fundamental group whether the monodromy is periodic or not. Monodromy of E is periodic if and only if $b_1(E) =$ rank of center of $\Pi_1(E)$. Moreover we need the next lemma due to K. Sakamoto and S. Fukuhara.

LEMMA 2.3 (K. Sakamoto and S. Fukuhara [13]). *If $A \in GL(2, Z)$ satisfies one of the following conditions,*

(1) $\det A=1$ *and* $|\operatorname{tr} A|\leq 2$

(2) $\det A=-1$ *and* $|\operatorname{tr} A|=0$

then A is conjugate to one and only one of the following matrices.

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \pm\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix},\quad \pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\ (n\geq 0),\quad \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

PROOF OF PROPOSITION 2.2. When $b_1(E)=b_1(E')=1$, the statement follows from Proposition 2.1. It is clear that $E\cong E'\cong T^4$, when $b_1(E)=b_1(E')=4$. When $b_1(E)=2$, we may assume that monodromy $M\in GL(3, Z)$ has the form

$$M=\begin{pmatrix}1 & p & q\\ 0 & & \\ 0 & N & \end{pmatrix}\qquad N=\begin{pmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\end{pmatrix}\in GL(2, Z)$$

namely

$$\Pi_1(E)=\left\langle\begin{array}{c|l}\alpha & [\alpha,\beta]=[\beta,\gamma]=[\gamma,\alpha]=1\\ \beta & \tau\alpha\tau^{-1}=\alpha\beta^p\gamma^q\\ \gamma & \tau\beta\tau^{-1}=\beta^{n_{11}}\gamma^{n_{12}}\\ \tau & \tau\gamma\tau^{-1}=\beta^{n_{21}}\gamma^{n_{22}}\end{array}\right\rangle$$

$G:=\ker(\Pi_1(E)\to H_1(E)/\textit{torsion})$ is generated by β,γ. If we denote by G' the centralizer of G in $\Pi_1(E)$, then $G'\cong Z^3$ and G' is generated by α,β,γ in the case when N is not periodic. Therefore G' determines the fiber direction uniquely. Thus in this case the proposition holds. In the case when N is periodic we need the next claim.

CLAIM. *If $b_1(E)=2$, N is periodic and M is not periodic, then M is conjugate to one of the following matrices.*

$$\begin{pmatrix}1 & p & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\quad \begin{pmatrix}1 & p & 1\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\quad \begin{pmatrix}1 & p & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\qquad p>0.$$

From the claim if we set

$$M=\begin{pmatrix}1 & p & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\qquad (p>0),$$

G' is generated by $\alpha, \beta, \gamma, \tau^2$. If we set $\langle \alpha^l \tau^{2m}, \beta, \gamma \rangle$ the fiber direction and $\langle \alpha^x \tau^y \rangle$ the base direction, then we may assume

$$\begin{pmatrix} l & 2m \\ x & y \end{pmatrix} \in SL(2, Z),$$

and calculation shows that

$$\begin{aligned} &(\alpha^x \tau^y)(\alpha^l \tau^{2m})(\alpha^x \tau^y)^{-1} = \alpha^l \tau^{2m} \beta^p \\ &(\alpha^x \tau^y)\beta(\alpha^x \tau^y)^{-1} = \beta \\ &(\alpha^x \beta^y)\gamma(\alpha^x \tau^y)^{-1} = \gamma^{-1}. \end{aligned}$$

That is to say, the monodromy is $\begin{pmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

When $M = \begin{pmatrix} 1 & p & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & p & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, the results are similar. Therefore the proposition is also valid in the case $b_1(E) = b_1(E') = 2$.

When $b_1(E) = b_1(E') = 3$, N has eigenvalue 1 and according to Lemma 2.3 M is conjugate to one of the following matrices

$$(1) \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad k > 0 \quad (2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (3) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For (1) $H_1(E) = Z^3 \oplus Z/k$ and E is orientable. For (2) $H_1(E) = Z^3 \oplus Z/2$ and E is non-orientable. For (3) $H_1(E) = Z^3$ and E is non-orientable. Thus the proposition holds also for $b_1(E) = 3$ this completes the proof of Proposition 2.2 modulo proving the claim. $\square$

PROOF OF CLAIM. When N is periodic, it may be assumed to be one of 7 matrices that appeared in Lemma 2.3. After a bit calculation we see that $b_1(E) \geq 3$ if $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and M is also periodic provided that $N = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Thus N can be $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Furthermore by making the conjugation with respect to $\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, we see that M is conjugate to one of three matrices in the claim. As eigenspace of M corresponding to eigenvalue 1 is 1 dimensional, A must be of the type $A=\begin{pmatrix} \pm 1 & * & * \\ 0 & & \\ 0 & & A' \end{pmatrix}$ so that AMA^{-1} has the form $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$.

Three matrices in the claim can not transform into one another by the conjugation with respect to these matrices. Thus the claim is proven. □

When $b_1(E)\geq 2$, monodromy can be expressed in the form of $\begin{pmatrix} 1 & p & q \\ 0 & & \\ 0 & & N \end{pmatrix}$ $(N\in GL(2, Z))$. If we denote E by $E(T^3; N: p, q)$ the following statement is shown.

PROPOSITION 2.4. *Let $E=E(T^3; N: p, q)$ and $E'=E(T^3; N': p', q')$, then*

(1) *When $b_1(E)=b_1(E')=2$*

$$E\cong_{(b)} E' \Longleftrightarrow \exists A\in GL(2, Z) \text{ such that}$$
$$\text{(i)}\quad N'=ANA^{-1} \quad \text{or} \quad N'^{-1}=ANA^{-1}$$
$$\text{(ii)}\quad (p', q')A\pm(p, q)\in\left\langle N-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\rangle.$$

(2) *When $b_1(E)=b_1(E')=3$*

$$E\cong_{(b)} E' \Longleftrightarrow H_*(E)\cong H_*(E').$$

PROOF. (2) can be proven in the same way as Proposition 2.2.
For (1) if we set

$$\begin{pmatrix} \pm 1 & a & b \\ 0 & & \\ 0 & & A \end{pmatrix}\begin{pmatrix} 1 & p & q \\ 0 & & \\ 0 & & N \end{pmatrix}\begin{pmatrix} \pm 1 & a & b \\ 0 & & \\ 0 & & A \end{pmatrix}^{-1}=\begin{pmatrix} 1 & l & m \\ 0 & & \\ 0 & & B \end{pmatrix}$$

then

$$B=ANA^{-1}$$
$$(l, m)A\pm(p, q)=(a, b)\left(N-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$$

and let

$$\begin{pmatrix}1 & p & q\\ 0 & & \\ 0 & & N\end{pmatrix}^{-1}=\begin{pmatrix}1 & x & y\\ 0 & & \\ 0 & & N^{-1}\end{pmatrix}$$

then $(x, y)N+(p, q)=0$, thus the assertion is proven. $\square$

Next we shall consider the fiber bundles over S^1 with fiber $\{0:(n_2, 2)\}$. (Let $F=\{0:(n_2, 2)\}$.)

$$\Pi_1(F)=\langle\alpha, \beta, \sigma \mid [\alpha, \beta]=1,\ \sigma\alpha\sigma^{-1}=\alpha^{-1},\ \sigma\beta\sigma^{-1}=\beta^{-1}\rangle.$$

We make the one to one correspondence between $f\in \mathrm{Aut}(\Pi_1(F))$ and a set consisting of a matrix $N:=(n_{ij})\in GL(2, Z)$, integers $p, q\in Z$, and signature $\varepsilon\in\{\pm 1\}$,

$$\text{i.e.}\quad \mathrm{Aut}(\Pi_1(F))\overset{1:1}{\longleftrightarrow}GL(2, Z)\times Z^2\times\{\pm 1\}$$

where f satisfies

$$f(\alpha)=\alpha^{n_{11}}\beta^{n_{12}}\qquad f(\beta)=\alpha^{n_{21}}\beta^{n_{22}}\qquad f(\sigma)=\alpha^p\beta^q\sigma^\varepsilon.$$

Therefore it is possible to give the expression $E(\{0:(n_2, 2)\};\ N:p, q:\varepsilon)$ to the fiber bundle over S^1 with fiber F. If no confusion may occur, we shall write $E(N:p, q:\varepsilon)$ for short.

$$\Pi_1(E)=\left\langle\begin{matrix}\alpha, \beta\\ \sigma, \tau\end{matrix}\;\middle|\;\begin{matrix}[\alpha, \beta]=1,\ \sigma\alpha\sigma^{-1}=\alpha^{-1},\ \sigma\beta\sigma^{-1}=\beta^{-1}\\ \tau\alpha\tau^{-1}=\alpha^{n_{11}}\beta^{n_{12}},\ \tau\beta\tau^{-1}=\alpha^{n_{21}}\beta^{n_{22}},\ \tau\sigma\tau^{-1}=\alpha^p\beta^q\sigma^\varepsilon\end{matrix}\right\rangle.$$

PROPOSITION 2.5. *Let E and E' be the fiber bundles over S^1 with fiber $\{0:(n_2, 2)\}$, then we see that the following three assertions are equivalent.*

$$(1)\quad \Pi_1(E)\cong\Pi_1(E')\qquad (2)\quad E\cong E'\qquad (3)\quad E\cong_b E'.$$

PROOF. As $b_1(F)=1$, $b_1(E)$ must be 1 or 2. If $b_1(E)=1$, the assertion follows from Proposition 2.1.

We shall consider the case $b_1(E)=2$. If E can be written as

$E(N:p,q,1)$, then $G:=\ker(\Pi(E)\to H_1(E)/torsion)\cong Z^2$ is generated by α and β. The elements of $\Pi(E)$ can be written uniquely in the form of $\alpha^x\beta^y\sigma^u\tau^v$ $(x,y,u,v\in Z)$. If we set

$$(\alpha^x\beta^y\sigma^u\tau^v)\alpha(\alpha^x\beta^y\sigma^u\tau^v)^{-1}=\alpha^k\beta^l$$
$$(\alpha^x\beta^y\sigma^u\tau^v)\beta(\alpha^x\beta^y\sigma^u\tau^v)^{-1}=\alpha^m\beta^n$$

then

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix}=(-1)^u N^v.$$

Therefore if N is not periodic, fiber direction is decided uniquely by α,β and σ, thus the proposition holds. If N is periodic, we need the next claim.

CLAIM. *When N is periodic, E is bundle isomorphic to one of the following fiber bundles.* (*Here* $b_1(E)=2$.)

(1) $E\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}:0,0,1\right)$ (2) $E\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}:1,0,1\right)$

(3) $E\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}:0,0,1\right)$ (4) $E\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}:1,0,1\right)$

(5) $E\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}:0,0,1\right)$ (6) $E\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0,0,1\right)$

(7) $E\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:1,0,1\right)$ (8) $E\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:1,1,1\right)$

(9) $E\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}:0,0,1\right)$ (10) $E\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}:1,0,1\right)$.

PROOF OF CLAIM. Let $\alpha^x\beta^y\sigma^u$ be some element in $\Pi_1(F)$, then

$$(\alpha^x\beta^y\sigma^u)\alpha^{n_{i1}}\beta^{n_{i2}}(\alpha^x\beta^y\sigma^u)^{-1}=\begin{cases}\alpha^{n_{i1}}\beta^{n_{i2}} & (u:\text{even})\\ \alpha^{-n_{i1}}\beta^{-n_{i2}} & (u:\text{odd})\end{cases}$$
$$(\alpha^x\beta^y\sigma^u)\alpha^p\beta^q\sigma(\alpha^x\beta^y\sigma^u)^{-1}=\begin{cases}\alpha^{p+2x}\beta^{q+2y}\sigma & (u:\text{even})\\ \alpha^{-p+2x}\beta^{-q+2y}\sigma & (u:\text{odd}).\end{cases}$$

Therefore it's sufficient to take

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as N and to take

$$(0,0),\quad (1,0),\quad (0,1),\quad (1,1)$$

as (p,q).

The case when $N=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Let f_1, f_2, g and $h\in \mathrm{Aut}(\Pi_1(F))$ be the elements corresponding to $(N:0,1,1)$, $(N:1,1,1)$, $\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:0,0,1\right)$ and $\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}:0,0,1\right)$ respectively. Then g^{-1} and h^{-1} correspond to $\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:0,0,1\right)$ and $\left(\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}:0,0,1\right)$ respectively. Therefore

$$\begin{array}{lll} g^{-1}f_1g(\alpha)=\alpha, & g^{-1}f_1g(\beta)=\beta, & g^{-1}f_1g(\sigma)=\alpha\sigma \\ h^{-1}f_2h(\alpha)=\alpha, & h^{-1}f_2h(\beta)=\beta, & h^{-1}f_2h(\sigma)=\alpha\sigma. \end{array}$$

Thus

$$E(N:1,0,1)\cong_b E(N:0,1,1)\cong_b E(N:1,1,1).$$

Each cases corresponding to $N=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ can be proven similarly, so the claim is proven.

Then we shall return to the proof of Proposition 2.5. It is sufficient to show that the fundamental groups of the ten fiber bundles in the claim are not isomorphic one another.

