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On the cylinder isomorphism associated to the family of lines on a hypersurface

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§ 1. Introduction.

Let X be a hypersurface of degree a in $\boldsymbol{P}^{n+1}$ defined over the complex number field $\boldsymbol{C}$. We assume that $n\geq 3$. Let $\mathcal{L}(X)$ denote the variety of all lines on X. Then we have the cylinder homomorphism

$$\Psi:\quad H_{n-2}(\mathcal{L}(X),\ \boldsymbol{Z}) \longrightarrow H_n(X,\ \boldsymbol{Z})$$
$$[\gamma] \qquad \longmapsto \Big[\bigcup_{L\in\gamma} L\Big]$$

where γ is a topological $(n-2)$-cycle on $\mathcal{L}(X)$ and $\bigcup_{L\in\gamma} L$ is the topological n-cycle of the union of all lines corresponding to the points on γ. In this paper, we shall show that, if n is odd or 4, and $a\leq n$, then Ψ is an isomorphism modulo torsion for a general X.

Let us state our result more precisely. Let Q be the variety of all hypersurfaces of degree a in $\boldsymbol{P}^{n+1}$, and G the Grassmannian variety of all lines in $\boldsymbol{P}^{n+1}$. Let Z be the incidence correspondence $Z:=\{(L, X)\in G\times Q|\ L\subset X\}$ with the natural projections $\alpha: Z\to G$, $\beta: Z\to Q$. Note that $\mathcal{L}(X)=\beta^{-1}(X)$ for $X\in Q$. It is easy to see that Z is smooth. On the other hand, it is known that β is surjective if $a\leq 2n-1$ (cf. [1]). Thus we have the maximal Zariski open dense subset Q' of Q such that β is smooth over Q'. It is easy to see that $\dim\mathcal{L}(X)=2n-a-1$ for $X\in Q'$. We denote by $\Psi\otimes\boldsymbol{Q}$ the cylinder homomorphism $H_{n-2}(\mathcal{L}(X), \boldsymbol{Q})\to H_n(X, \boldsymbol{Q})$. We put

$$V_{n-2}(\mathcal{L}(X), \boldsymbol{Z}) := \ker(H_{n-2}(\mathcal{L}(X), \boldsymbol{Z}) \longrightarrow H_{n-2}(G, \boldsymbol{Z})),$$
$$V_n(X, \boldsymbol{Z}) := \ker(H_n(X, \boldsymbol{Z}) \longrightarrow H_n(\boldsymbol{P}^{n+1}, \boldsymbol{Z})).$$

THEOREM. *Suppose that $n\geq 3$, $n\geq a$, and $X\in Q'$. Then $\Psi\otimes\boldsymbol{Q}$ is surjective. Moreover*

(1) *if n is odd, then Ψ is an isomorphism modulo torsion.*
(2) *If n is even, then*
(2-i) $\ker \Psi\otimes \boldsymbol{Q}$ *is contained in the image of the natural map* $H_{n+2a}(G, \boldsymbol{Q}) \to H_{n-2}(\mathcal{L}(X), \boldsymbol{Q})$, *and* $\dim\ker \Psi\otimes\boldsymbol{Q} \leq (n-2)/4$,
(2-ii) *if $n/2+2\geq a$, then Ψ is surjective.*
In particular, Ψ is an isomorphism modulo torsion if $n=4$.

It is known that, if X is a smooth cubic hypersurface, then $X\in Q'$ (cf. [1]). Thus we have

COROLLARY. *For a smooth cubic hypersurface X with* $\dim X\geq 3$, *Ψ is surjective. If* $\dim X$ *is odd, then Ψ is an isomorphism modulo torsion.*

It is known that Ψ is an isomorphism for a smooth X if $a=3$ and $n=3$ (cf. [5]), or $a=3$ and $n=4$ (cf. [2]). It is also known that $\Psi\otimes\boldsymbol{Q}$ is surjective for a general X if $a\leq n+1$ (cf. [4], [10]). If $\Psi\otimes\boldsymbol{Q}$ is surjective, the Hodge level of $H^n(X)$ must be less than n, hence the degree a must be $\leq n+1$. For the case $a=n+1$, it is known that $\Psi\otimes\boldsymbol{Q}$ is not an isomorphism (cf. [9]).

The tool of the proof of injectivity of $\Psi\otimes\boldsymbol{Q}$ is a higher dimensional analogue of the Clemens-Letizia method (cf. [3], [8]). To prove the surjectivity of Ψ, we use some ideas originated from [10]. The contents of this paper are as follows. In § 2, we investigate the degeneration of $\mathcal{L}(X)$. In § 3, we study the relation between vanishing cycles of $\mathcal{L}(X)$ and X. In § 4, we study the kernel of $\Psi\otimes\boldsymbol{Q}$, and prove that $\ker\Psi\otimes\boldsymbol{Q}$ is contained in the image of the natural map $H_{n+2a}(G, \boldsymbol{Q})\to H_{n-2}(\mathcal{L}(X), \boldsymbol{Q})$, and $\dim\ker\Psi\otimes\boldsymbol{Q}\leq(n-2)/4$. In § 5, we show that Ψ maps a vanishing cycle of $\mathcal{L}(X)$ to a vanishing cycle of X, and thus the image of $V_{n-2}(\mathcal{L}(X), \boldsymbol{Z})$ via Ψ is just $V_n(X, \boldsymbol{Z})$, and then prove (2-ii).

In this paper, we use the same symbol L for a line on X and the corresponding point on $\mathcal{L}(X)$, and write $L\in\mathcal{L}(X)$, for example. We also use the same symbol for a hypersurface and the corresponding point on Q.

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§ 2. Degenerations of $\mathcal{L}(X)$.

Let $\pi: V \to \Delta$ be a proper flat holomorphic map from a complex manifold V of dimension $m+1$ onto the unit disk Δ. This map is called a degeneration if π is smooth over the punctured disk $\Delta \setminus \{0\}$ and $V_t := \pi^{-1}(t)$ is irreducible for $t \neq 0$. Let $\operatorname{Sing} V_0$ denote the singular locus of V_0.

DEFINITION 1. A degeneration $\pi: V \to \Delta$ is called quadric of codimension k if $\operatorname{Sing} V_0$ is connected and, for every point $p \in \operatorname{Sing} V_0$, there exist local coordinates $(z_0, \cdots, z_m)$ of V around p such that $\pi = z_0^2 + \cdots + z_k^2$.

The following lemma is easy.

LEMMA 1. *The degeneration $\pi: V \to \Delta$ is quadric of codimension k if and only if* $\operatorname{Sing} V_0$ *is smooth, connected and, for each $p \in \operatorname{Sing} V_0$, there exists a smooth $(k+1)$-dimensional submanifold W_p of V defined in a small neighborhood of p such that* (a) *W_p intersects with* $\operatorname{Sing} V_0$ *transversely at p, and* (b) *p is a non-degenerate critical point of the function $\pi|_{W_p}$.* □

Let W'_p be an arbitrary $(k+1)$-dimensional submanifold of V which intersects with $\operatorname{Sing} V_0$ at p transversely. Then $p \in W'_p$ is a non-degenerate critical point of $\pi|_{W'_p}$. Let $[\sigma^+] \in H_k(W'_p \cap V_\varepsilon, \boldsymbol{Z})$ be the vanishing cycle for the non-degenerate critical point p of $\pi|_{W'_p}$, where $\varepsilon \in \Delta \setminus \{0\}$ is sufficiently small and $V_\varepsilon := \pi^{-1}(\varepsilon)$.

