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On the homotopy type of the moduli space of n-point sets of P^1

Dedicated to Professor Akio Hattori on his 60th birthday.

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Let X be a topological space. The space of all n-point sets in X with the natural topology will be denoted by B_nX. The fundamental group of B_nX is known as the n-string braid group of X.

In the present paper we confine ourselves to the case when X is the Riemann sphere $\boldsymbol{P}^1$, whose holomorphic automorphism group is the group of all linear fractional transformations $PSL(2, \boldsymbol{C})$. We abbreviate B_nX to B_n.

Our main purpose is to investigate the homotopy type of the quotient space

$$B_n/PSL(2, \boldsymbol{C}),$$

which is called the moduli space of n-point sets of $\boldsymbol{P}^1$. Here $PSL(2, \boldsymbol{C})$ is understood to act on B_n by the diagonal action.

The space $B_n/PSL(2, \boldsymbol{C})$ is interesting for several reasons:

(1) The "coordinate ring" of this space is the ring of invariants of binary forms. See Bolza [6], Igusa [8], and Shioda [14].

(2) For even n, $B_n/PSL(2, \boldsymbol{C})$ is the moduli space of hyperelliptic curves. In particular, for $n=6$, Igusa [8] proved that $B_6/PSL(2, \boldsymbol{C})$ is holomorphically isomorphic to the quotient space $\boldsymbol{C}^3/\boldsymbol{Z}_5$.

(3) Let F_n be the space of all sequences of n-points of $\boldsymbol{P}^1$. Then $F_n/PSL(2, \boldsymbol{C})$ is a smooth manifold and is the Eilenberg-MacLane space $K(\pi, 1)$, where π is the mapping class group of the 2-sphere with colored n-points. The space $B_n/PSL(2, \boldsymbol{C})$ is the quotient orbifold of $F_n/PSL(2, \boldsymbol{C})$ under the action of the n-th symmetric group S_n. The rational cohomology of $B_n/PSL(2, \boldsymbol{C})$ coincides with that of the mapping class group of the n-punctured 2-sphere, which is isomorphic to the quotient of the n-th braid group by its center $\{\pm 1\}$.

In § 2, we prove

THEOREM A. $H^*(B_n/PSL(2, \boldsymbol{C}); \boldsymbol{Q}) \cong H^*(pt; \boldsymbol{Q})$.

REMARK. For $n=6$, Lee and Weintraub [11] showed Theorem A. But it would seem difficult to investigate the action of the symmetric group on the cohomology ring $H^*(F_n/PSL(2, \boldsymbol{C}))$ only by their method.

Birman [5] proved the 1-connectivity of $B_n/PSL(2, \boldsymbol{C})$ (see also MacLachlan [12]). By her result and Serre's $\mathcal{C}$ theory [13] together with Theorem A above, we obtain the following

COROLLARY. $\pi_*(B_n/PSL(2, \boldsymbol{C}))\otimes \boldsymbol{Q}=0$.

Here arises a natural question: is $B_n/PSL(2, \boldsymbol{C})$ contractible? $B_n/PSL(2, \boldsymbol{C})$ has the homotopy type of an $(n-3)$-dimensional CW-complex (see (3.6)). Hence

THEOREM B. *For $n=4, 5$, the space $B_n/PSL(2, \boldsymbol{C})$ is contractible.*

By Igusa's theorem mentioned above, $B_6/PSL(2, \boldsymbol{C})$ has been known to be contractible. We remark $B_n/PSL(2, \boldsymbol{C})$ is the 1-point set for $n\leqq 3$.

In § 3, this CW-complex structure is obtained by regarding $B_n/PSL(2, \boldsymbol{C})$ as the moduli space of configurations of n "electrically charged particles" on $\boldsymbol{P}^1$. For $n=4, 5$, we determine all equilibria (see Figure 1) and show the "potential energy" of the configurations to be a "Morse function" of $B_n/PSL(2, \boldsymbol{C})$. As a corollary of this determination, we obtain another proof of Theorem B.

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§ 1. Notations and Preliminaries.

For a topological space (resp. a differentiable manifold, a holomorphic manifold) X, we define the configuration spaces as follows,

$$F_nX:=X^n-\bigcup_{\alpha\neq\beta}\{z_\alpha=z_\beta\}$$
$$B_nX:=F_nX/S_n,$$

where z_α denotes the α-th component, and the symmetric group S_n acts on F_nX by permuting the components. An element of B_nX will be denoted by $\{z_1, \cdots, z_n\}\in B_nX$, for $(z_1, \cdots, z_n)\in F_nX$. The product structure on X^n induces topological (resp. differentiable, holomorphic) structures on F_nX and B_nX.

Now set

$S^2 := \{\vec{z} = (x, y, h);\ \|\vec{z}\| := (x^2+y^2+h^2)^{1/2} = 1\}$
(the unit sphere of R^3, i.e., the 2-dimensional sphere)

$P^1 := CP^1$
(the complex projective line, i.e., the Riemann sphere)

$\varphi : P^1 \longrightarrow S^2$

(1.1) $$\varphi(z) := \left(\frac{2z}{1+|z|^2},\ \frac{1-|z|^2}{1+|z|^2}\right)$$
(the stereographic projection).

Denote $\varphi(z)$ by $\vec{z}$, and identify P^1 with S^2 using φ. We have

(1.2) $$\|\vec{z}-\vec{w}\|^2 = \frac{4|z-w|^2}{(1+|z|^2)(1+|w|^2)}.$$

Our interest is investigating the space related to B_nP^1. We abbreviate F_nP^1 and B_nP^1 to F_n and B_n, respectively.

The linear fractional transformation group will be denoted by G,

$$G := PSL(2, C).$$

The group of all orientation-preserving homeomorphisms is topologized by the compact-open topology. We denote it by $\mathscr{H}$,

$$\mathscr{H} := \text{Homeo}^+(S^2).$$

G and $\mathscr{H}$ act on F_n and B_n by the diagonal action

$$g(z_1, \cdots, z_n) = (gz_1, \cdots, gz_n)$$
$$g\{z_1, \cdots, z_n\} = \{gz_1, \cdots, gz_n\}$$

for $g \in G$ or $\mathscr{H}$, $(z_1, \cdots, z_n) \in F_n$ and $\{z_1, \cdots, z_n\} \in B_n$. The natural projection is denoted by

$$[\ ] : F_n \longrightarrow F_n/G$$
$$(z_1, \cdots, z_n) \longmapsto [z_1, \cdots, z_n].$$

(1.3) The holomorphic map
$F_n \longrightarrow (F_n/G) \times G$
$(z_1, \cdots, z_n) \longmapsto ([0, 1, \infty, g(z_4), \cdots, g(z_n)], g)$
where $g(z) = \dfrac{z-z_1}{z-z_3} : \dfrac{z_2-z_1}{z_2-z_3}$ (the cross ratio)

is a G-equivariant holomorphic isomorphism.

Especially F_n/G and B_n/G are the 1-pointed set for $n\leqq 3$. For the rest of this paper, let us assume

$$n\geqq 4.$$

By (1.3) we identify F_n/G with

$$F_{n-3}(\boldsymbol{P}^1-\{0,1,\infty\})\subset(\boldsymbol{P}^1)^{n-3}$$

and parametrize it using the coordinates $[0,1,\infty,u_4,\cdots,u_n]$.

So we have the following proposition (cf. Birman [4], p. 12. Th. 1.2.)

