J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
37 (1990), 289-295.

Uniform category

Dedicated to Professor Akio Hattori on his sixtieth birthday

By I. M. JAMES

1. Introduction

The purpose of this note is to introduce a uniform version of Lusternik-Schnirelmann category (see [12]) and use it to prove

THEOREM (1.1). *Let M be a complete connected C^2 Finsler manifold, without boundary, and suppose that M is not compact. Let ϕ be a C^2-real-valued function on M which satisfies the Palais-Smale condition and is bounded above and below. Then the critical set of ϕ is infinite.*

The ordinary Lusternik-Schnirelmann theorem is more general, of course, but it gives less information about the critical set. For example, take M to be the real line. When ϕ satisfies the above conditions the ordinary theorem tells us only that the critical set is non-empty. Of course the ordinary theorem also applies to functions which are unbounded above or below; for these the conclusion of (1.1) is obviously false.

2. Uniform homotopy

For simplicity we work in terms of metric spaces throughout, although some of the discussion applies just as well to uniform spaces generally. The term *uniform homotopy* is used in two different senses in the literature. Isbell [10], Gutiérrez-Burzaco [9] and others use it in the sense of uniformly continuous homotopy between uniformly continuous functions. However, Dowker [7] also Eilenberg and Steenrod [8], use it in a different sense, as follows, and in this article we follow Dowker, not Isbell.

DEFINITION (2.1). A homotopy $f_t: X \to Y$ $(0 \leq t \leq 1)$, where X and Y are metric spaces, is uniform if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|s-t| < \delta$ implies $\rho(f_s(x), f_t(x)) < \varepsilon$ for all $x \in X$.

Here ρ denotes the metric on Y. The condition is automatically satisfied when X is compact since then the cylinder $X \times [0, 1]$ is compact and so the homotopy, regarded as a continuous function on the cylinder, is uniformly continuous.

Uniform homotopy is obviously an equivalence relation between continuous functions. The equivalence classes are called *uniform homotopy classes*. The concepts of *uniform homotopy equivalence* and *uniform homotopy type* are defined in the usual way, using uniform homotopy instead of ordinary homotopy.

A metric space X is said to be *uniformly contractible* if X has the same uniform homotopy type as a point-space. Clearly X is uniformly contractible if and only if the identity function id_X is uniformly homotopic to a constant function.

DEFINITION (2.2). The subset A of the metric space X is uniformly categorical if the inclusion function $A \to X$ is uniformly homotopic to a constant function.

Uniform homotopy, in the sense of Dowker, arises naturally in the study of Čech cohomology based on finite covers. This cohomology theory is an invariant of uniform homotopy type, but not of ordinary homotopy type. Uniform homotopy theory has been studied by Calder [3], [4] and by Calder and Siegel [5], [6].

3. Uniform category

We now modify the well-known definition of topological category by substituting uniform homotopy for ordinary homotopy, as follows.

DEFINITION (3.1). The uniform category of the metric space X is the least cardinal number k such that X can be covered by k uniformly categorical closed subsets.

The number k is written $\mathrm{unicat}(X)$. Clearly $\mathrm{unicat}(X)=1$ if and only if X is uniformly contractible. Also $\mathrm{unicat}(X) \geq \mathrm{cat}(X)$, the topological category of X, and equality holds when X is compact.

PROPOSITION (3.2). *If the uniform category of the metric space X is finite then X is bounded.*

For let A be a uniformly categorical subset of X. Let $f_t : A \to X$ be a uniform deformation of the inclusion into the constant function at x_0, say. Given $\varepsilon > 0$ choose δ so that $|s-t| < \delta$ implies $\rho(f_s(x), f_t(x)) < \varepsilon$ for all $x \in A$. Then if $n\delta > 1$ there exists an ε-chain

$$x=x_1,\ \cdots,\ x_n=x_0$$

with n links, and so A is of finite diameter less than $2n\varepsilon$. Thus A is bounded and hence X is bounded if unicat(X) is finite.

In particular take $X=\boldsymbol{R}^k$, with the euclidean metric. The uniformly categorical subsets of X are precisely the compact subsets, and so unicat($\boldsymbol{R}^k$) is infinite.

Recall that a metric space X is said to be *totally bounded*, or *precompact*, if for each $\varepsilon>0$ there exists a finite subset $S\subset X$ such that $x\in X$ implies $\rho(x,\xi)<\varepsilon$ for some $\xi\in S$. Here, again, ρ denotes the metric. Also recall that X is uniformly locally compact if X admits a uniform covering by compact subsets. In that case, by (1.14) of [10], every uniform cover admits a star-finite uniform refinement. We prove

PROPOSITION (3.3). *Let X be a uniformly locally compact metric space. If* unicat(X) *is finite then X is totally bounded. If, further, X is complete then X is compact.*

In fact the argument used to prove (3.2) shows that X is finitely chainable, in the sense of Atsuji [1], and this implies that every star-finite uniform covering must be finite.

