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On the Hurder-Katok extension of the Godbillon-Vey invariant

Dedicated to Professor Akio Hattori on his sixtieth birthday

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§ 1. Introduction.

The Godbillon-Vey invariant ([8]) is defined for a codimension one foliation of class C^2 as a 3-dimensional de Rham cohomology class of the manifold. When the foliation is given by a 1-form ω, there exists a 1-form η such that $d\omega=\eta\wedge\omega$ by the integrability condition, and the Godbillon-Vey class is that of the closed 3-form $\eta\wedge d\eta$. This gives rise to the Godbillon-Vey 2-cocycle of the group $\mathrm{Diff}^2_+(S^1)$ of orientation preserving C^2-diffeomorphisms of the circle.

In [10], Hurder and Katok defined the Godbillon-Vey invariant for foliations of class $C^{1+\alpha}$, where $\alpha>1/2$. They asked whether one can define the Godbillon-Vey invariant for the foliations of class $C^{1+\alpha}$ for $0<\alpha\leq 1/2$. This paper answers to this question. Namely we assert that the Godbillon-Vey invariant is not nicely defined for the foliations of class $C^{1+\alpha}$ for $0<\alpha<1/2$. More precisely, we prove the following propositions.

PROPOSITION 1. *The Godbillon-Vey 2-cocycle defined in* $\mathrm{Diff}^\omega_+(S^1)$ *is not continuous in the* $C^{1+\alpha}$ *topology for* $0<\alpha<1/2$.

PROPOSITION 2. *For* $0<\alpha<1/2$, *there is a foliated* $\boldsymbol{R}$*-product* $\mathcal{F}$ *with compact support over a closed oriented surface* Σ *of class* $C^{1+\alpha}$ *with the following properties.* $\mathcal{F}$ *admits a partition into a countable number of saturated Borel sets* B_i *where the Godbillon-Vey invariants* $GV(\mathcal{F}, B_i)$ *are defined and* $\sum GV(\mathcal{F}, B_i)=\infty$.

Roughly speaking, Proposition 2 says that there is a foliated $\boldsymbol{R}$-product $\mathcal{F}$ with compact support of class $C^{1+\alpha}$ $(0<\alpha<1/2)$ whose Godbillon-Vey invariant is equal to infinity. Here a foliated $\boldsymbol{R}$-product with compact support is a foliation of $\Sigma\times\boldsymbol{R}$ transverse to the fibers of the projection $\Sigma\times\boldsymbol{R}\longrightarrow\Sigma$ whose leaves are horizontal out of a compact set.

For the fact that the Godbillon-Vey invariants are defined with respect to saturated Borel sets, see [12], [4], [5], [3], [9] and [7].

As is asked by Ghys ([6], [7]), it is interesting to know the natural domain of definition of the Godbillon-Vey invariant. We study this question in a future paper.

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§2. Proof of Proposition 1.

For the proof of Proposition 1, we use the construction of a family of foliated S^1-bundles with holonomy in the amalgamated group

$$SL(2, \boldsymbol{R}) * PSL(2, \boldsymbol{R}) / SO(2) = PSO(2)$$

giving a continuous variation of the Godbillon-Vey invariant (see [14], [2] and [15]). Since we need an estimate on the norm of the holonomy, we review here the construction.

In the hyperbolic plane $\boldsymbol{H}^2$, we take two geodesics intersecting normally at a point O $(\in \boldsymbol{H}^2)$. Let A, B, C and D be the points on these geodesics whose distances from O are equal to ε. We assume that A, B, C and D are in the positive cyclic order. Let φ and ψ be the orientation preserving isometries of $\boldsymbol{H}^2$ satisfying the following conditions.

$$\varphi(\overline{BC}) = \overline{AD} \quad \text{and} \quad \psi(\overline{DC}) = \overline{AB}.$$

