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Splitting off H-spaces and Conner-Raymond Splitting Theorem

dedicated to Akio Hattori

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Introduction

Conner and Raymond studied the action of the torus $T^k = S^1 \times S^1 \times \cdots \times S^1$ on a T^k-space. The *orbit map* $w: T^k \to X$ is induced by the action of T^k on a basepoint in X. The Conner-Raymond Splitting Theorem is first stated and proved in [CR, Theorem 3.1] and is restated and proved in [G3, Corollary 4] in the following form:

THE CONNER-RAYMOND SPLITTING THEOREM. *If a torus T^k acts on X so that evaluation at a point gives a map $w: T^k \to X$ so that $w_*(H_1(T^k))$ is a direct summand of $H_1(X)$ of rank k, then X is equivariantly homeomorphic to $T^k \times Y$ for some space Y where T^k acts on the product by $g(h, y) = (gh, y)$.*

The above statement will be referred to as the Conner-Raymond Splitting Theorem. This is a particular case of the original version [CR], in which the factor (L, ϕ) is the identity map of $\pi_1(X, x)$, the kernel H is trivial and π_1 is replaced by $H_1(\ , Z)$.

Oprea [O, Theorem 11] has independently of Gottlieb found a Conner-Raymond Splitting Theorem for the case that k equals 1.

In this paper we prove the following generalization of the Conner-Raymond Splitting Theorem. G is assumed to be a compact Lie group. X is a completely regular pathconnected G-space and $w: G \to X$ is the orbit map. X and G are assumed to be homotopy equivalent of CW complexes. $\underline{G}$ is the identity map of G. Let T_G be the functor from the homotopy category to the category of groups, sending a space X to the group $[X, G]$, and $w^\#: [X, G] \to [G, G]$ be the morphism induced by precomposing with w, sending a homotopy class $f: X \to G$ to $f \circ w: G \to G$.

THEOREM 3.1. *The following statements are equivalent*:

(i) *X is isomorphic as a G-space to $G\times(X/G)$, where G acts diagonally, by multiplication on the first factor and trivially on the second.*

(ii) *$w^{\#}$ has a right inverse, that is $w^{\#}$ is onto.*

(iii) *w has a left inverse $r: X\to G$.*

In the particular case $G=T^k$, Theorem 3.1 implies the Conner-Raymond Splitting Theorem, as explained in 3.2 below.

A proof of Theorem 3.1 follows from Theorem 1.3 below. This is a splitting theorem which characterizes when a given space is a cartesian product of an H-space. Again all spaces involved are assumed to be homotopy equivalent to CW complexes.

1.3. SPLITTING THEOREM. *Given spaces X, K and Y the following statements are equivalent:*

(i) *K is an H-space and there exists a space Y such that X is homotopy equivalent to $K\times Y$.*

(ii) *There is a class $i: K\to X$ in the generalized Gottlieb set $G(K, X)$ (as defined by Varadarajan* [V]*) such that $i^{\#}$ has a right inverse (that is $i^{\#}$ is onto).*

(iii) *There are classes i in $G(K, X)$ and $r: X\to K$ such that r is a left inverse for i (and has a fiber Y).*

1.7 below shows that when K equals $S^1\times\cdots\times S^1$, Theorem 1.3 implies the main theorem of [G3] (the theorem on page 216) and Theorem 10 of [O].

As a corollary, any of the above conditions and the fact that K is a finite non contractible H-space, both imply (Corollary 1.8 below) that the Euler characteristic of X is zero.

The proof of Theorem 1.3 involves the ideas of the evaluation subgroup, studied by Gottlieb [G1] [G2] and followed by the work of Varadarajan [V].

This work is organized as follows: Theorem 1.3 is the main result of Section 1. Section 2 contains the proof of Theorem 2.2, which is a dual of Theorem 1.3, and gives necessary and sufficient conditions for splitting co-H-spaces as wedge summands. Section 3 contains the proof of Theorem 3.1. Section 4 gives homology conditions for splitting Eilenberg-Maclane spaces off a space.