DEFINITION 2. We call the image of $[\sigma^+] \in H_k(W'_p \cap V_\varepsilon, \boldsymbol{Z})$ by the natural map $H_k(W'_p \cap V_\varepsilon, \boldsymbol{Z}) \to H_k(V_\varepsilon, \boldsymbol{Z})$ the vanishing cycle for the quadric degeneration of V_ε at $t=0$.

By a similar argument of [8] p. 482, we have

LEMMA 2. *Let $\pi: V \to \Delta$ be a quadric degeneration of codimension k. Then the space of vanishing k-cycles* $\ker(\mathrm{sp}_*: H_k(V_\varepsilon, \boldsymbol{Z}) \to H_k(V_0, \boldsymbol{Z})) \otimes \boldsymbol{Q}$ *is of dimension at most* 1 *and is generated by the vanishing cycle of the degeneration.* □

We use the notation in the introduction. In what follows, we always assume that $n \geq 3$, $n \geq a$. The lemma below will be proved together with Lemma 4.

LEMMA 3. *Every hypersurface contained in Q' is smooth.*

By this lemma, it is enough to prove Theorem for a general hypersurface X_∞ contained in Q'. Let $D \subset Q$ be a general line passing the point $X_\infty \in Q'$. Let t be an affine parameter on D such that the point $t=\infty$ is corresponding to X_∞. We may assume that the pencil $\{X_t\}_{t\in D}$ corresponding to the line D is a Lefschetz pencil. Let $X_{t_1}, \cdots, X_{t_N}$ be the singular members. By Bertini's theorem, we may also assume that $Z_D := \beta^{-1}(D)$ is smooth. Let β_D denote the restriction of β to Z_D. Our first goal is to prove the following proposition which will be used in the Clemens-Letizia argument in § 4.

PROPOSITION 1. *We put $D \setminus (D \cap Q') = \{t_1, \cdots, t_N, t_{N+1}, \cdots, t_{N+M}\}$. If we take the line D passing X_∞ general enough, then*
(a) *$\mathcal{L}(X_t)$ is smooth, connected and of dimension $2n-a-1$ for any $t \in D \cap Q'$,*
(b) *for $1 \le J \le M$, $\mathcal{L}(X_{t_{N+J}})$ has only isolated singularities, and*
(c) *for $1 \le J \le N$, β_D is a quadric degeneration of codimension $n-2$ at each t_J.*

PROOF. The assertion (a) is an easy consequence of the results of [1] and the definition of Q'.

PROOF OF (b). For a pair $(L, X) \in Z$, let $T_{L,\mathcal{L}(X)}$ be the Zariski tangent space of $\mathcal{L}(X)$ at $L \in \mathcal{L}(X)$. Then β is not smooth at (L, X) if and only if $\dim T_{L,\mathcal{L}(X)} > 2n-a-1$. We put

$Z_1 :=$ the closure in Z of the set $\{(L, X) \in Z \mid X$ is smooth along L, and $\dim T_{L,\mathcal{L}(X)} > 2n-a-1\}$.

Now Lemma 4 below implies that $\beta(Z_1) \subset Q$ is an irreducible subvariety of Q of codimension ≥ 1, and if the codimension is exactly 1, then $\beta|_{Z_1} : Z_1 \to \beta(Z_1)$ is generically finite. Thus (b) follows.

LEMMA 4. *If $a=2$ or 3, then Z_1 is empty. If $a \ge 4$, then Z_1 is an irreducible subvariety of Z of codimension $2n-a$.*

PROOF OF LEMMAS 3 AND 4. We fix a line $L \in G$. Let $X \in Q$ be a hypersurface which contains L. We have the canonical exact sequence $0 \to N_{L/X} \to N_{L/P^{n+1}} \xrightarrow{\varphi} N_{X/P^{n+1}}|_L$ of normal sheaves and the canonical isomorphism $T_{L,\mathcal{L}(X)} \simeq H^0(L, N_{L/X})$. We see that $\dim T_{L,\mathcal{L}(X)}$ is larger than $2n-a-1$ if and only if $H^0(\varphi) : H^0(L, N_{L/X}) \to H^0(L, N_{L/P^{n+1}})$ does not have the maximal rank. We choose homogeneous coordinates $(\xi_0 : \cdots : \xi_{n+1})$ of $\boldsymbol{P}^{n+1}$ such that L is defined by $\xi_1 = \cdots = \xi_n = 0$. Then the defining homogeneous equation F of X is written as follows;

$$F(\xi_0, \cdots, \xi_{n+1}) = \xi_1 \cdot \tilde{g}_1(\xi_0, \xi_{n+1}) + \cdots + \xi_n \cdot \tilde{g}_n(\xi_0, \xi_{n+1}) + (\text{terms which contain } \xi_1, \cdots, \xi_n \text{ with degree more than } 1).$$

The morphism $\varphi : N_{L/P^{n+1}}(\simeq \mathcal{O}(1)^{\oplus n}) \to N_{X/P^{n+1}}|_L(\simeq \mathcal{O}(a))$ is given by $(g_\nu)_{1\le\nu\le n}$, where $g_\nu = \tilde{g}_\nu|_L$. If we put

$$\tilde{g}_\nu(\xi_0, \xi_{n+1}) = \sum_{\mu=0}^{a-1} g_{\nu,\mu} \cdot \xi_0^{a-1-\mu} \cdot \xi_{n+1}^{\mu},$$

then the morphism $H^0(\varphi)$ is given by the matrix

$$\tilde{M}_F = \left[\begin{array}{c|c} M_F & 0 \cdots\cdots 0 \\ \hline 0 \cdots\cdots 0 & M_F \end{array}\right] \text{ where } M_F = \begin{bmatrix} g_{1,0} & g_{2,0} & \cdots & g_{n,0} \\ \vdots & \vdots & & \vdots \\ g_{1,a-1} & g_{2,a-1} & \cdots & g_{n,a-1} \end{bmatrix}.$$

Suppose that X is singular at a point $(\alpha : 0 : \cdots : 0 : \beta) \in L$. Then we have $\tilde{g}_1(\alpha, \beta) = \cdots = \tilde{g}_n(\alpha, \beta) = 0$, and we get a non-trivial linear relation $(\alpha^a, \alpha^{a-1}\beta, \cdots, \alpha\beta^{a-1}, \beta^a)\tilde{M}_F = (0, \cdots, 0)$ between the rows of $\tilde{M}_F$. Hence we get $\dim T_{L,\mathcal{L}(X)} > 2n - a - 1$. Now, for any singular hypersurface X of degree $a \le n$, there exists a line on X which passes through the singular locus of X (cf. proof of Lemma 1 in Lecture 4, [11]). This completes the proof of Lemma 3.

