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On propagation of regular singularities for solutions of nonlinear partial differential equations

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§0. Introduction.

In this paper we shall study in a complex domain Ω what sort of conditions on a hypersurface are necessary in order that a solution of a given nonlinear partial differential equation with holomorphic coefficients has regular singularities along this hypersurface and give a simple necessary condition. Moreover, we shall consider to construct a solution which has prescribed singularities on a given hypersurface satisfying the above condition.

In Chap. I we study the necessary condition. Let

$$P(u)=\sum_{\mu\in\mathcal{L}} a_{\mu}(z)((D^{\alpha}u))^{\mu}=0 \tag{0.1}$$

be a nonlinear equation for a partial differential operator $P(u)$ with differential order m and finite multiple order p for which we assume $a_{\mu}(z)\in\mathcal{O}(\Omega)$ for a domain $\Omega\subset C^{n}$ containing the origin. Let

$$u(z)=(\phi(z))^{\sigma(z)}(F_{0}(z)+F_{1}(z)) \tag{0.2}$$

be a solution of (0.1) with regular singularities of exponent $\sigma(z)$ and spiral exponent ω on a regular hypersurface $S=\{\phi(z)=0\}$ (Def. 1.4). Corresponding to the pair $(\sigma(z), \omega)$ we can associate, in general, with it a subset π (or π_{σ_r}) of $\mathcal{L}$ and a polynomial

$$p_{\pi}(s, z, \xi)=\sum_{\mu\in\pi} a_{\mu}(z)(([s:|\alpha|]\xi^{\alpha}))^{\mu}$$

$$(\text{or } p_{\sigma_r}(z, \xi, \chi)=\sum_{\mu\in\pi_{\sigma_r}} a_{\mu}(z)(([\sigma_r:|\alpha|]\xi^{\alpha}))^{|\mu|}\chi^{|\mu|})$$

of $s\in C$ and $\xi\in C^{n}$, so called the principal class for $(\sigma(z), \omega)$ (or principal class for a characteristic exponent σ_r) (Def. 1.1) and the characteristic polynomial for π (or π_{σ_r}) (Def. 1.2), respectively. Then S must satisfy

$$p_{\pi}(\sigma(z'), z', D\phi(z'))=0 \ (\text{or } p_{\sigma_r}(z', D\phi(z'), F_0(z'))=0); \quad z'\in S \tag{0.3}$$

(Theorem 1). §2 is devoted to the proof of Theorem 1. We believe that the above formula describes in generic the characteristic of propagation of regular singularities for solutions of nonlinear partial differential equations, because, similarly for linear case, that is a first order partial differential equation for $\phi(z)$ and, as we state later, we can find a plenty of solutions which have the prescribed singularities on $S=\{\phi(z)=0\}$ satisfying (0.3).

Tsuno [8] studied in a complex domain a continuation problem for quasilinear systems whether any solution u of a system can be extended beyond a real hypersurface S if it is holomorphic on one side of S, and gave a sufficient condition on S which contains our "non-charasteristic" condition in the case that u has some regularity on S. Kobayashi [5] determined the lower bound of such regularity of u and showed that it coincides with the maximal characteristic exponent. In the real domain Bony [1] investigated propagation of singularities of solutions with rather high regularity for quasilinear equations and showed that it is performed along bicharacteristic curves for the linearized operator microlocally, which may be transformed in our case.

On the other hand, we consider in Chap. II to construct solutions of the form (0.2) for a given hypersurfase $S=\{\phi(z)=0\}$ and $\sigma(z)$ which satisfies the characteristic equation (0.3) for a principal class π (or π_{σ_r}). Our solutions have the form

$$u(z)=(\phi(z))^{\sigma(z)}\sum_{k\in Z_+^M}\sum_{l=0}^{|k|}u_{k,l}(\phi(z))^{d(z)\cdot k}(\log\phi(z))^{|k|-l} \tag{0.4}$$

with an appropriate $M\in N$ and $d(z)\in(\mathcal{O}(\Omega))^M$ ((3.1), (3.2)). In §3 we introduce three cases, that is, Case A, Case B and Case C in which we can construct a solution of the form (0.4). In Case B the characteristic polynomial $p_\pi(s, z, \xi)$ has a particular structure and in Case C $\sigma(z)$ coincides with a characteristic exponent. Any discussions in Chap. II, except for §4 and §5, shall be performed for every these three cases. The form (0.4) of the solution enforces us to introduce the second characteristic polynomial and that at infinity (Def. 3.2). Then we can see that the characteristic polynomial and the second characteristic polynomial together with that at infinity are both invariant under any biholomorphic coordinate transformation on Ω (Lemma 2.1, Lemma 3.1). Let a triplet (π, σ, ω) satisfy conditions in one of above three cases. Then conditions on $\phi(z)$ which assure us to have a formal solution of the form (0.4) are given by the characteristic condition (Cond. I) and the noncharacteristic condition (Cond. II), the latter of which is described with respect to the second characteristic polynomial. In § 4 and § 5, we define for a given formal functional series (0.4) formal functional series

$$D^{\alpha}u=\phi^{\sigma-|\alpha|}\sum_{(k,l)\in D}u_{k,l}^{\alpha}\phi^{d\cdot k}(\log\phi)^{l*}, \tag{0.5}$$

$$((D^{\alpha}))^{\mu}=\phi^{\nu_{\mu}(\sigma)}\sum_{(k,l)\in D}u_{k,l}^{\mu}\phi^{d\cdot k}(\log\phi)^{l*}, \tag{0.6}$$

where $u_{k,l}^{\alpha}$ and $u_{k,l}^{\mu}$, $(k,l)\in D$, are determined so that both hand sides of (0.5) and (0.6) coincide really when the right hand side of (0.4) is convergent ((4.12), (5.7)). We can see that for any $(k,l)\in D$ $u_{k,l}^{\mu}$ is a polynomial of $u_{\kappa,\rho}$'s, $(\kappa,\rho)\leq(k,l)$, and their derivatives, and we count out, first, all terms in it which contain $u_{k,l}$ and, secondly in the case $D\sigma\equiv 0$, those containing $u_{\kappa,\rho}$'s, $(\kappa,\rho)\not<(k-f_1, l-1)$, (Lemma 5.1). In § 6 we consider the construction of formal solutions. Corresponding to each three cases, we define a formal functional series

$$P(u)=\phi^{\nu_{\pi(\text{or }\pi_{\sigma_r})}(\sigma)}\sum_{(k,l)\in D}w_{k,l}\phi^{d\cdot k}(\log\phi)^{l*}$$

((6.4), (6.10)). In Case A, we have

$$\begin{cases} w_{0,0}=p_{\pi}(\sigma(z),z,D\phi(z))(u_{0,0})^{|\pi|}, \\ w_{k,l}=(u_{0,0})^{|\pi|}s_{\pi}(\sigma(z),d(z)\cdot k,z,D\phi(z))u_{k,l}+R_{k,l}; \quad (k,l)\in D_{+}, \end{cases}$$

where $R_{k,l}$ is a polynomial of $u_{\kappa,\rho}$'s, $(\kappa,\rho)<(k,l)$, and their derivatives (Lemma 6.1). Take, for a given $\phi(z)$ satisfying Cond. I and Cond. II, an arbitrary $u_{0,0}\in\mathcal{O}(\Omega)$ such that $u_{0,0}(0)\neq 0$ and put

$$u_{k,l}=-\frac{R_{k,l}}{(u_{0,0})^{|\pi|-1}s_{\pi}(\sigma,d\cdot k,z,D\phi)}; \quad (k,l)\in D_{+}.$$

Then all $u_{k,l}$'s are determined uniquely from $u_{0,0}$ by induction, and we can obtain a formal solution of the form (0.4) of the equation (0.1). On the other hand we must search, in Case B, $w_{k,l}$ for all terms containing $u_{\kappa,\rho}$, $(\kappa,\rho)\not<(k-f_1, l-1)$ ((6.11), (6.12) and (6.13)). Assume that $\phi(z)$ satisfies Cond. I and Cond. II. Then we can see

$$w_{k,l}=0 \quad \text{for } (k,l) \text{ such that } (k)_1=0 \text{ or } l=0$$

(Cor. 6.4), and we have noting that the sum of all terms containing such $u_{\kappa,\rho}$ as $(\kappa,\rho)\not<(k,l)$ in $w_{k+f_1,l+1}$ vanishes (Lemma 6.3)

$$w_{k+f_1, l+1}=((u_{0,0})^{|\pi|-1}t_\pi(\sigma, 0, z, D\phi(z), D)+r_k(z, Du_{0,0}, u_{0,0}))u_{k,l}+\tilde{R}_{k+f_1, l+1},$$

where $r_k(z, \eta, u)$ is a polynomial of $\eta\in C^n$ and $u\in C$, especially, one of only u if $k=0$, and $\tilde{R}_{k+f_1, l+1}$ is a polynomial of such $u_{\kappa,\rho}$'s as $(\kappa, \rho)<(k, l)$ and their derivatives ((6.12), (6.13) and Lemma 6.2). Therefore we can utilize the equation

$$w_{k+f_1, l+1}=0, \tag{0.7}$$

which is a first order partial differential equation on $u_{k,l}$ to determine $u_{k,l}$. Let T be any hypersurface noncharacteristic to the equation (0.7) for any $(k, l)\in D$, whose existence is assured by Cond. II, and take an arbitrary $v_{k,l}\in\mathcal{O}(T)$ for every $(k, l)\in D$. Then we can solve the differential equation (0.7) with initial data

$$u_{k,l}|_T=v_{k,l}$$

inductively and can determine all $u_{k,l}$'s uniquely. In Case C we can treat similarly to Case A. On convergence of the formal solution we shall study in § 7. Our main estimate is majorant one on $u_{k,l}$, $(k, l)\in D\setminus\{(0, 0)\}$, such as

$$u_{k,l}\ll\{\exp(a(|k|+l)-b(|k|+l)^{1/2})\}(R-t)^{-|k|} \tag{0.8}$$

with positive constants a, b, R and $t=cz_1+\cdots+z_n$ for some constant $c\geq 1$ (Theorem 2). To prove (0.8) we derive in § 7 several a priori estimates (Lemma 7.8, Lemma 7.9) from a key lemma (Lemma 7.6). (0.8) is shown in § 8 for each three cases making another assumption on $v_{k,l}$ in Case B (Ass. B). Once (0.8) is established, it is not difficult to show that the formal solution is the genuin one which has regular singularities of exponent σ with spiral exponent ω, interplitting $\omega=0$ in Case C. Therefore we have Theorem 3 which shall be proved in § 9.

Koboyashi [6] was the first which remarked that much more hypersurfaces are admissible in the nonlinear equation to carry singularities of solutions than in the linear case. Ishii-Kobayashi [2] constructed for semilinear equations a solution with singularities of the maximal characteristic exponent. In the real domain Kobayashi-Nakamura [7] solved the same problem for semilinear hyperbolic equations with C^∞-coefficients. Ishii [3] and [4] are announcements of this work, among which [4] is concerned with Chap. I.

Chapter I. A necessary condition for propagation of regular singularities.

§ 1. Notations, definitions and a result.

We consider in a domain Ω of C^n containing the origin. We say simply a "subdomain" one containing the origin. Let $\mathcal{O}(\Omega)$ be the set of all holomorphic functions on Ω, $\mathcal{O}^0(\Omega)=\{\phi(z)\in\mathcal{O}(\Omega);\ \phi(0)=0,\ D\phi(z)\neq 0$ for any $z\in\Omega\}$ and $\mathcal{O}^1(\Omega)=\{\phi(z)\in\mathcal{O}(\Omega);\ \phi(z)\neq 0$ for any $z\in\Omega\}$. D^α stands for $D_1^{\alpha_1}\cdots D_n^{\alpha_n}=(\partial/\partial z_1)^{\alpha_1}\cdots(\partial/\partial z_n)^{\alpha_n}$ with $(\alpha_1,\cdots,\alpha_n)\in \boldsymbol{Z}_+^n$, denoting $\boldsymbol{Z}_+$ the set of all nonnegative integers, and $z=(z_1,\cdots,z_n)\in\Omega$. Now consider a nonlinear partial differential operator $P(u)$ of polynomial type with differential order m and multiple order $p\geq 2$, which we denote

$$P(u)=\sum_{\mu\in\mathcal{L}}a_\mu(z)((D^\alpha u))^\mu, \tag{1.1}$$

where $\mathcal{L}$ is a given subset of the set $\mathcal{L}^*=\{\mu=(\mu_\alpha)\in \boldsymbol{Z}_+^N;\ |\mu|=\sum_{|\alpha|\leq m}\mu_\alpha\leq p\}$ with $N=\#\{\alpha\in \boldsymbol{Z}_+^n;\ |\alpha|\leq m\}$, $a_\mu(z)\in\mathcal{O}(\Omega)$ and $((D^\alpha u))^\mu=\prod_{\alpha\in\operatorname{supp}\mu}(D^\alpha u)^{\mu_\alpha}$ with $\operatorname{supp}\mu=\{\alpha\in \boldsymbol{Z}_+^n;\ \mu_\alpha\neq 0\}$. We assume $|\mu|\geq 1$ for every $\mu\in\mathcal{L}$. For every $\mu\in\mathcal{L}$ we put

$$y_\mu(\sigma)=|\mu|\sigma-\sum_\alpha|\alpha|\mu_\alpha;\qquad \sigma\in \boldsymbol{C}.$$

For $\mu,\nu\in\mathcal{L}$ we denote $\mu\sim\nu$ if and only if $y_\mu=y_\nu$, and for a given $\tau\in\mathcal{L}/\sim$. y_τ or $|\tau|$ stands for the common y_μ or $|\mu|$ for any $\mu\in\tau$, respectively. Put

$$\eta(\rho)=\min_{\mu\in\mathcal{L}}y_\mu(\rho);\qquad \rho\in \boldsymbol{R}.$$

Then the graph of $\eta(\rho)$ describes a concave polygon which has a finite number, say J, of summits. We arrange the values of ρ corresponding to these summits in the order $\sigma_1<\cdots<\sigma_J$ and call them *characteristic exponents of the operator P*. Let $C^+=\{\omega=\exp(i\theta);\ -\pi/2<\theta<\pi/2\}\subset \boldsymbol{C}$.