The properties of uniform category are similar to those of topological category. For example

PROPOSITION (3.4). *The uniform category of a metric space is an invariant of uniform homotopy type.*

For let $\phi: Y\to X$ and $\psi: X\to Y$ be continuous functions, where X and Y are metric spaces. Suppose that $\psi\phi$ is uniformly homotopic to id_X. If A is a closed uniformly categorical subset of X then its preimage $\phi^{-1}A$ is a closed uniformly categorical subset of Y, and similarly with coverings. Hence unicat(X)$\geq$unicat(Y). The opposite inequality holds when $\psi\phi$ is uniformly homotopic to id_Y, therefore equality holds when ϕ is a uniform homotopy equivalence.

Other elementary results concerning uniform category, for example the product inequality, can be proved by similar straightforward modifications of the arguments used in the topological case.

An important lower bound (see [11]) for the topological category is provided by the index of nilpotency of the reduced cohomology ring of the space in question. In the case of uniform category the Čech cohomology based on finite covers provides an analogous lower bound which might, a priori, be an improvement on that provided by ordinary Čech cohomology. However it should be noted that Calder [3] has shown that for finite-dimensional CW complexes the natural functor from the finite Čech groups to the ordinary Čech groups is an isomorphism in all dimensions greater than one, so that no improvement is obtained for this class of spaces.

It might be thought that a more natural concept than uniform category would be compact category, i.e. the least number of topologically categorical compact subsets required to form a covering. Compact category is easily shown to be an invariant of proper homotopy type. However it is not at all clear whether there is a "proper" version of the Lusternik-Schnirelmann theorem involving compact category. Browder refers briefly to compact category in §8 of [2] but he uses the term in a different, although not unrelated, sense.

4. Relative uniform category

It is a routine exercise to relativize the notion of uniform category (indeed a functional version can be defined in a similar fashion).

Definition (4.1). The relative uniform category $\operatorname{unicat}_X(A)$ of a subspace A of the metric space X is the least cardinal number k such that A can be covered by k uniformly categorical closed subsets of X.

Now consider, for each positive integer k, the family

$$S_k = \{A \mid \operatorname{unicat}_X(A) \geq k\}$$

of subsets of X. Thus $S_1 \supset S_2 \supset \cdots \supset S_k \supset \cdots$. Note that if $f_t : X \to X$ is a uniform deformation of id_X then $A \in S_k$ implies $f_1 A \in S_k$. From now on let us suppose that X is complete and non-compact so that $\operatorname{unicat}(X)$ is infinite and S_k is non-empty for each positive integer k.

Given a continuous real-valued function ϕ on X we write $\phi^{-1}(-\infty, r] = X_r$ in the usual way for all real r. Suppose that ϕ is bounded both above and below. By the (compact) *minimax values* of ϕ we mean the monotone non-decreasing sequence of real numbers

$$m_k(\phi) = \inf_{A \in S_k} \sup(\phi A),$$

where $k = 1, 2, \cdots$. The basic Lusternik-Schnirelmann theorem concerns the relation between ϕ and a given subset K of X. Under certain conditions it can be shown first that each of the minimax values is achieved on K and secondly that if two minimax values are equal, say $m_k(\phi)=m_{k+1}(\phi)=r$, then $K\cap\phi^{-1}(r)$ is infinite. That being so we can conclude that K is infinite in any case, since if no two minimax values coincide then the number of minimax values is infinite, and so the number of points of K, at which these values are achieved, is also infinite.

5. Applications

In the applications we follow Palais [13], which is based on Schwartz [14], rather than Browder [2], since the arguments of the latter do not seem to be readily susceptible to the necessary modifications. Some familiarity with [13] is assumed, especially § 5.

So let X be a Finsler manifold and let ϕ be a C^1 real-valued function on X, bounded above and below. Recall that such a function is said to satisfy the Palais-Smale condition if for every subset A of X either $\|d\phi\|$ is bounded away from zero on A or the closure of A contains a critical point of ϕ. The condition implies that for each critical value c the corresponding critical set is compact. Obviously some condition is necessary if our main result is to be true since, for example, the uniform category of the real line is infinite while the function ϕ given by $\phi(x)=(1+x^2)^{-1}$ has only one critical point.

The proof of our main theorem is based on

LEMMA (5.1). *Let X be a complete connected C^2 Finsler manifold, without boundary. Let ϕ be a C^{2-} bounded real-valued function on X, satisfying the Palais-Smale condition. Let K be the critical set of ϕ and let U be a neighbourhood of $K_c=K\cap\phi^{-1}(c)$ for any c. Then for some $\varepsilon>0$ there exists a uniform isotopy f_t of* id_X *such that $f_1(X_{c+\varepsilon}-U)\subset X_{c-\varepsilon}$.*

In fact an isotopy f_t satisfying the last condition is constructed in (5.9) of [13]. Moreover by (5.12) of [13] the isotopy satisfies the condition

$$\rho(f_{t_1}(x),\ f_{t_2}(x))\leq 2(k+1)|t_1-t_2|,$$

where $k<\varepsilon^{-1/2}$, for all $x\in X$, and so is uniform.