All spaces considered are assumed to be homotopy equivalent to CW complexes. In this paper (co-)H-spaces do not necessarily have (co-)products which are (co-)associative.

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§ 1. Splitting H-spaces off a space

This Section reviews the definition of the Gottlieb set as defined by Varadarajan [V], and uses it to give necessary and sufficient conditions for the splitting of an H-space K. From 1.3 on, every space is assumed to be homotopy equivalent to a CW complex.

1.1 The Gottlieb set

Given spaces K and X the *Gottlieb set* denoted $G(K, X)$ is the subset of all the homotopy classes f in $[K, X]$ such that the diagram

$$\begin{array}{ccc} K\times X & \dashrightarrow & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ K\vee X & \xrightarrow{f\vee X} & X\vee X, \end{array}$$

(in which j is the inclusion and ∇ is the folding class) has a class $\mu: K\times X\to X$ making it commute. This definition of Varadarajan [V] specializes to the definitions of the evaluation group given by Gottlieb [G1, G2], when K is a sphere.

The following lemma is a corollary to [V, 2.2].

1.2 LEMMA. *A class f in the Gottlieb set has the property that the image of $\pi_1(f)$ lies in the center of $\pi_1(X)$.* □

Suppose X, K and Y are homotopy equivalent to CW complexes in the following Theorem.

A necessary and sufficient condition for splitting H-spaces

1.3 SPLITTING THEOREM. *The following conditions are equivalent:*

(i) *K is an H-space and there exists a space Y such that X is homotopy equivalent to $K\times Y$.*

(ii) *There is a class i in $G(K, X)$ such that $i^{\#}: [X, K]\to[K, K]$ has a right inverse, (that is $i^{\#}$ is onto).*

(iii) *There are classes i in $G(K, X)$ and $r: X \to K$ such that r is a left inverse for i (and has a fiber Y).*

PROOF. (i)$\Longrightarrow$(iii)

Assuming (i) there are classes $f: X \to K \times Y$ and $g: K \times Y \to X$ which are homotopy inverses of one another. The map i can be defined as the composition $K \xrightarrow{i_K} K \times Y \xrightarrow{g} X$ and r can be defined as $X \xrightarrow{f} K \times Y \xrightarrow{\pi_K} K$. Then $r \circ i$ equals $\pi_K \circ f \circ g \circ i_K$ which is $\underline{K}$. Thus i has r as a left inverse. Let $m: K \times K \to K$ be the product of K. There is the pairing $K \times K \xrightarrow{\underline{K} \times f} K \times K \times Y \xrightarrow{m \times \underline{Y}} K \times Y \xrightarrow{g} X$, which restricts to $i \vee \underline{X}$ on the wedge, and thus i is in $G(K, X)$.

(iii)$\Longrightarrow$(ii)

$r^{\#}$ is a right inverse for $i^{\#}$.

(ii)$\Longrightarrow$(i)

A right inverse $\bar{r}$ for $i^{\#}$ can be chosen, making $\bar{r}(\underline{K}) = r$ a left inverse to i. The map $h: Y \to X$ denotes the inclusion of the homotopy fiber of r. By [M], ΩK is homotopy equivalent to a CW complex. Thus in the fiber sequence $\Omega K \to Y \to X$ the base and fiber are homotopy equivalent to a CW complex, and hence by [S], Y is homotopy equivalent to a CW complex.

The fact that i is in $G(K, X)$ implies that there is a class $\mu: K \times X \to X$ making diagram in Section 1.1 commute.

The composition $r \circ \mu \circ (\underline{K} \times i)$ is a class $K \times K \to K$ establishing the fact that K is an H-space.

The composition $\mu \circ (\underline{K} \times h)$ is a class $K \times Y \to X$ denoted g in the following diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{i_Y} & K \times Y & \xrightarrow{\text{projection}} & K \\ \| & & \downarrow g & & \| \\ Y & \xrightarrow{h} & X & \xrightarrow{r} & K, \end{array}$$

in which i_Y denotes the composition $Y \hookrightarrow K \vee Y \xrightarrow{j} K \times Y$.