DEFINITION 1.1. (1) For a given $\sigma\in \boldsymbol{C}$ and a given $\omega\in \boldsymbol{C}^+$, a class $\pi\in\mathcal{L}/\sim$ is called *the principal class of the operator P for* (σ,ω) if and only if

$$\mathcal{R}e(\omega y_\pi(\sigma))<\mathcal{R}e(\omega y_\nu(\sigma))\qquad\text{for any }\nu\in\mathcal{L}\backslash\pi.$$

(2) For a characteristic exponent σ_r, $1\leq r\leq J$, the subset $\{\mu\in\mathcal{L};\ y_\mu(\sigma_r)=\eta(\sigma_r)\}$ of $\mathcal{L}$ is called *the principal class for the characteristic*

exponent σ_r and denoted by π_{σ_r}.

Note that for a characteristic exponent σ_r we have

$$\mathcal{R}e(\omega\eta(\sigma_r))<\mathcal{R}e(\omega y_\nu(\sigma_r))$$

for any $\nu\in\mathcal{L}\backslash\pi_{\sigma_r}$ and any $\omega\in C^+$. For $z\in C$ and $k\in Z$, put $[z;0]=1$, $[z;k]=z(z-1)\cdots(z-k+1)$ for $k>0$, and $[z;k]=0$ otherwise.

Definition 1.2. (1) Let π be a principal class for some $(\sigma,\omega)\in C\times C^+$. The polynomial

$$p_\pi(s,z,\xi)=\sum_{\mu\in\pi}a_\mu(z)(([s;|\alpha|]\xi^\alpha))^\mu$$

of $s\in C$ and $\xi\in C^n\backslash\{0\}$ is called *the characteristic polynomial for the class* π of the operator P.

(2) For every characteristic exponent σ_r, $1\leq r\leq J$, *the characteristic polynomial for the characteristic exponent* σ_r of P is defined by the polynomial

$$p_{\sigma_r}(z,\xi,\chi)=\sum_{\mu\in\pi_{\sigma_r}}a_\mu(z)(([\sigma_r;|\alpha|]\xi^\alpha))^\mu\chi^{|\mu|}$$

of $\xi\in C^n\backslash\{0\}$ and $\chi\in C$.

Let

$$S=\{\phi(z)=0\} \tag{1.2}$$

be a regular hypersurface in Ω through the origin with defining function $\phi(z)\in\mathcal{O}^0(\Omega)$. For a given $\omega\in C^+$ we design a $\psi(z)\in\mathcal{O}(\mathcal{R}(\Omega\backslash S))$, the set of all holomorphic functions on the covering space $\mathcal{R}(\Omega\backslash S)$ of $\Omega\backslash S$, by the relation

$$\phi(z)=(\psi(z))^\omega. \tag{1.3}$$

Definition 1.3. Let S be a regular hypersurface satisfying (1.2) and $\psi(z)$ be given by (1.3). For a given $z'\in S$, we say a sequence $\{z\}$ in $\Omega\backslash S$ is *spirally convergent of exponent* ω *to* z' if and only if $\{z\}$ tends to z' with constraint $|\arg\psi(z)|<K$ for some $K>0$, and denote $z\xrightarrow[(\omega,K)]{}z'$. Moreover, we denote $(\omega,K)\text{-}\lim_{z\to z'}f(z)=A$ if and only if $f(z)$ converges A for any sequence $\{z\}$ such that $z\xrightarrow[(\omega,K)]{}z'$.

Definition 1.4. Let S be given by (1.2) and $\psi(z)$ by (1.3). For a given $\omega\in C^+$ and a given $\sigma(z)\in\mathcal{O}(\Omega)\backslash\{0\}$, we say $u(z)$ has *regular singu-*

larities of exponent $\sigma(z)$ *and spiral exponent* ω *on* S if and only if $u(z)$ has the form

$$u(z)=(\phi(z))^{\sigma(z)}(F_0(z)+F_1(z)) \tag{1.4}$$

where $F_0(z)\in\mathcal{O}^1(\Omega)$ and $F_1(z)$ is holomorphic on $\mathcal{R}(\Omega\backslash S)\cap\{|\arg\phi|<K\}\cap\{0<|\phi|<\delta\}$ for any $K>0$ and some $\delta=\delta(K)>0$ and satisfies $(\omega, K)\text{-}\lim_{z\to z'} F_1(z)=0$ for any $z'\in S$.

Now we consider a solution $u(z)$ of the nonlinear partial differential equation

$$P(u)=0 \tag{1.5}$$

which has regular singularities of exponent $\sigma(z)$ and spiral exponent ω on S for some $\sigma(z)$ and ω. Then what conditions should be necessary on S?

THEOREM 1. *Suppose that the nonlinear partial differential equation* (1.5) *admits a solution which has regular singularities of exponent* $\sigma(z)$ *and spiral exponent* ω *on* $S=\{\phi(z)=0\}$ *for appropriate* $\phi(z)$, $\sigma(z)$, $F_0(z)$, $F_1(z)$ *and* ω. *Then we have*

(1) *If* $\sigma(z)\not\equiv\sigma_r$ *for any* $1\leq r\leq J$ *and if* π *is the principal class for* $(\sigma(0),\omega)$, *then we have*

$$p_\pi(\sigma(z), z, D\phi(z))=0 \qquad \text{on } S. \tag{1.6}$$

(2) *If* $\sigma(z)\equiv\sigma_r$ *is a characteristic exponent, then we have*

$$p_{\sigma_r}(z, D\phi(z), F_0(z))=0 \qquad \text{on } S. \tag{1.7}$$

REMARK 1. The assumption that $|\mu|\geq 1$ for any $\mu\in\mathcal{L}$ is not essential. Because, if $\tilde{P}(u)=0$ is any nonlinear P.D.E., we can find equivalent one which satisfies the above assumption, putting $P(u)=\tilde{P}(u+v)$, where v is an arbitrary holomorphic solution of $\tilde{P}(v)=0$.

REMARK 2. Theorem 1 shows that if $\sigma(z)$ does not equal to a characteristic exponent and if $p_\pi(\sigma(z), z, \xi)\not\equiv 0$ the surface $S=\{\phi(z)=0\}$ carrying the designated singularities of the solution must be an integral surface of the first order homogeneous equation $p_\pi(\sigma(z), z, D\phi(z))=0$ of homogeneous order $\sum_\alpha|\alpha|\mu_\alpha$ which is common to any $\mu\in\pi$. Then we can see any bicharacteristic curve issuing from S keeps staying on S. This phenomenon is similar to that of the linear equation but for dependency of S on $\sigma(z)$. On the other hand, if $\sigma(z)\equiv\sigma_r$ a characteristic exponent, the surface S, which is an integral surface of the inhomogeneous equation $p_{\sigma_r}(z, D\phi(z), F_0(z))=0$, intersects to any bicharacteristic curves transversally.

§ 2. Proof of Theorem 1.

We show, first, that the characteristic polynormial $p_\pi(s, z, \xi)$ or $p_{\sigma_r}(z, \xi, \chi)$ is invariant under biholomorphic coordinate transformation on Ω. Let

$$h:\Omega\longrightarrow\tilde{\Omega}\subset C^n \tag{2.1}$$

be a biholomorphic coordinate transformation and let $\tilde{z}=h(z)$. For any $\alpha\in Z_+^n$, define $p_{\alpha,\tilde{\alpha}}$ and $q_{\alpha,\tilde{\beta}}\in\mathcal{O}(\tilde{\Omega})$ with $\tilde{\alpha}, \tilde{\beta}\in Z_+^n$ such as $|\tilde{\alpha}|=|\alpha|$ and $|\tilde{\beta}|<|\alpha|$ by the relation

$$((\partial\tilde{z}/\partial z)\tilde{\xi})^\alpha=\sum_{|\tilde{\alpha}|=|\alpha|}p_{\alpha,\tilde{\alpha}}(\tilde{\xi})^{\tilde{\alpha}}\qquad\text{for } \tilde{\xi}\in C^n\backslash\{0\}, \tag{2.2}$$

$$D_z^\alpha u=\sum_{\tilde{\alpha}}p_{\alpha,\tilde{\alpha}}D_{\tilde{z}}^{\tilde{\alpha}}\tilde{u}+\sum_{|\tilde{\beta}|<|\alpha|}q_{\alpha,\tilde{\beta}}D_{\tilde{z}}^{\tilde{\beta}}\tilde{u} \tag{2.3}$$

with $\tilde{u}=u\circ h^{-1}$. Moreover, put for any $\mu\in\mathcal{L}$, first,

$$((\sum_{\tilde{\alpha}}p_{\alpha,\tilde{\alpha}}D_{\tilde{z}}^{\tilde{\alpha}}\tilde{u}))^\mu=\sum_{\tilde{\mu}}p_{\mu,\tilde{\mu}}((D_{\tilde{z}}^{\tilde{\alpha}}\tilde{u}))^{\tilde{\mu}} \tag{2.4}$$

and secondly,

$$((D_z^\alpha u))^\mu=\sum_{\tilde{\mu}}p_{\mu,\tilde{\mu}}((D_{\tilde{z}}^{\tilde{\alpha}}\, u))^{\tilde{\mu}}+\sum_{\tilde{\nu}}q_{\mu,\tilde{\nu}}((D^{\tilde{\beta}}u))^{\tilde{\nu}} \tag{2.5}$$

with some $p_{\mu,\tilde{\mu}}, q_{\mu,\tilde{\nu}}\in\mathcal{O}(\tilde{\Omega})$. Note that $|\mu|=|\tilde{\mu}|=|\tilde{\nu}|$, $\sum_{\tilde{\alpha}}|\tilde{\alpha}|\tilde{\mu}_{\tilde{\alpha}}=\sum_\alpha|\alpha|\mu_\alpha$ and $\sum_{\tilde{\beta}}|\tilde{\beta}|\tilde{\nu}_{\tilde{\beta}}<\sum_\alpha|\alpha|\mu_\alpha$. Now the nonlinear P.D.Op. $P(u)$ given by (1.1) is transformed into another one, say $\tilde{P}(\tilde{u})$, by h and, as easily seen, $\tilde{P}$ has the same differential order m and multiple order p as P. Put

$$\tilde{P}(\tilde{u})=\sum_{\tilde{\mu}\in\tilde{\mathcal{L}}}\tilde{a}_{\tilde{\mu}}(\tilde{z})((\tilde{D}^\alpha\,\tilde{u}))^{\tilde{\mu}}. \tag{2.6}$$

Lemma 2.1. *Let h be a biholomorphic coordinate transformation given by* (2.1) *and let $\tilde{P}(\tilde{u})$ be the operator obtained from $P(u)$ by the transformation* (2.1). *Then we have*:

(1) *Let π be the principal class for $(\sigma, \omega)\in C\times C^+$ of the operator P and let $\tilde{\pi}$ be the principal class of $\tilde{P}$ for the same (σ, ω). Moreover, let*

$\tilde{p}_{\tilde{\pi}}(\tilde{s}, \tilde{z}, \tilde{\xi})$ *be the characteristic polynomial for the class* $\tilde{\pi}$ *of* $\tilde{P}$. *Then we have*

$$\tilde{p}_{\tilde{\pi}}(s, \tilde{z}, \tilde{\xi}) = p_{\pi}(s, z, (\partial\tilde{z}/\partial z)\tilde{\xi}), \tag{2.7}$$

that is, $p_{\pi}(s, z, \xi)$ *is a function on* $T^*(\Omega)$, *cotangent space of* Ω, *with respect to* (z, ξ).

(2) *Any characteristic exponent* σ_r, $1 \le r \le J$, *is invariant under* h. *Further, we have*

$$\tilde{p}_{\sigma_r}(\tilde{z}, \tilde{\xi}, \chi) = p_{\sigma_r}(z, (\partial\tilde{z}/\partial z)\tilde{\xi}, \chi)$$

where $\tilde{p}_{\sigma_r}(\tilde{z}, \tilde{\xi}, \chi)$ *is the characteristic polynomial for the characteristic exponent* σ_r *of* $\tilde{P}$.

PROOF. We will show only (1) because (2) can be proved similarly. As it can be seen that

$$y_{\tilde{\mu}}(\sigma) = y_{\mu}(\sigma) \quad \text{or} \quad \mathcal{R}e(y_{\tilde{\nu}}(\sigma)) \le \mathcal{R}e(y_{\mu}(\sigma)) - 1$$

holds for each terms of the first or second sum of the right hand side of (2.5), respectively, it follows that

$$\sum_{\tilde{\mu} \in \tilde{\pi}} \tilde{a}_{\tilde{\mu}}(\tilde{z})((D^{\tilde{\alpha}} \tilde{u}))^{\tilde{\mu}} = \sum_{\mu \in \pi} a_{\mu} \circ h^{-1}(\tilde{z}) (\sum_{\tilde{\mu} \in \tilde{\pi}} p_{\mu, \tilde{\mu}}((D^{\alpha} \tilde{u}))^{\tilde{\mu}}). \tag{2.8}$$

Since $\tilde{p}_{\tilde{\pi}}(\sigma, \tilde{z}, \tilde{\xi})$ is obtained by substituting $[\sigma; |\tilde{\alpha}|](\tilde{\xi})^{\tilde{\alpha}}$ for $D_{\tilde{z}}^{\tilde{\alpha}} \tilde{u}$ in (2.8), we have

$$\begin{aligned}\tilde{p}_{\tilde{\pi}}(\sigma, \tilde{z}, \tilde{\xi}) &= \sum_{\mu \in \pi} a_{\mu} \circ h^{-1}(\tilde{z}) (\sum_{\tilde{\mu}} p_{\mu, \tilde{\mu}}(([\sigma; |\tilde{\alpha}|]\tilde{\xi}^{\tilde{\alpha}}))^{\tilde{\mu}}) \\ &= \sum_{\mu \in \pi} a_{\mu} \circ h^{-1}((\sum_{\tilde{\alpha}} p_{\alpha, \tilde{\alpha}}[\sigma : |\tilde{\alpha}|]\tilde{\xi}^{\tilde{\alpha}}))^{\mu} \\ &= \sum_{\mu \in \pi} a_{\mu} \circ h^{-1}(([\sigma : |\alpha|]((\partial\tilde{z}/\partial z)\tilde{\xi})^{\alpha}))^{\mu}.\end{aligned}$$

Hence we have (2.7) and this completes the proof.

Next, let us find principal behaviour of singularities of $P(u(z))$ when $u(z)$ has regular singularities on S.