Then $\varphi\psi\varphi^{-1}\psi^{-1}$ is the rotation around A by the angle equal to the area of the geodesic square $ABCD$. Now consider the space $T^2 - \operatorname{Int} D^2$ obtained from the 2-torus T^2 by deleting a small open disk in it. There is a presentation of $\pi_1(T^2 - \operatorname{Int} D^2)$ by the generators β and γ with their commutator $\beta\gamma\beta^{-1}\gamma^{-1}$ representing the homotopy class of the boundary $\partial(T^2 - \operatorname{Int} D^2)$. We can define a foliated S^1-bundle over $T^2 - \operatorname{Int} D^2$ by giving the holonomy homomorphism $\pi_1(T^2 - \operatorname{Int} D^2) \longrightarrow \operatorname{Diff}^{\omega}_{+}(S^1)$. Here a foliated S^1-bundle means a foliation of the total space of an S^1-bundle transverse to the fibers. Let $\mathcal{G}$ denote the foliated S^1-bundle over $T^2 - \operatorname{Int} D^2$ such that the holonomy along β and γ are equal to φ and ψ, respectively. Note that the isometries φ and ψ act real analytically on the boundary circle of the hyperbolic plane $\boldsymbol{H}$, hence we are considering φ and ψ as elements of $\operatorname{Diff}^{\omega}_{+}(S^1)$. Then the boundary of this foliated S^1-bundle is isomorphic to a linear foliation of $S^1 \times S^1$ with slope equal to the area of $ABCD$. There is a well-defined Godbillon-Vey number for this foliated S^1-bundle, which is also equal to the area of $ABCD$ (up to a non-zero real multiple) ([15]).

Now take another choice ε' of the distance and construct the geodesic square $A'B'C'D'$ in the negative cyclic order for this time. Then we have a foliated S^1-bundle over $T^2 - \operatorname{Int} D^2$ whose boundary is a linear foliation of $S^1 \times S^1$ with slope equal to the algebraic area of $A'B'C'D'$, that is, minus the absolute area of $A'B'C'D'$. Now consider the double cover along the fiber of this foliated S^1-bundle over $T^2 - \operatorname{Int} D^2$ (whose total space is in fact the product $(T^2 - \operatorname{Int} D^2) \times S^1$). Then the slope of the boundary of the new foliated S^1-bundle is the half of the algebraic area of $A'B'C'D'$. The Godbillon-Vey number is, however, equal to the double of the algebraic area of $A'B'C'D'$.

If the area of $ABCD$ is equal to the half of the area of $A'B'C'D'$, then there is an orientation reversing diffeomorphism from one boundary to the other preserving the S^1 fibration and the foliation, and we obtain a foliated S^1-bundle over the surface Σ_2 of genus 2. The Godbillon-Vey number of this foliated S^1-bundle is equal to -3 times the area of $ABCD$. By changing ε continuously, we obtain a family of foliations giving a smooth variation of the Godbillon-Vey number.

In this construction, the C^r-norm of the holonomy of the foliated S^1-bundle along the generators of $\pi_1(\Sigma_2)$ are estimated by the diameter of the square $ABCD$, hence by the square root of the area of $ABCD$. Note that this foliated S^1-bundle has in fact a structure of a foliated S^1-product, and we can give a foliated S^1-product structure such that the norm of this foliated product is also estimated by the square root of the area of $ABCD$. Here a foliated S^1-product over a space Y is a foliated S^1-bundle with a trivialization of the S^1-bundle structure, and we can define C^r-norms of a foliated S^1-products with smooth leaves as the supremum over the points y of Y of the norms of the linear maps from the tangent spaces T_yY to the space of the vector fields on S^1 with the C^r-topology defined by the foliated product structure (See [16].). Here the C^r-topology of the space of the vector fields on S^1 is of course defined by the following C^r-norm: For a vector field ξ,

$$|\xi|_r = \sum_{k=0}^{r} \sup_{x \in S^1} |(\partial/\partial x)^k \xi(x)|.$$

We also note that, for $0 < \alpha < 1$, the $C^{r+\alpha}$-norm is defined by

$$|\xi|_{r+\alpha} = |\xi|_r + \sup_{x \in S^1} |(\partial/\partial x)^r \xi(x_1) - (\partial/\partial x)^r \xi(x_0)| / |x_1 - x_0|^\alpha.$$

Hence we obtained a family $\{\mathcal{F}_t\}$ of foliated S^1-products such that, for any $r>0$,

$$|\mathcal{F}_t|_r \leq C_r t^{1/2} \quad \text{and} \quad GV(\mathcal{F}_t)=t \quad \text{for} \quad 0\leq t\leq T,$$

where C_r and T are positive real numbers.