When $N=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ or $\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$, E has the structure of T^3-bundle over S^1. For example $N=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, by taking $\langle\alpha,\beta,\tau\rangle$ to be the fiber direction and $\langle\sigma\rangle$ to be the base direction, it is regarded as T^3-bundle over S^1.

$$E\left(\{0:(n_2,2)\}:\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}:p,q,1\right)\cong E\left(T^3:-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}:p,q\right).$$

When $N=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, it is regarded as T^3-bundle over S^1 if we take $\langle\alpha,\beta,\sigma\tau^2\rangle$ as the fiber direction and $\langle\sigma\rangle$ to be the base direction.

$$E\left(\{0:(n_2,2)\}:\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}:p,q,1\right)\cong E\left(T^3:\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}:p,q\right).$$

When $N=\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$, it is regarded as T^3-bundle over S^1 if we take $\langle\alpha,\beta,\sigma\tau^3\rangle$ as the fiber direction and $\langle\sigma\rangle$ to be the base direction.

$$E\left(\{0:(n_2,2)\}:\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}:p,q,1\right)\cong E\left(T^3:\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}:p,q\right).$$

Proposition 2.4 tells us that in these cases the fundamental groups are not isomorphic one another.

On the other hand, when $N=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, it can not be T^3-bundle over S^1, because the centralizer G' of G in $\Pi_1(E)$ is generated by α,β,σ^2 and τ^2, so there exist no element of G' which is transformed into the prime element of $\Pi_1(E)\to H_1(E)/torsion\cong Z^2$. And in this cases the calculation shows the following table.

Table 1.

	$H_1(E)$	$H_2(E)$	center of $\Pi_1(E)$	Abel $(\Pi_1(E)/center)$
(6)	$Z^2\oplus(Z/2)^2$	$Z\oplus(Z/2)^3$	Z^2	$(Z/2)^4$
(7)	$Z^2\oplus Z/2$	$Z\oplus(Z/2)^3$	Z^2	$(Z/2)^3$
(8)	$Z^2\oplus Z/2$	$Z\oplus(Z/2)^3$	Z^2	$Z/2\oplus Z/4$
(9)	$Z^2\oplus Z/2$	$Z\oplus(Z/2)^2$	Z^2	$(Z/2)^3$
(10)	Z^2	$Z\oplus(Z/2)^2$	Z^2	$Z/2\oplus Z/4$

In every case $H_3(E)$ is $Z/2$, and $H_4(E)$ vanishes.

From the results above, the fundamental groups in the cases (6), (7), (8), (9), (10) are not isomorphic one another, and proof of Proposition 2.5 is accomplished. □

We prove the following proposition in connection with Proposition 2.4.

PROPOSITION 2.6. *The following two assertions are equivalent.*

(1) $E(\{0:(n_2,2)\}:N:p,q,\varepsilon)\cong_{(b)}E(\{0:(n_2,2)\}:N':p',q',\varepsilon')$

(2) $\varepsilon=\varepsilon'$ *and* $\exists A\in GL(2,Z)$ *such that*

(i) $ANA^{-1}=\pm N'$ *or* $ANA^{-1}=\pm N'^{-1}$

(ii) $(p',q')A-(p,q)\in\left\langle N-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}:\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\right\rangle.$

PROOF. First we remark that $\varepsilon=1$ implies $b_1(E)=2$ and $\varepsilon=-1$ implies $b_1(E)=1$. In considering bundle isomorphism, the degree of freedom in $\mathrm{Aut}(\Pi_1(F))$ consists of inner automorphism, conjugate in $\mathrm{Aut}(\Pi_1(F))$ and making inverse element.

For any element $\alpha^x\beta^y\sigma^z\in\Pi_1(F)$, we have

$$(\alpha^x\beta^y\sigma^z)\alpha^l\beta^m(\alpha^x\beta^y\sigma^z)^{-1}=\begin{cases}\alpha^l\beta^m & (z:\text{even})\\ \alpha^{-l}\beta^{-m} & (z:\text{odd})\end{cases}$$

$$(\alpha^x\beta^y\sigma^z)\alpha^p\beta^q\sigma^\varepsilon(\alpha^x\beta^y\sigma^z)^{-1}=\begin{cases}\alpha^{p+2x}\beta^{q+2y}\sigma^\varepsilon & (z:\text{even})\\ \alpha^{-p+2x}\beta^{-q+2y}\sigma^\varepsilon & (z:\text{odd}).\end{cases}$$

Thus if $N'\pm N=0$ and $(p,q)-(p',q')\in\left\langle\begin{pmatrix}2&0\\0&2\end{pmatrix}\right\rangle$, then

$$E(N:p,q,\varepsilon)\cong_{(b)}E(N':p',q',\varepsilon). \qquad \cdots(\mathrm{A})$$

Let $f\in\mathrm{Aut}(\Pi_1(F))$ (resp. $g\in\mathrm{Aut}(\Pi_1(F))$) be the element corresponding to $N=(n_{ij}),p,q,\varepsilon$ (resp. $A=(a_{ij}),l,m,\delta$). And if g^{-1} can be considered to correspond to $A^{-1}=(a'_{ij}),x,y,\delta$, then as $g^{-1}g(\sigma)=\sigma$, we have

$$(l,m)A^{-1}+(x,y)=0$$

$$\begin{aligned}g^{-1}fg(\sigma)&=g^{-1}f(\alpha^l\beta^m\sigma^\delta)\\&=g^{-1}(\alpha^{n_{11}l+n_{21}m+p}\beta^{n_{12}l+n_{22}m+q}\sigma^{\varepsilon\delta})\\&=\alpha^{a'_{11}(n_{11}l+n_{21}m+p)+a'_{21}(n_{12}l+n_{22}m+q)+x}\beta^{a'_{12}(n_{11}l+n_{21}m+p)+a'_{22}(n_{12}l+n_{22}m+q)+y}\sigma^\varepsilon.\end{aligned}$$

Thus $g^{-1}fg$ is the automorphism corresponding to ANA^{-1}, $\left\{(l,m)\left(N-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)+(p,q)\right\}A^{-1}$, ε. And we know from this the next matter.

If there exist $A\in GL(2,Z)$ such that

(i) $N'=ANA^{-1}$

(ii) $(p',q')A-(p,q)\in\left\langle N-\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\rangle$

then

$$E(N:p,q,\varepsilon)\cong_{(b)}E(N':p',q',\varepsilon). \qquad \cdots(\mathrm{B})$$

As f^{-1} corresponds to N^{-1}, $-(p,q)N^{-1}$, ε, when $N'=N^{-1}$, $(p',q')+(p,q)N^{-1}=(0,0)$, we see that

$$E(N:p,q,\varepsilon)\cong_{(b)}E(N':p',q',\varepsilon). \qquad \cdots(\mathrm{C})$$

Summing up (A), (B) and (C), Proposition 2.6 is obtained. □

Finally we consider the manifolds which have both the structure of fiber bundle over S^1 with fiber T^3 and the structure of fiber bundle over S^1 with fiber $\{0:(n_2,2)\}$.

PROPOSITION 2.7. *The following two assertions are equivalent*

(1) $E(\{0:(n_2,2)\}:N:p,q,\varepsilon)$ *has the structure as bundle over* S^1 *with fiber* T^3.

(2) $\varepsilon=1$ *and* N *or* $-N$ *is periodic and its period is not equal to* 2. *And in this case*

if $N=\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$ *then*

$$E(\{0:(n_2,2)\}:N:p,q,1)\cong E\left(T^3:\begin{pmatrix}-1&0\\0&-1\end{pmatrix}:p,q\right)$$

if $N\neq\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$ *then*

$$E(\{0:(n_2,2)\}:N:p,q,1)\cong E(T^3:N:p,q).$$

PROOF. We think in the similar way as Proposition 2.5. If N is not periodic, then the centralizer of α,β in $\Pi_1(E)$ is generated by α,β and σ^2. There exist no element which is transformed into prime element of $\Pi_1(E)\to H_1(E)/torsion\cong Z^2$. Therefore it has not the structure of T^3-bundle over S^1. The case when N is periodic is already mentioned in the proof of Proposition 2.5. □

3. Aspherical cases

We shall treat fiber bundles having aspherical small Seifert manifolds as fibers that are not in the range of §2.

1° The case when $F=\{b>0:(o_1,1)\}$

In this case, F is principal S^1-bundle over T^2.

$$\Pi_1(F)=\langle\alpha,\beta,\gamma\mid[\alpha,\gamma]=[\beta,\gamma]=1,\ [\alpha,\beta]=\gamma^b\rangle.$$

Here we can express arbitrary $f\in\mathrm{Aut}(\Pi_1(F))$ as follows:

$$f(\alpha)=\alpha^{n_{11}}\beta^{n_{12}}\gamma^{p} \qquad f(\beta)=\alpha^{n_{21}}\beta^{n_{22}}\gamma^{q} \qquad f(\gamma)=\gamma^{\varepsilon}$$
$$N:=(n_{ij})\in Gl(2,Z),\ \ p,q\in Z,\ \ \varepsilon=\det N.$$

By means of these relations, $\mathrm{Aut}(\Pi_1(F))$ corresponds to $GL(2,Z)\times Z^2$ by one to one. We shall denote the fiber bundle corresponding to f by $E(\{b:(o_1,1)\}:N:p,q)$ and we write $E(N:p,q)$ for short.

$$\Pi_1(E(N:p,q))=\left\langle \begin{matrix}\alpha,\beta\\ \gamma,\tau\end{matrix} \;\middle|\; \begin{matrix}[\alpha,\gamma]=[\beta,\gamma]=1,\ \ [\alpha,\beta]=\gamma^{b}\\ \tau\alpha\tau^{-1}=f(\alpha),\ \ \tau\beta\tau^{-1}=f(\beta),\ \ \tau\gamma\tau^{-1}=f(\gamma)\end{matrix}\right\rangle.$$

PROPOSITION 3.1. *The following two assertions are equivalent*

(1) $E(\{b:(o_1,1)\}:N:p,q)\cong_b E(\{b:(o_1,1)\}:N':p',q')\quad (b\neq 0)$

(2) $\exists A\in GL(2,Z)$ *such that*

(i) $A^{-1}NA=N'$ *or* $A^{-1}NA=N'^{-1}$

(ii) $(p,q)-(p',q')^{t}A\in\left\langle\begin{pmatrix}b&0\\0&b\end{pmatrix},\ {}^{t}N-(\det N)\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\rangle.$

PROOF. It is sufficient to consider inner automorphism, conjugate and inverse in $\mathrm{Aut}(\Pi_1(F))$. Arbitrary element in $\Pi_1(F)$ can be written uniquely as $\alpha^{x}\beta^{-y}\gamma^{z}$ $(x,y,z\in Z)$. By calculation, we have

$$(\alpha^{x}\beta^{-y}\gamma^{z})(\alpha^{n_{11}}\beta^{n_{12}}\gamma^{p})(\alpha^{x}\beta^{-y}\gamma^{z})^{-1}=\alpha^{n_{11}}\beta^{n_{12}}\gamma^{p+(n_{11}y+n_{12}x)b}$$
$$(\alpha^{x}\beta^{-y}\gamma^{z})(\alpha^{n_{21}}\beta^{n_{22}}\gamma^{q})(\alpha^{x}\beta^{-y}\gamma^{z})^{-1}=\alpha^{n_{21}}\beta^{n_{22}}\gamma^{q+(n_{21}y+n_{22}x)b}.$$

Let $g\in\mathrm{Aut}(\Pi_1(F))$ be the element corresponding to $B\in GL(2,Z)$, $x,y\in Z$, then we see the following. If we set

$$g^{-1}fg(\alpha)=\alpha^{m_{11}}\beta^{m_{12}}h^{l}$$
$$g^{-1}fg(\beta)=\alpha^{m_{21}}\beta^{m_{22}}h^{m}$$

then

$$\begin{pmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{pmatrix}=BNB^{-1}$$
$$\begin{pmatrix}p\\q\end{pmatrix}-(\det B)B^{-1}\begin{pmatrix}l\\m\end{pmatrix}=\left(N-(\det N)\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)B^{-1}\begin{pmatrix}x\\y\end{pmatrix}+u\begin{pmatrix}b\\0\end{pmatrix}+v\begin{pmatrix}0\\b\end{pmatrix}$$

for some $u,v\in Z$.

If f is the element of $\mathrm{Aut}(\Pi_1(F))$ that corresponds to N,p,q, then the inverse f^{-1} corresponds to N^{-1}, r, s such that $\begin{pmatrix}r\\s\end{pmatrix}+(\det N)N^{-1}\begin{pmatrix}p\\q\end{pmatrix}\in$

$\left\langle\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\right\rangle$. Summing up the results above, we have Proposition 3.1. $\square$

Needless to say, when $b_1(E)=b_1(E')=1$, Proposition 3.1 gives the classification in terms of diffeomorphism.

Next we will give the diffeomorphic classification of fiber bundles of which the 1st Betti numbers are greater than 1, and criterion for bundle isomorphism.

If $b_1(E)=2$ for $E=E(N:p,q)$, we may assume that N has the following form.

$$(*) \qquad \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad n>0.$$

PROPOSITION 3.2. *Let N and N' be the matrices of the form $(*)$. Then we have the following criteria.*

(1) $E(\{b:(o_1,1)\}:N:p,q)\cong_b E(\{b:(o_1,1)\}:N',p',q')$ $(b\neq 0)$ *if and only if*

(i) $N=N'=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ *and there exists* $l\in Z$ *such that*

$$(p,q)\pm(p',q')\begin{pmatrix} \pm 1 & 0 \\ l & 1 \end{pmatrix}\in\left\langle (n,0):\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\right\rangle$$

(ii) $N=N'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $(p,q)\pm(p',q')\in\left\langle (2,0):\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\right\rangle$

(iii) $N=N'=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and* $p'-q'\equiv\pm(p-q)$ *mod* b.