By the definition of g and the diagram in Section 1.1, the left square commutes, while the right square commutes after π_* is applied. Thus g induces an isomorphism of homotopy groups, and as all spaces involved are homotopy equivalent to CW complexes, it follows that g is a homotopy equivalence. $\square$

In the next lemma n is an integer, $\pi_1, \pi_2, \cdots, \pi_k$ are abelian groups, Π is $\pi_1 \oplus \cdots \oplus \pi_k$, T equals $K(\pi_1, n) \times K(\pi_2, n) \times \cdots \times K(\pi_k, n)$, X is a space homotopy equivalent to a CW complex, and i is a class $T \to X$.

1.4 LEMMA. *The following conditions are equivalent:*

(i) *There exists $r: X \to T$ which is the left inverse of i.*

(ii) *$H_n(i; \Pi)$ has a left inverse, (that is, it is a split injection).*

(iii) *$H_n(i; \pi_j)$ has a left inverse, for all j, $1 \leq j \leq k$.*

PROOF. (i)$\Longrightarrow$(ii) is trivial by functoriality.

(ii)$\Longrightarrow$(i).

A left inverse $\bar{r}$ to $H_n(i)$ is an element in $\mathrm{Hom}(H_n(X), \Pi)$. There is the following sequence of homomorphisms:

$$\mathrm{Hom}(H_n(X), \Pi) \xrightarrow{un} H^n(X, \Pi) \cong [X, T],$$

in which the homomorphism denoted un is part of the universal coefficient formula. Thus there is $r: X \to T$ so that $H_n(r; \Pi)$ equals $\bar{r}$. The fact that $\bar{r}$ is an inverse for $H_n(i; \Pi)$ implies that r is an inverse for i.

(ii)$\Longleftrightarrow$(iii) is clear. $\square$

In the following version of the splitting theorem, T is a finite product of Eilenberg-Maclane spaces $\times_{m=1}^{l} K(\pi_m, n_m)$.

1.5 SPLITTING THEOREM FOR T. *The following conditions are equivalent:*

(i) *X is homotopy equivalent to $T \times Y$ for some Y.*

(ii) *There is $i: T \to X$ in $G(T, X)$ such that for all $m, 1 \leq m \leq l$, $H_{n_m}(i; \pi_{n_m})$ is a split embedding.* $\square$

In 1.6 and 1.7 below, T^k is $S^1 \times \cdots \times S^1 = (S^1)^k$.

1.6 The Hurewicz rank and the toral number

The following definitions appear in [G3].

"Let G be a subgroup of $\pi_1(X)$. Define the *Hurewicz rank of* G as follows. Consider the image of G under the Hurewicz homomorphism h in the homology group. Then $h(G)$ may contain free summands of $H_1(X)$. We say that the *Hurewicz rank of* G is the maximum rank of these free summands. If there is no free summand in $h(G)$ then we say the Hurewicz rank of G is zero and if there is no maximum we say the Hurewicz rank of G is infinite."

The *toral number of* X is the biggest non negative integer k such that X is homotopy equivalent to $T^k \times Y$ for some space Y.

The following theorem is the main theorem of [G3], for a space X homotopy equivalent to a CW complex. T^k is the torus $S^1 \times S^1 \times \cdots \times S^1$. A generalization of this theorem appears in Theorem 4.9 below.