PROOF OF PROPOSITION 1. For a positive integer n, consider the n-fold cyclic cover along the fiber of the foliated S^1-product $\mathcal{F}_t$. Then we obtain the foliated S^1-product $\mathcal{F}_t^{(n)}$ over Σ_2. For the $C^{1+\alpha}$-norm and the Godbillon-Vey number of $\mathcal{F}_t^{(n)}$, we have the following estimates.

$$|\mathcal{F}_t^{(n)}|_{1+\alpha} \leq C_2 n^\alpha t^{1/2} \quad \text{and} \quad GV(\mathcal{F}_t^{(n)})=nt.$$

If $t=1/n$ and $\alpha<1/2$, then we have

$$|\mathcal{F}_{1/n}^{(n)}|_{1+\alpha} \leq C_2 n^{\alpha-(1/2)} \longrightarrow 0 \qquad \text{as} \quad n \longrightarrow \infty$$

while $GV(\mathcal{F}_t^{(n)})=1$. Thus GV is not continuous at the trivial foliated S^1-product in the $C^{1+\alpha}$-topology for $\alpha<1/2$.

REMARK. GV is not bounded in the neighborhood of the trivial foliated S^1-product. To show this, it is only necessary to put $t=n^{-\alpha'}$ in the above proof, where $2\alpha<\alpha'<1$.

§ 3. Proof of Proposition 2.

In order to construct an example showing Proposition 2, first we construct a family of foliated $\boldsymbol{R}$-products with support in a fixed interval which has properties similar to $\{\mathcal{F}_t^{(n)}\}$ in § 2. Here the support of a foliated $\boldsymbol{R}$-product over a space Y is the minimal closed subset K of $\boldsymbol{R}$ such that the leaves in $Y\times(\boldsymbol{R}-K)$ are horizontal. The construction is done by using the fragmentation homotopy ([1], [11], [16]).

LEMMA 3. *There exists a family of foliated* $\boldsymbol{R}$*-products* $\mathcal{H}_t^{(n)}$ $(0\leq t\leq T')$ *with support in* $[-2, 3]$ *such that*

$$|\mathcal{H}_t^{(n)}|_{1+\alpha} \leq C' n^\alpha t^{1/2} \quad \textit{and} \quad GV(\mathcal{H}_t^{(n)})=nt,$$

where T' *and* C' *are positive real numbers.*

PROOF. Take a cell decomposition of the closed orientable surface Σ_2 of genus 2 into squares with ordered edges. The order on the edges are given in such a way that $(0,0)<(1,0)<(1,1)$ and $(0,0)<(0,1)<(1,1)$ for the standard square $[0,1]^2$. In fact, Σ_2 has such a decomposition with 3 squares.

Take a smooth partition of unity $\{\mu_1, \mu_2, \mu_3\}$ of $S^1=\boldsymbol{R}/\boldsymbol{Z}$ such that

(0) $\mu_1+\mu_2+\mu_3=1$,
(1) $\mu_1=1$ on $[0, 1/6]$ and $\mu_1=0$ on $[2/6, 5/6]$ and
(2) $\mu_2(x)=\mu_1(x-1/3)$ and $\mu_3(x)=\mu_1(x-2/3)$.

Then we have the fragmentation homotopy using μ_1, μ_2 and μ_3 ([1], [11], [16]). For a foliated S^1-product sufficiently close to the trivial foliated S^1-product, by this homotopy we obtain a family of new foliated products. Each square of Σ_2 produces foliated products over 9 squares. The supports of the 9 foliated products are contained in the unions of supports of at most two functions of the partition of unity, hence they are contained in $[-1/6, 4/6]$, $[1/6, 6/6]$ or $[3/6, 8/6]$, and the supports of these 9 foliated products restricted over their edges are contained in the supports of one of these functions. The C^r-norm of the new foliated product is estimated by the C^r-norm of the old one ([16]).

Now we apply this homotopy to $\mathcal{F}_t$. Then we obtain a foliated S^1-product $\mathcal{G}_t$ such that

$$|\mathcal{G}_t|_r \leq C'_r t^{1/2} \quad \text{and} \quad GV(\mathcal{G}_t)=t \quad \text{for} \quad 0\leq t\leq T,$$

where C'_r and T' are positive real numbers, and the support of the foliated product over each small square is not the whole circle.