(2) $E(\{b:(o_1,1)\}:N:p,q)\cong E(\{b':(o_1,1)\}:N',p',q')$ $(b,b'\neq 0)$ *if and only if*

(i) $N=N'=\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ *and* $g.c.d.(b,q)=g.c.d.(b',q')$ *and* $p'\pm p\equiv 0$ *mod* $g.c.d.(b,q,n)$

(ii) $N=N'=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $g.c.d.(b,q)=g.c.d.(b',q')=:k$ *and* $\dfrac{b}{k}\equiv\dfrac{b'}{k}$ *mod* 2

when $\dfrac{b}{k}$ *is even,* $p\equiv p'$ *mod* 2

when $\dfrac{b'}{k}$ *is odd,* $p\equiv p'$ *mod* $g.c.d.(2,k)$

(iii) $N=N'=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ *and* $g.c.d.(b, p-q)=g.c.d.(b', p'-q')$.

PROOF. For (1) we get it using Proposition 3.1 and the fact that $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ $n>0$, $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ are not conjugate to each other in $GL(2, Z)$, and inverse of each matrix is conjugate to itself.

For (2) first we remark

$$\Pi_1(E)=\left\langle \begin{matrix}\alpha, \beta\\ \gamma, \tau\end{matrix} \middle| \begin{matrix}[\alpha, \gamma]=[\beta, \gamma]=[\tau, \gamma]=1, \ [\alpha, \beta]=\gamma^b\\ \tau\alpha\tau^{-1}=\alpha\beta^n\gamma^p, \ \tau\beta\tau^{-1}=\beta\gamma^q\end{matrix}\right\rangle.$$

If we set $k=g.c.d.(b, q)$, $x=\frac{q}{k}$ and $y=-\frac{b}{k}$, then x and y are prime each other. If we choose l and m so that $xm-ly=1$, E has the structure as T^3-bundle over S^1 by regarding $\langle\beta, \gamma, \alpha^x\tau^y\rangle$ to be the fiber direction and $\langle\alpha^l\tau^m\rangle$ to be the S^1 direction. By calculation, we have the following

$$E\left(\{b: (o_1, 1)\}: \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}: p, q\right)\cong E\left(T^3: \begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}: n, nb-p\right).$$

Similarly if we set $g.c.d.(b, q)=k$ when $N=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, then

when $\frac{b}{k}$ is even

$$E\left(\{b: (o_1, 1)\}: \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}: p, q\right)\cong E\left(T^3: -\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}: 0, p\right)$$

when $\frac{b}{k}$ is odd

$$E\left(\{b: (o_1, 1)\}: \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}: p, q\right)\cong E\left(\{0: (n_2, 2)\}: \begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}: 0, p, 1\right).$$

As $b\neq 0$ so $k\neq 0$, thus if $\frac{b}{k}$ is odd then E cannot be T^3-bundle over S^1.

If we set $k=g.c.d.(b, q-p)$ when $N=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, then we have

when $\frac{b}{k}$ is even

$$E\left(\{b:(o_1,1)\}:\begin{pmatrix}0&1\\1&0\end{pmatrix}:p,q\right)\cong E\left(T^3:-\begin{pmatrix}1&k\\0&1\end{pmatrix}:1,p\right)$$

when $\frac{b}{k}$ is odd

$$E\left(\{b:(o_1,1)\}:\begin{pmatrix}0&1\\1&0\end{pmatrix}:p,q\right)\cong E\left(\{0:(n_2,2)\}:\begin{pmatrix}1&k\\0&1\end{pmatrix}:-1,-p\right).$$

Here $k\neq 0$ also, thus if $\frac{b}{k}$ is odd then E cannot be the T^3-bundle over S^1.

We can finish the proof by using Proposition 2.4 and Proposition 2.6. □

Next we consider the case when $b_1(E)=3$ namely $N=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

PROPOSITION 3.3.

(1) $E\left(\{b:(o_1,1)\}:\begin{pmatrix}1&0\\0&1\end{pmatrix}:p,q\right)\cong_b E\left(\{b:(o_1,1)\}:\begin{pmatrix}1&0\\0&1\end{pmatrix}:p',q'\right)$ $(b\neq 0)$

if and only if $g.c.d.(p,q)\equiv\pm g.c.d.(p',q')$ *mod* b.

(2) $E\left(\{b:(o_1,1)\}:\begin{pmatrix}1&0\\0&1\end{pmatrix}:p,q\right)\cong E\left(\{b':(o_1,1)\}:\begin{pmatrix}1&0\\0&1\end{pmatrix}:p',q'\right)$

if and only if $g.c.d.(b,p,q)=g.c.d.(b',p',q')$.

PROOF. We can obtain (1) by using Proposition 3.1. For (2), if we set $k=g.c.d.(b,q)$, $x=\frac{q}{k}$ and $y=-\frac{b}{k}$, then x and y are prime each other and E has the structure as T^3-bundle over S^1 having $\langle\alpha^x\tau^y,\beta,\gamma\rangle$ as the fiber direction. We accomplish the proof of the proposition by utilizing Proposition 2.4. □

REMARK. By synthesizing Proposition 2.2, Proposition 3.2 and Proposition 3.3, we can conclude for the fiber bundles with fixed fiber $\{b:(o_1)\}$ $(b=0,1,2,3,4,6)$, there is no difference between bundle isomorphism and diffeomorphism. When b is not in this range, there is some difference between bundle isomorphism and diffeomorphism even if the fiber is fixed.

2° The case when $F=\{b>0:(n_2,2)\}$

In this case, F has the structure as T^2-bundle over S^1. $\left(\text{Monodromy is } \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}.\right)$

$$\Pi_1(F)=\langle \alpha,\beta,\sigma:\ [\alpha,\beta]=1,\ \ \sigma\alpha\sigma^{-1}=\alpha^{-1}\beta^{-b},\ \ \sigma\beta\sigma^{-1}=\beta^{-1}\rangle.$$

Here arbitrary element $f\in \mathrm{Aut}(\Pi_1(F))$ can be written as follows

$$f(\alpha)=\alpha^{\rho}\beta^{n},\ \ f(\beta)=\beta^{\varepsilon\rho},\ \ f(\sigma)=\alpha^{p}\beta^{q}\sigma^{\varepsilon} \qquad \rho,\varepsilon=\pm 1,\ \ n,p,q\in Z.$$

By the above equations, we can make one to one correspondence between $\mathrm{Aut}(\Pi_1(F))$ and $Z\times\{\pm 1\}\times\{\pm 1\}\times Z^2$ $((n,\rho,\varepsilon,p,q)\leftrightarrow f)$. Here the element with $\rho=-1$ can be modified into that with $\rho=1$ by means of inner automorphism. Therefore we shall consider only the ones with $\rho=1$. We set $f\in \mathrm{Aut}(\Pi_1(F))$ the element corresponding to $(n,1,\varepsilon,p,q)$, and denote the fiber bundle E corresponding to f by $E(\{b:(n_2,2):n,\varepsilon,p,q)$. We write $E(n,\varepsilon,p,q)$ for short if no confusion may occur. We remark that if $\varepsilon=1$ (resp. $\varepsilon=-1$) then $b_1(E)=2$ (resp. $b_1(E)=1$).

PROPOSITION 3.4. *The fiber bundle over S^1 with fiber $\{b>0:(n_2,2)\}$ is bundle isomorphic to one of the following.*

(1) *When b is even and $b_1(E)=1$,*

1) $E(1,-1,0,0)$	2) $E(1,-1,1,0)$	3) $E(0,-1,0,0)$
4) $E(0,-1,1,0)$	5) $E(0,-1,0,1)$	6) $E(0,-1,1,1)$.

(2) *When b is odd and $b_1(E)=1$,*

7) $E(1,-1,0,0)$	8) $E(1,-1,1,0)$	9) $E(1,-1,0,1)$
10) $E(1,-1,1,1)$	11) $E(0,-1,0,0)$	12) $E(0,-1,1,0)$.

(3) *When $b_1(E)=2$,*

13) $E(n,1,0,0)$	14) $E(n,1,1,0)$	15) $E(n,1,0,1)$ $(0\leq n\leq b$ *and* n : *even*)
16) $E(n,1,0,0)$	17) $E(n,1,1,0)$	$(0<n\leq b$ *and* n : *odd*).

PROOF. First we think of inner automorphism. In order to keep

$\rho=1$, we treat the inner automorphism by $\alpha^x\beta^y\sigma^{2z}$. Then we have

$$(\alpha^x\beta^y\sigma^{2z})\alpha\beta^n(\alpha^x\beta^y\sigma^{2z})^{-1}=\alpha\beta^{n+2zb}$$
$$(\alpha^x\beta^y\sigma^{2z})\beta^\varepsilon(\alpha^x\beta^y\sigma^{2z})^{-1}=\beta^\varepsilon$$
$$(\alpha^x\beta^y\sigma^{2z})\alpha^p\beta^q\sigma^\varepsilon(\alpha^x\beta^y\sigma^{2z})^{-1}=\alpha^{p+2x}\beta^{2zbp+q+2y+\varepsilon xb}\sigma^\varepsilon.$$

Therefore we may assume that $0\leq n<2b$, $p, q\in Z/2$.

The case when $\varepsilon=-1$.

If $g\in\mathrm{Aut}(\Pi_1(F))$ is the element such that $g(\alpha)=\alpha\beta^m$, $g(\beta)=\beta$ and $g(\sigma)=\sigma$, then $g^{-1}fg(\alpha)=\alpha\beta^{n-2m}$. Thus we may assume that n, p and q are all 0 or 1. Let h be the automorphism corresponding to $(0, 1, -1, 0)$, conjugate by h tells the following

$$E(1, -1, p, 1)\cong_b E(1, -1, p, 0) \quad (\text{when } b \text{ is even})$$
$$E(0, -1, p, 1)\cong_b E(0, -1, p, 0) \quad (\text{when } b \text{ is odd}).$$

It can be concluded that when $\varepsilon=-1$, it is bundle isomorphic to either (1) or (2). They don't transform one another by conjugate in $\mathrm{Aut}(\Pi_1(F))$, inner automorphism or inverse. Thus they are not bundle isomorphic one another.

The case when $\varepsilon=1$.

If $f\in\mathrm{Aut}(\Pi_1(F))$ is an element corresponding to $(n, 1, p, q)$, then f^{-1} corresponds to $(-n, 1, -p, np-q)$. Therefore we may assume that $0\leq n\leq b$. The rest of the proof is similar to the case when $\varepsilon=-1$. □

Notice the fact that the necessary and sufficient condition for bundle isomorphism cannot be written in a simple form when $\{b>0:(n_2, 2)\}$ is the fiber. Next we will give the classification by means of diffeomorphism for $\varepsilon=1$.

PROPOSITION 3.5.

$$E(\{b:(n_2, 2)\}:n, 1, p, q)\cong E(\{b':(n_2, 2)\}:n', 1, p', q')$$

if and only if $k:=g.c.d.(n, b)=g.c.d.(n', b')$ *and* $\dfrac{n}{k}\equiv\dfrac{n'}{k}$ *(mod 2) and*

(i) *when* $\dfrac{n}{k}$ *and* $\dfrac{n'}{k}$ *are even,*

$$(p', q')\pm(p, q)\begin{pmatrix}\pm1 & \alpha\\ 0 & 1\end{pmatrix}\in\left\langle\begin{pmatrix}2 & k\\ 0 & 2\end{pmatrix}\right\rangle \quad \textit{for some } \alpha\in Z$$

(ii) *when* $\frac{n}{k}$ *and* $\frac{n'}{k}$ *are odd,*

$$(p', q') \pm (p, q)\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in \left\langle (0, k) : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle \quad \textit{for some } \alpha \in Z.$$

PROOF. When $\frac{n}{k}$ is even, we have

$$E(\{b : (n_2, 2)\} : n, 1, p, q) \cong E\left(T^3 : -\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : q, p\right).$$

When $\frac{n}{k}$ is odd, we have

$$E(\{b : (n_2, 2)\} : n, 1, p, q) \cong E\left(\{0 : (n_2, 2)\} : \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : q, -p\right).$$

We obtain the result by combining these facts with Proposition 2.4 and Proposition 2.6. □

REMARK. In the case when fixed fiber is $\{b : (n_2, 2)\}$ ($|b| \leq 3$), there is no difference between bundle isomorphism and diffeomorphism. But otherwise diffeomorphism is different from bundle isomorphism.