PROPOSITION (1.4). *F_n/G is an Eilenberg-MacLane manifold $K(\pi,1)$.*

To clarify $\pi_1(F_n/G)$, we fix an arbitrary base point $\mathfrak{p}$ of F_n, and take $\{\mathfrak{p}\}\in B_n$ as a base point of B_n. We define the subgroups

$$\mathcal{H}_n:=\{f\in\mathcal{H};\ f\{\mathfrak{p}\}=\{\mathfrak{p}\}\}$$
$$\mathcal{H}_{n,F}:=\{f\in\mathcal{H};\ f(\mathfrak{p})=\mathfrak{p}\}$$

of $\mathcal{H}$. Their path component groups are called the mapping class groups, and act on the Teichmüller space of genus 0 with n punctures as the Teichmüller modular groups. The following hold:

PROPOSITION (1.5). $\pi_1(F_n/G)=\pi_0(\mathcal{H}_{n,F})$.

PROPOSITION (1.6). *The unitary path components of $\mathcal{H}_n$ and $\mathcal{H}_{n,F}$ are weakly contractible.*

PROOF OF (1.5) AND (1.6). In view of a theorem of Kneser [10], the inclusion

$$G=PSL(2,\boldsymbol{C})\longrightarrow\mathcal{H}=\mathrm{Homeo}^+(S^2)$$

is a homotopy equivalence. Hence the homotopy exact sequences of two fibrations

$$\mathcal{H}_{n,F}\longrightarrow\mathcal{H}\xrightarrow{\text{evaluation}}F_n$$
$$G\longrightarrow F_n\underset{(1.3)}{\cong}(F_n/G)\times G\xrightarrow{\text{1st. projection}}(F_n/G)$$

gives the isomorphism

$$\pi_q(F_n/G)\cong\pi_{q-1}(\mathcal{H}_{n,F})$$

for all $q \geqq 1$, which proves (1.5) and (1.6) for $\mathcal{H}_{n,F}$. By the continuity of the evaluation map, the unitary path components of $\mathcal{H}_n$ and $\mathcal{H}_{n,F}$ coincide, which completes the proof of the propositions.

F_n/G is related to the Teichmüller space as follows. Let T_n denote the universal covering of F_n/G,

(1.7) $$T_n := \{1\colon ([0,1],0) \longrightarrow (F_n/G, \mathfrak{p});\ \text{continuous paths}\}$$ modulo homotopy relative to $\{0,1\}$.

Proposition (1.4) implies the contractibility of T_n.

PROPOSITION (1.8) (cf. Bers [3]. p. 138). *The correspondence*

$T_n \longrightarrow$ *the Teichmüller space of genus* 0 *with* n *punctures*

$1 \longmapsto$ *the monodromy map of* F_n/G *along* $1: P^1 - \mathfrak{p} \rightarrow P^1 - l(1)$

defines a bijection.

The proof is immediate from the definition of T_n.

The bijection of (1.8) maps the covering transformation group of $T_n \rightarrow F_n/G \rightarrow B_n/G$ to the Teichmüller modular group $\pi_0(\mathcal{H}_n)$. In particular,

(1.9) $$T_n/\pi_0(\mathcal{H}_n) = B_n/G.$$

With coefficients in a field of characteristic 0, we have

(1.10) $$H^*(B_n/G) \cong H^*(B\pi_0(\mathcal{H}_n))$$

(1.11) $$\cong H^*(B\mathcal{H}_n)$$

(1.12) $$\cong H^*(F_n/G)^{S_n}.$$

Here, for a topological group Γ, $B\Gamma$ denotes the classifying space for principal Γ bundles.

PROOF. Since $\pi_0(\mathcal{H}_{n,F})$ acts freely on T_n, and $\pi_0(\mathcal{H}_n)/\pi_0(\mathcal{H}_{n,F}) \cong S_n$ is finite, so is the isotropy group of $\pi_0(\mathcal{H}_n)$ at each point of T_n. (1.10) follows the contractibility of T_n. Proposition (1.6) implies (1.11). The symmetric group S_n is finite, which proves (1.12).

§ 2. The rational contractibility of B_n/G.

We prove the Q-acyclicity of B_n/G. To prove it, we review some results of Arnol'd [1], [2].

For $1 \leqq \alpha \neq \beta \leqq n$, the holomorphic closed 1-form

$$\omega_{\alpha\beta} := \frac{dz_\alpha - dz_\beta}{z_\alpha - z_\beta}$$

is defined on $F_n C$, where z_α denotes the α-th component of $F_n C \subset C^n$.

Let $A(n)$ denote the exterior algebra over C of all holomorphic closed forms on $F_n C$ generated by the $\omega_{\alpha\beta}$'s. Then Arnol'd [1], [2] shows

(2.1) The de Rham isomorphism induces an isomorphism

$$A(n) \longrightarrow H^*(F_n C; C).$$

(2.2) The vector space of all S_n-invariant forms in $A(n)$ is spanned by

$$1 \quad \text{and} \quad \sum_{\alpha \neq \beta} \omega_{\alpha\beta}.$$

In view of (1.3) we identify

$$F_n/G = F_{n-3}(P^1 - \{0, 1, \infty\})$$

and parametrize it using the coordinates

$$[0, 1, \infty, u_4, \cdots, u_n].$$

Then, for $4 \leqq \alpha \neq \beta \leqq n$, we define the holomorphic closed 1-forms

$$\theta_{\alpha\beta} := \frac{du_\alpha - du_\beta}{u_\alpha - u_\beta} \qquad (2.3)$$

$$\theta'_\alpha := \frac{du_\alpha}{u_\alpha}$$

$$\theta''_\alpha := \frac{du_\alpha}{u_\alpha - 1}$$

on F_n/G, and denote by $A'(n)$ the exterior algebra over C of all holomorphic closed forms on F_n/G generated by the $\theta_{\alpha\beta}$'s, the θ'_α's, and the θ''_α's.

PROPOSITION (2.4). *The de Rham isomorphism induces a surjection*

$$A'(n) \longrightarrow H^*(F_n/G; C).$$

PROOF. Following Arnol'd [1], we consider the fibration

$$\pi: \quad F_n/G \longrightarrow F_{n-1}/G$$
$$[0, 1, \infty, u_4, \cdots, u_n] \longmapsto [0, 1, \infty, u_4, \cdots, u_{n-1}],$$

which admits a section

$$[0, 1, \infty, u_4, \cdots, u_{n-1}]$$
$$\longmapsto \frac{1}{n-2}(0+1+u_4+ \cdots +u_{n-1})$$
$$+2\cdot \max\{|u_\alpha - u_\beta|, |u_\alpha|, |u_\alpha - 1|; \ 4<\alpha\neq\beta<n-1\}$$

and the fundamental group of whose base acts trivially on the cohomology of the fiber, because the monodromy group of π preserves the orientation and the punctures of the fiber. Hence a "Gysin exact sequence"

$$0 \longrightarrow H^q(F_{n-1}/G) \xrightarrow{\pi^*} H^q(F_n/G)$$
$$\xrightarrow{\pi!} H^{q-1}(F_{n-1}/G;\ H^1(P^1-\{n-1 \text{ points}\})) \longrightarrow 0 \ \text{(exact)}$$

is obtained, which proves (2.4) inductively.

To investigate the action of the symmetric group S_n on $A'(n)$, we recall the natural projection

$$[\ ]: F_n \longrightarrow F_n/G$$

and the inclusion

$$i: F_nC \longrightarrow F_n = F_nP^1$$

are open maps. By means of straightforward computations we have

$$\text{(2.5)} \qquad \begin{aligned} i^*[\ ]^*\theta_{\alpha\beta} &= \omega_{\alpha\beta} - \omega_{3\alpha} - \omega_{3\beta} + \omega_{13} + \omega_{23} - \omega_{12} \\ i^*[\ ]^*\theta'_\alpha &= \omega_{1\alpha} - \omega_{3\alpha} + \omega_{23} - \omega_{12} \\ i^*[\ ]^*\theta''_\alpha &= \omega_{2\alpha} - \omega_{3\alpha} + \omega_{13} - \omega_{12}. \end{aligned}$$

So we have

(2.6) The S_n-equivariant homomorphism

$$i^*[\ ]^*: A'(n) \longrightarrow A(n)$$

is injective and its image does not contain the form

$$\sum_{\alpha\neq\beta} \omega_{\alpha\beta}.$$

Therefore (2.2) and Proposition (2.4) implies

$$H^*(F_n/G;\ \boldsymbol{C})^{S_n} \cong H^*(pt;\ \boldsymbol{C}). \tag{2.7}$$

Summing up (1.10)-(1.12) and (2.7), we have the following theorem which has been stated as Theorem A in the introduction.