1.7 THEOREM. *The Hurewicz rank of X equals the toral number of X.*

PROOF. By 1.5, the toral number of X equals to the biggest integer k such that there is an element i in $G(T^k, X)$ for which $H_1(i\,;\boldsymbol{Z})$ is a split injection. Thus it is left to show that this k equals the Hurewicz rank of X. Given that the Hurewicz rank of X is a number m, there exist m classes $i_\theta, 1 \le \theta \le m$, each lying in the Gottlieb group, and such that for all $\theta, H_1(i_\theta\,;\boldsymbol{Z})$ is a split injection into a different summand of $H_1(X\,;\boldsymbol{Z})$. There are classes $\mu_\theta, 1 \le \theta \le m$ which make the diagram in Section 1.1 commute. A class $\lambda : T^k \times X \to X$ can be defined as the composition $\mu_1 \circ (S^1 \times \mu_2) \circ \cdots \circ ((S^1)^{\times (k-1)} \times \mu_m)$. This class restricts to a class p on $T^m \times \{*\}$, and establishes the fact that p is in $G(T^m, X)$. As λ extends all μ_θ on the axes, it follows that $H_1(p\,;\boldsymbol{Z})$ equals $\bigoplus_{\theta=1}^{m} H_1(i_\theta\,;\boldsymbol{Z})$. Thus k is greater than or equal to m. The restricting of λ to μ_θ shows the other inequality and the proof follows. □

Using the known fact that every finite non contractible H-space has Euler characteristic equal to zero, the following corollary is obtained:

1.8 COROLLARY. *In the case that K is a finite non contractible H-space and any of the conditions of* 1.3 *holds, then it follows that the Euler characteristic of X is zero.* □

§ 2. Splitting co-H-spaces off a space

This Section reviews the definition of the dual Gottlieb set as defined by Varadarajan [V], and uses it to give necessary and sufficient conditions for splitting of a connected simply connected co-H-space K, which is homotopy equivalent to a CW complex. The proofs are dual to those of Section 1, and thus are only sketched. From 2.2 on, every space is assumed to be homotopy equivalent to a connected **simply connected** CW complex.

2.1 The dual Gottlieb set

Given spaces K and X the *dual Gottlieb set* denoted $DG(X, K)$ is the subset of all the homotopy classes f in $[X, K]$ such that the diagram

$$\begin{array}{ccc} X & & K\vee X \\ \Delta\downarrow & & \downarrow j \\ X\times X & \xrightarrow{f\times \underline{X}} & K\times X \end{array}$$

(in which j is the inclusion and Δ is the diagonal class) has a class $\rho: X\to K\vee X$ making it commute.

X and K are homotopy equivalent to connected and simply connected CW complexes in the following Theorem.

A necessary and sufficient condition for splitting co-H-spaces

2.2 THEOREM. *The following conditions are equivalent*:

(i) *K is a co-H-space and there exists a simply connected space Y such that X is homotopy equivalent to $K\vee Y$.*

(ii) *There is a class i in $DG(X, K)$ such that $i_\#: [K, X]\to[K, K]$ has a right inverse, that is $i_\#$ is onto.*

(iii) *There are classes i in $DG(X, K)$ and $r: K\to X$ such that r is a right inverse of i (and has a cofiber Y).*

PROOF. (i)$\Longrightarrow$(iii).

The assumptions in (i) provide $f: X\to K\vee Y$, $g: K\vee Y\to X$ and $\rho: K\to K\vee K$. The proof is dual to the proof of (i)$\Longrightarrow$(iii) in 1.3.

(iii)$\Longrightarrow$(ii)

$r_\#$ is a right inverse for $i_\#$.

(ii)$\Longrightarrow$(i)

The one sided inverse of $i_\#$ implies the existence of r. The fact that i is in $DG(X, K)$ implies the existence of the following diagram cofiber sequences:

$$\begin{array}{ccccc} Y & \xleftarrow{\text{projection}} & K\vee Y & \xleftarrow{i_K} & K \\ \| & & \uparrow{\scriptstyle g} & & \| \\ Y & \xleftarrow{h} & X & \xleftarrow{r} & K. \end{array}$$

Thus g implies homology isomorphism, and as all spaces are simply connected, the assertion follows. □

§ 3. The Conner-Raymond Splitting Theorem

From now on G is a compact Lie group acting on a completely regular path connected space X when both are homotopy equivalent to CW complexes. The orbit map $w: G \to X$ is the composition $G \to G \vee X \to G \times X \to X$ which by definition is in $G(G, X)$. In this Section a Theorem is proved which generalizes the Conner-Raymond Splitting Theorem appearing in the introduction.