3° The case when $F = \{1 : (o_2, 1)\}$

In this case, F is non-orientable S^1-bundle over T^2 and is also T^2-bundle over S^1. We think of three types of representations of $\Pi_1(F)$ A, B, C.

$$\begin{aligned} A &: \ \Pi_1(F) = \langle \alpha, \beta, \gamma \mid \alpha\gamma\alpha^{-1} = \gamma^{-1}, \beta\gamma\beta^{-1} = \gamma^{-1}, [\alpha, \beta] = \gamma \rangle \\ B &: \ \Pi_1(F) = \langle \alpha, \beta, \gamma \mid \alpha\gamma\alpha^{-1} = \gamma^{-1}, [\beta, \gamma] = 1, [\alpha, \beta] = \gamma \rangle \\ C &: \ \Pi_1(F) = \langle \alpha, \beta, \gamma \mid [\alpha, \gamma] = 1, \beta\gamma\beta^{-1} = \gamma^{-1}, [\alpha, \beta] = \gamma \rangle. \end{aligned}$$

In the representation of A, $f \in \mathrm{Aut}(\Pi_1(F))$ can be expressed by the following equalities.

$$f(\alpha) = \alpha^{n_{11}}\beta^{n_{12}}\gamma^{p} \quad f(\beta) = \alpha^{n_{21}}\beta^{n_{22}}\gamma^{q} \quad f(\gamma) = \gamma^{\varepsilon}$$
$$\text{where } N = (n_{ij}) \in GL(2, Z).$$

In this situation we denote the fiber bundle corresponding to f by $E(\{1 : (o_2, 1)\} : A : N : p, q, \varepsilon)$. We write $E(A : N : p, q, \varepsilon)$ for short, if no confusion occurs.

$$\Pi_1(E(A:N:p,q,\varepsilon))=\left\langle \begin{matrix}\alpha,\beta\\ \gamma,\tau\end{matrix} \;\middle|\; \begin{matrix}\alpha\gamma\alpha^{-1}=\gamma^{-1}, & \beta\gamma\beta^{-1}=\gamma^{-1}, & [\alpha,\beta]=\gamma\\ \tau\alpha\tau^{-1}=f(\alpha), & \tau\beta\tau^{-1}=f(\beta), & \tau\gamma\tau^{-1}=f(\gamma)\end{matrix} \right\rangle.$$

In the sequel, we consider the case when the 1st Betti number is larger than 1. By making the conjugate in $\mathrm{Aut}(\Pi_1(F))$, we may assume that N is one of the following three matrices.

$$\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} n\geq 0,\quad \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

If n is odd in $N=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ and the expression of $\Pi_1(F)$ is type A or C, then automorphism of $\Pi_1(F)$ cannot be constructed. That is to say it is not well defined. Therefore we only think of type B. If $n\neq 0$ is even in $N=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$, then A, B and C are all possible and in this case if $f\in\mathrm{Aut}(\Pi_1(F))$ has expression N and type C then f has also expression N and type A, but even if f has expression of type A (resp. B), f cannot have expression of type B (resp. A), thus we only think of type A and B. By similar way of thinking, we consider B only for $N=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, A, B and C for $N=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, and A only for $N=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

PROPOSITION 3.6. *When $b_1(E)\geq 2$, the fiber bundle E over S^1 with fiber $\{1:(o_2,1)\}$ is bundle isomorphic to one of the following. They are not diffeomorphic one another.*

(1) $E\left(A:\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}:0,0,1\right)$ $\quad n\neq 0$: *even*

(2) $E\left(B:\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}:0,0,1\right)$ $\quad n\neq 0$: *even*

(3) $E\left(B:\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}:0,0,1\right)$ $\quad n$: *odd*

(4) $E\left(A:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:0,0,1\right)$

(5) $E\left(B:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:0,0,-1\right)$

(6) $E\left(C:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:0,0,1\right)$

$$(7)\quad E\left(A:\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:0,0,-1\right)$$

$$(8)\quad E\left(B:\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}:0,0,1\right).$$

PROOF. Let $f\in \mathrm{Aut}(\Pi_1(F))$ be the element corresponding to $\left(A:\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}:p,q,\varepsilon\right)$ $n\neq 0$: even, then we have

$$\begin{aligned}[f(\alpha),f(\beta)]&=\alpha\beta^n\gamma^p\beta\gamma^q\gamma^{-p}\beta^{-n}\alpha^{-1}\gamma^{-q}\beta^{-1}\\ &=\alpha\beta\alpha^{-1}\beta^{-1}\gamma^{2q-2p}=\gamma^{1-2p+2q}.\end{aligned}$$

Therefore there are only two choices, i.e. $\varepsilon=1$ and $p=q$, or $\varepsilon=-1$ and $p=q+1$. If $\varepsilon=-1$, then we can assume $\varepsilon=1$ by means of inner automorphism induced by α. We may assume $p=q=1$ or $p=q=0$ by means of inner automorphism induced by γ. Moreover we have

$$(\alpha\beta)\alpha\beta^n\gamma(\alpha\beta)^{-1}=\alpha\beta^n \quad (\alpha\beta)\beta\gamma(\alpha\beta)^{-1}=\beta \quad (\alpha\beta)\gamma(\alpha\beta)^{-1}=\gamma.$$

Thus all the automorphisms of $\Pi_1(F)$ with $N=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ (n : even) can be considered to be the product of $\left(A:\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}:0,0,1\right)$ and inner automorphism. The other cases also can be transformed into desirable forms.

Next we intend to show that the cases from (1) to (8) are not diffeomorphic to one another.

So we list up the first homology group $H_1(E)$, the center of the fundamental group, the abelianization $\mathrm{Abel}(\Pi_1(E)/center)$, and the kernel of the mapping $\Pi_1(E)\to H_1(E)/torsion$ in the following table.

Table 2.

	$H_1(E)$	center	$\mathrm{Abel}(\Pi_1(E)/center)$	kernel
(1)	$Z^2\oplus Z/n$	Z	$Z^2\oplus Z/2$	$\Pi_1(K)$
(2)	$Z^2\oplus Z/n$	Z	$Z^2\oplus Z/2$	Z^2
(3)	$Z^2\oplus Z/n$	Z	Z^2	Z^2
(4)	$Z^2\oplus Z/2$	Z^2	$(Z/2)^3$	$\Pi_1(K)$
(5)	$Z^2\oplus Z/2$	Z^2	$(Z/2)^3$	Z^2
(6)	$Z^2\oplus Z/2$	Z^2	$(Z/2)^3$	$\Pi_1(K)$
(7)	Z^2	Z^2	$Z/4\oplus Z/2$	Z^2
(8)	Z^3	Z^3	$(Z/2)^2$	Z

Further consideration is needed for the distinction between (4) and (6). In (4), the center of $\Pi_1(E)$ is generated by α^2 and τ^2, and all the elements in the center can be written as a square of some element of $\Pi_1(E)$. On the other hand in (6) the element $\alpha^2\gamma$ in the center can not be a square of any element of $\Pi_1(E)$. Indeed, arbitrary element of $\Pi_1(E)$ can be written uniquely in the form of $\alpha^x\beta^y\gamma^u\tau^v$ $(x, y, u, v \in Z)$. Here $\ker(\Pi_1(E)\to H_1(E)/torsion)$ is generated by β and γ, so if $(\alpha^x\beta^y\gamma^u\tau^v)^2 = \alpha^2\gamma$, then $v=0$ and $x=1$. When y is even,

$$(\alpha\beta^y\gamma^u)^2 = \alpha\beta^y\alpha\beta^y\gamma^{2u} = \alpha^2\beta^{2y}\gamma^{2u}.$$

When y is odd,

$$(\alpha\beta^y\gamma^u)^2 = \alpha\beta^y\alpha\beta^y = \alpha\beta\alpha\beta^{2y-1} = \alpha^2\beta^{2y}\gamma^{-1}.$$

Thus $\alpha^2\gamma$ can not be a square of any element of $\Pi_1(E)$, and we have proven that the cases from (1) to (8) are not diffeomorphic one another. $\square$

The cases (2), (3) and (8) have the structure of T^3-bundle over S^1. The cases (5) and (7) have the structure of $\{0:(n_2, 2)\}$-bundle over S^1. We give the representations of them in the sequel.

Cases (2), (3) and (8)

$$E\left(\{1:(o_2, 1)\} : B : \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : 0, 0, 1\right) \cong E\left(T^3 : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : 0, n\right).$$

Case (5)

$$E\left(\{1:(o_2, 1)\} : B : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : 0, 0, -1\right) \cong E\left(\{0:(n_2, 2)\} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : 0, 0, 1\right).$$

Case (7)

$$E\left(\{1:(o_2, 1)\} : A : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : 0, 0, -1\right) \cong E\left(\{0:(n_2, 2)\} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : -1, 0, 1\right).$$

The cases (1), (4) and (6) have neither the structure of T^3-bundle over S^1 nor the structure $\{0:(n_2, 2)\}$-bundle over S^1.

Here we produce the invariant of the group which is used to show that (4) is not diffeomorphic to (6).

DEFINITION. Let G be a group. We define an invariant of the group G, $s(G) \in Z/2$ as follows.

$$s(G)=\begin{cases} 0: & \text{any element of the center of } G \text{ can be expressed as the square of the element of } G. \\ 1: & \text{there exist some element of the center of } G \text{ that can not be the square of the element of } G. \end{cases}$$

In the cases (4) and (6), we have the following

$$s\left(\Pi_1\left(E\left(A:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:0,0,1\right)\right)\right)=0$$

$$s\left(\Pi_1\left(E\left(C:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:0,0,1\right)\right)\right)=1.$$

4° The case when $F=\{0:(o_2,1)\}$

In this case, F is a non-orientable S^1-bundle over T^2 and it can also be regarded as T^2-bundle over S^1 or $K\times S^1$. As in the case of $\{1:(o_2,1)\}$, we consider three types A, B and C representations of $\Pi_1(F)$.

$$\begin{aligned} A:&\ \Pi_1(F)=\langle\alpha,\beta,\gamma\mid \alpha\gamma\alpha^{-1}=\gamma^{-1},\beta\gamma\beta^{-1}=\gamma^{-1},[\alpha,\beta]=1\rangle\\ B:&\ \Pi_1(F)=\langle\alpha,\beta,\gamma\mid \alpha\gamma\alpha^{-1}=\gamma^{-1},[\beta,\gamma]=[\alpha,\beta]=1\rangle\\ C:&\ \Pi_1(F)=\langle\alpha,\beta,\gamma\mid [\alpha,\gamma]=1,\beta\gamma\beta^{-1}=\gamma^{-1},[\alpha,\beta]=1\rangle. \end{aligned}$$

In the representation of A, $f\in\mathrm{Aut}(\Pi_1(F))$ can be written in the following equalities,

$$f(\alpha)=\alpha^{n_{11}}\beta^{n_{12}}\gamma^{p},\quad f(\beta)=\alpha^{n_{21}}\beta^{n_{22}}\gamma^{q},\quad f(\gamma)=\gamma^{\varepsilon}$$
$$\text{where } N:=(n_{ij})\in GL(2,Z).$$

In this situation $(A:N:p,q,\varepsilon)$ corresponds to f and we denote fiber bundle E corresponding to f by $E(\{0:(o_2,1)\}:A:N:p,q,\varepsilon)$. If no confusion takes place, we write $E(A:N:p,q,\varepsilon)$ for short.

$$\Pi_1(E(A:N:p,q,\varepsilon))=\left\langle \begin{matrix}\alpha,\beta\\ \gamma,\tau\end{matrix}\ \middle|\ \begin{matrix}\alpha\gamma\alpha^{-1}=\gamma^{-1},\beta\gamma\beta^{-1}=\gamma^{-1},[\alpha,\beta]=1\\ \tau\alpha\tau^{-1}=f(\alpha),\tau\beta\tau^{-1}=f(\beta),\tau\gamma\tau^{-1}=f(\gamma)\end{matrix}\right\rangle.$$

In the case when $b_1(E)\geq 2$, we have only to consider the following matrices as N

$$\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\ n\geq 0,\quad \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Similar to the case of $\{1:(o_2,1)\}$, when $N=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ and n is a non-

zero even integer we only think of the type A and B, when n is odd we consider the type B only, when $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we consider the type A, B and C, when $N=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we consider A only, and when $N=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we consider B only respectively.

PROPOSITION 3.7. *When $b_1(E)\geq 2$, the fiber bundle E over S^1 with fiber $\{0:(o_2,1)\}$ is bundle isomorphic to one of the following* 16 *fiber bundles, and they are not diffeomorphic to one another.*

$$\left.\begin{array}{ll}
(1) & E\left(A:\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}:0,0,1\right) \\
(2) & E\left(A:\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}:1,1,1\right) \\
(3) & E\left(B:\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}:0,0,1\right) \\
(4) & E\left(B:\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}:1,0,1\right)
\end{array}\right\} n\neq 0 : \textit{even}$$

$$\left.\begin{array}{ll}
(5) & E\left(B:\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}:0,0,1\right) \\
(6) & E\left(B:\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}:1,0,1\right)
\end{array}\right\} n : \textit{odd}$$

$$\begin{array}{ll}
(7) \quad E\left(A:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0,0,1\right) & (8) \quad E\left(A:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:1,1,1\right) \\
(9) \quad E\left(B:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0,0,1\right) & (10) \quad E\left(B:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:1,0,1\right) \\
(11) \quad E\left(C:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0,0,1\right) & (12) \quad E\left(C:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0,1,1\right) \\
(13) \quad E\left(A:\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}:0,0,1\right) & (14) \quad E\left(A:\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}:1,1,1\right) \\
(15) \quad E\left(B:\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}:0,0,1\right) & (16) \quad E\left(B:\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}:1,0,1\right).
\end{array}$$

PROOF. $f\in \mathrm{Aut}(\Pi_1(F))$ be the element corresponding to $\left(A:\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}:\right.$

$p, q, \varepsilon\Big)$ ($n \neq 0$ is even), then we have

$$[f(\alpha), f(\beta)] = \alpha\beta^n\gamma^p\beta\gamma^q\gamma^{-p}\beta^{-n}\alpha^{-1}\gamma^{-q}\beta^{-1} = \gamma^{2q-2p}.$$

Thus $p=q$. If $\varepsilon=-1$, then we can assume that $\varepsilon=1$ by means of inner automorphism induced by α. Furthermore we may assume $p=q=0$ or $p=q=1$ by means of inner automorphism induced by γ. The other cases can be transformed into the desirable form by means of inner automorphism also.