THEOREM (2.8). *For* $n>3$,

$$\begin{aligned} H^*(B_n/G;\ \boldsymbol{Q}) &\cong H^*(B\mathcal{H}_n;\ \boldsymbol{Q}) \\ &\cong H^*(B\pi_0(\mathcal{H}_n);\ \boldsymbol{Q}) \\ &\cong H^*(pt;\ \boldsymbol{Q}). \end{aligned}$$

COROLLARY (2.9). $\pi_*(B_n/G)\otimes\boldsymbol{Q}=0$.

The proof is obtained by Serre's $\mathcal{C}$-theory [13] and the 1-connectivity of B_n/G [5].

Now we can prove Theorem B stated in the introduction.

COROLLARY (2.10) (see Theorem B in the introduction). *B_4/G and B_5/G are contractible.*

PROOF. As is shown in Proposition (3.6), B_n/G has the homotopy type of an $(n-3)$-dimensional CW-complex. Birman [5] proves the 1-connectivity of B_n/G (see also MacLachlan [12]). Hence, by (2.8), B_4/G and B_5/G are 1-connected and $\boldsymbol{Z}$-acyclic, which proves (2.10).

§ 3. Configurations of electrically charged particles.

In this section, we regard each element of B_n as a configuration of "electrically charged particles" in $S^2\subset\boldsymbol{R}^3$. Then the "potential energy" is expected to be a Morse function and to give some information on the topology of B_n/G.

In the present paper, our "Coulomb's force" is inversely proportional to the distance in $\boldsymbol{R}^3$, and our mechanics is Aristotelean.

We determine all equilibria under our "Coulomb's law" and show the "potential energy" to be a Morse function on F_n/G, for $n=4, 5$. The proof is given in the next section. In consequence, an S_n-equivariant spine for F_4/G and F_5/G is obtained, and the contractibility of B_4/G and B_5/G is proved.

We begin by defining the energy E on F_n as follows;

$$E(z_1, z_2, \cdots, z_n) := -\log \prod_{1\leq\alpha<\beta\leq n} \|\vec{z}_\alpha - \vec{z}_\beta\|^2, \tag{3.1}$$
$$\text{for } (z_1, z_2, \cdots, z_n) \in F_n = F_n P^1,$$

where $\|\cdot\|$ denotes the usual norm on R^3, and $\vec{z}_\alpha$ denotes the stereographic image of $z_\alpha \in P^1$.

Clearly the following hold for $z=(z_1, z_2, \cdots, z_n) \in F_n$,

(3.2) $dE_z = 0$,

iff $\sum_{\beta\neq\alpha} \frac{\vec{z}_\alpha - \vec{z}_\beta}{\|\vec{z}_\alpha - \vec{z}_\beta\|^2}$ is parallel to $\vec{z}_\alpha$ for all $\alpha=1, 2, \cdots, n$,

iff $\sum_{\beta\neq\alpha} \frac{\vec{z}_\alpha - \vec{z}_\beta}{\|\vec{z}_\alpha - \vec{z}_\beta\|^2} = \frac{n-1}{2}\vec{z}_\alpha$ for all $\alpha=1, 2, \cdots, n$.

In the sequal, such a point $z \in F_n$ will be called an *equilibrium*.

Though we are interested in B_n/G, E is not $G=PSL(2, C)$-invariant. But in view of Kirwan [9] (p. 99, Th. 7.4., and p.115, (9.1)),

(3.3) the natural map

$$\mu^{-1}(0)/SO(3) \longrightarrow F_n/G$$

is a diffeomorphism, where $\mu: F_n \to R^3$ is given by

$$\mu(z_1, z_2, \cdots, z_n) := \sum_{\alpha=1}^{n} \vec{z}_\alpha = \sum_{\alpha=1}^{n} \varphi(z_\alpha),$$

which is the moment map for the action on F_n of $SO(3)$ embedded in $PSL(2, C)$ as $PSU(2)$.

Our "Coulomb's law" adapts itself to this situation as follows;

(3.4) For all $z \in F_n$,

$dE_z = 0$,

iff $\mu(z)=0$ and $d(E|_{\mu^{-1}(0)})_z = 0$.

The proof is given by easy calculations.

REMARK. Under the usual Coulomb's law, (3.4) does not hold.

Hence E induces a smooth function on F_n/G which is also denoted by E.

REMARK. Let $L \to P^1$ be the tautological line bundle on the complex projective line P^1. Let $p_\alpha : (P^1)^n \to P^1$ be the α-th projection, $\alpha=1, 2, \cdots, n$. When we put

$$s := \prod_{1 \leqq \alpha < \beta \leqq n} (z_\alpha - z_\beta),$$

s is regarded as a holomorphic section on $((p_1^* L^*) \otimes (p_2^* L^*) \otimes \cdots \otimes (p_n^* L^*))^{\otimes (n-1)}$, where L^* is the dual of L. $L \subset P^1 \times C^2$ possesses a canonical hermitian metric induced by the usual hermitian metric on C^2, so $((p_1^* L^*) \otimes (p_2^* L^*) \otimes \cdots \otimes (p_n^* L^*))^{\otimes (n-1)}$ possesses a positive hermitian metric h induced by this metric on L. Then our "potential energy" E coincides with

$$-\log h(s, s).$$

Hence, in view of Bott [7] § 4,

(3.5) the dimension of the positive eigenspace of the Hessian of E at each critical point on F_n is not smaller than n.

PROPOSITION (3.6). *B_n/G and F_n/G have the homotopy types of $(n-3)$-dimensional CW-complexes.*

PROOF. There exists a sufficiently small S_n-invariant function f on F_n/G such that $E+f$ is a Morse function and that the form $\frac{1}{2\pi i} d'' d' (E+f)$ on F_n is positive (cf. Bott [7] § 4). Thus $E+f$ gives an $(n-3)$-dimensional S_n-spine for F_n/G.

Next we study the Hessian of E. Because the Hessian is too complicated (cf. (3.7)), I failed to show E to be a Morse function for arbitrary n. But we give a sufficient condition for a equilibrium to be nondegenerate.