Suppose that X is smooth along L. If the rank of $\tilde{M}_F$ is not maximal, we get two linear relations

$$\begin{cases} (\gamma_0, \cdots, \gamma_{a-1})M_F = (0, \cdots, 0) \\ (\gamma_1, \cdots, \gamma_a)M_F = (0, \cdots, 0), \end{cases}$$

at least one of which is non-trivial. Then $(\gamma_0, \cdots, \gamma_{a-1})$ and $(\gamma_1, \cdots, \gamma_a)$ is linearly independent. In fact, if not, there would exist $(\alpha, \beta) \ne (0, 0)$ such that $(\gamma_0, \cdots, \gamma_a) = c \cdot (\alpha^a, \alpha^{a-1}\beta, \cdots, \alpha\beta^{a-1}, \beta^a)$, where $c \in \boldsymbol{C}^\times$. Hence $g_1, \cdots, g_n$ would have a common zero on L. Let $\mathcal{M}(a, n)$ be the variety of all $a \times n$ matrices. Let $\mathcal{M}_1$ be the variety of all $M \in \mathcal{M}(a, n)$ such that there exists a vector $(\gamma_0, \cdots, \gamma_a) \in \boldsymbol{C}^{a+1}$ which satisfies the following two conditions: 1) $(\gamma_0, \cdots, \gamma_{a-1})$ and $(\gamma_1, \cdots, \gamma_a)$ are linearly independent, and 2) $(\gamma_0, \cdots, \gamma_{a-1})M = (\gamma_1, \cdots, \gamma_a)M = (0, \cdots, 0)$. If $a \ge 3$, then $\mathcal{M}_1$ is irreducible and of codimension $2n-a$ in $\mathcal{M}(a,n)$. If $a=2$, then $\mathcal{M}_1=\{0\}$. Because the linear map

$$\begin{array}{ccc}\{F\in H^0(\boldsymbol{P}^{n+1},\mathcal{O}(a))\mid F|_L\equiv 0\} & \longrightarrow & \mathcal{M}(a,n)\\ F & \longmapsto & M_F\end{array}$$

is surjective, we have $\operatorname{codim} Z_1\leq 2n-a$. Let $X=\{F=0\}$ be a general hypersurface containing L such that $M_F\in\mathcal{M}_1$. The column vectors of M_F span an $(a-2)$-dimensional linear space in $\boldsymbol{C}^a\simeq\{g(\xi_0,\xi_{n+1})\mid g$ is homogeneous of degree $a\}$. If $a\geq 4$, then $g_1,\cdots,g_n$ do not have a common zero on L because each zero of g_1 defines a subspace in $\boldsymbol{C}^a$ of codimension $1<a-2$. Hence $X\in Z_1$, and $\operatorname{codim} Z_1=2n-a$. If $a=3$, then $g_1,\cdots,g_n$ are proportional to each other and X is singular at some points on L. Thus Z_1 is empty. □

PROOF OF (c). We fix a point $o\in\boldsymbol{P}^{n+1}$ and put

$$\begin{aligned} Q_{\text{sing}} &:=\{X\in Q\mid X \text{ is singular}\},\\ Q_{\text{sing},o} &:=\{X\in Q_{\text{sing}}\mid o\in X,\ X \text{ is singular at } o\},\quad \text{and}\\ \mathcal{L}(X,o) &:=\{L\in\mathcal{L}(X)\mid o\in L\}\quad \text{for } X\in Q_{\text{sing},o}.\end{aligned}$$

We take a general $X_0\in Q_{\text{sing},o}$ and consider the pencil $\{X_t\}_{t\in D}$ spanned by X_0 and X_∞. The two lemmas below prove (c).

LEMMA 5. (1) $\operatorname{Sing}(\mathcal{L}(X_0))$ *coincides with* $\mathcal{L}(X_0,o)$, *and*
(2) $\mathcal{L}(X_0,o)$ *is smooth, connected and of dimension* $n-a+1$.

LEMMA 6. *For each* $L_0\in\mathcal{L}(X_0,o)$, *we have a smooth* $(n-1)$*-dimensional submanifold* $S(L_0)$ *of* Z_D *defined in a small neighborhood of* $(L_0,X_0)\in Z_D$ *such that*
(1) $S(L_0)$ *intersects* $\mathcal{L}(X_0,o)$ *at* (L_0,X_0) *transversely,*
(2) *the restriction* $\beta_D|_{S(L_0)}$ *of* β_D *to* $S(L_0)$ *has a non-degenerate critical point at* $(L_0,X_0)\in S(L_0)$.

PROOF OF LEMMA 5. The assersion (2) can be proved by the same argument with the proof of Lemma 1 in Lecture 4, [11]. For (1), we consider the hypersurface $X'=\{\xi_1\xi_0^{a-1}+\xi_2\xi_{n+1}^{a-1}+\xi_1^a+\xi_2^a+\cdots+\xi_n^a=0\}$, which is smooth and contains the line $L'=\{\xi_1=\cdots=\xi_n=0\}$. It is easy to see that $\mathcal{L}(X')$ is singular at $L'\in\mathcal{L}(X')$ if $a\geq 4$. Hence we see that $X'\in\beta(Z_1)\setminus Q_{\text{sing}}$. Because $\beta(Z_1)$ and Q_{sing} are both irreducible and Q_{sing} is of codimension 1 in Q, we see that $\beta(Z_1)\cap Q_{\text{sing}}$ is of codimension ≥ 1 in Q_{sing}. Thus we have $\text{Sing}(\mathcal{L}(X))\subset\mathcal{L}(X,o)$ for a general $X\in Q_{\text{sing},o}$. The inclusion $\mathcal{L}(X,o)\subset\text{Sing}(\mathcal{L}(X))$ has been shown in the proof of Lemma 3. □

PROOF OF LEMMA 6. We fix affine coordinates $(x_0,\cdots,x_n)$ on an affine space A^{n+1} in P^{n+1} which contains the o as the origin. We may assume that the line $L_0\in\mathcal{L}(X_0,o)$ is defined by

$$L_0: x_1=\cdots=x_n=0. \tag{2.1}$$

Let $(u_1,\cdots,u_n,v_1,\cdots,v_n)$ be local coordinates of G around L_0 such that the line corresponding to $(u_1,\cdots,u_n,v_1,\cdots,v_n)\in G$ is given by $\{(\lambda,u_1\lambda+v_1,u_2\lambda+v_2,\cdots,u_n\lambda+v_n)\mid\lambda\in C\}\subset A^{n+1}$. Let f and g be the defining equations in A^{n+1} of X_0 and X_∞ respectively. The defining equation of X_t is $f+t\cdot g=0$. We put

[0] $f(v)+t\cdot g(v)=0,$ and for $\nu=1,\cdots,a,$

$$[\nu]\quad \sum_{i_1,\cdots,i_\nu=0}^{n} u_{i_1}\cdot\cdots\cdot u_{i_\nu}\cdot\left(\frac{\partial^\nu f}{\partial x_{i_1}\cdots\partial x_{i_\nu}}(v)+t\cdot\frac{\partial^\nu g}{\partial x_{i_1}\cdots\partial x_{i_\nu}}(v)\right)=0,$$

where $v:=(0,v_1,\cdots,v_n)$, $u_0:=1$. The local defining equations of $Z_D:=\beta^{-1}(D)$ in a small neighborhood of $(0,L_0)$ in $D\times G$ is [0], [1], $\cdots$, [a], and the subvariety $\mathcal{L}(X_0,o)$ of $D\times G$ is defined by the equations $t=v_1=\cdots=v_n=0$ and [2], [3], $\cdots$, [a] above. (Note that if $t=v_1=\cdots=v_n=0$, then the equations [0], [1] hold automatically because f does not have the homogeneous part of degree 1.) Let $T_{(0,L_0),D\times G}$ be the tangent space of $D\times G$ at $(0,L_0)\in D\times G$. We identify

$$\theta\frac{\partial}{\partial t}+\sum_{i=1}^{n}\zeta_i\frac{\partial}{\partial u_i}+\sum_{i=1}^{n}\eta_i\frac{\partial}{\partial v_i}\in T_{(0,L_0),D\times G}$$