Next we show that the bundles (1) to (16) are not diffeomorphic to one another. We show the 1st homology group and other groups for each cases in the following table. And we get Proposition 3.7. □

Table 3.

	H_1	center	Abel($\Pi_1(E)/center$)	kernel
(1)	$Z^2 \oplus Z/2 \oplus Z/n$	Z	$Z^2 \oplus (Z/2)^2$	$\Pi_1(K)$
(2)	$Z^2 \oplus Z/n$	Z	$Z^2 \oplus Z/2$	$\Pi_1(K)$
(3)	$Z^2 \oplus Z/2 \oplus Z/n$	Z	$Z^2 \oplus (Z/2)^2$	Z^2
(4)	$Z^2 \oplus Z/2n$	Z	$Z^2 \oplus Z/2$	Z^2
(5)	$Z^2 \oplus Z/2n$	Z	$Z^2 \oplus Z/2$	Z^2
(6)	$Z^2 \oplus Z/2n$	Z	Z^2	Z^2
(7)	$Z^2 \oplus (Z/2)^2$	Z^2	$(Z/2)^4$	$\Pi_1(K)$
(8)	$Z^2 \oplus Z/2$	Z^2	$(Z/2)^3$	$\Pi_1(K)$
(9)	$Z^2 \oplus (Z/2)^2$	Z^2	$(Z/2)^4$	Z^2
(10)	$Z^2 \oplus Z/2$	Z^2	$(Z/2)^3$	Z^2
(11)	$Z^2 \oplus (Z/2)^2$	Z^2	$(Z/2)^3$	$\Pi_1(K)$
(12)	$Z^2 \oplus Z/4$	Z^2	$Z/4 \oplus Z/2$	$\Pi_1(K)$
(13)	$Z^2 \oplus Z/2$	Z^2	$Z \oplus (Z/2)^2$	Z^2
(14)	$Z^2 \oplus Z/2$	Z^2	$Z/4 \oplus Z/2$	Z^2
(15)	$Z^3 \oplus Z/2$	Z^3	$(Z/2)^2$	Z
(16)	Z^3	Z^3	$(Z/2)^2$	Z

REMARK. $H_2(E) \cong Z \oplus Z/2 \oplus Z/4$ for (8).

The cases (3), (4), (5), (6), (15) and (16) are T^3-bundles over S^1, (9), (10), (13) and (14) are $\{0 : (n_2, 2)\}$-bundles over S^1. The expressions of these bundles are given below.

Cases (3), (4), (5), (6), (15) and (16). ($p=0, 1$)

$$E\left(\{0:(o_2,1)\}:B:\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}:p,0,1\right)\cong E\left(T^3:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:n,p\right).$$

Cases (9) and (10). $(p=0,1)$

$$E\left(\{0:(o_2,1)\}:B:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:p,0,1\right)\cong E\left(\{0:(n_2,2)\}:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:0,p,1\right).$$

Case (13).

$$E\left(\{0:(o_2,1)\}:A:\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:0,0,1\right)\cong E\left(\{0:(n_2,2)\}:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:1,0,1\right).$$

Case (14).

$$E\left(\{0:(o_2,1)\}:A:\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:1,1,1\right)\cong E\left(\{0:(n_2,2)\}:\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:1,1,1\right).$$

The bundles (1), (2), (7), (8), (11) and (12) have neither the structure of T^3-bundle over S^1 nor $\{0:(n_2,2)\}$-bundle over S^1.

5° The case when $F=\{1:(n_3,2)\}$

In this case, F has the structure of K-bundle over S^1.

$$\Pi_1(F)=\langle\alpha,\beta,\sigma\mid \alpha\beta\alpha^{-1}=\beta^{-1},\ \sigma\alpha\sigma^{-1}=\alpha^{-1}\beta,\ \sigma\beta\sigma^{-1}=\beta^{-1}\rangle.$$

We set $f_i\in\mathrm{Aut}(\Pi_1(F))$ $(i=1,\cdots,8)$ as follows

$$f_1:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\sigma^{-1}\end{cases}\quad f_2:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\sigma^{-1}\end{cases}\quad f_3:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\beta^{-1}\sigma^{-1}\end{cases}$$

$$f_4:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\beta^{-1}\sigma^{-1}\end{cases}\quad f_5:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\sigma\end{cases}\quad f_6:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\sigma\end{cases}$$

$$f_7:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\beta^{-1}\sigma\end{cases}\quad f_8:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\beta^{-1}\sigma.\end{cases}$$

We shall denote the fiber bundle corresponding to f_i by $E(\{1:(n_3,2)\}:f_i)$. If there is no possibility of confusion, we write $E(f_i)$ for short.

$$\Pi_1(E(f_i))=\left\langle \begin{matrix}\alpha,\beta\\ \sigma,\tau\end{matrix} \;\middle|\; \begin{matrix}\alpha\beta\alpha^{-1}=\beta^{-1}, & \sigma\alpha\sigma^{-1}=\alpha^{-1}\beta, & \sigma\beta\sigma^{-1}=\beta^{-1}\\ \tau\alpha\tau^{-1}=f_i(\alpha), & \tau\beta\tau^{-1}=f_i(\beta), & \tau\sigma\tau=f_i(\sigma)\end{matrix}\right\rangle.$$

PROPOSITION 3.8. *The fiber bundle E over S^1 with fiber $\{1:(n_3,2)\}$ is isomorphic to one of the bundles $E(f_i)$ $(i=1,\cdots,8)$ and these bundles are not diffeomorphic to one another.*

PROOF. Any element $f\in \mathrm{Aut}(\Pi_1(F))$ can be written in the following form.

$$f(\alpha)=\alpha^{\varepsilon}\beta^{x} \qquad f(\beta)=\beta^{\delta} \qquad f(\sigma)=\alpha^{2p}\beta^{q}\sigma^{\gamma}$$
$$\varepsilon,\delta,\gamma=\pm 1, \quad p,q,x\in Z, \quad 2x+2q+\delta=1.$$

If $\varepsilon=-1$, then we may assume $\varepsilon=1$ by means of inner automorphism induced by σ. If $\delta=-1$, then δ can be transformed into 1 in keeping $\varepsilon=1$, by means of inner automorphism induced by α. We may assume $p=1$ or 0 by means of inner automorphism induced by α^2, and $x=-q=1$ or 0 by means of inner automorphism induced by β. Hence we have only to consider f_i $(i=1,\cdots,8)$.

It is easy to verify that f_i cannot be transformed into the other one by inner automorphism and $f_i^{-1}=f_i$ in $\mathrm{Out}(\Pi_1(F))$. If we set

$$f(\alpha)=\alpha\beta^{x} \qquad f(\beta)=\beta \qquad f(\sigma)=\alpha^{2p}\beta^{-x}\sigma^{\gamma}$$
$$g(\alpha)=\alpha^{\varepsilon'}\beta^{x'} \qquad g(\beta)=\beta^{\delta'} \qquad g(\sigma)=\alpha^{2p'}\beta^{q'}\sigma^{\gamma'}$$

then we have

$$g^{-1}fg(\alpha)=\alpha\beta^{\delta x} \qquad g^{-1}fg(\beta)=\beta \qquad g^{-1}fg(\sigma)=\alpha^{2\varepsilon' p}\beta^{-\delta' x}\sigma^{\gamma}.$$

Therefore E is bundle isomorphic to only one of the bundles $E(f_i)$ $(i=1,\cdots,8)$. When $b_1(E)=1$, classification by bundle isomorphism means that by diffeomorphism, so we shall show that if $b_1(E)=2$ then $E(f_5)$,

Table 4.

	$H_1(E)$	center	Abel($\Pi_1(E)$/*center*)	$s(\Pi_1(E))$
$E(f_5)$	$Z^2\oplus Z/4$	Z^2	$Z/2\oplus Z/4$	
$E(f_6)$	$Z^2\oplus Z/2$	Z^2	$(Z/2)^3$	0
$E(f_7)$	$Z^2\oplus Z/2$	Z^2	$(Z/2)^3$	1
$E(f_8)$	$Z^2\oplus Z/2$	Z^2	$(Z/2)^3$	0

$H_2(E(f_6))\cong Z\oplus Z/2\oplus Z/4$
$H_2(E(f_8))\cong Z\oplus Z/2\oplus Z/2$

$E(f_6)$, $E(f_7)$, $E(f_8)$ are not diffeomorphic to one another. We show the 1st homology group and others in Table 4 and thus $E(f_i)$ are not diffeomorphic to one another. Then the proof is complete. □

6° The case when $F=\{0:(n_3,2)\}$

In this case also, F is K-bundle over S^1.

$$\Pi_1(F)=\langle\alpha,\beta,\sigma \mid \alpha\beta\alpha^{-1}=\beta^{-1},\ \sigma\alpha\sigma^{-1}=\alpha^{-1},\ [\sigma,\beta]=1\rangle.$$

We set $f_i\in\mathrm{Aut}(\Pi_1(F))$ $(i=1,\cdots,8)$ as follows.

$$f_1:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\sigma^{-1}\end{cases}\qquad f_2:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\sigma^{-1}\end{cases}\qquad f_3:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\sigma^{-1}\end{cases}$$

$$f_4:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\sigma^{-1}\end{cases}\qquad f_5:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\sigma\end{cases}\qquad f_6:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\sigma\end{cases}$$

$$f_7:\begin{cases}\alpha\mapsto\alpha\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\sigma\end{cases}\qquad f_8:\begin{cases}\alpha\mapsto\alpha\beta\\ \beta\mapsto\beta\\ \sigma\mapsto\alpha^2\sigma.\end{cases}$$

We shall denote the fiber bundle corresponding to f_i by $E(\{0:(n_3,2)\}:f_i)$. If there is no possibility of confusion, we write $E(f_i)$ for short.

$$\Pi_1(E(f_i))=\left\langle\begin{matrix}\alpha,\beta\\ \sigma,\tau\end{matrix}\;\middle|\;\begin{matrix}\alpha\beta\alpha^{-1},\ \sigma\alpha\sigma^{-1}=\alpha^{-1},\ [\sigma,\beta]=1\\ \tau\alpha\tau^{-1}=f_i(\alpha),\ \tau\beta\tau^{-1}=f_i(\beta),\ \tau\sigma\tau^{-1}=f_i(\sigma)\end{matrix}\right\rangle.$$

PROPOSITION 3.9. *The fiber bundle E over S^1 with fiber $\{0:(n_3,2)\}$ is bundle isomorphic to one of the bundles $E(f_i)$ $(i=1,\cdots,8)$ and there are not diffeomorphic to one another.*

Table 5.

	$H_1(E)$	center	$\mathrm{Abel}(\Pi_1(E)/center)$	$s(\Pi_1(E))$
$E(f_5)$	$Z^2\oplus(Z/2)^2$	Z^2	$(Z/2)^3$	
$E(f_6)$	$Z^2\oplus Z/2$	Z^2	$(Z/2)^3$	1
$E(f_7)$	$Z^2\oplus(Z/2)^2$	Z^2	$(Z/2)^4$	
$E(f_8)$	$Z^2\oplus Z/2$	Z^2	$(Z_2)^3$	0
$H_2(E(f_8))\cong Z\oplus(Z/2)^2$				

PROOF. Similar to the case of $\{1:(n_3, 2)\}$, we can show that E is bundle isomorphic to one of the bundles $E(f_i)$. We show H_1 and so on for the case of $b_1(E)=2$ i.e. $E(f_5), E(f_6), E(f_7)$ and $E(f_8)$ on Table 5. Thus they are not diffeomorphic to one another. $\square$

4. Main results in the aspherical cases.

We shall summarize the results from § 1 to § 4 and mention some applications in this section. Let F and F' be aspherical small Seifert manifolds. Let E and E' be fiber bundles over S^1 with fiber F and F' respectively. Moreover we denote the Seifert obstruction class of those fibers by b. ($b \in Z$ and $b \geq 0$, or $b \in Z/2$.) The invariant $s(G)$ is already defined in § 3.