3.1 THEOREM. *The following statements are equivalent:*

(i) *X is isomorphic as a G space to $G \times (X/G)$, where G acts diagonally, by multiplication on the first factor and trivially on the second.*

(ii) *$w^{\#}$ has a right inverse, that is $w^{\#}$ is onto.*

(iii) *w has a left inverse $r: X \to G$.*

PROOF. The proof of (i)$\Longrightarrow$(iii) is trivial, as in this case w is the inclusion of G in $G \times (X/G)$ and r is the projection of $G \times (X/G)$ on G.

The proof of (iii)$\Longrightarrow$(ii) is trivial by applying $[\ \ , G]$.

(ii)$\Longrightarrow$(i)

The right inverse of $w^{\#}$ implies the existence of a left inverse r for w, forming the following diagram:

$$\begin{array}{ccccc} & & G & & \\ & & \uparrow{\scriptstyle r} & & \\ G & \xrightarrow{w} & X & \xrightarrow{q} & X/G, \\ & & \uparrow{\scriptstyle h} & & \\ & & Y & & \end{array}$$

in which $h: Y \to X$ is the inclusion of the homotopy fiber of r and q maps X to the quotient under the action of G. As $r \circ w$ is the identity class, and G is a compact group it follows that the isotropy group at the basepoint is the trivial subgroup, because if the isotropy subgroup were the non trivial H, then $H_n(w): H_n(G, Z) \to H_n(G/H, Z)$ would have had no inverse, where n is the dimension of G. The fact that X is path connected implies that the isotropy group is trivial at every point. Thus by [B, II, 5.8] the quotient map $X \xrightarrow{q} X/G$ is a principal bundle with fiber G.

Two continuations of the proof are presented. One that uses Theorem 1.3 above and one that does not. Suppose we have a short exact sequence of abelian groups $G \xrightarrow{w} X \xrightarrow{q} X/G$. Then a one sided inverse to either w or q implies that the group X is the direct sum of the groups G and X/G. In the topological setup, the proof using 1.3 has an inverse to q while the other proof uses an inverse to w in some sense.

Using the fact that w is in the Gottlieb set and that $r \circ w$ is the identity class, the assumptions of Theorem 1.3 (iii) hold, and thus X is homotopy equivalent to $G \times Y$. Thus $\pi_*(X)$ equals $\pi_*(G) \oplus \pi_*(Y)$. This can be plugged into the long exact sequence of the fibration $q: X \twoheadrightarrow X/G$. It follows that $q \circ h: Y \to X/G$ induces an isomorphism on homotopy groups. As X and G are homotopy equivalent to CW complexes, so is Y, by [M] and by [S]. Also it follows that X is an ANR. By [H], being an ANR is a local property so that this property is preserved in X/G, which is therefore homotopy equivalent to a CW complex. Thus $q \circ h$ is a homotopy equivalence. Let β be a homotopy inverse, then $h \circ \beta: X/G \to X$ is a homotopy section of q, and by the covering homotopy property, there is a cross section $\gamma: X/G \to X$. Thus the principal bundle X is isomorphic to trivial principal bundle $G \times (X/G)$.

Another way to see that $q: X \twoheadrightarrow X/G$ is a trivial principal bundle consists of observing that X homotopy retracts into its fiber creating the following diagram

$$\begin{array}{ccccc} G & \xrightarrow{w} & X & \xrightarrow{q} & X/G \\ \| & & \downarrow{\scriptstyle r\times q} & & \| \\ G & \xrightarrow{i_G} & G \times X/G & \xrightarrow{\pi} & X/G, \end{array}$$

in which i_G is the inclusion and π is the projection. Then $r \times q$ can be shown to be a fiber homotopy equivalence, by using the homotopy exact sequence and Dold's theorem [Do, Theorem 6.3] on fiber homotopy equivalences. This implies isomorphism between the two bundles. □

In the following corollary, π is an abelian group, n is an integer and G is the group $K(\pi, n)^{\times k}$, denoted T^k. The proof is trivial using 1.3 and 1.4.