For $\vec{z} \in S^2$, identify $T_{\vec{z}} S^2$ with the orthogonal complement of $\vec{z}$ in $\boldsymbol{R}^3$;

$$T_{\vec{z}} S^2 = \{ \vec{v} \in \boldsymbol{R}^3;\ \vec{v} \perp \vec{z} \}.$$

Then the exponential map of $S^2 \subset \boldsymbol{R}^3$ is given by

$$\mathrm{Exp}_{\vec{z}} \vec{v} = \vec{z} \cdot \cos \|\vec{v}\| + \vec{v} \cdot \frac{\sin \|\vec{v}\|}{\|\vec{v}\|}, \text{ for } \vec{z} \in S^2 \text{ and } \vec{v} \in T_{\vec{z}} S^2.$$

For $z = (z_1, z_2, \cdots, z_n) \in F_n$ and $\vec{u} = (\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)$, $\vec{v} \in T_z F_n = T_{\vec{z}_1} S^2 \times T_{\vec{z}_2} S^2 \times \cdots \times T_{\vec{z}_n} S^2$, the Hessian of E is given by

$$(3.7) \qquad H_z(\vec{u}, \vec{v}) = \left(\frac{d}{ds} \right)_{s=0} \left(\frac{d}{dt} \right)_{t=0} E(\mathrm{Exp}_z(s\vec{u} + t\vec{v}))$$

$$= -2\sum_{\alpha<\beta}\frac{(\vec{u}_\alpha-\vec{u}_\beta, \vec{v}_\alpha-\vec{v}_\beta)}{\|\vec{z}_\alpha-\vec{z}_\beta\|^2}$$
$$-2\sum_{\alpha<\beta}\left(\frac{\vec{z}_\alpha-\vec{z}_\beta}{\|\vec{z}_\alpha-\vec{z}_\beta\|^2}, -\vec{z}_\alpha(\vec{u}_\alpha, \vec{v}_\alpha)+\vec{z}_\beta(\vec{u}_\beta, \vec{v}_\beta)\right)$$
$$+2\sum_{\alpha<\beta}\frac{2}{\|\vec{z}_\alpha-\vec{z}_\beta\|^4}(\vec{z}_\alpha-\vec{z}_\beta, \vec{u}_\alpha-\vec{u}_\beta)(\vec{z}_\alpha-\vec{z}_\beta, \vec{v}_\alpha-\vec{v}_\beta),$$

where (,) denotes the usual inner product on R^3;

$$((x_1, y_1, h_1), (x_2, y_2, h_2))=x_1x_2+y_1y_2+h_1h_2,$$
$$\text{for } (x_1, y_1, h_1), (x_2, y_2, h_2)\in R^3.$$

By (3.2), for any equilibria $z\in F_n$,

(3.8) $$H_z(\vec{u}, \vec{v})=(H_z(\vec{u}), \vec{v}) = \sum_{\alpha=1}^{n}(H_z(\vec{u})_\alpha, \vec{v}_\alpha),$$

where

(3.9) $$H_z(\vec{u}) := (H_z(\vec{u})_\alpha)_{\alpha=1}^n\in(R^3)^n$$
$$H_z(\vec{u})_\alpha := (n-1)\vec{u}_\alpha-2\sum_{\beta\neq\alpha}\frac{\vec{u}_\alpha-\vec{u}_\beta}{\|\vec{z}_\alpha-\vec{z}_\beta\|^2}+4\sum_{\beta\neq\alpha}\frac{(\vec{z}_\alpha-\vec{z}_\beta, \vec{u}_\alpha-\vec{u}_\beta)}{\|\vec{z}_\alpha-\vec{z}_\beta\|^4}(\vec{z}_\alpha-\vec{z}_\beta).$$

Here, by (3.2), we have

(3.10) $$H_z(T_{\vec{z}}\mu^{-1}(0))\subset T_{\vec{z}}\mu^{-1}(0),$$

where

$$T_{\vec{z}}\mu^{-1}(0)=\left\{(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)\in(R^3)^n;\ \sum_{\alpha=1}^{n}\vec{u}_\alpha=0, \text{ and } \vec{u}_\alpha\perp\vec{z}_\alpha \text{ for } \alpha=1, 2, \cdots, n\right\}.$$

Hence, for any equilibria $z\in F_n$, the following holds;

(3.11) $[z]\in F_n/G$ is a non-degenerate critical point of E on F_n/G, if and only if

$$\mathrm{Ker}(H_z : T_zF_n \longrightarrow T_zF_n)=T_z(SO(3)\cdot z).$$

LEMMA (3.12). *Let $z=(z_1, z_2, \cdots, z_n)\in F_n$ be an equilibrium. Suppose the vectors $\vec{z}_1, \vec{z}_2, \cdots, \vec{z}_n$ generate R^3, and the* $\left(\frac{1}{2}n(n-1)\right)\times\left(\frac{1}{2}n(n-1)\right)$

matrix A (defined below) is non-singular, then $[z]\in F_n/G$ is a non-degenerate critical point of E on F_n/G, where $A=(A_{(\alpha\beta)(\gamma\delta)})$ is defined as follows. ((α,β) and (γ,δ) run over the set of all 2-point sets of $\{1,2,\cdots,n\}$.)

$$A_{(\alpha\beta)(\alpha\beta)}=n-1+\frac{2}{\|\vec{z}_\alpha-\vec{z}_\beta\|^2}-\sum_{\gamma\neq\alpha}\frac{1}{\|\vec{z}_\alpha-\vec{z}_\gamma\|^2}-\sum_{\gamma\neq\beta}\frac{1}{\|\vec{z}_\beta-\vec{z}_\gamma\|^2}$$

$$A_{(\alpha\beta)(\beta\gamma)}=\frac{1}{\|\vec{z}_\alpha-\vec{z}_\beta\|^2}-\frac{2}{\|\vec{z}_\beta-\vec{z}_\gamma\|^4}(\vec{z}_\beta-\vec{z}_\gamma,\vec{z}_\alpha)$$

if α, β, γ are distinct.

$$A_{(\alpha\beta)(\gamma\delta)}=0 \qquad \text{if } \alpha, \beta, \gamma, \delta \text{ are distinct.}$$

PROOF. We reduce (3.12) to (3.11). Remark

$$T_z(SO(3)\cdot z)=\{\vec{u}=(\vec{u}_1,\vec{u}_2,\cdots,\vec{u}_n);\ \text{There exists an } \vec{a}\in \boldsymbol{R}^3 \text{ s.t., } \vec{u}_\alpha=\vec{a}\times\vec{z}_\alpha \text{ for all } \alpha=1,2,\cdots,n\},$$

where $\times$ denotes the vector product on the Euclid space $\boldsymbol{R}^3$. Easily, we have the following;

(3.13) Suppose $n\geqq 3$,

$$\vec{z}_1,\vec{z}_2,\cdots,\vec{z}_n\in S^2 \text{ generate } \boldsymbol{R}^3,$$

$$\text{and } \vec{v}_1,\vec{v}_2,\cdots,\vec{v}_n\in\boldsymbol{R}^3\ \vec{v}_\alpha\perp\vec{z}_\alpha \text{ for } \alpha=1,2,\cdots,n.$$

If, for all $\alpha,\beta=1,2,\cdots,n$,

$$(\vec{z}_\alpha-\vec{z}_\beta,\vec{v}_\alpha-\vec{v}_\beta)=0,$$

then there is a unique $\vec{a}\in\boldsymbol{R}^3$, such that

$$\vec{v}_\alpha=\vec{a}\times\vec{z}_\alpha \quad \text{for all } \alpha=1,2,\cdots,n.$$

If we set, for $\vec{u}\in T_zF_n$ and $\alpha,\beta=1,2,\cdots,n$,

$$X_{\alpha\beta}=(\vec{z}_\alpha-\vec{z}_\beta,\vec{u}_\alpha-\vec{u}_\beta),$$

then from (3.2), (3.8) and (3.9) follows

$$(H_z(\vec{u})_\alpha,\vec{z}_\beta)+(H_z(\vec{u})_\beta,\vec{z}_\alpha)=-2\sum_{(\gamma\delta)}A_{(\alpha\beta)(\gamma\delta)}X_{\gamma\delta}.$$

So if $\vec{u} \in \operatorname{Ker} H_z$, then

$$\sum_{(\gamma\delta)} A_{(\alpha\beta)(\gamma\delta)} X_{\gamma\delta} = 0,$$

which implies together with the assumption A is non-singular

$$X_{\gamma\delta} = 0 \qquad \text{for all } (\gamma\delta).$$

Hence by (3.13)

$$\vec{u} \in T_z(SO(3)\cdot z),$$

which completes the proof of the lemma.