Let $\mathcal{G}_t^{(\infty)}$ denote the foliated $\boldsymbol{R}$-product over Σ_2 which is the infinite cyclic cover of $\mathcal{G}_t$. For each square Q, let $\mathcal{G}_Q^{(\infty)}$ denote the foliated $\boldsymbol{R}$-product $\mathcal{G}_t^{(\infty)}$ restricted over the square Q. The support of $\mathcal{G}_Q^{(\infty)}$ is contained in $\cup_{k\in \boldsymbol{Z}}[k+a, k+b]$, where $[a, b]$ is one of the above three intervals containing the support of $\mathcal{G}_Q$. Let $\mathcal{G}_Q^{(n)}$ be the foliated $\boldsymbol{R}$-product over Q defined as the restriction of $\mathcal{G}_Q^{(\infty)}$ to $[a, n-1+b]$ (in the direction of the fiber $\boldsymbol{R}$); that is, $\mathcal{G}_Q^{(n)}$ coincides with $\mathcal{G}_Q^{(\infty)}$ on $Q\times[a, n-1+b]$ and the leaves of $\mathcal{G}_Q^{(n)}$ in $Q\times(\boldsymbol{R}-[a, n-1+b])$ are horizontal. Note that these $\mathcal{G}_Q^{(n)}$ might not match up along the all edges of the squares. However, for each edge of Q, there is another edge of another square Q' such that the foliated $\boldsymbol{R}$-product over these edges are isomorphic by the translation by ± 1 or 0.

Let $\mathcal{H}_Q^{(n)}$ denote the foliated $\boldsymbol{R}$-product over Q obtained from $\mathcal{G}_Q^{(n)}$ by applying the similarity transformation by $1/n$ in the direction of the fiber $\boldsymbol{R}$. Let ρ denote a C^∞ vector field on $\boldsymbol{R}$ such that

$$\rho(x)=1 \quad \text{on} \quad [-1,2] \quad \text{and} \quad \rho(x)=0 \quad \text{on} \quad \boldsymbol{R}-[-2,3].$$

Let $\tau(s)$ denote the time s map of ρ. Let σ and σ' be the pair of foliated $\boldsymbol{R}$-products over the edges of the disjoint union of $\mathcal{H}_Q^{(n)}$ such that σ' is isomorphic to σ translated by $1/n$ in the direction of the fiber $\boldsymbol{R}$. Let σ also denote the smooth map $\sigma:[0,1]\longrightarrow \mathrm{Diff}_c^\infty(\boldsymbol{R})$ such that the leaf through $(0,x)\in[0,1]\times\boldsymbol{R}$ is the graph of $t\longmapsto\sigma(t)(x)$. For this pair we consider the foliated $\boldsymbol{R}$-product $\mathcal{G}_{\tau\sigma}$ over $[0,1]^2$ given by

$$(s,t)\longmapsto\tau(s/n)\sigma(t).$$

Here a C^∞ foliated $\boldsymbol{R}$-product with compact support over $[0,1]^2$ is defined by a continuous map $Q:[0,1]^2\longrightarrow\mathrm{Diff}_c^\infty(\boldsymbol{R})$ and the leaf of this foliated product passing through (y,x) is given by

$$\{(z, Q(z)Q(y)^{-1}(x));\ z\in[0,1]^2\}.$$

Note that, the foliated $\boldsymbol{R}$-product $\mathcal{G}_{\tau\sigma}$ restricted over $[0,1]\times\{0\}$ and that restricted over $[0,1]\times\{1\}$ are the same, and $\mathcal{G}_{\tau\sigma}$ restricted over $\{0\}\times[0,1]$ and $\{1\}\times[0,1]$ are isomorphic to σ and σ', respectively. Hence $\mathcal{G}_{\tau\sigma}$ can be considered as a foliated $\boldsymbol{R}$-product over $[0,1]\times S^1$ and we attach this to the disjoint union of $\mathcal{H}_Q^{(n)}$ along σ and σ'.