LEMMA 2.2. *Let* $u(z)$ *have regular singularities of exponent* $\sigma(z)$ *and spiral exponent* ω *on a regular hypersurface* $S = \{\phi(z) = 0\}$ *with* $\phi(z) \in \mathcal{O}^0(\Omega)$, *satisfying* (1.4) *for some* $F_0(z)$ *and* $F_1(z)$. *Then we have*

(1) *If* $\sigma(z) \not\equiv \sigma_r$, $1 \le r \le J$, *and if* π *is the principal class for* $(\sigma(0), \omega)$ *of the operator* P, *then there exists a subdomain* Ω' *such that*

(2.9) $$(\omega, K)\text{-}\lim_{z\to z_0} (\phi(z))^{-y_\pi(\sigma(z))}P(u(z))=p_\pi(\sigma(z_0), z_0, D\phi(z_0))(F_0(z_0))^{|\pi|};\ z_0\in S\cap \Omega'$$

for any $K>0$.

(2) *If* $\sigma(z)\equiv\sigma_r$ *for some* $1\leq r\leq J$, *we have*

$$(0, K)\text{-}\lim_{z\to z_0} (\phi(z))^{-\eta(\sigma_r)}P(u(z))=p_{\sigma_r}(z_0, D\phi(z_0), F(z_0))$$

for any $z_0\in S$ *and* $K>0$.

The proof of the above lemma is accomplished by making use of the following lemma.

Lemma 2.3. *Let* $\omega\in C^+$ *and put* $t=s^\omega$ *for* $t\in \boldsymbol{C}\backslash\{0\}$. *Let* $f(t)$ *be a holomorphic function on the domain* $\mathcal{R}(0<|t|<d)\cap\{|\arg s|<K\}\cap\{0<|s|<\delta\}$ *for some positive constants* K, d *and* δ *such that* $A=(\omega, K)\text{-}\lim_{t\to 0} f(t)$ *exists. Then we have*

(2.10) $$(\omega, K-\varepsilon)\text{-}\lim_{t\to 0} t^n f^{(n)}(t)=0$$

for any $\varepsilon>0$ *and* $n=1, 2, \cdots$.

Proof of Lemma 2.3. Let us show this lemma by induction with respect to $n\in N$. Put, first, for any $0<\varepsilon<K/2$ $\omega_\varepsilon=\{s\in \boldsymbol{C};\ 0<|s|<d, |\arg s|<K-\varepsilon\}$ and for any $s\in\omega_\varepsilon$ let $C_{\varepsilon,s}$ be a circle centred at s with radius $\varepsilon|s|$. Since $C_{\varepsilon,s}\subset\{|\arg s|<K\}$ for sufficiently small $|s|$, it follows by Cauchy's integral formula

(2.11) $$tf'(t)=\frac{s}{2\pi\omega i}\oint_{C_{\varepsilon,s}}\frac{f(\xi^\omega)-A}{(\xi-s)^2}d\xi\,;$$

$t=s^\omega$ and sufficiently small $|s|$.

Then, estimating the absolute value of the right hand side of (2.11), we can obtain (2.10) for $n=1$. Next, assume that (2.10) holds for any $1\leq n\leq N$. Then, as

$$t^{N+1}f^{(N+1)}(t)=t(t^Nf^{(N)})'-Nt^Nf^{(N)}$$

holds, we have (2.10) for $n=N+1$ by the assumption of the induction and by application of (2.10) for $n=1$ to $t^Nf^{(N)}$. This completes the proof.

Proof of Lemma 2.2. Let π be the principal class for $(\sigma(0), \omega)$ of P. Then we can find a subdomain Ω' such that π is the principal class for $(\sigma(z), \omega)$ for any $z \in \Omega'$. Now, by virtue of Lemma 2.1, we may assume $\phi(z)=z_1$ and then we have

$$(2.12)\qquad (\phi(z))^{-\nu_\pi(\sigma(z))}P(u(z))=\sum_{\mu\in\mathcal{L}} a_\mu(z)z_1^{-\nu_\pi(\sigma(z))+\nu_\mu(\sigma(z))}((z_1^{-\sigma(z)+|\alpha|}D^\alpha(z_1^{\sigma(z)}F(z))))^\mu$$

with $F(z)=F_0(z)+F_1(z)$. For a given $z_0=(0, z_0')=(0, z_2^0, \cdots, z_n^0) \in S\cap\Omega'$ we associate a multi-circle $C_{\delta,z_0'}=C_{\delta,z_2^0}\times\cdots\times C_{\delta,z_n^0}$ in C^{n-1} with so small $\delta>0$ as $\{0<|z_1|<\delta\}\times C_{\delta,z_0'}\subset\Omega'\backslash S$, where C_{δ,z_j^0} denotes a circle in C with radius δ centred at z_j^0. Represent $F(z)$, $z_1\neq 0$, as an iterated integral on $\{z_1\}\times C_{\delta,z_0'}$ using Cauchy's integral formula. Then, since we can see by this representation

$$(\omega, K-\varepsilon)\text{-}\lim_{z\to z_0} D'^{\gamma'}F_1(z)=0 \qquad \text{for any } \varepsilon>0 \text{ and any } \gamma'\in Z_+^{n-1}$$

with $D'=(D_2, \cdots, D_n)$, we have applying Lemma 2.3 to the integrand

$$(\omega, K-\varepsilon)\text{-}\lim_{z\to z_0} z_1^{\gamma_1}D^\gamma F(z)=\delta_{\gamma_1,0}D^{\gamma'}F_0(z_0)$$

for any $K>0$, $0<\varepsilon<K/2$ and $\gamma=(\gamma_1, \gamma')\in Z_+^n$, where $\delta_{m,n}$ denotes Kronecker's delta. Therefore, as it can be easily seen

$$(\omega, K)\text{-}\lim_{z\to z_0} z_1^{-\sigma(z)+|\alpha|-\alpha_1+\beta_1}D^\beta z_1^{\sigma(z)}=\delta_{|\alpha|,\alpha_1}[\sigma(z_0):\beta_1];$$
$$\alpha=(\alpha_1, \alpha'),\ \beta=(\beta_1, \beta')\in Z_+^n \text{ such that } 0\le\beta\le\alpha,$$

it follows from Leibnitz's rule that

$$(2.13)\qquad \begin{aligned}&(\omega, K-\varepsilon)\text{-}\lim_{z\to z_0} z_1^{-\sigma(z)+|\alpha|}D^\alpha(z_1^{\sigma(z)}F(z))\\ &=(\omega, K-\varepsilon)\text{-}\lim_{z\to z_0}\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}(z_1^{-\sigma(z)+|\alpha|-\alpha_1+\beta_1}D^\beta z_1^{\sigma(z)})\times\\ &\quad\times(z_1^{\alpha_1-\beta_1}D^{\alpha-\beta}F(z))\\ &=\delta_{|\alpha|,\alpha_1}[\sigma(z_0):|\alpha|]F_0(z_0).\end{aligned}$$

Operate $(\omega, K-\varepsilon)\text{-}\lim_{z\to z_0}$ on the both hand side of (2.12). Then using the equality (2.13) to the right hand side, we can obtain the equality (2.9). The proof of (2) is left to the reader because it can be shown similarly to (1) and this completes the proof.

PROOF OF THEOREM 1. Since (1.5) holds and $F_0(0)\neq 0$, the theorem can be easily proved by Lemma 2.2.

Chapter II. Construction of a solution with given singularities.

§ 3. Preliminary considerations.

Hereafter we give and fix an arbitrary $\sigma(z) \in \mathcal{O}(\Omega)\backslash\{0\}$ and a subset $\pi^* \subset \mathcal{L}$ so as to satisfy the conditions of one of the following three cases.

Case A. (1) For some $\omega \in C^+$, π^* coincides with the principal class π for $(\sigma(0), \omega)$. (2) We can find some $\mu = (\mu_\alpha)$, $\mu' = (\mu'_\alpha) \in \pi$ and an integer i, $0 \leq i \leq m$, such that

$$\sum_{|\alpha|=i} \mu_\alpha \neq \sum_{|\alpha|=i} \mu'_\alpha$$

holds.

Case B. (1) $\sigma(z) \equiv \sigma$, a constant, and for some $\omega \in C^+$ π^* equals to the principal class π for (σ, ω). (2) For any $\nu \in \mathcal{L}\backslash\pi$ it holds

$$\mathcal{R}e(\omega y_\pi(\sigma)) \leq \mathcal{R}e(\omega(y_\nu(\sigma)-1)).$$

(3) For every integer i, $0 \leq i \leq m$, the value

$$p_i = \sum_{|\alpha|=i} \mu_\alpha$$

is common to any $\mu = (\mu_\alpha) \in \pi$.

Case C. $\sigma(z) \equiv \sigma_r$ for some $1 \leq r \leq J$ and $\pi^* = \pi_{\sigma_r}$.

Our aim in this chapter is to construct a solution of the equation (1.5) satisfying (1.4) in the form

$$u(z) = (\phi(z))^{\sigma(z)} \sum_{k \in Z_+^M} \sum_{l=0}^{|k|} u_{k,l}(z)(\phi(z))^{d(z)\cdot k}(\log \phi(z))^{l^*}, \tag{3.1}$$

where $\phi(z) \in \mathcal{O}(\Omega)$, $M \in N$, $d(z) \in (\mathcal{O}(\Omega))^M$, $l^* = |k| - l$ and $u_{k,l}(z) \in \mathcal{O}(\Omega)$ for any (k, l) with $u_{0,0}(z) \in \mathcal{O}^1(\Omega)$. M and $d(z)$ are given in each case as follows. Let $y_{\pi^*}(\sigma(z))$ be the function $y_\pi(\sigma(z))$ (resp. the value $y_\pi(\sigma)$ or the value $\eta(\sigma_r)$) in Case A (resp. Case B or Case C). We classify the subset $\{y_\mu(\sigma(z)) - y_{\pi^*}(\sigma(z)); \mu \in \mathcal{L}\backslash\tilde{\pi}\} \cup \{1\}$ of $\mathcal{O}(\Omega)$ (resp. C or C) modulo integers. Let M be the cardinal number of the classified set. From each class we pick out the element whose real part at $z=0$ is minimal and denote them $d_1 \equiv 1, d_2(z), \cdots, d_M(z)$ (resp. $d_1 \equiv 1, d_2, \cdots, d_M$ or the same as the last). Put

(3.2) $d(z)=(d_1, d_2(z), \cdots, d_M(z))$ (resp. $(d_1, d_2-1, \cdots, d_M-1)$ or $(d_1, d_2, \cdots, d_M)$).

On $\phi(z)$ we impose two conditions, so called the characteristic condition and the noncharacteristic condition, respectively, the former of which corresponds to (1.6) or (1.7). Let us introduce still more several characteristic polynomials in addition to those in Definition 1.2.

DEFINITION 3.1. In Case B the polynomial

$$q_\pi(z, \xi)=\sum_{\mu\in\pi} a_\mu(z)((\xi^\alpha))^\mu$$

of $\xi\in C^n\backslash\{0\}$ is called *the characteristic polynomial of P for the class π in Case* B.

Note that it holds in Case B

$$p_\pi(s, z, \xi)=(\prod_{i=1}^m [s:i]^{p_i})q_\pi(z, \xi).$$

DEFINITION 3.2. We call the polynomial

$$s_\pi(s, \lambda, z, \xi)=\sum_{\mu\in\pi} a_\mu(z)\Big(\sum_{\alpha\in\operatorname{supp}\mu} \mu_\alpha\frac{[s+\lambda:|\alpha|]}{[s:|\alpha|]}\Big)(([s:|\alpha|]\xi^\alpha))^\mu$$

of $s, \lambda\in C$ and $\xi\in C^n\backslash\{0\}$,

$$t_\pi(s, \lambda, z, \xi, \eta)=\Big(\prod_{i=1}^m [s:i]^{p_i}\Big)\Big[\sum_{\mu\in\pi} a_\mu(z)((\xi^\alpha))^\mu\Big\{\sum_{i=1}^m \frac{[s+\lambda:i-1]}{[s:i]}\times \times(D_\xi\log\prod_{|\alpha|=i}(\xi)^{\mu_\alpha\alpha})\cdot\eta\Big\}\Big]$$

of $s, \lambda\in C$ and $\xi, \eta\in C^n\backslash\{0\}$, or

$$s_{\sigma_r}(\lambda, z, \xi, \chi)=\sum_{\mu\in\pi_{\sigma_r}} a_\mu(z)\Big(\sum_{\alpha\in\operatorname{supp}\mu} \mu_\alpha\frac{[\sigma_r+\lambda:|\alpha|]}{[\sigma_r:|\alpha|]}\Big)\times \times(([\sigma_r:|\alpha|]\xi^\alpha))^\mu\chi^{|\mu|-1}$$

of $\lambda, \chi\in C$ and $\xi\in C^n\backslash\{0\}$ *the second characteristic polynomial of P for the principal class π, π or π_{σ_r}* in Case A, B or C, respectively. Moreover, the polynomial

$$s_{\pi,\infty}(s, z, \xi)=[s:m]^{-1}\sum_{\mu\in\pi, |\operatorname{supp}\mu|=m} a_\mu(z)(\sum_{|\alpha|=m}\mu_\alpha)(([s:|\alpha|]\xi^\alpha))^\mu,$$

$$t_{\pi,\infty}(s, z, \xi, \eta)=([s:m]^{-1}\prod_{i=1}^{m}[s:i]^{p_i})\{\sum_{\mu\in\pi,|\mathrm{supp}\,\mu|=m} a_\mu(z)\times$$
$$\times((\xi^\alpha))^\mu(D_\xi \log \prod_{\alpha\in\mathrm{supp}\,\mu,|\alpha|=m} \xi^{\mu_\alpha\alpha})\cdot\eta\}$$

or

$$s_{\sigma_r,\infty}(z, \xi, \chi)=[\sigma_r:m]^{-1}\sum_{\mu\in\pi_{\sigma_r},|\mathrm{supp}\,\mu|=m} a_\mu(z)(\sum_{|\alpha|=m}\mu_\alpha)(([\sigma_r:|\alpha|]\xi^\alpha))^\mu\chi^{|\mu|-1}$$

with $|\mathrm{supp}\,\mu|=\max_{\alpha\in\mathrm{supp}\,\mu}|\alpha|$ is called *the second characteristic polynomial at infinity* for the class π, π or π_{σ_r} in the Case A, B or C, respectively.

Note that the relation between $s_\pi(s, \lambda, z, \xi)$ and $s_{\pi,\infty}(s, z, \xi)$ is given by

$$s_{\pi,\infty}(s, z, \xi)=\lim_{\lambda\to\infty}\lambda^{-m}s_\pi(s, \lambda, z, \xi),$$

and that similar relations hold also for the other two cases.