From this we can deduce that the minimax values of ϕ are achieved on the critical set K, as follows. Suppose to obtain a contradiction, that

$m_k(\phi) \notin \phi K$ for some k. Then $c = m_k(\phi)$ is regular and so, by the lemma (with U empty), there exists an $\varepsilon > 0$ and a uniform isotopy f_t of id_X such that $f_1 X_{c+\varepsilon} \subset X_{c-\varepsilon}$. Since c is the infimum of the set of numbers $\sup(\phi A)$ for $A \in S_k$, there exists a member A of S_k such that

$$c \leq \sup(\phi A) \leq c + \varepsilon.$$

Thus $f_1 A \subset X_{c-\varepsilon}$, since $A \subset X_{c+\varepsilon}$, and so

$$\sup(\phi f_1 A) \leq c - \varepsilon < c.$$

But $f_1 A \in S_k$, and so we have our contradiction.

Now let c be critical. I assert that K_c is infinite when two minimax values are equal to c. For suppose, to obtain a contradiction, that K_c is a finite set. Since X is locally uniformly contractible each point x_j of K_c admits a uniformly contractible neighbourhood V_j in X. Since each V_j can be shrunk while remaining uniformly contractible we may assume that the neighbourhoods V_j are pairwise disjoint. Since the union V of these neighbourhoods is a neighbourhood of K_c itself there exists a uniform isotopy f_t of id_X such that $f_1(X_{c+\varepsilon} - V) \subset X_{c-\varepsilon}$. Then $f_1 X_{c+\varepsilon} \subset U \cup X_{c-\varepsilon}$ where $U = f_1 V$ is uniformly categorical since $f_1 V_j$ is uniformly contractible for each j.

Suppose then that $m_k(\phi) = m_{k+1}(\phi) = c$ for some value of k. Then $\mathrm{unicat}_X(A) \geq k+1$ and $\sup(\phi A) \leq c + \varepsilon$ for some subset A of X. We have $\mathrm{unicat}_X(f_1 A) \geq k+1$. However, since $A \subset X_{c+\varepsilon}$ we also have $f_1 A \subset U \cup X_{c-\varepsilon}$ and so

$$\mathrm{unicat}_X(f_1 A) \leq \mathrm{unicat}_X(U) + \mathrm{unicat}_X(X_{c-\varepsilon}) = 1 + \mathrm{unicat}_X(X_{c-\varepsilon}) \leq k,$$

by subadditivity. Thus we have our contradiction and the proof of the theorem is complete.

References

[1] Atsuji, M., Uniform continuity of continuous functions of metric spaces, Pacific J. Math. **8** (1958), 11-16.

[2] Browder, F. E., Nonlinear eigenvalue problems and group invariance, Functional Analysis and Related Fields, Proc. Conf. for M. Stone, Univ. of Chicago, 1968, Springer Verlag, New York, 1970, 1-58.

[3] Calder, A., Cohomology of finite covers, Trans. Amer. Math. Soc. **218** (1976), 349-352.

[4] Calder, A., Uniform homotopy, Fund. Math. **102** (1979), 91-99.

[5] Calder, A. and J. Siegel, Homotopy and uniform homotopy I, Trans. Amer. Math. Soc. **235** (1978), 245-270.

[6] Calder, A. and J. Siegel, Homotopy and uniform homotopy II, Proc. Amer. Math. Soc. **78** (1980), 288-290.
[7] Dowker, C. H., Mapping theorems for non-compact spaces, Amer. J. Math. **69** (1947), 200-242.
[8] Eilenberg, S. and N. Steenrod, Foundations of Algebraic Topology, Princeton Univ. Press, Princeton, 1952.
[9] Gutiérrez-Burzaco, M., Extension of uniform homotopies, Indag. Math. **22** (1960), 67-73.
[10] Isbell, J., On finite-dimensional uniform spaces, Pacific J. Math. **9** (1959), 107-121.
[11] James, I. M., On category, in the sense of Lusternik-Schnirelmann, Topology **17** (1978), 331-348.
[12] Lusternik, L. and L. Schnirelmann, Méthodes Topologiques dans les Problèmes Variationnels, Herman, Paris, 1934.
[13] Palais, R. S., Lusternik-Schnirelmann category on Banach manifolds, Topology **5** (1966), 115-132.
[14] Schwartz, J. T., Generalizing the Lusternik-Schnirelmann theory of critical points, Comm. Pure Appl. Math. **17** (1964), 307-315.

(Received October 20, 1989)

Mathematical Institute
University of Oxford
24-29 St. Giles
Oxford OX1 3LB
England