3.2 COROLLARY. *The following statements are equivalent*:

(i) *X is isomorphic as a T^k space to $T^k \times (X/T^k)$, where T^k acts diagonally, by multiplication on the first factor and trivially on the second.*

(ii) *X is homotopy equivalent to $T^k \times (X/T^k)$.*

(iii) *$H_n(w; \pi)$ induces a split embedding of $H_n(T^k; \pi)$ into $H_n(X; \pi)$.* □

3.3 Remarks about conditions for 3.1

The classical Conner-Raymond Splitting Theorem, follows from 3.2 in the case that n equals one and π equals $\boldsymbol{Z}$.

The restriction that G is a compact Lie group can not removed. For example the group of the real numbers acts on $\boldsymbol{S}^1$, and any map $r: \boldsymbol{S}^1 \to \boldsymbol{R}$ is an inverse for w, but it is not true that $\boldsymbol{S}^1$ is homeomorphic to $\boldsymbol{R}^1 \times Y$ for some Y.

§ 4. The rank theorem

The purpose of this section is to generalize Theorem 1.7 above so that $K(\pi, n)$ replaces $\boldsymbol{S}^1$. Thus an integer k is determined for X, such that k is biggest with the property that X is homotopy equivalent to $T^k \times Y$ for some Y, where T^k is the kth fold cartesian power of $K(\pi, n)$ and π is an abelian group. All spaces are assumed to be homotopy equivalent to CW complexes.

4.1 The extending group

For a given compact space X homotopy equivalent to a CW complex, the space of self maps of X forms an associative monoid, denoted X^X. This space is homotopy equivalent to a CW complex by [M]. The subspace of all self maps of X which are homotopy equivalences is a submonoid denoted X^X_{he}, and the subset of all self maps homotopic to $\underline{X}$ is the connected component of $\underline{X}$ in X^X, which is also a submonoid homotopy equivalent to a CW complex, denoted $X^X_{\underline{X}}$. It follows that $X^X_{\underline{X}}$ is a grouplike space, [W, p. 461].

It follows that the set of homotopy classes $[K, X^X_{\underline{X}}]$ is a group, for every connected space K. Since X is compactly generated, this group the is isomorphic to the subset of $[K\times X, X]$ which we denote $E(K, X)$ and call the *extending group*, consisting of all classes $\mu: K\times X\to X$ such that μ restricts on $\{*\}\times X$ to the class $\underline{X}$, with $\mu+\nu$ equaling:

$$K\times X\xrightarrow{\Delta\times\underline{X}} K\times K\times X\xrightarrow{\underline{K}\times\nu} K\times X\xrightarrow{\mu} X.$$

The existence of the extending group does not imply that the Gottlieb set is a group, because of the fact that a given class f in $G(K, X)$ may have many classes μ in $E(K, X)$ extending $f\vee\underline{X}$.

In the following, the image of the Gottlieb set under functors is considered, as in [Du].

4.2 The image of the Gottlieb set under a functor

Given a cofunctor $\mathcal{F}: HOM\to\mathcal{C}$ from the homotopy category, the subset $\{\mathcal{F}(f); f\in G(K, X)\}$ of $HOM_{\mathcal{C}}(\mathcal{F}(K), \mathcal{F}(X))$ is called the $\mathcal{F}$ *image of the Gottlieb set* and will be denoted by $G(K, X, \mathcal{F})$.

4.3 Lemma. (i) *If $\mathcal{F}$ carries products to products, it carries $E(K, X)$ to $E(\mathcal{F}(K), \mathcal{F}(X))$.*

(ii) *If for any two classes μ, ν in $E(K, X)$ extending $f\vee\underline{X}$ it holds that $\mathcal{F}(\mu)$ equals $\mathcal{F}(\nu)$, then it follows that $G(K, X, \mathcal{F})$ is a monoid subset of $HOM_{\mathcal{C}}(\mathcal{F}(K), \mathcal{F}(X))$.*

(iii) *If in addition to the assumptions of* (i) *and* (ii), *$\mathcal{F}(X)$ is a group, and for every μ extending $f\vee\underline{X}$ it holds that $\mathcal{F}(\mu)$ equals $\mathcal{F}(f)+\overline{\mathcal{F}(X)}$ where the addition is taken in $\mathcal{F}(X)$, then it follows that $G(K, X, \mathcal{F})$ is a subgroup of $HOM_{\mathcal{C}}(\mathcal{F}(K), \mathcal{F}(X))$.*

Proof. Proof of (i) is trivial.