By attaching all the foliated $\boldsymbol{R}$-products $\mathcal{H}_Q^{(n)}$ and $\mathcal{G}_{\tau\sigma}$ along all pairs of edges where the foliated $\boldsymbol{R}$-products are isomorphic, we obtain the foliated $\boldsymbol{R}$-product $\mathcal{H}_t^{(n)}$ over a certain closed surface with support in $[-2,3]$. Since the norm of $\tau(s/n)$ is estimated by a constant times $1/n$, for the $C^{1+\alpha}$-norm of $\mathcal{H}_t^{(n)}$, we have the following estimate as before.

$$|\mathcal{H}_t^{(n)}|_{1+\alpha}\leq C'n^\alpha t^{1/2}.$$

As for the Godbillon-Vey number of $\mathcal{H}_t^{(n)}$, we can calculate it as follows. We have the universal defining 1-form ω for the foliated $\boldsymbol{R}$-products.

$$\omega=dx-F_{u_1}du_1-F_{u_2}du_2=F_{x_0}dx_0,$$

where $F(u_1,u_2,x_0)=Q(u_1,u_2)Q(0,0)^{-1}(x_0)$ for the foliated $\boldsymbol{R}$-product corresponding to $Q:[0,1]^2\longrightarrow\mathrm{Diff}_c^\infty(\boldsymbol{R})$. Then we obtain the universal Godbillon-Vey form $\eta\wedge d\eta$ as follows (see [13]).

$$d\omega=(F_{x_0u_1}du_1+F_{x_0u_2}du_2)\wedge dx_0.$$

$$\eta=\frac{F_{x_0u_1}}{F_{x_0}}du_1+\frac{F_{x_0u_2}}{F_{x_0}}du_2.$$

$$\eta\wedge d\eta=\frac{1}{(F_{x_0})^2}\begin{vmatrix} F_{x_0u_1} & F_{x_0u_2} \\ F_{x_0x_0u_1} & F_{x_0x_0u_2}\end{vmatrix} du_1\wedge du_2\wedge dx_0.$$

If $F(u_1, u_2, x_0)=\tau(u_1/n)\sigma(u_2)(x_0)$, then

$$F=\begin{cases} \sigma(u_2)(x_0)+u_1/n & \text{on } [-1+1/n, 2-1/n] \\ \tau(u_1/n)(x_0) & \text{on } (-\infty, -1+1/n]\cup[2-1/n, \infty). \end{cases}$$

Hence for $\mathcal{G}_{\tau\sigma}$, the first column or the second column of the matrix in $\eta\wedge d\eta$ is zero. Thus the Godbillon-Vey form restricted to the added foliated $\boldsymbol{R}$-products is zero and we have

$$GV(\mathcal{H}_t^{(n)})=nt.$$

PROOF OF PROPOSITION 2. Let $x_0, x_1, \cdots$, be a strictly increasing sequence of real numbers such that

$$x_0=0 \qquad \text{and} \qquad \lim_{m\to\infty} x_m=1.$$

For each interval $[x_{m-1}, x_m]$, we choose a foliated $\boldsymbol{R}$-product $\mathcal{H}_{1/n_m}^{(n_m)}$ given in Lemma 3 in such a way that

$$(x_m-x_{m-1})^{-\alpha}\cdot(n_m)^{\alpha-1/2}\longrightarrow 0 \qquad \text{as} \quad m\longrightarrow\infty.$$

Let $\mathcal{H}$ be the foliated $\boldsymbol{R}$-product defined as the union of $f_m\mathcal{H}_{1/n_m}^{(n_m)}$, where f_m is an affine map of $\boldsymbol{R}$ sending $[-2, 3]$ onto $[x_{m-1}, x_m]$. That is, $\mathcal{H}$ is a foliation of $\Sigma\times\boldsymbol{R}$ such that

$$\mathcal{H}|\Sigma\times[x_{m-1}, x_m]=f_{m*}\mathcal{H}_{1/n_m}^{(n_m)}|\Sigma\times[x_{m-1}, x_m]$$

and the leaves in $\mathcal{H}|\Sigma\times((-\infty, 0]\cup[1, \infty))$ are horizontal. Then $\mathcal{H}$ is a foliated $\boldsymbol{R}$-product of class $C^{1+\alpha}$ and there is a partition of $\Sigma\times\boldsymbol{R}$ into saturated sets $\Sigma\times(x_{m-1}, x_m]$, $\Sigma\times(-\infty, 0]$ and $\Sigma\times[1, \infty)$. By the construction, for the Godbillon-Vey number, we have $GV(\mathcal{H}, \Sigma\times(x_{m-1}, x_m])=1$.

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