THEOREM 1. (1) *If* $b_1(E)=b_1(E')=1$, *or* $F \cong F'$ *and* $b=0, 1, 2, 3$, *or* $F \cong F' \cong \{4:(o_1, 1)\}$ *or* $\{6:(o_1, 1)\}$, *then the following three assertions are equivalent*

$$\text{(a)} \quad \Pi_1(E) \cong \Pi_1(E') \qquad \text{(b)} \quad E \cong E' \qquad \text{(c)} \quad E \cong_b E'.$$

(2) *If* $F \cong F' = \{b \neq 0 : (o_1, 2)\}$, *then the following two assertions are equivalent*

(a) $E(F:N:p, q) \cong_b E(F':N':p', q')$

(b) *there exist some* $A \in GL(2, Z)$ *such that*

(i) $A^{-1}NA = N'$ *or* $A^{-1}NA = N'^{-1}$

(ii) $(p, q) - (p', q')^t A \in \left\langle \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, {}^tN - (\det N)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle.$

(3) *If* $F \cong F' = \{b \neq 0 : (n_2, 2)\}$ *and* $b_1(E)=b_1(E')=2$, *the bundle isomorphism is reduced to the following three types.*

(a) *If there exist* $\varepsilon \in \{\pm 1\}$ *s.t.*

$$n' \equiv n \mod 2b$$

$$(p', q') - (p, q)\begin{pmatrix} 1 & \frac{1-\varepsilon}{2}(n+b) \\ 0 & 1 \end{pmatrix} \in \left\langle \begin{pmatrix} 2 & b \\ 0 & 2 \end{pmatrix} \right\rangle$$

then $E(n, 1, p, q) \cong_b E(n', 1, p', q')$.

(b) *If* $0 \leq n \leq b$, *then* $E(n, 1, 1, 1) \cong_b E(n, 1, 1, 0)$.

(c) *If* $0 \leq n \leq b$ *and* n *is odd, then* $E(n, 1, 0, 1) \cong_b E(n, 1, 0, 0)$.

REMARK. All the cases when $F \cong F'$ are non-orientable are included in (1).

If $b_1(E) \geq 2$ then $\ker(\Pi_1(E) \to H_1(E)/torsion)$ with Z^2 or $\Pi_1(K)$ or Z or trivial.

THEOREM 2. *$E \cong E'$ if and only if $\Pi_1(E) \cong \Pi_1(E')$ and more precisely,*
(1) *If* $\mathrm{Ker}(\Pi_1(E) \to H_1(E)/torsion) \cong Z$, *then the diffeomorphism type of E is determined uniquely by $H_*(E)$.*
(2) *If* $\mathrm{Ker}(\Pi_1(E) \to H_1(E)/torsion) \cong \Pi_1(K)$, *then the diffeomorphism type of E is determined by $H_*(E)$, center of $\Pi_1(E)$,* $\mathrm{Abel}(\Pi_1(E)/center)$, *and* $s(\Pi_1(E))$.
(3) *If* $\mathrm{Ker}(\Pi_1(E) \to H_1(E)/torsion) \cong Z^2$, *then the diffeomorphism type of E is determined by the three procedure below*
(i) *to give the representation as the bundle over S^1.*
(ii) *to transform the representation into that of T^3-bundle over S^1 or fiber bundle over S^1 with fiber $\{0:(n_2, 2)\}$.*
(iii) *to apply Propositions* 2.4, 2.6, 2.7.
(4) *If* $\mathrm{Ker}(\Pi_1(E) \to H_1(E)/torsion)$ *is trivial, then $E \cong T^4$.*

REMARK. In (3), it is easy to give the representation of E as a bundle over S^1. The transformation of representation into that of T^3-bundle over S^1 or $\{0:(n_2, 2)\}$ bundle over S^1 is already given in § 3.

PROOF OF THEOREM 2. As (1), (3) and (4) are shown in § 2 and § 3, it is necessary to show (2) only. The cases for (2) are exhausted in Tables 2, 3, 4 and 5.

The case when $F = \{1:(n_3, 2)\}$ (Table 4).

Setting the fiber direction $\langle \alpha, \beta, \tau \rangle$ and base direction $\langle \sigma \rangle$, we introduce the new fiber bundle structure over S^1. Then we have fiber $\{0:(o_2, 1)\}$ for $E(f_5)$ and $E(f_6)$, and fiber $\{1:(o_2, 1)\}$ for $E(f_7)$ and $E(f_8)$. We can derive following relations from Tables 2, 3 and 4.

$$E(\{1:(n_3, 2)\}:f_5) \cong E\left(\{0:(o_2, 1)\}:C:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0, 1, 1\right)$$

$$E(\{1:(n_3, 2)\}:f_6) \cong E\left(\{0:(o_2, 1)\}:A:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:1, 1, 1\right)$$

$$E(\{1:(n_3, 2)\}:f_7) \cong E\left(\{1:(o_2, 1)\}:C:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0, 0, 1\right)$$

$$E(\{1:(n_3, 2)\}:f_8) \cong E\left(\{1:(o_2, 1)\}:A:\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}:0, 0, 1\right).$$

The case when $F=\{0:(n_3,2)\}$ (Table 5).

Setting the fiber direction $\langle\alpha,\beta,\tau\rangle$ and base direction $\langle\sigma\rangle$, we introduce the new fiber bundle structure over S^1. Then we have fiber $\{0:(o_2,1)\}$ for $E(f_5)$ and $E(f_6)$, and fiber $\{1:(o_2,1)\}$ for $E(f_7)$ and $E(f_8)$. We can derive following relations from Tables 2, 3 and 5.

$$E(\{0:(n_3,2)\}:f_5)\cong E\left(\{0:(o_2,1)\}:C:\begin{pmatrix}1&0\\0&-1\end{pmatrix}:0,0,1\right)$$
$$E(\{0:(n_3,2)\}:f_6)\cong E\left(\{1:(o_2,1)\}:C:\begin{pmatrix}1&0\\0&-1\end{pmatrix}:0,0,1\right)$$
$$E(\{0:(n_3,2)\}:f_7)\cong E\left(\{0:(o_2,1)\}:A:\begin{pmatrix}1&0\\0&-1\end{pmatrix}:0,0,1\right)$$
$$E(\{1:(n_3,2)\}:f_8)\cong E\left(\{1:(o_2,1)\}:A:\begin{pmatrix}1&0\\0&-1\end{pmatrix}:0,0,1\right).$$

Therefore it is only necessary to consider the case when $F=\{b:(o_2,1)\}$ $b\in Z/2$. $E\left(\{1:(o_2,1)\}:A:\begin{pmatrix}1&n\\0&1\end{pmatrix}:0,0,1\right)$ and $E\left(\{0:(o_2,1)\}:A:\begin{pmatrix}1&n\\0&1\end{pmatrix}:1,1,1\right)$ $(n\neq 0:\text{even})$ remain to be undistinguishable in Tables 2 and 3. Setting the fiber direction $\langle\beta,\gamma,\tau\rangle$ and base direction $\langle\alpha\rangle$, $E\left(\{1:(o_2,1)\}:A:\begin{pmatrix}1&n\\0&1\end{pmatrix}:0,0,1\right)$ can be regarded as fiber bundle over S^1 with fiber $\{0:(o_2,1)\}$. Therefore

$$E\left(\{1:(o_2,1)\}:A:\begin{pmatrix}1&n\\0&1\end{pmatrix}:0,0,1\right)\cong E\left(\{0:(o_2,1)\}:A:\begin{pmatrix}1&n\\0&1\end{pmatrix}:1,1,1\right).$$

Thus the proof of Theorem 2 is completed. □

Here the classification of fiber bundle over S^1 with aspherical small Seifert manifold as fiber by means of bundle isomorphism and diffeomorphism is completed.

Theorem 3 in [13] tells that T^2-bundle over T^2 has structure of T^3-bundle over S^1 or $\{0:(n_2,2)\}$-bundle over S^1. As the base space of K-bundle over T^2 and K-bundle over K is S^1-bundle over S^1, they have the structure of fiber bundle over S^1 with fiber K-bundle over S^1. So we have the following corollary.

COROLLARY 1. *Let M and M' be either T^2- or K-bundle over T^2 or K-bundle over K, then the following two assertions are equivalent.*

$$(1)\quad \Pi_1(M)\cong\Pi_1(M') \qquad (2)\quad M\cong M'.$$

Three kinds of complex surfaces of class VII_0 with zero 2nd Betti number, that are neither elliptic surface nor Hopf surface, are known and they are called Inoue surfaces $S_M^{(\pm)}$, $S_{N,p,q,r,t}^{(+)}$, $S_{N,p,q,r}^{(-)}$ [7].

The 2nd Betti number of Inoue surface may be positive. Class VII_0 complex surfaces already known with positive Betti numbers that do not coincide with neither elliptic surface nor Hopf surface are diffeomorphic to $(S^3\times S^1)\#CP^2\#\cdots\#CP^2$([9]).

$S_M^{(+)}$ is diffeomorphic to $S_M^{(-)}$. If surface S satisfies $\Pi_1(S)\cong\Pi_1(S_M^{(+)})$, then S is known to be analytically isomorphic to $S_M^{(+)}$ or $S_M^{(-)}$ ([8]). But nothing is known about $S_{N,p,q,r,t}^{(+)}$ and $S_{N,p,q,r}^{(-)}$. We shall pay attention to topological properties of $S_M^{(\pm)}$, $S_{N,p,q,r,t}^{(+)}$ and $S_{N,p,q,r}^{(-)}$. $S_M^{(\pm)}$ is T^3 bundle over S^1 of which monodromy is M. $S_{N,p,q,r,t}^{(+)}$ and $S_{N,p,q,r}^{(-)}$ have the structure as fiber bundle over S^1 with fiber $\{r:(o_1,1)\}$. Here t of $S_{N,p,q,r,t}^{(+)}$ represents the deformation then $S_{N,p,q,r,t}^{(+)}\cong S_{N,p,q,r,t'}^{(+)}$ and we shall write $S_{N,p,q,r}$ for short neglecting the suffix "$\pm$" which denotes the signature of $\det N$. Then as a corollary of Theorem 1, we have the following.

COROLLARY 2. *The following three assertions are equivalent*

(1) $S_{N,p,q,r}\cong S_{N',p',q',r'}$

(2) $\Pi_1(S_{N,p,q,r})\cong\Pi_1(S_{N',p',q',r'})$

(3) $r=\pm r'$ *and there exists* $A\in GL(2,Z)$ *such that*

(i) $A^{-1}NA=N'$ *or* $A^{-1}NA=N'^{-1}$

(ii) $(p,q)-(p',q')^tA\in\left\langle\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix}:{}^tN-(\det N)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\rangle.$

5. Non-aspherical cases

Following manifolds are non-aspherical small Seifert manifolds.

S^3, lens spaces, $S^2\times S^1$, $S^2\tilde{\times}S^1$, $P^2\times S^1$, $RP^3\#RP^3$, prism manifolds, and Seifert manifolds with 3 singular fibers over S^2 of which orders are either $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$.

The results in this section owe to the results in [1], [2], [3], [4], [10] and [12]. If two diffeomorphisms are homotopic in the noted manifolds, then they are isotopic.

First we shall consider $S^2\times S^1$, $S^2\tilde{\times}S^1$ and $P^2\times S^1$. The fact that $\Pi_1(\mathrm{Diff}(S^2\times S^1))\cong Z/2\oplus Z/2\oplus Z/2$ is known. If we set

f_1: To rotate S^2 once along S^1.
$f_2 = f_2' \times \mathrm{id}_{S^1}$: f_2' is the mapping to reverse the orientation of S^2.
$f_3 = \mathrm{id}_{S^2} \times f_3'$: f_3' is the mapping to reverse the orientation of S^1,

then f_1, f_2 and f_3 generate $\Pi_1(\mathrm{Diff}(S^2 \times S^1))$. $\Pi_1(\mathrm{Diff}(S^2 \tilde{\times} S^1)) \cong Z/2 \oplus Z/2$ is also known. If we set

g_1 : To rotate S^2 once along S^1.
g_2 : To invert S^1 direction,

then g_1 and g_2 generate $\Pi_1(\mathrm{Diff}(S^2 \tilde{\times} S^1))$. Furthermore $\Pi_1(\mathrm{Diff}(P^2 \times S^1)) \cong Z/2 \oplus Z/2$ is known and it is generated by

h_1 : To rotate P^2 once along S^1.
$h_2 = \mathrm{id}_{P^2} \times f_3'$.

Denote the fiber bundle over S^1 with monodromy f by E_f, we show the homology groups and the types of intersection form (type I or II) on $Z/2$ in Table 6. It is easy to see that E_{f_2} and $E_{\mathrm{id}_{S^2} \tilde{\times} S^1}$ (resp. $E_{f_1 f_2}$ and E_{g_1}) are diffeomorphic. We cannot distinguish E_{h_2} from $E_{h_1 h_2}$ by Table 6 only, but the orientable double coverings of these are $E_{f_2 f_3}$ and $E_{f_1 f_2 f_3}$

Table 6.