with $(\theta,\zeta_1,\cdots,\zeta_n,\eta_1,\cdots,\eta_n)\in C^{2n+1}$. Then the tangent space $T_{(0,L_0),\mathcal{L}(X_0,o)}\subset T_{(0,L_0),D\times G}$ is given by

$$\left[\begin{array}{c|c|c} 1 & 0\cdots 0 & 0\cdots 0\\ \hline * & A & *\\ \hline 0 & 0 & I_n \end{array}\right]\begin{bmatrix}\theta\\ \zeta_1\\ \vdots\\ \zeta_n\\ \eta_1\\ \vdots\\ \eta_n\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ \vdots\\ 0\end{bmatrix},$$

where A is an $(a-1)\times n$ matrix defined as follows: Let f_ν be the homogeneous part of degree ν of f. We put

$$f_\nu = x_0^{\nu-1}\cdot(\alpha_{\nu 1}x_1+\alpha_{\nu 2}x_2+\cdots+\alpha_{\nu n}x_n)$$
$$+(\text{terms which contain } x_0 \text{ with degree less than } \nu-1).$$

The matrix A is

$$A=\begin{bmatrix} \alpha_{21} & \alpha_{22} & \cdots\cdots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \cdots\cdots & \alpha_{3n} \\ \vdots & \vdots & & \vdots \\ \alpha_{a1} & \alpha_{a2} & \cdots\cdots & \alpha_{an} \end{bmatrix}.$$

If X_0 is general in $Q_{\text{sing},o}$, the matrix A has the maximal rank for every $L_0\in\mathcal{L}(X_0,o)$, because $\mathcal{L}(X_0,o)$ is smooth by Lemma 5. Hence we can change the affine coordinates $(x_0,\cdots,x_n)$ linearly so that (2.1) still holds and

$$A=\left[\, I_{a-1} \,\middle|\, 0 \,\right]. \tag{2.2}$$

We also change the local coordinates $(u_1,\cdots,u_n,v_1,\cdots,v_n)$ of G in accord with $(x_0,\cdots,x_n)$. We define $M\subset D\times G$ by $u_a=\cdots=u_n=0$. Then M intersects with Z_D transversely in a smooth manifold $S:=M\cap Z_D$ of dimension $n-1$, and S meets with $\mathcal{L}(X_0,o)$ transversely at $(0,L_0)$. We shall prove that the restriction $\beta_D|_S$ of β_D to S has a non-degenerate critical point at $(0,L_0)$, thus S is the desired submanifold $S(L_0)$. It is easy to see that

(2.3) $w_2:=v_2|_S,\cdots,w_n:=v_n|_S$ define local coordinates on S,

(2.4) $v_1|_S$ has a critical point at $(0,L_0)$ (i.e., at $w=0$).

Note that $\beta_D|_S$ is nothing but $t|_S$. By the equation [0], we see

$$t|_S=-\left.\frac{f(0,v_1,\cdots,v_n)}{g(0,v_1,\cdots,v_n)}\right|_S=-\frac{f(0,v_1|_S(w),w_2,\cdots,w_n)}{g(0,v_1|_S(w),w_2,\cdots,w_n)}. \tag{2.5}$$

Because f_2 is of the form $x_0\cdot x_1+$(terms not containing x_0) by (2.2), the non-degeneracy of the symmetric bilinear form defined by f_2 assures that the form of $(n-1)$-variables defined by $f_2(0,0,w_2,\cdots,w_n)$ is also non-degenerate. By (2.4), (2.5), we see that $\beta_D|_S=t|_S$ has a non-degenerate critical point at $w=0$. □

§ 3. Relation between the vanishing cycles of X and $\mathcal{L}(X)$.

We continue to consider the Lefschetz pencil $\{X_t\}_{t\in D}$ with $o\in X_0$ the node. Let ε be a non-zero, sufficiently small number. It is well known that there exists a vanishing cycle $[\Sigma^+]\in H_n(X_\varepsilon, \boldsymbol{Z})$ of X_ε, uniquely determined up to sign, for the node $o\in X_0$. By Lemma 6, we also have a vanishing cycle $[\sigma^+]\in H_{n-2}(\mathcal{L}(X_\varepsilon), \boldsymbol{Z})$ for the quadric degeneration of $\mathcal{L}(X_\varepsilon)$ at $t=0$. First, we give an explicit description of the topological cycles in X_ε and $\mathcal{L}(X_\varepsilon)$ which represent $[\Sigma^+]$ and $[\sigma^+]$, and next, we shall study the relation between $[\Sigma^+]$ and $[\sigma^+]$. The main result of this section is Proposition 2.

Let $(w)=(w_2, \cdots, w_n)$ be the local coordinates on $S(L_0)$ defined by (2.3), and $\tilde{v}_1(w)$ be the restriction to $S(L_0)$ of the function v_1 on $D\times G$. We consider the embedding

$$\begin{array}{rccl} \iota: & S(L_0) & \hookrightarrow & A^{n+1} \\ & (w_2, \cdots, w_n) & \longmapsto & (0, \tilde{v}_1(w), w_2, \cdots, w_n), \end{array}$$

where A^{n+1} is the affine space with the affine coordinates $(x_0, \cdots, x_n)$ which we have used in Proof of (c) in § 2. Let $R(L_0)$ denote its image. We put $\tau:=f/g$. The local defining equation of X_ε is $\tau(x)=\varepsilon$. We see from (2.5) that

$$t|_{S(L_0)}=\tau|_{R(L_0)}\circ\iota. \tag{3.1}$$

Because $t|_{S(L_0)}{}^{-1}(\varepsilon)=S(L_0)\cap\mathcal{L}(X_\varepsilon)$, we have the isomorphism

$$\iota_\varepsilon: S(L_0)\cap\mathcal{L}(X_\varepsilon) \xrightarrow{\sim} R(L_0)\cap X_\varepsilon \tag{3.2}$$

induced from ι, for ε small enough. We put

$$P(L_0):=\bigcup_{L\in S(L_0)} L\subset \boldsymbol{P}^{n+1}.$$

Then $P(L_0)$ is a smooth hypersurface in a small neighborhood of $o\in\boldsymbol{P}^{n+1}$. In fact, $P(L_0)\cap A^{n+1}$ is the image of the map

$$\begin{array}{l} \boldsymbol{C}\times S(L_0)\ni(\lambda, w_2, \cdots, w_n) \\ \quad\downarrow \tilde{\iota} \qquad\qquad\quad \downarrow \\ A^{n+1}\ni(\lambda, \tilde{u}_1(w)\lambda+\tilde{v}_1(w), \tilde{u}_2(w)\lambda+w_2, \tilde{u}_3(w)\lambda+w_3, \cdots, \tilde{u}_n(w)\lambda+w_n), \end{array}$$

where $\tilde{u}_i(w)$ is the restriction to $S(L_0)$ of the function u_i. It is obvious that $\tilde{\iota}$ is an embedding in a small neighborhood of $(\lambda, w)=(0, 0, \cdots, 0) \in \boldsymbol{C}\times S(L_0)$. Let $P'(L_0)$ be a small neighborhood of o in $P(L_0)$. We have a canonical projection $\pi : P'(L_0)\to R(L_0)$ which is compatible with the projection $\boldsymbol{C}\times S(L_0)\to S(L_0)$ via ι and $\tilde{\iota}$. For $p \in R(L_0)$, the fibre $\pi^{-1}(p)$ is a segment of the line corresponding to $\iota^{-1}(p) \in S(L_0)$, which is contained in $X_{\tau(p)}$. Hence we have

$$\iota_\varepsilon(L)=R(L_0)\cap L \qquad \text{for } L\in S(L_0)\cap \mathcal{L}(X_\varepsilon), \tag{3.3}$$

and

$$\tau|_{P'(L_0)}=\tau|_{R(L_0)}\circ\pi. \tag{3.4}$$

LEMMA 7. *There is an analytic local coordinate system* $(z_0, \cdots, z_n)$ *of* $\boldsymbol{P}^{n+1}$ *around* o *such that we have coordinate descriptions as follows*;

$$\tau=z_0^2+\cdots+z_n^2,$$
$$P'(L_0) : z_0+\sqrt{-1}\,z_1=0, \quad R(L_0) : z_0=z_1=0,$$
$$\pi : (z_0, \sqrt{-1}\,z_0, z_2, \cdots, z_n) \longmapsto (0, 0, z_2, \cdots, z_n).$$