Lemma 3.1. *Let $h:\Omega\longrightarrow\tilde{\Omega}$ be a biholomorphic transformation with $\tilde{z}=h(z)$ and let $\tilde{P}(\tilde{u})$ be given by* (2.6). *Let π, π or $\pi_{\sigma_r}\subset\mathcal{L}$ satisfy conditions of Case* A, *Case* B *or Case* C *for $P(u)$ for some $(\sigma(z), \omega)$, (σ, ω) or σ_r, respectively, and $\tilde{\pi}$, $\tilde{\pi}$ or $\tilde{\pi}_{\sigma_r}\subset\tilde{\mathcal{L}}$ be the corresponding subset for $\tilde{P}$ for the same $(\sigma(z), \omega)$, (σ, ω) or σ_r. Then we have*

$$\text{(3.3)}\qquad \begin{cases}\tilde{s}_{\tilde{\pi}}(s, \lambda, \tilde{z}, \tilde{\xi})=s_\pi(s, \lambda, z, (\partial\tilde{z}/\partial z)\tilde{\xi})\\ \tilde{s}_{\tilde{\pi},\infty}(s, \tilde{z}, \xi)=s_{\pi,\infty}(s, \tilde{z}, (\partial\tilde{z}/\partial z)\xi),\end{cases}$$

$$\text{(3.4)}\qquad \begin{cases}\tilde{t}_{\tilde{\pi}}(s, \lambda, \tilde{z}, \tilde{\xi}, \tilde{\eta})=t_\pi(s, \lambda, z, (\partial\tilde{z}/\partial z)\tilde{\xi}, (\partial\tilde{z}/\partial z)\tilde{\eta})\\ t_{\tilde{\pi},\infty}(s, \tilde{z}, \tilde{\xi}, \tilde{\eta})=t_{\pi,\infty}(s, z, (\partial\tilde{z}/\partial z)\tilde{\xi}, (\partial\tilde{z}/\partial z)\tilde{\eta})\end{cases}$$

or

$$\begin{cases}\tilde{s}_{\sigma_r}(\lambda, \tilde{z}, \tilde{\xi}'\chi)=s_{\sigma_r}(\lambda, z, (\partial\tilde{z}/\partial z)\tilde{\xi}, \chi)\\ \tilde{s}_{\sigma_r,\infty}(\tilde{z}, \tilde{\xi}, \chi)=s_{\sigma_r,\infty}(z, (\partial\tilde{z}/\partial z)\tilde{\xi}, \chi)\end{cases}$$

where $\tilde{s}_{\tilde{\pi}}$ $(\tilde{s}_{\tilde{\pi},\infty})$, $\tilde{t}_{\tilde{\pi}}$ $(\tilde{t}_{\tilde{\pi},\infty})$ or $\tilde{s}_{\sigma_r}$ $(\tilde{s}_{\sigma_r,\infty})$ is the second characteristic polynomial (at infinity) of $\tilde{P}$ for the class $\tilde{\pi}$, $\tilde{\pi}$ or $\tilde{\pi}_{\sigma_r}$, respectively.

Proof. We will prove only equalities on s_π and t_π and others are left to the reader because they can be seen similarly. Let us use notations in (2.2)$\sim$(2.5). First, since $|\tilde{\alpha}|=|\alpha|$ for any $\tilde{\alpha}$ of the right hand side of (2.2), it follows for any $\mu=(\mu_\alpha)$ that

$$\text{(3.5)}\qquad \sum_{|\alpha|=i}\mu_\alpha=\sum_{|\tilde{\alpha}|=i}\tilde{\mu}_{\tilde{\alpha}}$$ for any $\tilde{\mu}=(\tilde{\mu}_{\tilde{\alpha}})$ of the right hand side of (2.4) and any $0\le i\le m$.

Since, therefore, we have

$$\sum_{\alpha\in\operatorname{supp}\mu}\mu_\alpha\frac{[s+\lambda:|\alpha|]}{[s:|\alpha|]}=\sum_{i=0}^{m}(\sum_{|\alpha|=i}\mu_\alpha)\frac{[s+\lambda:i]}{[s:i]}$$
$$=\sum_{\alpha\in\operatorname{supp}\tilde\mu}\tilde\mu_{\tilde\alpha}\frac{[s+\lambda:|\tilde\alpha|]}{[s:|\tilde\alpha|]},$$

we can obtain the first equality of (3.3) by similar method as that of the proof of Lemma 2.1. Next, if π satisfies conditions of Case B, we can see from (3.5) that class $\tilde\pi$ satisfies also those for the same (σ,ω). Set for any $\mu\in\pi$

$$\prod_{\alpha\in\operatorname{supp}\mu,|\alpha|=j}(\sum_{\tilde\alpha}p_{\alpha,\tilde\alpha}D_{\tilde z}^{\tilde\alpha}\tilde u)^{\mu_\alpha}=\sum_{\tilde\mu_j=(\tilde\mu_{\tilde\alpha})\in\tilde{\mathcal{L}}_j}p_{\mu,\tilde\mu_j}^{(j)}\prod_{|\tilde\alpha|=j}(D_{\tilde z}^{\tilde\alpha}\tilde u)^{\tilde\mu_{\tilde\alpha}}$$

where $\tilde{\mathcal{L}}_j$ is the set of all multi-indices $\tilde\mu_j=(\tilde\mu_{\tilde\alpha})$ with suffices $|\tilde\alpha|=j$ satisfying $\sum_{\tilde\alpha}\tilde\mu_{\tilde\alpha}=\sum_{|\alpha|=j}\mu_\alpha$ and $p_{\mu,\tilde\mu_j}^{(j)}\in\mathcal{O}(\tilde\Omega)$. Since

$$((\sum_{\tilde\alpha}p_{\alpha,\tilde\alpha}D_{\tilde z}^{\tilde\alpha}\tilde u))^\mu=\sum_{\tilde\mu=(\tilde\mu_j)\in\otimes_{j=0}^m\tilde{\mathcal{L}}_j}\prod_{j=0}^{m}\{p_{\mu,\tilde\mu_j}^{(j)}\prod_{|\tilde\alpha|=j}(D_{\tilde z}^{\tilde\alpha}\tilde u)^{\tilde\mu_{\tilde\alpha}}\}$$
$$=\sum_{\tilde\mu}(\prod_{j=0}^{m}p_{\mu,\tilde\mu_j}^{(j)})((D^{\tilde\alpha}\tilde u))^{\tilde\mu}$$

holds, it follows that

$$\tilde t_{\tilde\pi}(s,\lambda,\tilde z,\tilde\xi,\tilde\eta)=(\prod_{i=1}^{m}[s:i]^{p_i})\Big[\sum_{\tilde\mu\in\tilde\pi}(\sum_{\mu\in\pi}a_\mu\circ h^{-1}(\tilde z)\prod_{j=0}^{m}p_{\mu,\tilde\mu_j}^{(j)})\times$$
$$\times\Big\{\sum_{i=1}^{m}\frac{[s+\lambda:i-1]}{[s:i]}(\prod_{j\neq i}\prod_{|\tilde\alpha|=j}(\tilde\xi^{\tilde\alpha})^{\tilde\mu_{\tilde\alpha}})(D_{\tilde\xi}\prod_{|\tilde\alpha|=i}\tilde\xi^{\tilde\mu_{\tilde\alpha}\tilde\alpha}\cdot\tilde\eta)\Big\}\Big]$$
$$=(\prod_{i=1}^{m}[s:i]^{p_i})\Big(\sum_{\mu\in\pi}a_\mu(z)\Big\{\sum_{i=1}^{m}\frac{[s+\lambda:i-1]}{[s:i]}\times$$
$$\times\{\prod_{j\neq i}((\partial\tilde z/\partial z)\tilde\xi^{\tilde\mu_{\tilde\alpha}\tilde\alpha})((D_{\tilde\xi}\prod_{|\tilde\alpha|=i}((\partial\tilde z/\partial z)\tilde\xi)^{\tilde\mu_{\tilde\alpha}\tilde\alpha})\cdot\tilde\eta)\}\Big\}\Big)$$
$$=t_\pi(s,\lambda,z,(\partial\tilde z/\partial z)\tilde\xi,(\partial\tilde z/\partial z)\tilde\eta).$$

Hence we have the first equality of (3.4) and this completes the proof.

Now we can state conditions on $\phi(z)$.

CONDITION I (Characteristic condition). $\phi(z)$ satisfies

(3.6) $$p_\pi(\sigma(z),z,D\phi(z))=0 \quad \text{in Case A,}$$

(3.7) $$q_\pi(z,D\phi(z))=0 \quad \text{in Case B,}$$

or that the equation

$$p_{\sigma_r}(z, D\phi(z), \chi)=0 \tag{3.8}$$

has at least one solution $\chi=u_0(z)\in\mathcal{O}^1(\Omega)$ in Case C, respectively.

Condition II (Noncharacteristic condition).
Case A: (1) The equation

$$s_\pi(\sigma(0), \lambda, 0, D\phi(0))=0$$

of λ has no solutions such that $\lambda=\boldsymbol{d}(0)\cdot\boldsymbol{k}$ for any $\boldsymbol{k}\in\boldsymbol{Z}_+^M\backslash\{0\}$.
(2) It holds

$$s_{\pi,\infty}(\sigma(0), 0, D\phi(0))\neq 0.$$

Case B: We can find an $\eta\in\boldsymbol{C}^n\backslash\{0\}$ satisfying following properties.
(1) The equation

$$t_\pi(\sigma, \lambda, 0, D\phi(0), \eta)=0$$

of λ has no solutions such that $\lambda=\boldsymbol{d}\cdot\boldsymbol{k}$ for any $\boldsymbol{k}\in\boldsymbol{Z}_+^M$.
(2) It holds

$$t_{\pi,\infty}(\sigma, 0, D\phi(0), \eta)\neq 0.$$

Case C: (1) There exists a solution $\chi=u_0(z)\in\mathcal{O}^1(\Omega)$ of the equation (3.8) for which the equation

$$s_{\sigma_r}(\lambda, 0, D\phi(0), u_0(0))=0$$

of λ has no solutions such that $\lambda=\boldsymbol{d}\cdot\boldsymbol{k}$ for any $\boldsymbol{k}\in\boldsymbol{Z}_+^M\backslash\{0\}$.
(2) It holds

$$s_{\sigma_r,\infty}(0, D\phi(0), u_0(0))\neq 0.$$

Note that Condition II, (2) implies in each case that the principal class π or π_{σ_r} contains at least one $\mu=(\mu_\alpha)$ such that $|\mathrm{supp}\,\mu|=m$.

Remark. The condition of Case A, (2) is necessary for Cond. I and Cond. II to be consistent. However, it is not sufficient, because $s_\pi(s, \lambda, z, \xi)$ can be divisible by $p_\pi(s, z, \xi)$ as a polynomial of ξ under particular choice of $a_\mu(z)$, for example

$$P(u)=u\,D_1^2u+u\,D_2^2u+(D_1u)^2+(D_2u)^2.$$

§ 4. Some differential formulae and its application to formal functional series.

In this and the next sections $\sigma(z)$ and $\phi(z)$ denote any functions in $\mathcal{O}(\Omega)$ not necessarily satisfying conditions in the previous section. For any multi-index $\boldsymbol{k}=(k_1, \cdots, k_N), \boldsymbol{l}=(l_1, \cdots, l_N) \in \boldsymbol{Z}_+^N$, we denote $\boldsymbol{k}\leq \boldsymbol{l}$ if and only if $k_i \leq l_i, i=1, \cdots, N$, and $\boldsymbol{k}<\boldsymbol{l}$ if and only if $\boldsymbol{k}\leq \boldsymbol{l}$ and $\boldsymbol{k}\neq \boldsymbol{l}$. We associate for any $\alpha \in \boldsymbol{Z}_+^n$, $0\leq\beta\leq\alpha$ and $0\leq i, j\leq|\alpha|$ a polynomial $N_{i,j}^{\alpha,\beta}=N_{i,j}^{\alpha,\beta}(\sigma(z), l, \phi(z))$ of $D^\gamma\sigma, 0\leq\gamma\leq\alpha, l\in \boldsymbol{Z}_+$ and $D^\delta\phi$ with $0\leq\delta\leq\alpha$, inductively with respect to $(\alpha, \leq)$, as follows:

$$N_{0,0}^{0,0}\equiv 1, \tag{4.1}$$

$$\begin{aligned} N_{i,j}^{\alpha+e_k,\beta} = &(\sigma-j+1)N_{i-1,j-1}^{\alpha,\beta-e_k} D_k\phi + (l+i-j+1)N_{i,j-1}^{\alpha,\beta-e_k} D_k\phi + \\ &+ N_{i-1,j}^{\alpha,\beta-e_k} D_k\sigma + D_k N_{i,j}^{\alpha,\beta-e_k} + N_{i,j}^{\alpha,\beta}, \end{aligned} \tag{4.2}$$

where $e_k=(0, \cdots, \overset{k}{\check{1}}, \cdots, 0), k=1, \cdots, n$, is the k-th unit vector in $\boldsymbol{Z}^n$. In (4.2) and from now we put conventionally

$$N_{i,j}^{\alpha,\beta}\equiv 0 \qquad \text{for any } \alpha, \beta \in \boldsymbol{Z}^n \text{ and } i, j \in \boldsymbol{Z} \text{ not satisfying } 0\leq\beta\leq\alpha \text{ and } 0\leq i, j\leq|\alpha|. \tag{4.3}$$

LEMMA 4.1. *$N_{i,j}^{\alpha,\beta}$ is determined uniquely by $\sigma, \phi \in \mathcal{O}(\Omega)$ and $l \in \boldsymbol{Z}_+$ for every $\alpha\geq 0$, $0\leq\beta\leq\alpha$ and $0\leq i, j\leq|\alpha|$, and we have for any $u(z) \in \mathcal{O}(\Omega)$*

$$\begin{aligned} D^\alpha((\phi(z))^{\sigma(z)}(\log\phi(z))^l u(z)) = &\sum_{i=0}^{|\alpha|}\sum_{j=0}^{|\alpha|}\phi^{\sigma-j}(\log\phi)^{l+i-j}\times \\ &\times\Big(\sum_{0\leq\beta\leq\alpha} N_{i,j}^{\alpha,\beta}(\sigma, l, \phi) D^{\alpha-\beta}u\Big). \end{aligned}$$

PROOF. We can prove this lemma by elementary calculus inductively with respect to $\alpha\geq 0$, using (4.2), and leave it to the reader.

For any $z\in \boldsymbol{C}$ and $k, l\in \boldsymbol{Z}$ put

$$[z:k, l]=\sum_{0\leq i_1<i_2<\cdots<i_l\leq k-1}(z-i_1)(z-i_2)\cdots(z-i_l); \qquad k\geq l\geq 1,$$

$[z:k, 0]=1$ for any $k\geq 0$ and $[z:k, l]=0$ otherwise. Then the following lemma is clear.