Next we study the equilibrium $z=(z_1, z_2, \cdots, z_n) \in F_n$ corresponding to the regular polygon on the equator;

$$z_\alpha = \zeta^\alpha, \quad \zeta = \exp\left(\frac{2\pi i}{n}\right), \qquad \alpha = 1, 2, \cdots, n.$$

Clearly z is a critical point of E, and the isotropy group of S_n at $[z] \in F_n/G$ is the dihedral group of order $2n$, D_n.

LEMMA (3.14). *Let $z=(z_1, z_2, \cdots, z_n) \in F_n$ be as above. Denote by S the unit sphere of the negative eigenspace of the Hessian of E at $[z] \in F_n/G$. Then $[z] \in F_n/G$ is a non-degenerate critical point of E. Furthermore the quotient S/D_n is contractible if and only if $n=4, 5, 6$.*

PROOF. Denote by H the Hessian of E on F_n at z. Since E on F_n is degenerate in the direction of the $SO(3)$-action, if we find an $(n-3)$-dimensional subspace in $T_z\mu^{-1}(0)$ generated by negative eigenvectors of H, then E is non-degenerate at $[z] \in F_n/G$ and the subspace is isomorphic to the negative eigenspace of the Hessian of E at $[z]$ through the projection $[\ \ ] : F_n \longrightarrow F_n/G$, by (3.5).

For $2 \leqq k,\ l \leqq n-2$, we set

$$\begin{aligned}
u_k &:= ((u_k)_\alpha)_{\alpha=1}^n \in T_z\mu^{-1}(0) \subset (\boldsymbol{R}^3)^n \\
v_k &:= ((v_k)_\alpha)_{\alpha=1}^n \in T_z\mu^{-1}(0) \subset (\boldsymbol{R}^3)^n \\
(u_k)_\alpha &:= \left(0, 0, \cos\frac{2\pi k\alpha}{n}\right) \\
(v_k)_\alpha &:= \left(0, 0, \sin\frac{2\pi k\alpha}{n}\right),
\end{aligned}$$

then, by (3.9), when H extends to $T_z\mu^{-1}(0)\otimes \boldsymbol{C}$ complex bilinearly,

$$
\begin{aligned}
&H(u_k+iv_k, u_l+iv_l)\\
&=(n-1)\sum_{\alpha=1}^{n}((u_k+iv_k)_\alpha, (u_l+iv_l)_\alpha)\\
&\quad-\sum_{\alpha\neq\beta}\frac{1}{\|\vec{z}_\alpha-\vec{z}_\beta\|^2}((u_k+iv_k)_\alpha-(u_k+iv_k)_\beta, (u_l+iv_l)_\alpha-(u_l+iv_l)_\beta)\\
&=(n-1)\sum_{\alpha=1}^{n}\zeta^{\alpha k}\zeta^{\alpha l}-\sum_{\alpha\neq\beta}\frac{1}{|\zeta^\alpha-\zeta^\beta|^2}(\zeta^{\alpha k}-\zeta^{\beta k})(\zeta^{\alpha l}-\zeta^{\beta l}).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
(3.15)\qquad H(u_k+iv_k, u_l+iv_l)&=-n &&\text{if } k+l=n\\
&=0 &&\text{otherwise,}
\end{aligned}
$$

which implies, for $2\leqq k,\ l\leqq\frac{n}{2}$,

$$
\begin{aligned}
H(u_k, u_l)&=H(v_k, v_l)=-n\delta_{kl}\\
H(u_k, v_l)&=H(v_k, u_l)=0,
\end{aligned}
$$

where δ_{kl} denotes the Kronecker's delta.

We denote by N the vector subspace in $T_z\varphi^{-1}(0)$ generated by

$$
\left\{u_k\,;\,2\leqq k\leqq\frac{n}{2}\right\}\cup\left\{v_k\,;\,2\leqq k<\frac{n}{2}\right\},
$$

which is linearly independent. Hence E is nondegenerate at $[z]$, and N may be regarded as the negative eigenspace of the hessian of E at $[z]$. S is the unit sphere of N.

We identify N with $\boldsymbol{R}^\varepsilon\oplus\boldsymbol{C}^{[(n-3)/2]}$ ($\varepsilon=0$ if n is odd, 1 if n is even.) by

$$
x_0u_{n/2}+\sum_{2\leqq k<n/2}x_ku_k+y_kv_k\longmapsto x_0e_0+\sum_{2\leqq k<n/2}(x_k+iy_k)e_k,
$$

for $x_0, x_k, y_k\in\boldsymbol{R}$, where e_k, $2\leqq k<\frac{n}{2}$, denote the standard basis of $\boldsymbol{C}^{[(n-3)/2]}$ and e_0 that of $\boldsymbol{R}^\varepsilon$.

When we put

$$
r:=\begin{pmatrix}1 & 2 & \cdots & n\\ n & 1 & \cdots & n-1\end{pmatrix}\in S_n
$$

$$
s:=\begin{pmatrix}1 & 2 & \cdots & n-1 & n\\ n-1 & n-2 & \cdots & 1 & n\end{pmatrix}\in S_n,
$$

r and s generate the isotropy group of S_n at $[z]\in F_n/G$. r and s act N as follows; for all $(x_0, w_k)\in R^{\epsilon}\oplus C^{[(n-3)/2]}$,

$$(3.16)\qquad r(x_0e_0+\sum w_ke_k)=-x_0e_0+\sum w_k\zeta^ke_k$$

$$s(x_0e_0+\sum \overline{w_k}e_k)=-x_0e_0-\sum \overline{w_k}e_k.$$

We can now investigate the homotopy type of S/D_n.

Case 1 $n=4$. Then $N=Re_0$. (3.16) implies $S/D_4=\{pt\}$.

Case 2 $n=5$. Then $N=Ce_2$. (3.16) implies $S/\langle r\rangle=S^1$ and under this identification, s acts $S^1\subset C$ as a reflection $z\longmapsto -\bar{z}$. Hence $S/D_5=[0, 1]$ is contractible.

Case 3 $n=6$. Then $N=R\oplus C$. $\langle r^3\rangle$ is a normal subgroup in D_6 and by (3.16), for $(x_0, w_2)\in R\oplus C$, we have

$$r^3(x_0e_0+w_2e_2)=-x_0e_0+w_2e_2.$$

Hence $S/\langle r^3\rangle=$the cone of S^1. Under this identification, (3.16) implies $D_6/\langle r^3\rangle$ acts the cone of S^1 preserving the cone structure. Hence S/D_6 is contractible.

Case 4 $n\geqq 7$. $S=S^{n-4}$ has a D_n-equivariant triangulation so that the fixed point set of D_n is a subcomplex. Since the fixed point set of D_n has the codimension >1 in this case, D_n acts freely on the $(n-4)$-simplices and the $(n-5)$-simplices. Therefore S/D_n has a mod 2 "fundamental class" $\in H_{n-4}(S/D_n; Z/2)$. This implies S/D_n is not contractible.

This completes the proof of the lemma.