PROOF. It is obvious that we have local coordinates $(y_0, \cdots, y_n)$ of $\boldsymbol{P}^{n+1}$ with the origin o such that

$$P'(L_0) : y_0=0,\ R(L_0) : y_0=y_1=0, \qquad \text{and} \tag{3.5}$$
$$\pi : (0, y_1, y_2, \cdots, y_n) \longmapsto (0, 0, y_2, \cdots, y_n).$$

From (3.1) and Lemma 6, we see that $\tau|_{R(L_0)}=\tau(0, 0, y_2, \cdots, y_n)$ has a non-degenerate critical point at $(y_2, \cdots, y_n)=(0, \cdots, 0)$. Hence we can change the coordinates $(y_0, y_1, \cdots, y_n)$ so that we have $\tau(0, 0, y_2, \cdots, y_n)=y_2^2+\cdots+y_n^2$, and the descriptions (3.5) still hold. From (3.4), we have $\tau(y)=y_2^2+\cdots+y_n^2+\sum_{i=0}^{n} h_i(y)\cdot y_0\cdot y_i$. Because o is a non-degenerate critical point of τ, we have $h_1(0, \cdots, 0)\neq 0$. Now we can get easily the desired local coordinates $(z_0, \cdots, z_n)$ from $(y_0, \cdots, y_n)$ by a suitable coordinate transformation. □

Using the coordinates in Lemma 7, we put, for a small positive real number r, $B_r:=\{(z_0, \cdots, z_n)\in \boldsymbol{P}^{n+1} \mid |z_0|^2+\cdots+|z_n|^2<r^2\}$. We also put

$$D\Sigma=\{(u, v)\in \boldsymbol{R}^{n+1}\times \boldsymbol{R}^{n+1} \mid |u|=1,\ |v|<1,\ \langle u, v\rangle=0\}$$
$$\Sigma=\{(u, 0)\in \boldsymbol{R}^{n+1}\times \boldsymbol{R}^{n+1} \mid |u|=1\}.$$

Let ε be a small *positive real* number. By [7] p. 37, there is a diffeomorphism between $B_r \cap X_\varepsilon$ and $D\Sigma$, which is given by

$$u=\frac{\operatorname{Re} z}{|\operatorname{Re} z|}, \quad v=\{(r^2-\varepsilon)/2\}^{-1/2}\cdot \operatorname{Im} z.$$

We will consider $\Sigma \subset D\Sigma$ as a submanifold embedded in $B_r \cap X_\varepsilon$ by this diffeomorphism. It is well known that Σ with an orientation $+$ is a topological cycle which represents the vanishing cycle in $H_n(X_\varepsilon, \boldsymbol{Z})$ for the node $o \in X_0$. From Lemma 7, we can see that

(3.6) Σ intersects with $P(L_0)$ transversely in a $(n-2)$-sphere,

(3.7) the $(n-2)$-sphere $\Sigma \cap P(L_0)$ is contained in $R(L_0) \cap X_\varepsilon$,

(3.8) $B_r \cap R(L_0) \cap X_\varepsilon$ is diffeomorphic to the space $D(\Sigma \cap P(L_0))$ of all tangent vectors of length <1 of the sphere $\Sigma \cap P(L_0)$.

Since $P'(L_0) \cap X_\varepsilon$ is a complex submanifold of X_ε of codimension 1, we see from (3.6) that there is a orientation $+$ of $\Sigma \cap P(L_0)$ *canonically* induced from that of Σ^+ and the complex structure of $P'(L_0) \cap X_\varepsilon$ and X_ε. By the fact (3.7) and the local isomorphism (3.2), we have a topological cycle

$$\sigma^+ := \iota_\varepsilon^{-1}((\Sigma \cap P(L_0))^+)$$

in $S(L_0) \cap \mathcal{L}(X_\varepsilon)$. By (3.8), this cycle σ^+ represents the vanishing cycle of $S(L_0) \cap \mathcal{L}(X_\varepsilon)$ for the node $(0, L_0) \in S(L_0) \cap \mathcal{L}(X_0)$. Hence $[\sigma^+] \in H_{n-2}(\mathcal{L}(X_\varepsilon), \boldsymbol{Z})$ is the vanishing cycle of $\mathcal{L}(X_\varepsilon)$ for the quadric degeneration at $t=0$.

We embed Z_D in a projective space $\boldsymbol{P}^I$, and let Y be the intersection of Z_D with a general plane in $\boldsymbol{P}^I$ of codimension $n-a+1$. We may assume that $Y_\varepsilon := Y \cap \mathcal{L}(X_\varepsilon)$ is smooth for ε which is non-zero and sufficiently small, and that Y intersects with $\mathcal{L}(X_0, o)$ transversely at points $(0, L_1), \cdots, (0, L_r) \in D \times G$. Then $Y_0 := Y \cap \mathcal{L}(X_0)$ has only nodes $(0, L_1), \cdots, (0, L_r)$ as its singularities. We move Y continuously so that Y coincides, in a small neighborhood of $(0, L_i)$, with the submanifold $S(L_i)$ which we constructed in Lemma 6. For each i, we have a topological $(n-2)$-cycle σ_i^+ on Y_ε constructed as above.

PROPOSITION 2. *For* $[\rho] \in H_{n-2}(Y_\varepsilon, \boldsymbol{Z})$, *let* $[\rho]'$ *be the image of* $[\rho]$ *by the natural map* $H_{n-2}(Y_\varepsilon, \boldsymbol{Z}) \to H_{n-2}(\mathcal{L}(X_\varepsilon), \boldsymbol{Z})$. *Then the intersection number* $[\rho]\cdot([\sigma_1^+]+\cdots+[\sigma_r^+])$ *on* Y_ε *equals with the intersection number*

$\Psi([\rho]')\cdot[\Sigma^+]$ *on* X_ε.