Proof of (ii). By the assumption for any f in $G(K, X)$, $\mathcal{F}(f)$ can be presented as $\mathcal{F}(K\xrightarrow{i_K}K\times X\xrightarrow{\mu}X)$ and this presentation does not depend on the choice of μ in $E(K, X)$ extending $f\vee\underline{X}$. The operation in $G(K, X, \mathcal{F})$ can be defined by $\mathcal{F}(f)+\mathcal{F}(g)=\mathcal{F}(K\xrightarrow{i_K}K\times X\xrightarrow{\mu+\nu}X)$, where $\mu+\nu$ is the class resulting by adding μ and ν in the monoid $E(K, X)$ (mentioned in 4.1). It is easy to check that $G(K, X, \mathcal{F})$ is an associative monoid.

Proof of (iii). By definition $\mathcal{F}(f+g)$ equals $\mathcal{F}(K)\xrightarrow{\mathcal{F}(i_K)}\mathcal{F}(K)\times \mathcal{F}(X)\xrightarrow{\mathcal{F}(\Delta)\times\mathcal{F}(\underline{X})}\mathcal{F}(K)\times\mathcal{F}(K)\times\mathcal{F}(X)\xrightarrow{\mathcal{F}(\underline{K})\times\mathcal{F}(\nu)}\mathcal{F}(K)\times\mathcal{F}(X)\xrightarrow{\mathcal{F}(\mu)}\mathcal{F}(X)$, which can be presented as $\mathcal{F}(K)\xrightarrow{\mathcal{F}(\Delta)}\mathcal{F}(K)\times\mathcal{F}(K)\xrightarrow{\mathcal{F}(i)}\mathcal{F}(K)\times\mathcal{F}(K)\times$

$\mathcal{F}(X)\xrightarrow{\mathcal{F}(\underline{K})\times(\mathcal{F}(g)+\underline{\mathcal{F}(X)})}\mathcal{F}(K)\times\mathcal{F}(X)\xrightarrow{\mathcal{F}(f)+\underline{\mathcal{F}(X)}}\mathcal{F}(X)$, which is the same as $\mathcal{F}(K)\xrightarrow{\mathcal{F}(\Delta)}\mathcal{F}(K)\times\mathcal{F}(K)\xrightarrow{\mathcal{F}(f)\times\mathcal{F}(g)}\mathcal{F}(X)\times\mathcal{F}(X)\xrightarrow{\text{action}}\mathcal{F}(X)$, which is the addition in $HOM_c(\mathcal{F}(K),\mathcal{F}(X))$. The inverse of $\mathcal{F}(f)$ is the composition $\mathcal{F}(K)\xrightarrow{\mathcal{F}(f)}\mathcal{F}(X)\xrightarrow{\text{inverse}}\mathcal{F}(X)$. □

4.4 *Examples.* (i) Let $\mathcal{F}$ be π_n. Then for every μ extending $f\vee\underline{X}$ we have that $\pi_n(\mu)$ equals $\pi_n(f)\oplus\underline{\pi_n(X)}$. Thus it follows that $G(K, X, \pi_n)$ is a subgroup of $\mathrm{Hom}(\pi_n(K), \pi_n(X))$.

(ii) Let K be an n-connected space and $\mathcal{F}$ be $H_n(\ ;\pi)$. Then for every μ extending $f\vee\underline{X}$, $H_n(\mu;\pi)$ equals $H_n(f;\pi)\oplus\underline{H_n(X;\pi)}$. Thus it follows that $G(K, X, H_n(\ ;\pi))$ is a subgroup of $\mathrm{Hom}(H_n(K;\pi), H_n(X;\pi))$.