THEOREM (3.17). *For $n=4, 5$, E is a Morse function on F_n/G and all its critical points are S_n-equivalent to the following.*

$n=4$

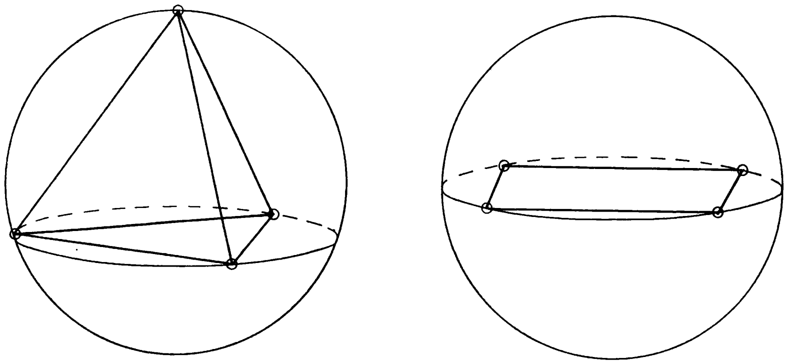
(1) $\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\omega, \frac{1}{\sqrt{2}}\omega^2, \infty\right]$ index$=0$, $\omega=\exp\frac{2\pi i}{3}$

the tetrahedra type *cf. Figure* 1(1)

(2) $[1, i, -1, -i]$ index$=1$

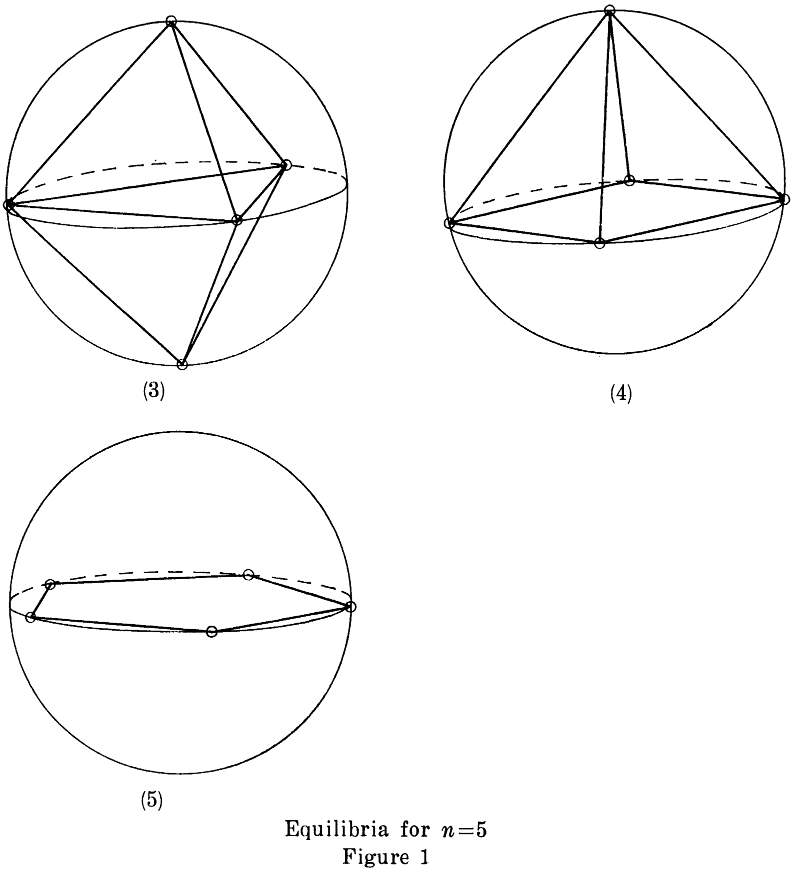
the square type *cf. Figure* 1(2)

$n=5$



(1) (2)

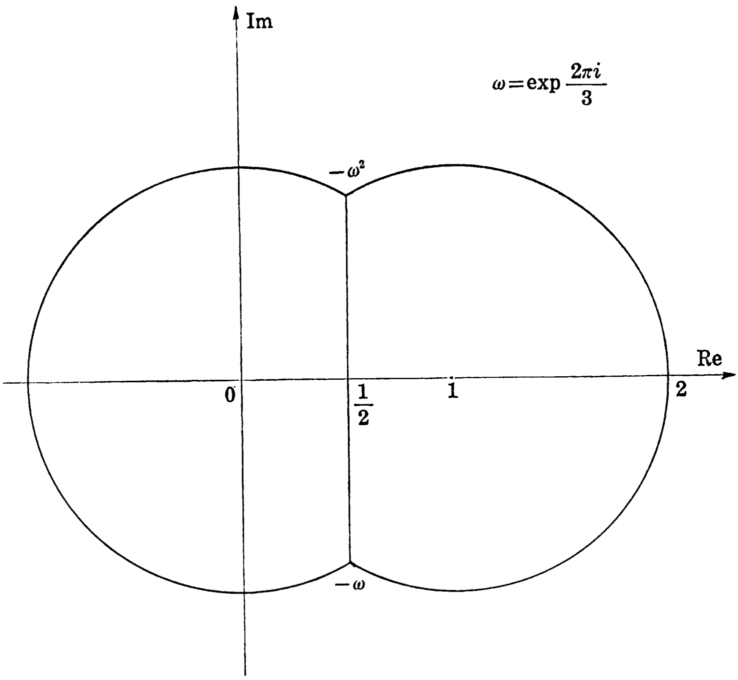
Equilibria for $n=4$
Figure 1



(3) (4)

(5)

Equilibria for $n=5$
Figure 1



S_4-spine for $F_4/G=P^1-\{0,1,\infty\}$
Figure 2

(3) $[0, \infty, 1, \omega, \omega^2]$ index$=0$
cf. Figure 1(3)

(4) $\left[\frac{\sqrt{15}}{5}, \frac{\sqrt{15}}{5}i, -\frac{\sqrt{15}}{5}, -\frac{\sqrt{15}}{5}i, \infty\right]$ index$=1$
cf. Figure 1(4)

(5) $[1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4]$ index$=2$, $\zeta_5=\exp\frac{2\pi i}{5}$
the pentagon type *cf. Figure* 1(5).

The proof will be given in § 4.

REMARK. By (1.4), F_4/G is identified with $P^1-\{0,1,\infty\}$. Theorem (3.17) gives a S_4-equivariant spine of F_4/G as Figure 2.

COROLLARY (3.18). *B_4/G and B_5/G are contractible. (See Theorem*

B in the introduction.)

PROOF. We prove (3.18) for $n=5$. Lemma (3.14) implies the 2-cells which the pentagon type critical points induce are contractible relative to boundary in B_5/G. Hence B_5/G has a homotopy type of 1-dimensional finite cell complex. But Theorem (2.8) implies $b_1(B_5/G)=0$, which proves B_5/G is contractible.

§ 4. Proof of Theorem (3.17).

We must show the following

(1) All critical points are given in Figure 1.

(2) The critical points given in Figure 1 are non-degenerate.

(2) follows Lemma (3.12) and Lemma (3.14). The index of each critical point is easily found. For the rest we prove (1).

First, by easy computations, the following hold;

(4.1) Let $(z_1, z_2, \cdots, z_n) \in F_n$ be a equilibrium. Then we have, for $\alpha, \beta = 1, 2, \cdots, n$,

$$1+(\vec{z}_\alpha, \vec{z}_\beta)+\sum_{\gamma\neq\alpha,\beta}\frac{(\vec{z}_\alpha, \vec{z}_\gamma)-(\vec{z}_\alpha, \vec{z}_\beta)(\vec{z}_\beta, \vec{z}_\gamma)}{1-(\vec{z}_\beta, \vec{z}_\gamma)}=0.$$

The proof is omitted.