PROOF. For the open small ball B_r of o in P^{n+1}, we put $B_r^\sim=\{L\in G|L\cap B_r\neq\emptyset\}\subset G$. Then $(D\times B_r^\sim)\cap Z_D$ is a neighborhood of $\mathcal{L}(X_0,o)$ in Z_D. By taking the radius r of B_r sufficiently small, we may assume that $Y\cap(D\times B_r^\sim)=$disjoint union of $U_1, U_2, \cdots, U_r$, where U_i is a small neighborhood of $(0, L_i)$ in Y. By construction, the support σ_i of σ_i^+ is contained in U_i. Let ρ be the topological cycle in Y_ε which represents $[\rho]$. We put $\rho\cap U_i=\rho_i$. We see that, if $L\in\rho\setminus\bigcup_{i=1}^r\rho_i\subset Y_\varepsilon\subset\mathcal{L}(X_\varepsilon)$, then $L\cap\Sigma=\emptyset$. We see from (3.7) that, if $L\cap\Sigma\neq\emptyset$ for $L\in S(L_i)\cap\mathcal{L}(X_\varepsilon)$, then this intersection $L\cap\Sigma$ is a single point $L\cap R(L_i)$. By (3.3), we have $L=\iota_\varepsilon^{-1}(L\cap R(L_i))$, and since $L\cap R(L_i)\in\Sigma\cap P(L_i)$, we have $L\in\sigma_i$. On the other hand, if $L\in\sigma_i$, then L and Σ meets at $\iota_\varepsilon(L)$. By moving ρ in Y_ε, we may assume that ρ_i meets σ_i^+ transversely at μ points $a_1, \cdots, a_\mu\in\sigma_i$. Then the intersection of $\bigcup_{L\in\rho_i}L$ and Σ consists of the points $\iota_\varepsilon(a_1), \cdots, \iota_\varepsilon(a_\mu)$. By (3.6), $\bigcup_{L\in\rho_i}L$ and Σ meets at these points transversely. Note that the orientation of the topological n-cycle $\bigcup_{L\in\rho}L$ is obtained canonically from the orientation of ρ and the complex structure of each $L\in\rho$. By the definition of the orientation $+$ of σ^+, we see that the sign of the local intersection numbers at a_j and $\iota_\varepsilon(a_j)$ are same. □

§4. The kernel of $\Psi\otimes Q$.

PROPOSITION 3. *The kernel of* $\Psi\otimes Q$ *is contained in the image of the natural map* $H_{n+2a}(G,Q)\to H_{n-2}(\mathcal{L}(X),Q)$. *In particular,* $\Psi\otimes Q$ *is injective if n is odd. If n is even, we have* $\dim\ker\Psi\otimes Q\leq(n-2)/4$.

PROOF. We consider the Lefschetz pencil $\{X_t\}_{t\in D}$ in Proposition 1. The fundamental group $\pi_1(D\setminus\{t_1, \cdots, t_{N+M}\}, \infty)$ acts on $H_{n-2}(\mathcal{L}(X_\infty), Q)$ and $H^{n-2}(\mathcal{L}(X_\infty), Q)$.

LEMMA 8. *The space of invariant cocycles* $I^{n-2}(\mathcal{L}(X_\infty),Q)\subset H^{n-2}(\mathcal{L}(X_\infty),Q)$ *under this monodromy action is the image of the natural map* $H^{n-2}(G,Q)\to H^{n-2}(\mathcal{L}(X_\infty),Q)$.

PROOF. By [7](7.4.1), the natural map $\pi_1(D\setminus\{t_1, \cdots, t_{N+M}\},\infty)\to\pi_1(Q',\infty)$ is surjective. Hence $I^{n-2}(\mathcal{L}(X_\infty),Q)$ is the space of invariant cocycles of $\pi_1(Q',\infty)$. Let β' be the restriction of β to $Z':=\beta^{-1}(Q')$. By Théorème (4.1.1) (ii) of [6], the natural map

$$H^{n-2}(Z, Q) \longrightarrow H^{n-2}(Z', Q) \longrightarrow H^0(Q', R^{n-2}\beta'_* Q) \simeq I^{n-2}(\mathcal{L}(X_\infty), Q)$$

is surjective. Consider the commutative diagram

$$\begin{array}{ccc} Z & \overset{\eta}{\hookrightarrow} & G \times Q \\ {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle\tilde{\alpha}} & \\ G & & \end{array},$$

where $\tilde{\alpha}$ is the natural projection and η is the inclusion. Since α is smooth and every fibre of α is a linear subspace of $Q \simeq \boldsymbol{P}^K$ of codimension $a+1$, we have natural isomorphisms $R^i\tilde{\alpha}_* Q \xrightarrow{\sim} R^i\alpha_* Q$ and $H^j(G, R^i\tilde{\alpha}_* Q) \xrightarrow{\sim} H^j(G, R^i\alpha_* Q)$ for $i \leq 2(K-a-1)$. Because the Leray spectral sequences with respect to α and $\tilde{\alpha}$ degenerate at E_2, we see that $H^{n-2}(G \times Q, Q) \xrightarrow{\eta^*} H^{n-2}(Z, Q)$ is an isomorphism. (It is clear that $n-2 \leq 2(K-a-1)$.) We have the Künneth decomposition $H^{n-2}(G \times Q, Q) = \bigoplus_{i+j=n-2} H^j(G, Q) \otimes H^i(Q, Q)$. It is obvious that

$$H^j(G, Q) \otimes H^i(Q, Q) \hookrightarrow H^{n-2}(G \times Q, Q) \xrightarrow{\sim} H^{n-2}(Z, Q) \longrightarrow H^0(Q', R^{n-2}\beta'_* Q)$$

is a zero-map unless $j = n-2$, $i = 0$. It is also easy to see that

$$\begin{aligned} H^{n-2}(G, Q) &\xrightarrow{\sim} H^{n-2}(G, Q) \otimes H^0(Q, Q) \hookrightarrow H^{n-2}(G \times Q, Q) \xrightarrow{\sim} H^{n-2}(Z, Q) \\ &\longrightarrow H^0(Q', R^{n-2}\beta'_* Q) \hookrightarrow H^{n-2}(\mathcal{L}(X_\infty), Q) \end{aligned}$$

is the natural restriction map induced from $\mathcal{L}(X_\infty) \hookrightarrow G$. □

Following the argument of [8] p. 483, we see that $\ker \Psi \otimes Q$ is contained in the space $I_{n-2}(\mathcal{L}(X_\infty), Q)$ of invariant cycles, using Proposition 1 and the surjectivity of $\Psi \otimes Q$ (cf. [10] or see § 5). The fundamental group $\pi_1(D \setminus \{t_1, \cdots, t_{N+M}\}, \infty)$ acts on $H^{\cdot}(G)$ and $H_{\cdot}(G)$ trivially. We have a commutative diagram of π_1-equivariant homomorphisms

$$\begin{array}{ccccc} H^{n-2}(G, Q) & \underset{\text{P.D.}}{\xrightarrow{\sim}} & H_{3n+2}(G, Q) & \underset{\cdot[H]^{n-a+1}}{\hookrightarrow} & H_{n+2a}(G, Q) \\ \downarrow & & \downarrow & & \downarrow \\ H^{n-2}(\mathcal{L}(X_\infty), Q) & \underset{\text{P.D.}}{\xrightarrow{\sim}} & H_{3n-2a}(\mathcal{L}(X_\infty), Q) & \underset{\cdot[H]^{n-a+1}}{\xrightarrow{\sim}} & H_{n-2}(\mathcal{L}(X_\infty), Q) \end{array}$$

where P.D. denotes Poincaré duality and $\cdot[H]^{n-a+1}$ denotes intersection product with $(n-a+1)$-st power of the homology class of a hyperplane section. From Lemma 8, we see that $I_{n-2}(\mathcal{L}(X_\infty), Q)$ coincides with the image of $H_{n+2a}(G, Q) \to H_{n-2}(\mathcal{L}(X_\infty), Q)$. If n is odd, then $\dim I_{n-2}(\mathcal{L}(X_\infty), Q) = \dim I^{n-2}(\mathcal{L}(X_\infty), Q) \leq \dim H^{n-2}(G, Q) = 0$. Suppose that n is even. Let $I_n(X_\infty, Q) \subset H_n(X_\infty, Q)$ be the one-dimensional subspace of invariant cycles. Because $\Psi \otimes Q$ is surjective, the map $\Psi \otimes Q|_{I_{n-2}(\mathcal{L}(X_\infty))} : I_{n-2}(\mathcal{L}(X_\infty), Q) \to I_n(X_\infty, Q)$ must be surjective. Thus $\dim \ker \Psi \otimes Q = \dim I^{n-2}(\mathcal{L}(X_\infty), Q) - 1$. Since $\dim H^{n-2}(G, Q) = [(n-2)/4] + 1$, we have $\dim \ker \Psi \otimes Q \leq (n-2)/4$. □

§ 5. The image of Ψ.