LEMMA 4.2. *For any $k, l\in \boldsymbol{Z}$ we have*

$$[z:k, k]=[z:k]$$

$$(4.4)\qquad (z-k)[z:k,l-1]+[z:k,l]=[z:k+1,l] \qquad \textit{if } (k,l)\neq(-1,0).$$

Let us calculate $N_{i,j}^{\alpha,\beta}(\sigma(z),l,\phi(z))$ in some special cases.

Lemma 4.3. (1) *We have*

$$(4.5)\qquad N_{i,j}^{\alpha,\beta}=0 \qquad \textit{if} \quad |\beta|<i \textit{ or } |\beta|<j,$$

$$(4.6)\qquad N_{i,|\alpha|}^{\alpha,\alpha}(\sigma,l,\phi)=[\sigma:|\alpha|,i][l:|\alpha|-i](D\phi)^{\alpha},$$

$$(4.7)\qquad N_{i,|\alpha|-1}^{\alpha,\alpha-e_k}=\alpha_k[\sigma:|\alpha|-1,i][l:|\alpha|-1-i](D\phi)^{\alpha-e_k}; \qquad k=1,\cdots,n.$$

(2) *If* $D\sigma\equiv 0$, *we have*

$$(4.8)\qquad N_{i,j}^{\alpha,\beta}=0 \qquad \textit{if} \quad i>j$$

$$(4.9)\qquad N_{i,|\alpha|-1}^{\alpha,\alpha}=[\sigma:|\alpha|-1,i][l:|\alpha|-1-i]\Big(\sum_{|\gamma|=2}\binom{\alpha}{\gamma}(D\phi)^{\alpha-\gamma}D^{\gamma}\phi\Big)$$
$$\textit{for } |\alpha|\geq 2 \textit{ and } i\leq|\alpha|-1.$$

Proof. Since these formulae can be proved inductively with respect to α by elementary calculus, we show here only (4.6) and (4.9) and leave others to the reader. First, (4.6) is trivial for $\alpha=0$ and $i=0$, because (4.1) holds by definition. If for a given $\alpha>0$ (4.6) holds for any $\beta\leq\alpha$, we have using (4.2), (4.3) and (4.4)

$$\begin{aligned}N_{i,|\alpha|+1}^{\alpha+e_k,\alpha+e_k}(\sigma,l,\phi)&=(\sigma-|\alpha|)N_{i-1,|\alpha|}^{\alpha,\alpha}D_k\phi+(l+i-|\alpha|)N_{i,|\alpha|}^{\alpha,\alpha}D_k\phi\\&=\{(\sigma-|\alpha|)[\sigma:|\alpha|,i-1][l:|\alpha|-i+1]+(l+i-|\alpha|)[\sigma:|\alpha|,i]\times\\&\quad\times[l:|\alpha|-i]\}(D\phi)^{\alpha+e_k}\\&=[\sigma:|\alpha|+1,i][l:|\alpha|-i+1](D\phi)^{\alpha+e_k}.\end{aligned}$$

Hence we have (4.6) for $\alpha+e_k$. Next, if $D\sigma\equiv 0$, we have

$$N_{i,0}^{e_k,e_k}=0 \qquad \text{for any } i$$

from (4.2). For a given $\alpha\geq 0$ with $|\alpha|\geq 2$ let the equality (4.9) hold for any $\beta\leq\alpha$. Then it follows from (4.2), (4.3), (4.6) and the assumption

$$\begin{aligned}N_{i,|\alpha|}^{\alpha+e_k,\alpha+e_k}&=(\sigma-|\alpha|+1)N_{i-1,|\alpha|-1}^{\alpha,\alpha}D_k\phi+(l+i-|\alpha|+1)N_{i,|\alpha|-1}^{\alpha,\alpha}D_k\phi+D_kN_{i,|\alpha|}^{\alpha,\alpha}\\&=(\sigma-|\alpha|+1)[\sigma:|\alpha|-1,i-1][l:|\alpha|-i]\Big(\sum_{|\gamma|=2}\binom{\alpha}{\gamma}(D\phi)^{\alpha+e_k-\gamma}D^{\gamma}\phi\Big)+\\&\quad+[\sigma:|\alpha|-1,i][l:|\alpha|-i]\Big(\sum_{|\gamma|=2}\binom{\alpha}{\gamma}(D\phi)^{\alpha+e_k-\gamma}D^{\gamma}\phi\Big)+\end{aligned}$$

$$+[\sigma:|\alpha|, i][l:|\alpha|-i](\sum_{p=1}^{n}\alpha_p(D\phi)^{\alpha-e_p}D_kD_p\phi).$$

Using the facts

$$\sum_{p=1}^{n}\alpha_p(D\phi)^{\alpha-e_p}D_kD_p\phi=\sum_{|\gamma|=2}\binom{\alpha}{\gamma-e_k}(D\phi)^{\alpha+e_k-\gamma}D^\gamma\phi$$

and

$$\binom{\alpha}{\gamma}+\binom{\alpha}{\gamma-e_k}=\binom{\alpha+e_k}{\gamma},$$

we have (4.9) for $\alpha+e_k$ and this completes the proof.

LEMMA 4.4. *$N_{i,j}^{\alpha,\beta}(\sigma, l, \phi)$ is a polynomial of $D^\gamma\sigma, l$ and $D^\delta\phi$ such that $0\leq\gamma\leq\alpha$, $|\gamma|\leq|\beta|-j$, $0\leq\delta\leq\alpha$ and $|\delta|\leq|\beta|-j+1$ with coefficients in Z. If $\prod_\gamma(D^\gamma\sigma)^{p_\gamma}l^q\prod_\delta(D^\delta\phi)^{r_\delta}$ with nonzero coefficient appears in this polynomial, then we have:*

$$\sum_\gamma p_\gamma+q\leq|\beta|$$
$$\sum_\gamma(|\gamma|+1)p_\gamma+\sum_\delta(|\delta|+1)r_\delta\leq|\beta|+i+j.$$

PROOF. The proof of this lemma can be assured inductively with respect to α by elementary calculus using the definition of $N_{i,j}^{\alpha,\beta}$ and we leave it to the reader.

Now consider a formal functional series

$$u=(\phi(z))^{\sigma(z)}\sum_{(k,l)\in D}u_{k,l}\,\phi^{d(z)\cdot k}(\log\phi)^{l*} \tag{4.10}$$

where $D=\{(k, l)\in Z_+^M\times Z_+;\, l\leq|k|\}$, $u_{k,l}\in\mathcal{O}(\Omega)$ and $d(z)\in(\mathcal{O}(\Omega))^M$ satisfies $(d(z))_1\equiv1$. Moreover, put conventionally $u_{k,l}=0$ for any $(k, l)\notin D$. We associate u another formal functional series $D^\alpha u$, $\alpha\in Z_+^n$, such that

$$D^\alpha u=\phi^{\sigma-|\alpha|}\sum_{(k,l)\in D}u_{k,l}^\alpha\,\phi^{d\cdot k}(\log\phi)^{l*} \tag{4.11}$$

with

$$u_{k,l}^\alpha=\sum_{i=0}^{|\alpha|}\sum_{j=0}^{|\alpha|}\sum_{0\leq\beta\leq\alpha}N_{i,j}^{\alpha,\beta}(\sigma+d\cdot k-|\alpha|+j, |k|-l+j-i, \phi)\times \times D^{\alpha-\beta}u_{k-(|\alpha|-j)f_1, l-|\alpha|+i} \tag{4.12}$$

denoting $f_k=(0,\cdot\cdot,\overset{k}{\overset{\vee}{1}},\cdot\cdot,0)$, $k=1,\cdots,M$, the k-th unit vector of Z^M. As easily seen, if the right hand side of (4.10) converges to a holomorphic function $u(z)$ at a $z\in\mathcal{R}(\Omega\backslash S)$, then the right hand side of (4.11) converges at the same point and coincides actually to $D^{\alpha}u(z)$. Let us consider to divide each $u_{k,l}^{\alpha}$ into the "principal part" and the remainder in the following three ways. We put, first,

$$u_{k,l}^{\alpha}=[\sigma+\boldsymbol{d}\cdot\boldsymbol{k}:|\alpha|](D\phi)^{\alpha}u_{k,l}+R_{k,l}^{\alpha}, \tag{4.13}$$

secondly,

$$u_{k,l}^{\alpha}=S_{k,l}^{|\alpha|}(D\phi)^{\alpha}+Q_{k,l}^{\alpha}, \tag{4.14}$$

where we denote

$$S_{k,l}^{N}=\sum_{i=0}^{N}[|\boldsymbol{k}|-l+i:i][\sigma+\boldsymbol{d}\cdot\boldsymbol{k}:N,N-i]u_{k,l-i}\qquad \text{for } N\in\boldsymbol{Z}_{+}, \tag{4.15}$$

and, thirdly,

$$u_{k,l}^{\alpha}=S_{k,l}^{|\alpha|}(D\phi)^{\alpha}+[\sigma+\boldsymbol{d}\cdot\boldsymbol{k}-1:|\alpha|-1](D\phi)^{\alpha}D_{\alpha,\phi}u_{k-f_1,\,l-1}+\tilde{R}_{k,l}^{\alpha}, \tag{4.16}$$

where $(D\phi)^{\alpha}D_{\alpha,\phi}$ denotes a partial differential operator

$$(D\phi)^{\alpha}D_{\alpha,\phi}=\sum_{k=1}^{n}\alpha_{k}(D\phi)^{\alpha-e_{k}}D_{k}+\sum_{|\gamma|=2}\binom{\alpha}{\gamma}(D\phi)^{\alpha-\gamma}D^{\gamma}\phi.$$

LEMMA 4.5. *(1) $R_{k,l}^{\alpha}$ is a linear form of such $D^{\beta}u_{\kappa,\rho}$'s as $\beta\leq\alpha$ and $(\kappa,\rho)<(\boldsymbol{k},l)$ with coefficients in $\mathcal{O}(\Omega)$.*
(2) If $D\sigma\equiv 0$,

$$Q_{k,l}^{\alpha}=\tilde{R}_{k,l}^{\alpha}=0\qquad \textit{for any } (\boldsymbol{k},l) \textit{ such that } (\boldsymbol{k})_1=0 \textit{ or } l=0$$

and, for $(\boldsymbol{k},l)\geq(\boldsymbol{f}_1,1)$, $Q_{k,l}^{\alpha}$ or $\tilde{R}_{k,l}^{\alpha}$ is a linear form of such $D^{\beta}u_{\kappa,\rho}$'s as $\beta\leq\alpha$ and $(\kappa,\rho)\leq(\boldsymbol{k}-\boldsymbol{f}_1,l-1)$, or $\beta\leq\alpha$ and $(\kappa,\rho)<(\boldsymbol{k}-\boldsymbol{f}_1,l-1)$ with coefficients in $\mathcal{O}(\Omega)$, respectively.

PROOF. Denote by $(4.12)_r$ the right hand side of (4.12).
(1) If a term in $(4.12)_r$ contains $D^{\alpha-\beta}u_{k,l}$, then we can see $i=j=|\alpha|$. Noting that for $\beta<\alpha$ $N_{|\alpha|,|\alpha|}^{\alpha,\beta}=0$ by (4.5), we can conclude that the only term containing $u_{k,l}$ in $(4.12)_r$ is $N_{|\alpha|,|\alpha|}^{\alpha,\alpha}(\sigma+\boldsymbol{d}\cdot\boldsymbol{k},|\boldsymbol{k}|-l,\phi)u_{k,l}$ and hence the assertion follows from (4.6).
(2) Count out, first, all terms in $(4.12)_r$ containing $D^{\alpha-\beta}u_{k,l-|\alpha|+i}$, $i=0,1,\cdots,|\alpha|$, for which we can see $j=|\alpha|$. Then, since it holds by (4.5)

$$N_{i,|\alpha|}^{\alpha,\beta}=0\qquad \text{for } \beta<\alpha,$$

we can see from (4.6) the sum of all such terms equals to $S_{k,l}^{|\alpha|}(D\phi)^{\alpha}$. Next, as it can be seen that the coefficient of $D^{\alpha-\beta}u_{k-f_1,l}$ in $(4.12)_r$

vanishes from the fact $i=|\alpha|$ and $j=|\alpha|-1$ and from (4.8), we have the first half of the assertion. On terms containing $D^{\alpha-\beta}u_{k-f_1,l-1}$ in $(4.12)_r$, we can see $i=j=|\alpha|-1$ and that $\beta=\alpha$ or $\beta=\alpha-e_k$, $k=1, \cdots, n$. Calculating coefficients of these terms using (4.9) or (4.7), for $\beta=\alpha$ or $\beta=\alpha-e_k$, respectively, we can ascertain the latter half of the assertion and this completes the proof.

COROLLARY 4.6. (1) *We have*

$$u_{0,0}^{\alpha}=[\sigma:|\alpha|](D\phi)^{\alpha}u_{0,0}, \tag{4.17}$$

$$u_{f_1,1}^{\alpha}=S_{f_1,1}^{|\alpha|}(D\phi)^{\alpha}+[\sigma:|\alpha|-1](D\phi)^{\alpha}D_{\alpha,\phi}u_{0,0} + [\sigma:|\alpha|-1,|\alpha|-2]\sum_{k=1}^{n}\alpha_k(D\phi)^{\alpha-e_k}D_k\sigma\cdot u_{0,0}, \tag{4.18}$$

$$u_{k,l}^{\alpha}=S_{k,l}^{|\alpha|}(D\phi)^{\alpha} \quad \text{for } k=(0,k')\in Z_{+}^{M}. \tag{4.19}$$

(2) *If* $D\sigma\equiv 0$, *we have*

$$u_{f_1,0}^{\alpha}=[\sigma+1:|\alpha|](D\phi)^{\alpha}u_{f_1,0}, \tag{4.20}$$

$$u_{k,0}^{\alpha}=S_{k,0}^{|\alpha|}(D\phi)^{\alpha}. \tag{4.21}$$

PROOF. (4.17), (4.20) and (4.21) can be seen directly from the above lemma. On $u_{f_1,1}^{\alpha}$, as we may consider only terms corresponding to $(i,j)=(|\alpha|,|\alpha|)$, $(|\alpha|-1,|\alpha|)$ or $(|\alpha|-1,|\alpha|-1)$ of the right hand side of (4.12), we have (4.18) by direct calculus using Lemma 4.3. If $k=(0,k')$, any terms except for $j=|\alpha|$ of the right hand side of (4.12) vanish, because for such k we have $u_{\kappa,\rho}=0$, $(\kappa,\rho)\leq(k-f_1,l)$, by definition. Hence we have $Q_{k,l}=0$ and obtain (4.19). This completes the proof.