4.5 COROLLARY. *The union HI of the images of all $H_n(f;\pi)$ as f varies in $G(K(\pi, n), X)$ is a subgroup of $H_n(X;\pi)$.*

PROOF. Given a subgroup S of $\mathrm{Hom}(H_n(K;\pi), H_n(X;\pi))$, the union of all the images of all maps in S is a subgroup of $H_n(X;\pi)$. □

4.6 The Hurewicz image of $G(K(\pi, n), X)$

HI is called the *Hurewicz image of* $G(K(\pi, n), X)$.

From now on the image of $H_n(f;\pi)$ will be denoted $\mathrm{Im}(f)$.

4.7 LEMMA. *Given a subgroup SG of HI which is of the form* $\mathrm{Im}(f_1)\oplus\cdots\oplus\mathrm{Im}(f_k)$, *then*

(i) *There is a class g in $G(T^k, X)$ so that* $\mathrm{Im}(g)$ *equals SG.*

(ii) *If all $H_n(f_i;\pi)$ are one to one maps, so is $H_n(g;\pi)$.*

(iii) *If all $H_n(f_i;\pi)$ are injective maps, so is $H_n(g;\pi)$.*

PROOF. Each f_i has a class μ_i in $E(K(\pi, n), X)$ establishing the fact that f_i is in $G(\pi, n, X)$. Thus $\mu: K(\pi, n)^{\times k}\times X\to X$ can be defined as $\mu_1+\mu_2+\cdots+\mu_k$ in $E(K(\pi, n)^{\times K}, X)$. It has a restriction $g: K(\pi, n)^{\times k}\times\{*\}\to X$ which is in $G(K(\pi, n)^{\times k}, X)$, and $H_n(g;\pi)$ equals $\bigoplus_{i=1}^{k} H_n(f_i;\pi)$. A direct sum of maps is one to one, provided that the direct summands are. Given left inverses for the summands, they imply the existence of left inverses r_i for f_i respectively. The sum of the r_i's in $[X, K(\pi, n)]$ induces a left inverse for $H_n(g;\pi)$. □

4.8 The injective Hurewicz rank of $G(K(\pi, n), X)$ and the (π, n) splitting number of X

The *injective Hurewicz image* is the biggest subgroup of HI which is a direct sum of $\pi\otimes\pi$'s, all of which split in $H_n(X;\pi)$. The number of copies of $\pi\otimes\pi$ is called *the injective Hurewicz rank of* $G(K(\pi, n), X)$. By 4.7 there is some g in $G(K(\pi, n)^{\times k}, X)$ so that $H_n(g;\pi)$ has a left inverse, and $\mathrm{Im}(g)$ equals $(\pi\otimes\pi)^{\oplus \text{the injective Hurewicz rank}}$.

In the case when $n=1$ and $\pi=\boldsymbol{Z}$, this specializes to the Hurewicz rank of $G(X)$ as in [G3], mentioned in 1.6 above. A similar invariant using fundamental group and 4.4 (i) above is mentioned in [L].

The (π, n) *splitting number of* X is the biggest integer k such that there exists a space Y and a homotopy equivalence of X with $K(\pi, n)^{\times k}\times Y$.

In the case when $n=1$ and $\pi=\boldsymbol{Z}$, this specializes to the toral number of [G3], as mentioned in 1.7 above.

The following rank theorem generalizes the main theorem of [G3].

4.9 THE RANK THEOREM. *The* (π, n) *splitting number of* X *equals the injective Hurewicz rank of* $G(K(\pi, n), X)$.

PROOF. By 1.3 the (π, n) splitting number of X is the biggest number k such that there exists a class $i: T^k=K(\pi, n)^{\times k}\rightarrow X$ in $G(T^k, X)$ with a left inverse. By 1.4 this equals the biggest number k such that there exists i in $G(T^k, X)$ and $H_n(i;\pi)$ has a left inverse. By 4.7 this is the injective Hurewicz rank of $G(K(\pi, n), X)$. □

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