	$H_1(E)$	$H_2(E)$	$H_3(E)$	$H_4(E)$	type
$E_{\mathrm{id}_{S^2 \times S^1}}$	Z^2	Z^2	Z^2	Z	II
E_{f_1}	Z^2	Z^2	Z^2	Z	I
E_{f_2}	Z^2	$Z \oplus Z/2$	$Z/2$	0	II
$E_{f_1 f_2}$	Z^2	$Z \oplus Z/2$	$Z/2$	0	I
E_{f_3}	$Z \oplus Z/2$	Z	$Z \oplus Z/2$	0	II
$E_{f_1 f_3}$	$Z \oplus Z/2$	Z	$Z \oplus Z/2$	0	I
$E_{f_2 f_3}$	$Z \oplus Z/2$	$Z/2$	Z	Z	II
$E_{f_1 f_2 f_3}$	$Z \oplus Z/2$	$Z/2$	Z	Z	I
$E_{\mathrm{id}_{S^2 \tilde{\times} S^1}}$	Z^2	$Z \oplus Z/2$	$Z/2$	0	II
E_{g_1}	Z^2	$Z \oplus Z/2$	$Z/2$	0	I
E_{g_2}	$Z \oplus Z/2$	$Z/2$	$Z/2$	0	II
$E_{g_1 g_2}$	$Z \oplus Z/2$	$Z/2$	$Z/2$	0	I
$E_{\mathrm{id}_{P^2 \times S^1}}$	$Z^2 \oplus Z/2$	$Z \oplus (Z/2)^2$	$Z/2$	0	II
E_{h_1}	$Z^2 \oplus Z/2$	$Z \oplus (Z/2)^2$	$Z/2$	0	I
E_{h_2}	$Z \oplus (Z/2)^2$	$(Z/2)^2$	$Z/2$	0	I
$E_{h_1 h_2}$	$Z \oplus (Z/2)^2$	$(Z/2)^2$	$Z/2$	0	I

respectively. So E_{h_2} is not homotopy equivalent to $E_{h_1h_2}$.

In the sequel we denote non-aspherical small Seifert manifolds by F or F', and fiber bundles over S^1 with fiber F (resp. F') by E (resp. E'). The main result in non-aspherical case is the following.

THEOREM 3. *Except for* $E_{f_2}\cong E_{\mathrm{id}_{S^2\tilde{\times}S^1}}$ *and* $E_{f_1f_2}\cong E_{g_1}$, *the following three assertions hold.*

(1) $E\cong E'$ *if and only if* $E\cong_b E'$.

(2) *When* F *is not lens space,* $E\cong_{(b)}E'$ *if and only if* $E\simeq E'$.

(3) *When* $F\cong F'$, $E\cong_{(b)}E'$ *if and only if* $E\simeq E'$.

REMARK. In (2), if $E\simeq E'$ then we have $F\cong F'$ and $E\cong_{(b)}E'$. Therefore (3) mentions the case when $F\cong F'$ is a lens space essentially.

The rest of § 5 will be devoted to the proof of Theorem 3. Before starting the proof, some preparations will be necessary. When $b_1(F)\geq 1$, F coincides with either $S^2\times S^1$, $S^2\tilde{\times}S^1$ or $P^2\times S^1$, all the cases when $b_1(E)\geq 2$ are listed in Table 6. We only have to show Theorem 3 when $b_1(E)=b_1(E')=1$.

LEMMA 5.1. *Under the assumption that* $b_1(E)=b_1(E')=1$,

(1) $E\simeq E'$ *leads to* $F\simeq F'$.

(2) $E\cong E'$ *leads to* $F\cong F'$.

PROOF. As $b_1(E)=b_1(E')=1$, the infinite cyclic covering of E and E' is uniquely determined to be $F\times R$ and $F'\times R$ respectively. If $E\simeq E'$ then homotopy equivalence map lifts up and $F\simeq F'$ is shown. Especially if F is not a lens space, then $F\cong F'$. If $E\cong E'$ then the diffeomorphism lifts up in $F\times R$ and $F'\times R$, so F and F' are h-cobordant and then we have $F\cong F'$. $\square$

According to Lemma 5.1, we may fix fiber F. In the sequel we assume that $F=F'$ and set the monodromy of E (resp. E') to be f (resp. f') and $b_1(E)=b_1(E')=1$. The following lemma is quite evident in this situation.

LEMMA 5.2. (1) $E\simeq E'$ *if and only if there exists an auto homotopy equivalence* $g: F\to F$ *such that* $gf'^{\pm 1}\simeq fg$.

(2) $E\cong_b E'$ *if and only if there exists* $g\in \mathrm{Diff}(E)$ *such that* $gf'^{\pm 1}\simeq fg$.

COROLLARY 5.3. *When the diffeotopy of* F *is at most* $Z/2$, *then the following two assertions are equivalent.*

$$(1)\quad E\simeq E' \qquad (2)\quad E\cong_b E'.$$

PROOF OF THEOREM 3. The cases when fiber is $S^2\times S^1$, $S^2\tilde{\times}S^1$ or $P^2\times S^1$ are already treated. It is known that the diffeotopies of S^3 and Seifert manifolds with 3 singular fibers over S^2 of which orders are $(2,3,4)$ and $(2,3,5)$ is at most $Z/2$. ([2], [3]) Therefore if these manifolds are fibers in $E\simeq E'$, then $E\cong_b E'$ is concluded, i.e. in this case Theorem 3 holds.

When the order of singular fiber over S^2 is $(2,3,3)$, then the diffeotopies of $\{b:(o_1,0):(2,1),(3,1),(3,2)\}$ and $\{-1:(o_1,0):(2,1),(3,1),(3,1)\}$ are $Z/2$ and that of $\{b\neq -1:(o_1,0):(2,1),(3,1),(3,1)\}$ is $Z/2\oplus Z/2$ ([3]). If we denote

$$\Pi_1(\{b_1\neq -1:(o_1,0):(2,1),(3,1),(3,1)\})=\left\langle \begin{matrix} q_1, q_2 \\ q_3, h \end{matrix} \middle| \begin{matrix} [q_i,h]=1 & q_1^2h=1 & q_2^3h=1 \\ q_3^3h=1 & q_1q_2q_3=h^b & \end{matrix} \right\rangle$$

then the diffeotopies in this case are generated by auto diffeomorphism f and g such that

$$f_*(q_1)=q_2^{-1}q_1^{-1}q_2 \quad f_*(q_2)=q_2^{-1} \quad f_*(q_3)=q_3^{-1} \quad f_*(h)=h^{-1}$$
$$g_*(q_1)=q_1 \quad g_*(q_2)=q_3 \quad g_*(q_3)=q_3^{-1}q_2q_3 \quad g_*(h)=h.$$

The homology groups of fiber bundles corresponding to id, f, g and fg are

$$\begin{aligned} &H_1(E_{\mathrm{id}})\cong Z\oplus Z/3(6b+7)\\ &H_1(E_f)\cong Z\\ &H_1(E_g)\cong Z\oplus Z/(6b+7)\\ &H_1(E_{fg})\cong Z\oplus Z/3. \end{aligned}$$

Therefore if singular fiber has order $(2,3,3)$, then the Theorem 3 holds.

The diffeotopy of $RP^3\# RP^3$ is known to be $Z/2\oplus Z/2$ ([10]). When we denote $\Pi_1(RP^3\# RP^3)=\langle\alpha,\beta\mid \alpha^2=\beta^2=1\rangle$, the diffeotopy is generated by the auto diffeomorphism f which preserves the orientation $f_*(\alpha)=\beta$ and $f_*(\beta)=\alpha$, and auto diffeomorphism g which reverses the orientation $g_*=\mathrm{id}$. The homology groups corresponding to each fiber bundles are

$$H_1(E_{\mathrm{id}})\cong H_1(E_g)\cong Z\oplus(Z/2)^2 \qquad H_1(E_f)\cong H_1(E_{fg})\cong Z\oplus Z/2$$
$$H_4(E_{\mathrm{id}})\cong H_4(E_f)\cong Z \qquad H_4(E_g)\cong H_4(E_{fg})\cong 0.$$

So the theorem is also valid in this case.

We will treat the case when the fiber is lens space $L_{p,q}$. Their diffeotopies are studied in [4]. $(p>q>0, g.c.d.(p,q)=1.)$

THEOREM (F. Bonahon [4]).

$$\Pi_0(\mathrm{Diff}(L_{p,q}))\cong\begin{cases} Z/2\oplus Z/2 & (q^2\equiv 1 \ (mod\ p)\ and\ q\not\equiv\pm 1\ (mod\ p)) \\ Z/4 & (q^2\equiv -1\ (mod\ p)\ and\ p\neq 2) \\ Z/2 & (otherwise). \end{cases}$$

Thus we have only to consider the case when $q^2\equiv 1$ (mod p) and $q\not\equiv\pm 1$ (mod p) and the case when $q^2\equiv -1$ (mod p) and $p\neq 2$. If we write $\Pi(L_{p,q})=H_1(L_{p,q})=\langle\alpha\,|\,\alpha^p=1\rangle$, then the diffeotopy is generated by auto diffeomorphisms f and g such that $f_*(\alpha)=\alpha^{-1}$, and $g_*(\alpha)=\alpha^{-q}$ when $q^2\equiv 1$ (mod p) and $q\not\equiv\pm 1$ (mod p). We will show that $E_{\mathrm{id}_{L_{p,q}}}$, E_f, E_g and E_{fg} are not homotopy equivalent to each other.

First we see that $E_{\mathrm{id}_{L_{p,q}}}$ is not homotopy equivalent to the others as in Corollary 5.3. If we assume $E_f\simeq E_g$, then from Lemma 5.2 the homotopy equivalence h exists such that $hf\simeq gh$. When $h_*(\alpha)=\alpha^x$ $(g.c.d.(p,x)=1)$, we have

$$h_*f_*(\alpha)=\alpha^{-x} \qquad g_*h_*(\alpha)=\alpha^{-qx}.$$

Therefore

$$x(1-q)\equiv 0 \quad (\mathrm{mod}\ p)$$

and this leads to the contradiction $q\equiv 1$ (mod p).

If we assume $E_f\simeq E_{fg}$, then we have the contradiction $q\equiv -1$ (mod p) similarly.

If we assume $E_g\simeq E_{fg}$, then we have $2q\equiv 0$ (mod p) similarly and $0<q<p$ leads to $2q=p$. On the other hand $q^2\equiv 1$ (mod p), so $\frac{p^2}{4}\equiv 1$ (mod p) and $4\equiv 0$ (mod p). Thus p must be 2 or 4. If $p=2$ we have the contradiction $q\equiv 1$ (mod p), and if $p=4$ then we have the contradiction $g.c.d.(p,q)=2$. Therefore $E_{\mathrm{id}_{L_{p,q}}}$, E_f, E_g and E_{fg} are not homotopy equivalent to each other.

Next we consider the case when $q^2\equiv -1$ (mod p) and $p\neq 2$. The diffeotopy is generated by f such that $f_*(\alpha)=\alpha^q$. As $f^{-1}\simeq f^3$, we have only to say that E_f is not homotopy equivalent to E_{f^2}.

If $E_f\simeq E_{f^2}$, then from Lemma 5.2 there exists a homotopy equivalence h of $L_{p,q}$ such that $hf\simeq f^2h$. If we set $h_*(\alpha)=\alpha^x$, then $qx\equiv -x$

(mod p). As $g.c.d.(x, p)=1$ we have $q\equiv -1$ (mod p) and this contradict $q^2\equiv -1$ (mod p). The above argument shows that Theorem 3 is valid for lens spaces also.

Finally we consider the case when the fiber is prism manifold. The diffeotopies of prism manifolds are studied in [1], [12]. Every prism manifold corresponds to the pair of integers $m>1$ and $n>1$ such that $g.c.d.(m, n)=1$, we will write $F_{m,n}$.

THEOREM (K. Asano [1] J. H. Rubinstein [12]).

$$\Pi_0(\mathrm{Diff}(F_{m,n}))\cong\begin{cases} Z/2\times Z/2 & (m\neq 2, n\neq 1)\\ Z/2 & (m\neq 2, n=1)\\ S_3\times Z/2 & (m=2, n\neq 1)\\ S_3 & (m=2, n=1).\end{cases}$$

Here we can write $\Pi_1(F_{m,n})=\langle\alpha, \beta \mid \beta^{-1}\alpha\beta=\alpha^{-1},\ \alpha^m\beta^{2n}=1\rangle$ and every element of $\Pi_1(F_{m,n})$ can be expressed in the form $\beta^x\alpha^y$ $0\leq x<2n$ $0\leq y<2m$ ([1]) and the order of β (resp. α) is $4n$ (resp. $2m$). When $m\neq 2$ and $n=1$, Theorem 3 holds according to Corollary 5.3. So we shall consider the other cases.

When $m=2$ and $n=1$.
The mapping below p is an isomorphism ([12]).

$$p:\Pi_0(\mathrm{Diff}(F_{2,1}))\xrightarrow{\cong}\mathrm{Aut}(H_1(F_{2,1}:Z/2)).$$

Setting $f, g\in\mathrm{Diff}(F_{2,1})$ and $E_f\simeq E_g$, then from Lemma 5.3 there exists an auto homotopy equivalence h of $F_{2,1}$ such that $hgh^{-1}\simeq f$. Let $\tilde{h}$ be the diffeomorphism which induces the same automorphism on $H_1(F_{2,1}:Z/2)$ that h induces, then $\tilde{h}g\tilde{h}^{-1}$ and f induce the same automorphism on $H_1(F_{2,1}:Z/2)$. This implies $E_f\cong_b E_g$. Thus Theorem 3 is valid when $m=2$ and $n=1$.

When $m\neq 2$ and $n\neq 1$.
We can write the elements $\mathrm{id}_{F_{m,n}}$, f_1, f_2 and f_3 of diffeotopy as follows ([12]).

$$\begin{aligned} f_{1*}(\alpha)&=\alpha^{-1} & f_{1*}(\beta)&=\beta^{-1}\\ f_{2*}(\alpha)&=\alpha & f_{2*}(\beta)&=\alpha\beta\\ f_{3*}(\alpha)&=\alpha^{-1} & f_{3*}(\beta)&=(\alpha\beta)^{-1}.\end{aligned}$$

We calculate the 1st homology groups corresponding to each element.