Next we remark the electric field the vectors $\vec{z}_1, \vec{z}_2, \cdots, \vec{z}_{n-1} \in S^2-\vec{\infty}$ determine at $\vec{\infty}$ (under our "Coulomb's law") is given by

$$(4.2)\qquad \left(2\sum_{\alpha=1}^{n-1} z_\alpha,\ *\right)\in \boldsymbol{R}^3.$$

Now suppose $(z_1, z_2, \cdots, z_n) \in F_n$ is a equilibrium, then, since $\dfrac{i}{\sqrt{1+|z_\alpha|^2}} \times \begin{pmatrix} \bar{z}_\alpha & 1 \\ 1 & -z_\alpha \end{pmatrix} \in SU(2)$, (4, 2) implies, for each $\alpha=1, 2, \cdots, n$,

$$\sum_{\beta\neq\alpha}\frac{\bar{z}_\alpha z_\beta+1}{z_\beta-z_\alpha}=0,$$

that is,

$$(4.3)\qquad \frac{\bar{z}_\alpha}{1+|z_\alpha|^2}=\frac{1}{n-1}\sum_{\beta\neq\alpha}\frac{1}{z_\alpha-z_\beta}$$

$$=A_\alpha(z_1, z_2, \cdots, z_n),$$

where we set

$$A_\alpha = A_\alpha(z_1, z_2, \cdots, z_n) := \frac{1}{n-1} \sum_{\beta \neq \alpha} \frac{1}{z_\alpha - z_\beta}.$$

On the other hand, (1.1) implies

$$(\vec{z}_\alpha, \vec{z}_\beta) = \frac{2}{(1+|z_\alpha|^2)(1+|z_\beta|^2)} (z_\alpha \bar{z}_\beta + \bar{z}_\alpha z_\beta) + \frac{1-|z_\alpha|^2}{1+|z_\alpha|^2} \frac{1-|z_\beta|^2}{1+|z_\beta|^2}.$$

From (4.3) follows

$$(\vec{z}_\alpha, \vec{z}_\beta) = 2(1-(z_\alpha - z_\beta)A_\alpha)(1-(z_\beta - z_\alpha)A_\beta) - 1.$$

Hence we have

$$(\vec{z}_\alpha, \vec{z}_\beta) = \frac{2}{(n-1)^2} \Big(\sum_{\gamma \neq \alpha} \frac{z_\beta - z_\gamma}{z_\alpha - z_\gamma} \Big) \Big(\sum_{\gamma \neq \beta} \frac{z_\alpha - z_\gamma}{z_\beta - z_\gamma} \Big) - 1, \tag{4.4}$$

where for $z_\alpha = \infty$, (4.4) may be interpreted as follows;

$$(\vec{\infty}, \vec{z}_\beta) = \frac{2}{(n-1)^2} \Big(\sum_{\gamma \neq \alpha} z_\beta - z_\gamma \Big) \Big(\sum_{\gamma \neq \beta} \frac{1}{z_\beta - z_\gamma} \Big) - 1. \tag{4.5}$$

We remark the RHS's of (4.4) and (4.5) are invariant under the affine transformations;

$$z \longmapsto rz + s \qquad r \in C^*, \ s \in C.$$

Hence, by (4.1) (4.4) (4.5), we have

LEMMA (4.6). *Let $[0, 1, \infty, u_4, \cdots, u_n] \in F_n/G$ be a critical point of E. Then, for $\alpha, \beta = 1, 2, \cdots, n$, $\alpha \neq \beta$,*

$$1 + I_{\alpha\beta} + \sum_{\gamma \neq \alpha, \beta} \frac{I_{\alpha\gamma} - I_{\alpha\beta} I_{\beta\gamma}}{1 - I_{\beta\gamma}} = 0,$$

where

$$\begin{aligned} I_{\alpha\beta} &= I_{\alpha\beta}(z_1, z_2, \cdots, z_n) \\ &:= \frac{2}{(n-1)^2} \Big(\sum_{\gamma \neq \alpha} \frac{z_\beta - z_\gamma}{z_\alpha - z_\gamma} \Big) \Big(\sum_{\gamma \neq \beta} \frac{z_\alpha - z_\gamma}{z_\beta - z_\gamma} \Big) \Big|_{(z_1, \cdots, z_n) = (0, 1, \infty, u_4, \cdots, u_n)}. \end{aligned}$$

Here each $I_{\alpha\beta}$ is holomorphic on F_n/G. When we set

$$\begin{aligned} V_{\alpha\beta} &= V_{\alpha\beta}(z_1, z_2, \cdots, z_n) \\ &:= \Big(1 + I_{\alpha\beta} + \sum_{\gamma \neq \alpha, \beta} \frac{I_{\alpha\gamma} - I_{\alpha\beta} I_{\beta\gamma}}{1 - I_{\beta\gamma}} \Big) \prod_{\gamma \neq \alpha, \beta} (1 - I_{\beta\gamma}) \end{aligned}$$

for $\alpha, \beta=1, 2, \cdots, n$, (4.6) implies

(4.7) All critical points of E are included in the complex analytic subset

$$\bigcap_{\alpha,\beta\neq 1}^{n}\left\{V_{\alpha\beta}=0\right\}$$

of $F_n/G\subset(\boldsymbol{P}^1)^{n-3}$.

Case $n=4$.
Put $u_4=u$, then $I_{\alpha\beta}=I_{\beta\alpha}=I_{\alpha\beta}(u)$ is given by

$\beta\backslash\alpha$	1	2	3	4
1	1	$I\left(\frac{u}{u-1}\right)$	$I(u)$	$I(1-u)$
2		1	$I(1-u)$	$I(u)$
3			1	$I\left(\frac{u}{u-1}\right)$
4				1 ,

where $I(u)=I_{13}(u)=\frac{2}{9}(1+u)\left(1+\frac{1}{u}\right)-1$. Hence by a straightforward computation,

$$V_{13}(u)=\frac{2^3(u+1)^2(2u-1)(u-2)(u^2-u+1)}{3^5u^2(u-1)^2},$$

which proves Theorem 2 for $n=4$.

Case $n=5$.
Put $u=u_4$, $v=u_5$, then $I_{\alpha\beta}=I_{\beta\alpha}=I_{\alpha\beta}(u, v)$ is given by

$\beta\backslash\alpha$	1	2	3	4	5
1	1	$I\left(\frac{u}{u-1}, \frac{v}{v-1}\right)$	$I(u, v)$	$I\left(1-u, \frac{v-u}{v}\right)$	$I\left(1-v, \frac{u-v}{u}\right)$
2		1	$I(1-u, 1-v)$	$I\left(u, \frac{v-u}{v-1}\right)$	$I\left(v, \frac{u-v}{u-1}\right)$

3			1	$I\left(\frac{u-1}{u}, \frac{u-v}{u}\right)$	$I\left(\frac{v-1}{v}, \frac{v-u}{v}\right)$
4				1	$I\left(\frac{u}{v}, \frac{1-u}{1-v}\right)$
5					1 ,

where $I(u, v)=I_{13}(u, v)=\frac{1}{8}(1+u+v)\left(1+\frac{1}{u}+\frac{1}{v}\right)-1.$

Under the identification $F_5/G=F_2(P^1-\{0, 1, \infty\})\subset P^1\times P^1$, denote by $D_{\alpha\beta}$ the effective divisor which the zeros of $V_{\alpha\beta}$ determine on $P^1\times P^1$.

We claim

(A) All symmetric (i.e., whose isotropy groups of S_5 are non-trivial) critical points of E on F_5/G are given in Figure 1.

(B) The intersection number of the divisors D_{13} and D_{31} is $2\cdot 9^2=162$.

(C) The divisors D_{13} and D_{31} have no common components.

If (A), (B) and (C) are proved, Theorem 2 for $n=5$ is proved as follows.

First we remark the total number of the critical points in Figure 1 is 62. ((3): 20, (4): 30, (5): 12.) Hence, by (A), if E has other critical points on F_5/G than in Figure 1, then the total number of the critical points is not smaller than $62+5!=182$. But (4.6) Lemma 3 and (B), (C) imply that the total number of the critical points of E is not larger than 162. Therefore all critical points of E are given in Figure 1.
We prove (A), (B) and (C):

PROOF OF (A). Every non-trivial element of the symmetric group S_5 is conjugate to one of the following;

$$(12),\ (13)(45),\ (123),\ (12)(345),\ (1425),\ (12345).$$

Since the linear fractional transformation fixes 3 distinct points is trivial, (12) and (12)(345) $(((12)(345))^3=(12))$ have no fixed points. Easy calculations show the fixed point sets of (123), (1425) and (12345) are contained in that of (13)(45), which is equal to $\{uv=1\}$.