§ 5. Differential-productive formulae for formal functional series.

In this section we define a formal functional series

$$((D^{\alpha}u))^{\mu}=(\phi(z))^{\nu_{\mu}(\sigma(z))}\sum_{(k,l)\in D}u_{k,l}^{\mu}\phi^{d(z)\cdot k}(\log\phi)^{l*}$$

from formal functional series $D^{\alpha}u$ given by (4.11) and (4.12) and consider two kinds of representations of $u_{k,l}^{\mu}$, $(k,l)\in D\backslash\{(0,0)\}$, which shall be used to construct a formal solution of the equation (1.5) in each three cases. Consider formal power series

$$V(X, Y)=\sum_{(k,l)\in D} v_{k,l}\, X^k Y^{l*}$$

with indeterminants $X=(X_1, \cdots, X_M)$ and Y and $v_{k,l}\in\mathcal{O}(\Omega)$. We define multiplication between K such power series

(5.1) $$V_i(X, Y)=\sum_{(k,l)\in D} v^i_{k,l} X^k Y^{l*}; \qquad i=1, \cdots, K,$$

as

(5.2) $$\prod_{i=1}^{K} V_i(X, Y)=\sum_{(k,l)\in D} \hat{v}_{k,l}\, X^k Y^{l*},$$

where

(5.3) $$\hat{v}_{k,l}=\sum_{((k_i,l_i))_{1\le i\le K}\in\mathcal{C}(K;k,l)} \prod_{i=1}^{K} v^i_{k_i,l_i}$$

with

(5.4) $$\mathcal{C}(K; k, l)=\{((k_i, l_i))_{1\le i\le K}\in D^K; \sum_{i=1}^{K}(k_i, l_i)=(k, l)\}.$$

Now let us introduce two subsets of $\mathcal{C}(K; k, l)$ such as

$$\mathcal{C}_1(K; k, l)=\{((k_i, l_i))_{1\le i\le K}\in\mathcal{C}(K; k, l); (k_{i_0}, l_{i_0})=(k, l) \text{ for some } i_0\},$$

$$\mathcal{C}_2(K; k, l)=\mathcal{C}\backslash\mathcal{C}_1$$

$$=\{((k_i, l_i))_i\in\mathcal{C}; (k_i, l_i)<(k, l) \text{ for any } i\},$$

and denote summation of $\prod_{i=1}^{K} v^i_{k_i,l_i}$'s over all $((k_i, l_i))_i\in\mathcal{C}(K; k, l)$ or $\mathcal{C}_\nu(K; k, l)$, $\nu=1, 2$, simply

$$^{\sigma}\mathcal{C}(\prod_{i=1}^{K} V_i(X, Y; k, l) \quad\text{or}\quad ^{\sigma}\mathcal{C}_\nu(\prod_{i=1}^{K} V_i(X, Y); k, l),$$

respectively. Note that it holds

$$\hat{v}_{k,l}={}^{\sigma}\mathcal{C}(\prod_{i=1}^{K} V_i; k, l)$$

$$^{\sigma}\mathcal{C}(\prod_{i=1}^{K} V_i; k, l)={}^{\sigma}\mathcal{C}_1+{}^{\sigma}\mathcal{C}_2.$$

When $V_1(X, Y)=\cdots=V_K(X, Y)$, we provide particular notations of sets of multi-indices to describe $(V(X, Y))^K$. Put

(5.5) $$\mathcal{B}(K; k, l)=\{(j_i, k_i, l_i))_{1\le i\le r}\in(N\times D)^r; 1\le r\le K, \sum_{i=1}^{r} j_i=K,$$

$$(\mathbf{0}, 0) \leqq (\boldsymbol{k}_1, l_1) \prec \cdots \prec (\boldsymbol{k}_r, l_r) \leqq (\boldsymbol{k}, l), \ \sum_{i=1}^{r} j_i (\boldsymbol{k}_i, l_i) = (\boldsymbol{k}, l)\},$$

$$\mathscr{B}_1(K; \boldsymbol{k}, l) = \{((j_i, \boldsymbol{k}_i, l_i))_i \in \mathscr{B}(K; \boldsymbol{k}, l) \,;\, (\boldsymbol{k}_{i_0}, l_{i_0}) = (\boldsymbol{k}, l) \text{ for some } i_0\},$$

$$\mathscr{B}_2(K; \boldsymbol{k}, l) = \mathscr{B} \backslash \mathscr{B}_1$$

$$= \{((j_i, \boldsymbol{k}_i, l_i))_i \in \mathscr{B}(K; \boldsymbol{k}, l) \,;\, (\boldsymbol{k}_i, l_i) < (\boldsymbol{k}, l) \text{ for any } i\},$$

where we use another order $\prec$ on $Z_+^M \times Z_+$ such that for any $\boldsymbol{m}, \boldsymbol{n} \in Z_+^{M+1}$

$$\boldsymbol{m} = (m_1, \cdots, m_{M+1}) \prec \boldsymbol{n} = (n_1, \cdots, n_{M+1}) \Longleftrightarrow \boldsymbol{m} \neq \boldsymbol{n} \text{ and (a) } |\boldsymbol{m}| < |\boldsymbol{n}| \text{ or}$$

$$\text{(b) } |\boldsymbol{m}| = |\boldsymbol{n}| \text{ and } m_{i_0} < n_{i_0} \text{ with } i_0 = \min\{i \,;\, m_i \neq n_i\}.$$

As easily seen, it holds

$$^{\sigma}\mathcal{C}(\text{or } \mathcal{C}_\nu, \nu = 1, 2)(V(X, Y)^K; \boldsymbol{k}, l) =$$

$$= \sum_{((j_i, \boldsymbol{k}_i, l_i))_{1 \leq i \leq r} \in \mathscr{B}(\text{or } \mathscr{B}_\nu)(K; \boldsymbol{k}, l)} \begin{pmatrix} K \\ j_1 \cdots j_r \end{pmatrix} \prod_{i=1}^{r} (v_{\boldsymbol{k}_i, l_i})^{j_i},$$

where we note

$$\begin{pmatrix} K \\ j_1 \cdots j_r \end{pmatrix} = \frac{K!}{j_1! \cdots j_r!}.$$

Now for a given formal functional series (4.10) and for any $\alpha \in Z_+^n$ we set

$$U^\alpha(X, Y) = \sum_{(\boldsymbol{k}, l)} u_{\boldsymbol{k}, l}^\alpha X^{\boldsymbol{k}} Y^{l*} \tag{5.6}$$

with $u_{\boldsymbol{k},l}^\alpha$ given by (4.12) and put for any $\mu \in \mathcal{L}^*$

$$((U^\alpha(X, Y)))^\mu = \sum_{(\boldsymbol{k}, l) \in D} u_{\boldsymbol{k}, l}^\mu X^{\boldsymbol{k}} Y^{l*}, \tag{5.7}$$

using our definition (5.2) with (5.3) of multiplication between formal power series. Then it is clear that

$$u_{\boldsymbol{k}, l}^\mu = {}^{\sigma}\mathcal{C}(((U^\alpha(X, Y)))^\mu; \boldsymbol{k}, l)$$

$$= {}^{\sigma}\mathcal{C}_1 + {}^{\sigma}\mathcal{C}_2.$$

Since it can be seen from (4.13) and (4.17) that

$$^{\sigma}\mathcal{C}_1(((U^\alpha))^\mu; \boldsymbol{k}, l) = \sum_{\alpha \in \operatorname{supp} \mu} \mu_\alpha ((u_{\mathbf{0}, 0}^\alpha))^{\mu - \varepsilon_\alpha} u_{\boldsymbol{k}, l}^\alpha$$

$$=\Big(\sum_{\alpha}\mu_\alpha\frac{[\sigma+d\cdot k:|\alpha|]}{[\sigma:|\alpha|]}\Big)(([\sigma:|\alpha|](D\phi)^\alpha))^\mu(u_{0,0})^{|\mu|-1}u_{k,l}+$$

$$+\sum_{\alpha}\mu_\alpha((u^\alpha_{0,0}))^{\mu-\varepsilon_\alpha}R^\alpha_{k,l}\qquad \text{for } (k,l)\in D_+$$

where $\varepsilon_\alpha=(\delta_{\alpha\alpha'})_{\alpha'}$, $\alpha\in Z^n_+$, with $\delta_{\alpha\alpha'}=1$ for $\alpha'=\alpha$ and $\delta_{\alpha\alpha'}=0$ for $\alpha'\neq\alpha$ is the α-th unit vector of $\mathcal{L}^*$ and $D_+=D\backslash\{(0,0)\}$, we have

$$u^\mu_{0,0}=(([\sigma:|\alpha|](D\phi)^\alpha))^\mu(u_{0,0})^{|\mu|}, \tag{5.8}$$

$$u^\mu_{k,l}=\Big(\sum_{\alpha\in\operatorname{supp}\mu}\mu_\alpha\frac{[\sigma+d\cdot k:|\alpha|]}{[\sigma:|\alpha|]}\Big)(([\sigma:|\alpha|](D\phi)^\alpha))^\mu(u_{0,0})^{|\mu|-1}u_{k,l}+R^\mu_{k,l};\quad (k,l)\in D_+ \tag{5.9}$$

with

$$R^\mu_{k,l}=\sum_{\alpha\in\operatorname{supp}\mu}\mu_\alpha((u^\alpha_{0,0}))^{\mu-\varepsilon_\alpha}R^\alpha_{k,l}+{}^\sigma\mathcal{C}_2(((U^\alpha))^\mu;k,l). \tag{5.10}$$

Note that $R^\mu_{k,l}$ is a polynomial of such $D^\beta u_{\kappa,\rho}$'s as $(\kappa,\rho)<(k,l)$ and $\beta\leq\alpha$ of degree $|\mu|$ with coefficients in $\mathcal{O}(\Omega)$, by Lemma 4.5 (1).

Next, under the condition $D\sigma\equiv0$ let us find out in $u^\mu_{k,l}$ two kinds of terms, that is, those which contain $u_{k-f_1,l-1}$ and their derivatives or $u_{k,\rho}$, $\rho\leq l$, respectively. Let $\tilde{Q}^\alpha(X,Y)$, $\alpha\in Z^n_+$, and $S^N(X,Y)$, $N\in Z_+$, be given by

$$\tilde{Q}^\alpha(X,Y)=\sum_{(k,l)\in D}Q^\alpha_{k+f_1,\,l+1}X^kY^{l*},$$

$$S^N(X,Y)=\sum_{(k,l)\in D}S^N_{k,l}X^kY^{l*}$$

and $S^\mu_{k,l}$, $\mu\in\mathcal{L}^*$, be given by

$$((S^{|\alpha|}(X,Y)))^\mu=\sum_{(k,l)\in D}S^\mu_{k,l}X^kY^{l*}.$$

As it holds under the condition $D\sigma\equiv0$ that

$$U^\alpha(X,Y)=S^{|\alpha|}(X,Y)(D\phi)^\alpha+\tilde{Q}^\alpha(X,Y)X_1$$

from (4.14) and Lemma 4.5 (2), we have

$$((U^\alpha(X,Y)))^\mu=((S^{|\alpha|}(X,Y)(D\phi)^\alpha+\tilde{Q}^\alpha(X,Y)X_1))^\mu$$

$$=((S^{|\alpha|}(D\phi)^\alpha))^\mu+\sum_{\alpha\in\operatorname{supp}\mu}\mu_\alpha((S^{|\alpha|}(D\phi)^\alpha))^{\mu-\varepsilon_\alpha}\tilde{Q}^\alpha X_1$$

$$+\sum_{\substack{\mu'+\mu''=\mu\\|\mu''|\geq2}}\binom{\mu}{\mu'}((S^{|\alpha|}(D\phi)^\alpha))^{\mu'}((\tilde{Q}^\alpha))^{\mu''}X_1^{|\mu''|}$$

and

$$u_{k,l}^{\mu}=S_{k,l}^{\mu}(((D\phi)^{\alpha}))^{\mu}+\sum_{\alpha}\mu_{\alpha}{}^{\sigma}\mathcal{C}(((S^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}}\tilde{Q}^{\alpha};\boldsymbol{k}-\boldsymbol{f}_1,l-1)+$$

$$+\sum_{\{\substack{\mu'+\mu''=\mu\\|\mu''|\geq 2}}\binom{\mu}{\mu'}{}^{\sigma}\mathcal{C}(((S^{|\alpha|}(D\phi)^{\alpha}))^{\mu'}((\tilde{Q}^{\alpha}))^{\mu''};\boldsymbol{k}-|\mu''|\boldsymbol{f}_1,l-|\mu''|).$$

Therefore we can conclude

(5.11) $\qquad u_{k,l}^{\mu}=S_{k,l}^{\mu}(((D\phi)^{\alpha}))^{\mu}$ for any $(\boldsymbol{k},l)$ such that $(\boldsymbol{k})_1=0$ or $l=0$.