PROPOSITION 4. *The homomorphism* Ψ *maps* $V_{n-2}(\mathcal{L}(X), Z)$ *onto* $V_n(X, Z)$ *surjectively.*

PROOF. Note that the image of $V_{n-2}(\mathcal{L}(X), Z)$ via Ψ is contained in $V_n(X, Z)$ because we have a commutative diagram

$$\begin{array}{ccc} H_{n-2}(\mathcal{L}(X), Z) & \xrightarrow{\Psi} & H_n(X, Z) \\ \downarrow & & \downarrow \\ H_{n-2}(G, Z) & \longrightarrow & H_n(P^{n+1}, Z). \end{array}$$

It is obvious that vanishing cycles in $H_{n-2}(\mathcal{L}(X_\infty), Z)$ of quadratic degenerations at $t = t_1, \cdots, t_N$ are contained in $V_{n-2}(\mathcal{L}(X_\infty), Z)$. On the other hand, $V_n(X_\infty, Z)$ is generated by vanishing cycles in $H_n(X_\infty, Z)$ (cf. [7]). Let $[\Sigma^+] \in V_n(X_\varepsilon, Z)$ and $[\sigma^+] \in V_{n-2}(\mathcal{L}(X_\varepsilon), Z)$ be the vanishing cycles at $t=0$, as in § 3. Because the cylinder map commutes with specialization map, we have $\Psi([\sigma^+]) = m_0 \cdot [\Sigma^+]$ ($m_0 \in Z$). Now it is enough to show that $m_0 = \pm 1$, because all vanishing cycles in $H_n(X_\infty, Z)$ are conjugate by the action of the global monodromy (cf. [7]). First, suppose that n is even. In this case, we see from [7] p. 40 that

$$[\sigma_1^+] \cdot ([\sigma_1^+] + \cdots + [\sigma_r^+]) = [\sigma_1^+] \cdot [\sigma_1^+] = \pm 2 \quad \text{in } H_{2n-4}(Y_\varepsilon, Z)$$

where $[\sigma_i^+] \in H_{n-2}(Y_\varepsilon, Z)$ ($i = 1, \cdots, r$) are the cycles in Proposition 2. Thus we have $\Psi([\sigma^+]) \cdot [\Sigma^+] = m_0 \cdot [\Sigma^+] \cdot [\Sigma^+] = \pm 2$. Comparing this with $[\Sigma^+] \cdot [\Sigma^+] = \pm 2$ in $H_{2n}(X_\varepsilon, Z)$, we get $m_0 = \pm 1$.

Next, we shall treat the odd dimensional case. Let $\{X_t\}_{t \in D}$ be the Lefschetz pencil with $n = \dim X_t$ is *even* and $n \geq a+1$. Let $\{H_s\}_{s \in P^1}$ be a general pencil of hyperplanes in P^{n+1} such that $o \in H_0$ and $o \notin H_\infty$, where $o \in X_0$ is a node. We denote by $W_{t,s}$ the hyperplane section $X_t \cap H_s$. Fix a non-zero, sufficiently small number ε, and consider the pencil $\{W_{\varepsilon,s}\}_{s \in P^1}$.

We shall prove Proposition 4 for a general member of this pencil. By Lemma 3, 4 in [10], we have a small neighborhood Δ of 0 in $\boldsymbol{P}^1$ and two points $s_{+1}, s_{-1} \in \Delta$ such that

(a) $W_{\varepsilon,s}$ is smooth for all $s \in \Delta \setminus \{s_{+1}, s_{-1}\}$, and
(b) W_{ε,s_i} $(i=\pm 1)$ has one and only one singular point p_i which is a node.

We take a path $\gamma : [-1, 1] \to \Delta$ which satisfies the three conditions in Lemma 5 of [10], and apply this Lemma to our situation. We get the following facts:

(c) there is an $(n-1)$-sphere $S_{\varepsilon,\gamma(v)}$ in $W_{\varepsilon,\gamma(v)}$ for each $v \in (-1, 1)$, which represents, with an orientation $+$, the vanishing cycle $[S_\varepsilon^+] \in H_{n-1}(W_{\varepsilon,\gamma(v)}, \boldsymbol{Z})$ for both of the two nodes $p_{-1} \in W_{\varepsilon,\gamma(-1)}$ and $p_{+1} \in W_{\varepsilon,\gamma(+1)}$, and
(d) $\{p_{-1}\} \cup \bigcup_{v \in (-1,1)} S_{\varepsilon,\gamma(v)} \cup \{p_{+1}\}$ is an n-sphere in X_ε, which represents, with an appropriate orientation, the vanishing cycle $[\Sigma^+] \in H_n(X_\varepsilon, \boldsymbol{Z})$.

It is enough to show that

$$\operatorname{im} \Psi_{W_{\varepsilon,\gamma(v)}} \ni [S^+_{\varepsilon,\gamma(v)}]\,. \tag{5.1}$$

Let $\tilde{\mathcal{L}}_\varepsilon := \{(s, L) \in \boldsymbol{P}^1 \times G \mid L \subset W_{\varepsilon,s}\}$ be the incidence correspondence with the natural projection $\pi : \tilde{\mathcal{L}}_\varepsilon \to \boldsymbol{P}^1$. By Proposition 1, we may assume that

(a)′ there are finitely many points $\{s_{(1)}, \cdots, s_{(M)}\} \subset \Delta \setminus \{s_{+1}, s_{-1}\}$ such that π is smooth over $\Delta \setminus \{s_{+1}, s_{-1}, s_{(1)}, \cdots, s_{(M)}\}$, and
(b)′ π is a quadric degeneration of codimension $n-3$ at $s=s_{+1}$ and $s=s_{-1}$.