By a straightforward computation, we obtain

(4.8)
$$V_{13}\left(u, \frac{1}{u}\right)=\frac{3(u^2+u+1)^2(u^2-u+1)(u^2+1)(u^2+5u+1)(u^2-u-1)(u^2+u-1)}{2^9},$$

where the points corresponding to the roots of $u^2+5u+1=0$ satisfy $I_{13}=1$, which contradicts (4.4). Hence (4.8) implies (A).

PROOF OF (B). We need the following results;

$$\lim_{u\to 0} u^2 V_{13}(u, v)=\frac{3(v+1)^2(2v^2+7v+2)}{2^{10}v} \tag{4.9}$$

$$\lim_{v\to 0} v^2 V_{13}(u, v)=\frac{3(u+1)^2(2u^2+7u+2)}{2^{10}u}$$

$$\lim_{u\to 1} (1-u)^2 V_{13}(u, v)=\frac{3(v-2)(v+2)(2v^2-11v+11)}{2^{10}(1-v)}$$

$$\lim_{v\to 1} (1-v)^2 V_{13}(u, v)=\frac{3(u-2)(u+2)(2u^2-11u+11)}{2^{10}(1-u)}$$

$$\lim_{u\to\infty} \frac{1}{u^3} V_{13}(u, v)=\frac{3(v-2)(2v-1)(v+1)}{2^{10}(v-1)^2v^2}$$

$$\lim_{v\to\infty} \frac{1}{v^3} V_{13}(u, v)=\frac{3(u-2)(2u-1)(u+1)}{2^{10}(u-1)^2u^2}$$

$$\lim_{v\to u} (v-u)^2 V_{13}(u, v)=\frac{3(2u-1)(2u+1)(11u^2-11u+2)}{2^{10}u(1-u)},$$

which are obtained by straightforward computations.

Since V_{13} is holomorphic on F_5/G, (4.9) implies that the divisor determined by the poles of V_{13} is equal to

$$-2\{u=0\}\ -2\{u=1\}\ -3\{u=\infty\}\ -2\{u=v\}$$
$$-2\{v=0\}\ -2\{v=1\}\ -3\{v=\infty\}.$$

Hence the homology class induced by D_{13} is

$$9[\boldsymbol{P}^1{}_u]+9[\boldsymbol{P}^1{}_v]\in H_2(\boldsymbol{P}^1\times\boldsymbol{P}^1;\ \boldsymbol{Z}),$$

where $[\boldsymbol{P}^1{}_u]$ and $[\boldsymbol{P}^1{}_v]$ are the homology classes induced by $\boldsymbol{P}^1\times\{0\}$ and $\{0\}\times\boldsymbol{P}^1$, respectively. Since the transformation (13) : $(u, v)\longmapsto\left(\frac{1}{u}, \frac{1}{v}\right)$ extends to $\boldsymbol{P}^1\times\boldsymbol{P}^1$ holomorphically and permutes D_{13} and D_{31}, the homology class induced by D_{31} is equal to

$$9[\boldsymbol{P}^1{}_u]+9[\boldsymbol{P}^1{}_v].$$

Hence the intersection number of D_{13} and D_{31} is

$$2\cdot 9^2=162,$$

which proves (B).

PROOF OF (C). To factorize V_{13}, we need the following

$$(4.10)\qquad V_{13}(u,\ -1-u)=0$$

$$V_{13}(u,1-u)=V_{13}(u,u-1)=V_{13}(u,u+1)=0,$$

which are obtained by straightforward computations. The same results hold for V_{31}. Hence

$$D' :=D_{13}-\{u+v+1=0\}-\{u+v-1=0\}$$
$$-\{u-v-1=0\}-\{u-v+1=0\}$$
$$D'' :=D_{31}-\{u+v+uv=0\}-\{u+v-uv=0\}$$
$$-\{u-v-uv=0\}-\{u-v+uv=0\}$$

are both effective divisors on $P^1\times P^1$.

By (4.9), the intersections of D' and $u=0,1,\infty$ are given as follows;

$$(4.11)\qquad \begin{array}{ll} u=0 & v=\gamma,\ \gamma^*,\ 0,\ 0,\ 0. \\ u=1 & v=1-\gamma,\ 1-\gamma^*,\ 1,\ 1,\ 1. \\ u=\infty & v=2,\ -1,\ \frac{1}{2},\ \infty,\ \infty, \end{array}$$

where γ and γ^* denote the roots of $2v^2+7v+2=0$. The same results hold for D''.

Suppose D_{13} and D_{31} have some non-empty common components C. Then C is contained in D' and D''. (4.11) implies the intersections of C and $u=0,1,\infty$ are contained in the following;

$$(4.12)\qquad \begin{array}{ll} u=0 & v=0,\ 0. \\ u=1 & v=1,\ 1,\ 1. \\ u=\infty & v=\infty,\ \infty. \end{array}$$

Now since D_{13} and D_{31} are (45)-invariant, we may suppose C is (45)-invariant. Hence the homology class induced by C is

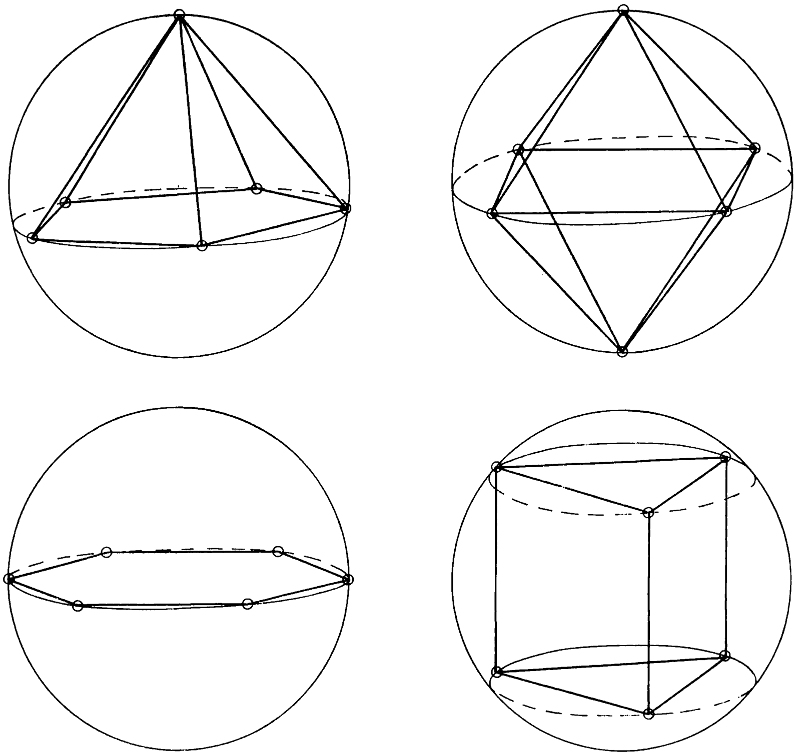


Figure 3

$$[P^1_u]+[P^1_v] \quad \text{or} \quad 2[P^1_u]+2[P^1_v].$$

As is well known, such divisor C is the zeros of a linear combination of 1, u, v, u^2, v^2 and v^2. Hence, by (4.12), C is equal to $\{u-v=0\}$ or $2\{u-v=0\}$, which contradicts the definitions of D_{13} and D_{31}.

Thus the proof of Theorem (3.17) is completed.

REMARK. For $n=6$, all critical points of E on F_6/G are expected to be given in Figure 3. Indeed, the alternative sum of the critical points given in Figure 3 is equal to the Euler number of F_6/G. If this is true, an alternative proof of the contractibility of B_6/G is obtained by the same method for $n=5$. But I do not know whether this is true or not.

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