On the other hand, since it holds for any $\alpha\in\operatorname{supp}\mu$

$${}^{\sigma}\mathcal{C}(((S^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}}\tilde{Q}^{\alpha};\boldsymbol{0},0)=((S_{0,0}^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}}Q_{f_1,1}^{\alpha},$$

$${}^{\sigma}\mathcal{C}_1(((S^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}}\tilde{Q}^{\alpha};\boldsymbol{k},l)=((S_{0,0}^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}}Q_{k+f_1,l+1}^{\alpha}+$$

$$+\sum_{\beta\in\operatorname{supp}\mu,\beta\neq\alpha}\mu_{\beta}((S_{0,0}^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}-\varepsilon_{\beta}}Q_{f_1,1}^{\alpha}S_{k,l}^{|\beta|}(D\phi)^{\beta}+$$

$$+(\mu_{\alpha}-1)((S_{0,0}^{|\alpha|}(D\phi)^{\alpha}))^{\mu-2\varepsilon_{\alpha}}Q_{f_1,1}^{\alpha}S_{k,l}^{|\alpha|}(D\phi)^{\alpha};\quad(\boldsymbol{k},l)>(\boldsymbol{0},0),$$

we have using (4.14)$\sim$(4.16) and Cor. 4.6

$$u_{f_1,1}^{\mu}=S_{f_1,1}^{\mu}(((D\phi)^{\alpha}))^{\mu}+(u_{0,0})^{|\mu|-1}(([\sigma:|\alpha|](D\phi)^{\alpha}))^{\mu}\times \tag{5.12}$$

$$\times\Big(\sum_{\alpha\in\operatorname{supp}\mu}\mu_{\alpha}\frac{[\sigma:|\alpha|-1]}{[\sigma:|\alpha|]}D_{\alpha,\phi}u_{0,0}\Big)$$

$$u_{k+f_1,l+1}^{\mu}=S_{k+f_1,l+1}^{\mu}(((D\phi)^{\alpha}))^{\mu}+(u_{0,0})^{|\mu|-1}(([\sigma:|\alpha|](D\phi)^{\alpha}))^{\mu}\times \tag{5.13}$$

$$\times\Big\{\sum_{\alpha\in\operatorname{supp}\mu}\mu_{\alpha}\frac{[\sigma+\boldsymbol{d}\cdot\boldsymbol{k}:|\alpha|-1]}{[\sigma:|\alpha|]}D_{\alpha,\phi}+\sum_{\alpha,\beta\in\operatorname{supp}\mu}\frac{\mu_{\alpha}(\mu_{\beta}-\delta_{\alpha,\beta})}{\sigma-|\alpha|+1}\times$$

$$\times\frac{[\sigma+\boldsymbol{d}\cdot\boldsymbol{k}:|\beta|]}{[\sigma:|\beta|]}\frac{D_{\alpha,\phi}u_{0,0}}{u_{0,0}}\Big\}u_{k,l}+\tilde{R}_{k+f_1,l+1}^{\mu};\qquad(\boldsymbol{k},l)\in\boldsymbol{D}_+$$

putting

$$\tilde{R}_{k+f_1,l+1}^{\mu}=\sum_{\alpha}\mu_{\alpha}((S_{0,0}^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}}\tilde{R}_{k+f_1,l+1}^{\alpha}+ \tag{5.14}$$

$$+\sum_{\alpha,\beta}\mu_{\alpha}(\mu_{\beta}-\delta_{\alpha,\beta})((S_{0,0}^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}-\varepsilon_{\beta}}Q_{f_1,1}^{\alpha}\tilde{S}_{k,l}^{|\beta|}(D\phi)^{\beta}+$$

$$+\sum_{\alpha}\mu_{\alpha}{}^{\sigma}\mathcal{C}_2(((S^{|\alpha|}(D\phi)^{\alpha}))^{\mu-\varepsilon_{\alpha}}\tilde{Q}^{\alpha};\boldsymbol{k},l)+\sum_{\{\substack{\mu'+\mu''=\mu\\|\mu''|\geq 2}}\binom{\mu}{\mu'}\times$$

$$\times{}^{\sigma}\mathcal{C}(((S^{|\alpha|}(D\phi)^{\alpha}))^{\mu'}((\tilde{Q}^{\alpha}))^{\mu''};\boldsymbol{k}-(|\mu''|-1)\boldsymbol{f}_1,l-|\mu''|+1)$$

with

$$\begin{aligned}\tilde{S}^N_{k,l}&=S^N_{k,l}-[\sigma+\boldsymbol{d}\cdot\boldsymbol{k}:N]u_{k,l}\\&=\sum_{i=1}^{N}[|\boldsymbol{k}|-l+i:i][\sigma+\boldsymbol{d}\cdot\boldsymbol{k}:N,N-i]u_{k,l-i};\quad N\in\boldsymbol{Z}_+.\end{aligned}$$

Then Lemma 4.5 (2) assures us that $\tilde{R}^{\mu}_{k+f_1,l+1}$ is a polynomial of such $D^{\beta}u_{\kappa,\rho}$'s as $(\boldsymbol{\kappa},\rho)<(\boldsymbol{k},l)$ and $|\beta|\leq|\mathrm{supp}\,\mu|$ of degree $|\mu|$ with coefficients in $\mathcal{O}(\Omega)$. Hence we have the following lemma.

LEMMA 5.1. *Let* $u^{\mu}_{k,l}$, $(\boldsymbol{k},l)\in\boldsymbol{D}$, *be given by* (5.7). *Then we have*:
(1) $u^{\mu}_{k,l}$ *satisfies* (5.9) *with* $R^{\mu}_{k,l}$ *given by* (5.10). $R^{\mu}_{k,l}$ *is a polynomial of such* $D^{\beta}u_{\kappa,\rho}$*'s as* $(\boldsymbol{\kappa},\rho)<(\boldsymbol{k},l)$ *and* $|\beta|\leq|\mathrm{supp}\,\mu|$ *of degree* $|\mu|$ *with coefficients in* $\mathcal{O}(\Omega)$.
(2) *If* $D\sigma\equiv0$, $u^{\mu}_{k,l}$ *still more satisfies* (5.11), (5.12) *and* (5.13) *with* $\tilde{R}^{\mu}_{k+f_1,l+1}$ *given by* (5.14). $\tilde{R}^{\mu}_{k+f_1,l+1}$ *is a polynomial of such* $D^{\beta}u_{\kappa,\rho}$ *as* $(\boldsymbol{\kappa},\rho)<(\boldsymbol{k},l)$ *and* $|\beta|\leq|\mathrm{supp}\,\mu|$ *of degree* $|\mu|$ *with coefficients in* $\mathcal{O}(\Omega)$.

Note that, if (4.10) converges at some point $z\in\Omega$, we have a formula

$$((D^{\alpha}u(z)))^{\mu}=(\phi(z))^{y_{\mu}(\sigma(z))}\sum_{(\boldsymbol{k},l)\in\boldsymbol{D}}u^{\mu}_{k,l}(\phi(z))^{\boldsymbol{d}(z)\cdot\boldsymbol{k}}(\log\phi(z))^{l^*}$$

whose right hand side is also convergent at the same point.

§ 6. Construction of a formal solution.

In this section we revise notations and conditions for $\sigma(z)$, π^*, $\boldsymbol{d}(z)$ and $\phi(z)$ which were introduced in § 3 and consider to construct a formal solution of the equation (1.5) in the form (4.10) in every Case A, B and and C.

Case A: Suppose that $\sigma(z)$ and $\pi^*=\pi$ satisfy conditions in Case A, that $\phi(z)$ fulfils both of the characteristic condition and noncharacteristic condition, that is, Cond. I and Cond. II and that $\boldsymbol{d}(z)$ is given by (3.2) for Case A. For every $\mu\in\mathcal{L}\backslash\pi$ we can find unique i_{μ}, $1\leq i_{\mu}\leq M$, and unique $n_{\mu}\in\boldsymbol{Z}_+$ such that

$$y_{\mu}(\sigma(z))-y_{\pi}(\sigma(z))=d_{i_{\mu}}(z)+n_{\mu} \tag{6.1}$$

where $d_{i_{\mu}}(z)$ is the i_{μ}-th component of $\boldsymbol{d}(z)$. Put

$$\boldsymbol{h}_{\mu}=\boldsymbol{f}_{i_{\mu}}+n_{\mu}\boldsymbol{f}_1. \tag{6.2}$$

Now define a formal functional series $P(u)$ for a given formal functional series (4.10) by

$$P(u)=(\phi(z))^{\nu_\pi(\sigma(z))}\sum_{(k,l)\in D} w_{k,l}\,\phi^{d(z)\cdot k}(\log\phi)^{l^*} \tag{6.3}$$

with

$$w_{k,l}=\sum_{\mu\in\pi} a_\mu(z)u^\mu_{k,l}+\sum_{\mu\in\mathcal{L}\setminus\pi} a_\mu(z)u^\mu_{k-h_\mu,l-|h_\mu|}. \tag{6.4}$$

Then we have using (5.8) and (5.9)

$$w_{0,0}=p_\pi(\sigma(z),z,D\phi(z))(u_{0,0})^{|\pi|} \tag{6.5}$$

$$w_{k,l}=(u_{0,0})^{|\pi|-1}s_\pi(\sigma(z),d(z)\cdot k,z,D\phi(z))u_{k,l}+R_{k,l}\,;\quad (k,l)\in D_+ \tag{6.6}$$

with

$$R_{k,l}=\sum_{\mu\in\pi} a_\mu R^\mu_{k,l}+\sum_{\mu\in\mathcal{L}\setminus\pi} a_\mu u^\mu_{k-h_\mu,l-|h_\mu|}. \tag{6.7}$$

LEMMA 6.1. *$R_{k,l}$ is a polynomial of such $D^\alpha u_{\kappa,\rho}$'s as $(\kappa,\rho)<(k,l)$ and $|\alpha|\le m$ of degree at most p with coefficients in $\mathcal{O}(\Omega)$.*

PROOF. This lemma can be seen easily by Lemma 5.1 (1) and the fact that $h_\mu>0$ for any $\mu\in\mathcal{L}\setminus\pi$ and hence the proof is left to the reader.

Now let us find a $u_{k,l}$ holomorphic on some subdomain so that

$$w_{k,l}=0 \qquad \text{for any } (k,l)\in D \tag{6.8}$$

is satisfied. First, give and fix any $u_{0,0}\in\mathcal{O}^1(\Omega)$. Then, owing to Cond. I and (6.5), we have always

$$w_{0,0}=0.$$

Next, for any $(k,l)\in D_+$ put

$$u_{k,l}=-\frac{R_{k,l}}{(u_{0,0})^{|\pi|-1}s_\pi(\sigma(z),d(z)\cdot k,z,D\phi(z))}. \tag{6.9}$$

Then Cond. II and Lemma 6.1 assure us that all $u_{k,l}$'s, $(k,l)\in D_+$, are determined uniquely from $u_{0,0}$ by induction with respect to $\{(k,l),>\}$ and belong to $\mathcal{O}(\Omega')$ for some subdomain Ω'. Since (6.8) holds trivially, we call the formal functional series (4.10) with coefficients $u_{k,l}$'s given above a formal solution of the equation (1.5).

Case B: Assume that $\sigma(z)\equiv\sigma$, a constant, and $\pi^*=\pi$ satisfy conditions in Case B and that d is given by (3.2) for the Case B. Set for any $\mu\in\mathcal{L}\setminus\pi$

$$\tilde{h}_\mu = h_\mu \qquad \text{for } i_\mu = 1,$$

$$\tilde{h}_\mu = h_\mu + f_1 \qquad \text{for } 2 \le i_\mu \le M$$

with h_μ given by (6.2). Then we define a formal functional series $P(u)$ by the right hand side of (6.3) with

$$w_{k,l} = \sum_{\mu\in\pi} a_\mu(z) u^\mu_{k,l} + \sum_{\mu\in\mathcal{L}\setminus\pi} a_\mu(z) u^\mu_{k-h_\mu, l-|\tilde{h}_\mu|}\,; \qquad (k, l) \in D_+ \tag{6.10}$$

instead of (6.4). First, we can see from (5.11)

$$w_{k,l} = \sum_{\mu\in\pi} a_\mu(z) S^\mu_{k,l}(((D\phi)^\alpha))^\mu \qquad \text{for any } (k, l) \text{ such that } (k)_1 = 0 \text{ or } l = 0. \tag{6.11}$$

Next, let us apply the formula (6.10) to $w_{k+f_1,l+1}$, $(k, l) \in D$. Then, substituting the right hand side of (5.12) or (5.13) for $u^\mu_{k+f_1,l+1}$, $\mu \in \pi$, and that of (5.8) or (5.9) for $u^\mu_{k,l}$, $\mu \in \pi_1 = \{\mu \in \mathcal{L}\setminus\pi\,; y_\mu(\sigma) = y_\pi(\sigma) + 1\}$, in the right hand side of (6.10) for $(k, l) = (0, 0)$ or $\neq (0, 0)$, respectively, we have

$$\begin{aligned} w_{f_1,1} = {} & ((u_{0,0})^{|\pi|-1} t_\pi(\sigma, 0, z, D\phi(z), D) + \\ & + r^0_\pi(\sigma, z, D\phi(z), D^\gamma\phi(z), u_{0,0})) u_{0,0} + \sum_{\mu\in\pi} a_\mu(z) S^\mu_{f_1,1}(((D\phi)^\alpha))^\mu, \end{aligned} \tag{6.12}$$

$$\begin{aligned} w_{k+f_1,l+1} = {} & ((u_{0,0})^{|\pi|-1} t_\pi(\sigma, d\cdot k, z, D\phi(z), D) + \\ & + r_\pi(\sigma, d\cdot k, z, D\phi(z), D(\log u_{0,0}), D^\gamma\phi(z), u_{0,0})) u_{k,l} + \\ & + \sum_{\mu\in\pi} a_\mu(z) S^\mu_{k+f_1,l+1}(((D\phi)^\alpha))^\mu + \tilde{R}_{k+f_1,l+1}\,; \qquad (k, l) \in D_+, \end{aligned} \tag{6.13}$$

where we put

$$r^0_\pi(s, z, \xi, \zeta, u) = u^{|\pi|-1} \alpha_\pi(s) \beta^2_\pi(s, 0, z, \xi, \zeta) + p_{\pi_1}(s, z, \xi, u),$$

$$\begin{aligned} r_\pi(s, \lambda, z, \xi, \eta, \zeta, u) = {} & u^{|\pi|-1} \alpha_\pi(s) (\beta^1_\pi(s, \lambda, z, \xi, \eta) + \beta^2_\pi(s, \lambda, z, \xi, \zeta) + \\ & + \beta^3_\pi(s, \lambda, z, \xi, \zeta)) + s_{\pi_1}(s, \lambda, z, \xi, u); \quad s, \lambda \in C, \ z \in \Omega, \ \xi, \eta \in C^n\setminus\{0\}, \\ & \zeta = (\zeta_\gamma), \ \gamma \in Z^n_+ \text{ and } |\gamma| = 2, \ \in C^{n(n+1)/2} \text{ and } u \in \mathcal{O}(\Omega), \end{aligned}$$

and

$$\begin{aligned} \tilde{R}_{k+f_1,l+1} = {} & \sum_{\mu\in\pi} a_\mu(z) \tilde{R}^\mu_{k+f_1,l+1} + \sum_{\mu\in\pi_1} a_\mu(z) R^\mu_{k,l} + \\ & + \sum_{\mu\in\mathcal{L}\setminus(\pi\cup\pi_1)} a_\mu(z) u^\mu_{k+f_1-\tilde{h}_\mu, l+1-|\tilde{h}_\mu|}; \qquad (k, l) \in D_+ \end{aligned} \tag{6.14}$$