When m is even

$$H_1(E_{\mathrm{id}_{F_{m,n}}})\cong Z\oplus Z/2\oplus Z/2n$$
$$H_1(E_{f_1})\cong Z\oplus Z/2\oplus Z/2$$
$$H_1(E_{f_2})\cong Z\oplus Z/2n$$
$$H_1(E_{f_3})\cong Z\oplus Z/2.$$

When m is odd

$$H_1(E_{\mathrm{id}_{F_{m,n}}})\cong Z\oplus Z/4n$$
$$H_1(E_{f_1})\cong Z\oplus Z/2$$
$$H_1(E_{f_2})\cong Z\oplus Z/2n$$
$$H_1(E_{f_3})\cong\begin{cases} Z\oplus Z/2 & (n:\text{even})\\ Z\oplus Z/4 & (n:\text{odd}).\end{cases}$$

At this stage we can not distinguish E_{f_1} from E_{f_3} when m is odd and n is even (note that $n\neq 1$). For arbitrary element $\sigma\in\mathrm{Aut}(\Pi_1(F_{m,n}))$, we show that $f_{3*}\sigma\neq\sigma f_{1*}$ in $\mathrm{Out}(\Pi_1(F_{m,n}))$. We set

$$\sigma(\alpha)=\beta^{n_{11}}\alpha^{n_{12}}\qquad \sigma(\beta)=\beta^{n_{21}}\alpha^{n_{22}}$$
$$(0\leq n_{11},n_{21}<2n,\quad 0\leq n_{12},n_{22}<2m)$$

then we have

$$1=\sigma(\beta^{-1})\sigma(\alpha)\sigma(\beta)\sigma(\alpha)$$
$$=\begin{cases} (\beta^{n_{11}}\alpha^{n_{12}})^2=\sigma(\alpha^2) & (n_{11},n_{21}:\text{even})\\ \beta^{2n_{11}} & (n_{11}:\text{even}\quad n_{21}:\text{odd})\\ \beta^{2n_{11}}\alpha^{-2n_{22}} & (n_{11}:\text{odd},\quad n_{21}:\text{even})\\ \beta^{2n_{11}}\alpha^{2n_{12}-2n_{22}} & (n_{11},n_{21}:\text{odd}).\end{cases}$$

This implies that if n_{11} and n_{21} are even then $\alpha^2=1$ and this contradicts $m\geq 3$, and that if n_{11} is odd then $\beta^{2n_{11}}\alpha^l=1$ for some $l\in Z$ and this contradicts that n is even. Therefore n_{11} must be even and n_{21} odd. Here we have the following relations of $f_{3*}\sigma$ and σf_{1*}.

$$f_{3*}\sigma(\alpha)=\alpha^{-n_{12}}\qquad f_{3*}\sigma(\beta)=\beta^{-n_{21}}\alpha^{-n_{21}-1}$$
$$\sigma f_{1*}(\alpha)=\alpha^{-n_{12}}\qquad \sigma f_{1*}(\beta)=\beta^{-n_{21}}\alpha^{n_{22}}.$$

If $f_{3*}\sigma=\sigma f_{3*}$ in $\mathrm{Out}(\Pi_1(F_{m,n}))$ and we assume that $f_{3*}\sigma$ transforms into σf_{1*} by inner automorphism induced by $\beta^x\alpha^y$, we have

$$(\beta^x\alpha^y)(\beta^{-n_{21}}\alpha^{-n_{22}-1})(\beta^x\alpha^y)^{-1}=\begin{cases}\beta^{-n_{21}}\alpha^{-n_{22}-2y-1} & (x:\text{even})\\ \beta^{-n_{21}}\alpha^{n_{22}+2y+1} & (x:\text{odd})\end{cases}$$

and

$$\pm(n_{22}+2y+1)-n_{22}\equiv 0 \pmod{2m}.$$

The left side of the above equality is odd so this is a contradiction. We see the theorem is valid when $m\neq 2$ and $n\neq 1$.

When $m=2$ and $n\neq 1$.

We remark that n is odd. While $\Pi_0(\mathrm{Diff}(F_{2,n}))\cong S_3\times Z/2$, the S_3 part stands for the action on $H_1(F_{2,n}:Z/2)$ similar to $F_{2,1}$. Therefore we identify S_3 with $GL(2,Z/2)$. If we denote the element of $\Pi_0(\mathrm{Diff}(F_{2,n}))$ by (N,ε) $N\in GL(2,Z/2)$, $\varepsilon\in Z/2$, then the fiber bundle over S^1 with fiber $F_{2,n}$ is bundle isomorphic to one of those that corresponds to $f_1,\cdots,f_6$ $\in\Pi_0(\mathrm{Diff}(F_{2,n}))$.

$$f_1=\mathrm{id}_{F_{2,n}}\qquad f_2=\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},1\right)\qquad f_3=\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},0\right)$$

$$f_4=f_2f_3\qquad f_5=\left(\begin{pmatrix}1&1\\1&0\end{pmatrix},0\right)\qquad f_6=f_2f_5.$$

As in the case when $m=2$ and $n=1$, if the matrix part of $f_1,\cdots,f_6$ differ, then the corresponding fiber bundles are not homotopy equivalent. We have only to show that E_{f_1} and E_{f_2} (resp. E_{f_3} and E_{f_4}, E_{f_5} and E_{f_6}) are not homotopy equivalent.

As for E_{f_1} and E_{f_2}, while $f_1=\mathrm{id}_{F_{2,n}}$ they are not homotopy equivalent similar to the case in Corollary 5.3.

As for E_{f_5} and E_{f_6}, the order of f_5 in $\Pi_0(\mathrm{Diff}(F_{2,n}))$ is 3 and that of f_6 is 6. Let $g:F_{2,n}\to F_{2,n}$ be a homotopy equivalence and $gf_5^{\pm1}\simeq f_6g$, then we have $f_6\simeq gf_5^{\pm1}g^{-1}$ and then $f_6^3\simeq(gf_5^{\pm1}g^{-1})^3\simeq gf_5^{\pm3}g^{-1}\simeq\mathrm{id}_{F_{2,n}}$. Therefore f_6 is isotopic to $\mathrm{id}_{F_{2,n}}$ that is a contradiction. Therefore E_{f_5} and E_{f_6} are not homotopy equivalent.

As for E_{f_3} and E_{f_4}, we know that $f_{2*}(\alpha)=\alpha^{-1}$, $f_{2*}(\beta)=\beta^{-1}$. Furthermore we may assume

$$f_{3*}(\alpha)=\beta^x\alpha^y\qquad f_{3*}(\beta)=\beta^p\alpha^q$$

$$0\leq x,p<2n\quad 0\leq y,q<4\quad x,q:\text{odd}\quad y,p:\text{even}.$$

Because $f_{3*}(\beta^{-1}\alpha\beta\alpha)=\beta^{2x}\alpha^{-2q}=1$, $x=n$. Moreover

$$f_{4*}(\alpha)=f_{2*}f_{3*}(\alpha)=\beta^{-n}\alpha^{-y}$$

$$f_{4*}(\beta)=f_{2*}f_{3*}(\beta)=\beta^{-p}\alpha^{-q}.$$

We shall show that $f_{3*}g \neq gf_{4*}$ in $\mathrm{Out}(\Pi_1(F_{2,n}))$ for arbitrary $g \in \mathrm{Aut}(\Pi_1(F_{2,n}))$. If $f_{3*}g = gf_{4*}$ in $\mathrm{Out}(\Pi_1(F_{2,n}))$ and the isomorphism on $H_1(F_{2,n} : Z/2)$ induced by g be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore if we set

$$\begin{aligned} &g(\alpha) = \beta^{n_{11}}\alpha^{n_{12}} && g(\beta) = \beta^{n_{21}}\alpha^{n_{22}} \\ &(0 \leq n_{11}, n_{21} < 2n && 0 \leq n_{12}, n_{22} < 4) \end{aligned}$$

then n_{11} and n_{22} are even and n_{12} and n_{21} are odd, otherwise n_{11} and n_{22} are odd and n_{12} and n_{21} are even.

$$g(\beta^{-1}\alpha\beta\alpha) = g(\beta^{-1})g(\alpha)g(\beta)g(\alpha) = \beta^{2n_{11}} = 1 \qquad \text{and then } n_{11} = 0$$

i.e. we may write

$$g(\alpha) = \alpha^{n_{12}} \qquad g(\beta) = \beta^{n_{21}}\alpha^{n_{22}}.$$

Calculation tells us that

$$\begin{aligned} &f_{3*}g(\alpha) = \beta^{nn_{12}}\alpha^{y} && f_{3*}g(\beta) = \beta^{pn_{21}}\alpha^{n_{22}+qn_{21}} \\ &gf_{4*}(\alpha) = \beta^{-nn_{21}}\alpha^{n_{22}-yn_{12}} && gf_{4*}(\beta) = \beta^{-pn_{21}}\alpha^{-qn_{12}}. \end{aligned}$$

If we assume that the inner automorphism induced by $\beta^l\alpha^m$ transforms $f_{3*}g$ into gf_{4*}, then

$$(\beta^l\alpha^m)(\beta^{pn_{21}}\alpha^{n_{22}+qn_{21}})(\beta^l\alpha^m)^{-1} = \beta^{pn_{21}}\alpha^{\pm(n_{22}+qn_{21})} = \beta^{-pn_{21}}\alpha^{-qn_{12}}.$$

Thus $2pn_{21} \equiv 0 \pmod{2n}$. As n is odd and p is even, $pn_{21} \equiv 0 \pmod{2n}$.

Here $f_{3*}g$ is an isomorphism, so there exist s and t such that

$$f_{3*}g(\beta^s\alpha^t) = \beta.$$

While

$$\begin{aligned} f_{3*}g(\beta^s\alpha^t) &= (\beta^{pn_{21}}\alpha^{n_{22}+qn_{21}})^s(\beta^{nn_{12}}\alpha^{y})^t \\ &= \begin{cases} \beta^{spn_{21}+tnn_{12}}\alpha^{s(n_{22}+qn_{21})} & (t : \text{even}) \\ \beta^{spn_{21}+tnn_{12}}\alpha^{-s(n_{22}+qn_{21})+y} & (t : \text{odd}). \end{cases} \end{aligned}$$

If t is even then $spn_{21}+tnn_{12}$ is also even, and this cannot be β. If t is odd then $spn_{21}+tnn_{12}=n+2nk$ for some $k\in Z$, and this cannot be β either. This leads to contradiction.

When n_{11} and n_{22} are odd and n_{12} and n_{21} are even, we have $n_{11}=n$ similar to the case of f_{3*}, i.e. we may write

$$g(\alpha)=\beta^n\alpha^{n_{12}} \qquad g(\beta)=\beta^{n_{21}}\alpha^{n_{22}}.$$

Calculation tells us that

$$\begin{aligned} f_{3*}g(\alpha)&=\alpha^{p+n_{12}+nq} & f_{3*}g(\beta)&=\beta^{pn_{21}+nn_{22}}\alpha^{y-qn_{21}} \\ gf_{4*}(\alpha)&=\alpha^{y+n_{21}-nn_{22}} & gf_{4*}(\beta)&=\beta^{-pn_{21}-qn}\alpha^{pn_{22}+n_{12}}. \end{aligned}$$

If $f_{3*}g$ transforms into gf_{4*} by the inner automorphism induced by $\beta^l\alpha^m$, we have

$$(\beta^l\alpha^m)(\beta^{pn_{21}+nn_{22}}\alpha^{y-qn_{21}})(\beta^l\alpha^m)^{-1}=\beta^{pn_{21}+nn_{22}}\alpha^{\pm(y-qn_{21})}=\beta^{-pn_{21}-qn}\alpha^{pn_{22}+n_{12}}.$$

This implies that $pn_{21}+nn_{22}\equiv -pn_{21}-qn \pmod{2n}$, thus $2pn_{21}+n(n_{22}+q)\equiv 0 \pmod{2n}$. As $n_{22}+q$ is even here, $2pn_{21}\equiv 0 \pmod{2n}$. While n is odd and p and n_{21} are even, we have $pn_{21}\equiv 0 \pmod{4n}$. Therefore $f_{3*}g(\beta)=\beta^{nn_{22}}\alpha^{y-qn_{21}}$ and

$$\begin{aligned} f_{3*}g(\beta^s\alpha^t)&=(\beta^{nn_{22}}\alpha^{y-qn_{21}})^s(\alpha^{p+n_{12}+nq})^t \\ &=\begin{cases}\beta^{snn_{22}}\alpha^{(p+n_{12}+nq)t} & (s:\text{even}) \\ \beta^{snn_{22}}\alpha^{y-qn_{21}+(p+n_{12}+nq)t} & (s:\text{odd}).\end{cases} \end{aligned}$$

If we set $f_{3*}g(\beta^s\alpha^t)=\beta$, then snn_{21} is a multiple of n and this leads to contradiction.

The above discussion shows that Theorem 3 holds for prism manifolds also and here the proof of Theorem 3 is completed. □

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(Received December 4, 1989)

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