We take the path γ which avoids $s_{(1)}, \cdots, s_{(M)}$. Recall that $\sigma^+ \subset \mathcal{L}(X_\varepsilon)$ is an $(n-2)$-sphere with an orientation which represents the vanishing cycle $[\sigma^+] \in H_{n-2}(\mathcal{L}(X_\varepsilon), \boldsymbol{Z})$ for the quadric degeneration of $\mathcal{L}(X_\varepsilon)$ at $t=0$. We shall construct an $(n-3)$-dimensional sphere $s_{\varepsilon,\gamma(v)}$ in $\mathcal{L}(W_{\varepsilon,\gamma(v)})$ for each $v \in (-1, 1)$ which has the following properties;

(c)′ with an orientation $+$, $s^+_{\varepsilon,\gamma(v)}$ represents the vanishing cycle for both of the quadric degenerations of $\mathcal{L}(W_{\varepsilon,\gamma(v)})$ at $s=s_{+1}$ and $s=s_{-1}$, and
(d)′ there is a point $q_i \in \mathcal{L}(W_{\varepsilon,\gamma(i)})$ $(i=\pm 1)$ such that the union $\{q_{-1}\} \cup \bigcup_{v \in (-1,1)} s_{\varepsilon,\gamma(v)} \cup \{q_{+1}\}$ in $\mathcal{L}(X_\varepsilon)$ is an $(n-2)$-sphere which, with an appropriate orientation, represents $[\sigma^+] \in H_{n-2}(\mathcal{L}(X_\varepsilon), \boldsymbol{Z})$. (Note that $\mathcal{L}(W_{\varepsilon,\gamma(v)})$ can be naturally embedded in $\mathcal{L}(X_\varepsilon)$.)

The assertion (5.1) follows from this construction as follows. By (c) and (c)$'$, we see that $\Psi_{W_{\varepsilon,\gamma(v)}}([s^+_{\varepsilon,\gamma(v)}])=m\cdot([S^+_{\varepsilon,\gamma(v)}])$ where m is an integer which does not depend on $v\in(-1,1)$. Then by (d) and (d)$'$, we have $\Psi_{X_\varepsilon}([\sigma^+])=m\cdot[\Sigma^+]$. By what we have proved in the previous paragraph, we have $m=m_0=\pm1$, which implies (5.1).

Now we construct $s^+_{\varepsilon,\gamma(v)}$. We use the coordinate description in Proof of Lemma 6. We use the affine coordinates $(x_0,\cdots,x_n)$ which is *not* arranged so that (2.2) holds. We take the hyperplane pencil $\{H_s\}_{s\in P^1}$ given by $H_s=\{x_n=s\}$, and put $G_s:=\{L\in G|\ L\subset H_s\}$. The defining equations of $D\times G_s$ in $D\times G$ is given by $u_n=0$, $v_n=s$. Let M be a submanifold in a small neighborhood of $(0,L_0)$ in $D\times G$ which is of codimension $n-a+1$ and is defined by linear equations $u_n=0$ and $\sum_{j=1}^n\beta_{ij}u_j+\sum_{j=1}^n\gamma_{ij}v_j=0$ $(i=1,\cdots,n-a)$. We put $S:=Z_D\cap M$. If X_0 is general and M is general, then we have

(1) M intersects with $\mathcal{L}(X_0,o)$ transversely, and S is smooth near $(0,L_0)$,
(2) $v_n|_S$ has no critical points in a neighborhood of $(0,L_0)$,
(3) $t|_S$ and $v_n|_S$ satisfy the condition (♯) in Lemma 3 of [10].

In fact, these three conditions are open conditions for X_0 and M. Hence it is enough to show that there is at least one example. If we choose X_0 whose defining equation f satisfies

$$f_2=x_0x_1+x_2^2+\cdots+x_n^2,\qquad A=\left[\,I_{a-1}\;\middle|\;0\,\right],$$

and M which is defined by $u_a=\cdots=u_n=0$, then the conditions (1), (2), (3) above are satisfied. Note that, on S, $S\cap\mathcal{L}(X_\varepsilon)$ is defined by $t|_S=\varepsilon$, and $S\cap\mathcal{L}(W_{\varepsilon,s})$ is defined by $t|_S=\varepsilon$, $v_n|_S=s$. By Lemma 4 of [10], we have two values $s'_{+1},s'_{-1}\in\Delta$ such that $M\cap\mathcal{L}(W_{\varepsilon,s})$ is singular for $s=s'_{+1},s'_{-1}$. If $\mathcal{L}(W_{\varepsilon,s})$ is smooth, then $M\cap\mathcal{L}(W_{\varepsilon,s})$ is smooth for a general M. Hence s'_{+1} and s'_{-1} must coincide with s_{+1} and s_{-1}. Now by Lemma 5 of [10], we get the desired cycle $s^+_{\varepsilon,\gamma(v)}$ and two points q_{+1},q_{-1}. □

Next we prove the assertion (2-ii) of Theorem. We suppose that $n=\dim X$ is even. We put $m=n/2$. Since we have already shown that the image of Ψ containes $V_n(X,\boldsymbol{Z})$, it is enough to show that the composition $H_{n-2}(\mathcal{L}(X),\boldsymbol{Z})\xrightarrow{\Psi}H_n(X,\boldsymbol{Z})\xrightarrow{i_*}H_n(\boldsymbol{P}^{n+1},\boldsymbol{Z})$ is surjective for a general X. We fix an m-dimensional linear subspace $P\simeq\boldsymbol{P}^m$ of $\boldsymbol{P}^{n+1}$, and a point q on P. Let $G(q,P)\subset G$ be the variety of all lines which pass q and are contained in P. Let $Q_P\subset Q$ be the variety of all hypersurfaces of degree a which contain P.

CLAIM. *Suppose that $m+2\geq a$. Then, for a general $X\in Q_P$, the morphism $\beta: Z\rightarrow Q$ is smooth at every point of $\alpha^{-1}(G(q, P))\cap\beta^{-1}(X)$, i.e. at every point of $G(q, P)\subset\mathcal{L}(X)\subset G$.*

Note that $G(q, P)\simeq \boldsymbol{P}^{m-1}$ defines a topological $(n-2)$-cycle in $\mathcal{L}(X)$ for $X\in Q_P$. The claim above implies that this cycle $G(q, P)$ can be deformed to a topological $(n-2)$-cycle Γ contained in $\mathcal{L}(X')$ for a general $X'\in Q$. It is obvious that $i_*\circ\Psi([\Gamma])$ is the generator of $H_n(\boldsymbol{P}^{n+1}, \boldsymbol{Z})\simeq\boldsymbol{Z}$, hence (2-ii) follows. The Claim can be proved by the argument similar to the proof of Lemma 4. □

References

[1] Barth, W. and A. Van de Ven, Fano varieties of lines on hypersurfaces, Arch. Math. **31** (1978), 94-104.

[2] Beauville, A. and R. Donagi, La variété des droites d'une hypersurface cubique de dimension 4, C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), 703-706.

[3] Clemens, H., On surjectivity of Abel-Jacobi mappings, Ann. of Math. **117** (1983), 71-76.

[4] Clemens, H., The local geometry of the Abel-Jacobi mapping, Proc. Sympos. Pure Math. vol. 46, Amer. Math. Soc., Providence, R. I., 1987, 223-233.

[5] Clemens, C. H. and P. A. Griffiths, The intermediate Jacobian of the cubic threefold, Ann. of Math. **95** (1972), 281-356.

[6] Deligne, P., Théorie de Hodge, II, Publ. Math. I.H.E.S. **40** (1971), 5-58.

[7] Lamotke, K., The topology of complex projective varieties after S. Lefschetz, Topology **20** (1981), 15-51.

[8] Letizia, M., The Abel-Jacobi mapping for the quartic threefold, Invent. Math. **75** (1984), 477-492.

[9] Lewis, J. D., The cylinder correspondence for hypersurfaces of degree n in $\boldsymbol{P}^n$, Amer. J. Math. **110** (1988), 77-114.

[10] Shimada, I., On the cylinder homomorphisms of Fano complete intersections, J. Math. Soc. Japan **42** (1990), 719-738.

[11] Tyurin, A. N., Five lectures on three-dimensional varieties, Russian Math. Surveys **27** (1972), 1-53.

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