with

$$\alpha_{\pi}(s)=\prod_{i=1}^{m}[s:i]^{p_i},$$

$$\beta_{\pi}^{1}(s,\lambda,z,\xi,\eta)=\sum_{\mu\in\pi}a_{\mu}(z)((\xi^{\alpha}))^{\mu}\left\{\sum_{i=1}^{m}\sum_{j=0}^{m}\frac{(p_j-\delta_{i,j})}{(s-i+1)}\frac{[s+\lambda:j]}{[s:j]}\times\right.$$
$$\left.\times(D_{\xi}\log\prod_{|\alpha|=i}(\xi^{\alpha})^{\mu_{\alpha}})\cdot\eta\right\},$$

$$\beta_{\pi}^{2}(s,\lambda,z,\xi,\zeta)=\sum_{\mu\in\pi}a_{\mu}(z)((\xi^{\alpha}))^{\mu}\left\{\sum_{\alpha\in\operatorname{supp}\mu}\mu_{\alpha}\frac{[s+\lambda:|\alpha|-1]}{[s:|\alpha|]}\times\right.$$
$$\left.\times\left(\sum_{|\gamma|=2}\binom{\alpha}{\gamma}\xi^{-\gamma}\zeta_{\gamma}\right)\right\},$$

$$\beta_{\pi}^{3}(s,\lambda,z,\xi,\zeta)=\sum_{\mu\in\pi}a_{\mu}(z)((\xi^{\alpha}))^{\mu}\left\{\sum_{\alpha\in\operatorname{supp}\mu}\sum_{j=0}^{m}\mu_{\alpha}\frac{(p_j-\delta_{|\alpha|,j})}{s-|\alpha|+1}\times\right.$$
$$\left.\times\frac{[s+\lambda:j]}{[s:j]}\left(\sum_{|\gamma|=2}\binom{\alpha}{\gamma}\xi^{-\gamma}\zeta_{\gamma}\right)\right\},$$

$$p_{\pi_1}(s,z,\xi,u)=\sum_{\mu\in\pi_1}a_{\mu}(z)(([s:|\alpha|]\xi^{\alpha}))^{\mu}u^{|\mu|-1},$$

$$s_{\pi_1}(s,\lambda,z,\xi,u)=\sum_{\mu\in\pi_1}a_{\mu}(z)(([s:|\alpha|]\xi^{\alpha}))^{\mu}\left(\sum_{\alpha\in\operatorname{supp}\mu}\mu_{\alpha}\frac{[s+\lambda:|\alpha|]}{[s:|\alpha|]}u^{|\mu|-1}\right),$$

putting $R_{0,0}^{\mu}=0$. Since $\tilde{h}_{\mu}>f_1$ for any $\mu\in\mathcal{L}\backslash(\pi\cup\pi_1)$, the following lemma is clear from Lemma 5.1.

LEMMA 6.2. *Let $D\sigma\equiv0$ and let $\tilde{R}_{k+f_1,l+1}$ be given by* (6.14). *Then $\tilde{R}_{k+f_1,l+1}$ is a polynomial of degree at most p of such $D^{\alpha}u_{\kappa,\rho}$'s as $(\kappa,\rho)<(k,l)$ and $|\alpha|\leq m$.*

LEMMA 6.3. *Let $\phi(z)$ satisfy Cond.* I *for Case* B. *Then we have*

$$\sum_{\mu\in\pi}a_{\mu}(z)(((D\phi)^{\alpha}))^{\mu}S_{k,l}^{\mu}=0\qquad\textit{for any }(k,l)\in\boldsymbol{D}.$$

PROOF. Owing to the condition (3) of Case B, we have

$$((S^{|\alpha|}(X,Y)))^{\mu}=\prod_{i=0}^{m}(S^{i}(X,Y))^{p_i};\qquad\mu\in\pi.$$

Hence we can see that $S_{k,l}^{\mu}$, the coefficients of X^kY^{l*} of the above formal power series, are common to any $\mu\in\pi$ for every $(k,l)\in\boldsymbol{D}$. If we set this common value $S_{k,l}^{\pi}$, we have using the characteristic condition (3.7)

$$\sum_{\mu\in\pi}a_{\mu}(z)(((D\phi)^{\alpha}))^{\mu}S_{k,l}^{\mu}=S_{k,l}^{\pi}q_{\pi}(z,D\phi(z))=0.$$

This completes the proof.

Corollary 6.4. *Under the same condition as in Lemma* 6.3 *we have*

$$w_{k,l}=0 \qquad \text{for any } (\boldsymbol{k}, l) \in \boldsymbol{D} \text{ such that } (\boldsymbol{k})_1=0 \text{ or } l=0.$$

Proof. We can show easily the above formula from (6.11) and Lemma 6.3 and leave the proof to the reader.

Now let us construct a formal solution in this case. Choose an $\eta \in \boldsymbol{C}^n \backslash \{0\}$ satisfying Cond. II for Case B and take a $\phi(z) \in \mathcal{O}^0(\Omega)$ so that

$$D\phi(0)=\eta$$

for which we put $T=\{\phi(z)=0\} \cap \Omega$. For every $(\boldsymbol{k}, l) \in \boldsymbol{D}$ we consider the following initial data problem for $u_{k,l}$ for an arbitrary initial data $v_{k,l} \in \mathcal{O}(T)$ but for $v_{0,0} \in \mathcal{O}^1(T)$ on T:

$$\text{(6.15)} \qquad \begin{cases} ((u_{0,0})^{|\pi|-1} t_\pi(\sigma, 0, z, D\phi, D)+r_\pi^0(\sigma, z, D\phi, D^r\phi, u_{0,0}))u_{0,0}=0, \\ u_{0,0}|_T=v_{0,0}\,; \end{cases}$$

$$\text{(6.16)} \qquad \begin{cases} ((u_{0,0})^{|\pi|-1} t_\pi(\sigma, \boldsymbol{d}\cdot\boldsymbol{k}, z, D\phi, D)+ \\ \quad +r_\pi(\sigma, \boldsymbol{d}\cdot\boldsymbol{k}, z, D\phi, D\log u_{0,0}, D^r\phi, u_{0,0}))u_{k,l}+\tilde{R}_{k+f_1,l+1}=0, \\ u_{k,l}|_T=v_{k,l} \qquad \text{for } (\boldsymbol{k}, l) \in \boldsymbol{D}_+. \end{cases}$$

Then, since we have a solution $u_{0,0} \in \mathcal{O}^1(\Omega')$ for some subdomain Ω', we can determine inductively every $u_{k,l} \in \mathcal{O}(\Omega'')$ for some subdomain Ω'' by Lemma 6.2. Moreover, these $u_{k,l}$'s give us a formal solution of the form (4.10), because we can see by Lemma 6.3 that (6.8) follows from (6.15) and (6.16) for any $(\boldsymbol{k}+f_1, l+1)$, $(\boldsymbol{k}, l) \in \boldsymbol{D}$, using the representation (6.12) and (6.13) of $w_{k+f_1,l+1}$.

Case C: Let $\sigma(z)\equiv\sigma_r$ and $\pi^*=\pi_{\sigma_r}$ satisfy the condition in Case C, and take any $\phi(z) \in \mathcal{O}^0(\Omega)$ and $\chi=u_0(z) \in \mathcal{O}^1(\Omega)$ so that (3.8) is satisfied. Let $\boldsymbol{d} \in \boldsymbol{C}^M$ be given by (3.2) for Case C. Exchanging π for π_{σ_r} in (6.1) and (6.3), a formal functional series $P(u)$ in this case is defined by the right hand side of (6.3) with

$$w_{k,l}=\sum_{\mu \in \pi_{\sigma_r}} a_\mu(z) u_{k,l}^\mu+\sum_{\mu \in \mathcal{L} \setminus \pi_{\sigma_r}} a_\mu(z) u_{k-h_\mu, l-|h_\mu|}^\mu.$$

Then we have from (5.8) and (5.9)

$$\text{(6.17)} \qquad w_{0,0}=p_{\sigma_r}(z, D\phi, u_{0,0})$$

and

$$w_{k,l}=s_{\sigma_r}(d\cdot k, z, D\phi, u_{0,0})u_{k,l}+R_{k,l} \qquad \text{for } (k, l)\in D_+$$

with

(6.18) $$R_{k,l}=\sum_{\mu\in\pi_{\sigma_r}} a_\mu(z)R^\mu_{k,l}+\sum_{\mu\in\mathcal{L}\setminus\pi_{\sigma_r}} a_\mu(z)u^\mu_{k-h_\mu,l-|h_\mu|}.$$

The following lemma is clear.

LEMMA 6.5. *$R_{k,l}$ is a polynomial of degree at most p of such $D^\alpha u_{\kappa,\rho}$'s as $(\kappa,\rho)<(k,l)$ and $|\alpha|\le m$ with coefficients in $\mathcal{O}(\Omega)$.*

To construct a formal solution in this case put, first,

(6.19) $$u_{0,0}=u_0.$$

Then from (6.17) we have automatically

$$w_{0,0}=0.$$

Next, for any $(k, l)\in D_+$, set

(6.20) $$u_{k,l}=-\frac{R_{k,l}}{s_{\sigma_r}(d\cdot k, z, D\phi(z), u_0(z))}.$$

Then, owing to Cond. II for Case C, we can determine inductively any $u_{k,l}\in\mathcal{O}(\Omega')$, $(k, l)\in D_+$, for some subdomain Ω' by the above formula. Since it is clear that these $u_{k,l}$'s satisfy (6.8) for any (k, l), we have a formal solution.

§ 7. Convergence of the formal solution—preliminary estimates.

As any formal solution, obtained in the previous section corresponding to each case, is a genuin one if it is convergent, we shall investigate convergence of the formal solution from this section. Our aim of this section is to obtain the following majorant inequalities for any formal solution:

$(M_{k,l})$ $$u_{k,l}\ll\{\exp(a(|k|+l)-b(|k|+l)^{1/2})\}(R-t)^{-|k|} \qquad \text{for } (k, l)\in D_+$$

with some positive constants a, b and R and

(7.1) $$t=cz_1+z_2+\cdots+z_n$$

with some constant $c\ge 1$. Put

(7.2) $$s=z_1+z_2+\cdots+z_n.$$

Then we can find positive constants A_1 and L such that the estimates

(7.3) $$a_\mu(z) \text{ for any } \mu\in\mathcal{L},\ \sigma(z) \text{ and } \phi(z)\ll A_1\phi_0(s)$$

with

(7.4) $$\phi_0(s)=(L-s)^{-1}; \qquad s\in \boldsymbol{C}.$$

Set

(7.5) $$\phi_k(s)=(R-s)^{-k}; \qquad s\in \boldsymbol{C},\ k\in \boldsymbol{N}$$

with

(7.6) $$0<R<L/2 \text{ and } 0<R\leq 1.$$

Then we can see easily the following lemma.

Lemma 7.1. *Let* $\phi_0(s)$ *and* $\phi_k(s)$, $k\in \boldsymbol{N}$, *be given by* (7.4) *and* (7.5) *with* (7.6), *respectively. Then we have*

(7.7) $$\phi_0(s)\ll\phi_1(t),$$

(7.8) $$\phi_0(s)\phi_k(t)\ll(L-R)^{-1}\phi_k(t),$$

(7.9) $$\phi_k(t)\ll\phi_{k+1}(t),$$

(7.10) $$D_{z_1}\phi_k(t)=ck\phi_{k+1}(t); \qquad k\in \boldsymbol{N}$$

with t *and* s *given by* (7.1) *and* (7.2), *respectively.*

Put

(7.11) $$\begin{aligned} g_0(x)&=ax-bx^{1/2},\\ g_1(x)&=-bx^{-1/2}+m(\log x)/x,\\ g_2(x)&=-(b/2)x^{-1/2}+m(\log x)/x; \quad x>0\end{aligned}$$

with positive constants a, b satisfying

(7.12) $$a-b\geq 0.$$

(7.13) $$b\geq 16((M+1)(p-1)+m).$$

Then the following lemma can be seen by elementary calculus and hence the proof is left to the reader.

Lemma 7.2. *Let* $g_i(x)$, $i=0, 1, 2$, *be given by* (7.11). *Then we have:*

(1) *If positive constants* a, b *satisfy the inequality* (7.12), $g_0(x)$ *is increasing on* $x \geq 1$.

(2) *If* b *satisfies* (7.13), *then* $g_1(x)$ *and* $g_2(x)$ *are increasing and concave for* $x \geq 1$.

Set for constants $p \geq 0$, $q > 0$ and $r > 0$

$$h(x) = x^p \exp(-qx^r); \quad x \geq 1. \tag{7.14}$$

Then we have the following lemma which can be shown by elementary calculus and is left to the reader.

LEMMA 7.3. *Let* $h(x)$ *be given by* (7.14). *Then we have:*

(1) *If* $p < qr$, $h(x)$ *is decreasing for* $x \geq 1$.

(2) *If* $qr > p + 1 - r$, *it holds*

$$\int_1^\infty h(x)dx \leq (qr - p + r - 1)^{-1} \exp(-q).$$

PROOF. We show only (2). Putting $t = x^r$ and integrating by parts iteratively, we have

$$\int_1^\infty h(x)dx = \int_1^\infty r^{-1} t^{(p-q+1)/r} \exp(-qt)dt$$
$$\leq \sum_{i=0}^\infty (rq)^{-1} \{(p-r+1)/rq\}^i.$$

Consequently we have the result and this completes the proof.

For a given $(\boldsymbol{k}, l) \in \boldsymbol{D}$ and $0 \leq j \leq |\boldsymbol{k}| + l$ put

$$N_j(\boldsymbol{k}, l) = \#\{(\kappa, \rho);\ (\boldsymbol{0}, 0) \leq (\kappa, \rho) \leq (\boldsymbol{k}, l),\ |\kappa| + \rho = j\}. \tag{7.15}$$

LEMMA 7.4. *Let* $N_j(\boldsymbol{k}, l)$ *be given by* (7.15) *and* $0 \leq j < |\boldsymbol{k}| + l$. *Then we have*

$$N_j(\boldsymbol{k}, l) \leq A_2(|\boldsymbol{k}| + l - j)^M$$

with a positive constant A_2 *which depends only on* M.

PROOF. Setting $\tilde{\kappa} = \boldsymbol{k} - \kappa$ and $\tilde{\rho} = l - \rho$, we can see it holds

$$N_j(\boldsymbol{k}, l) = \#\{(\tilde{\kappa}, \tilde{\rho});\ (\boldsymbol{0}, 0) \leq (\tilde{\kappa}, \tilde{\rho}) \leq (\boldsymbol{k}, l),\ |\tilde{\kappa}| + \tilde{\rho} = |\boldsymbol{k}| + l - j\}$$
$$\leq \#\{(\tilde{\kappa}, \tilde{\rho});\ (\boldsymbol{0}, 0) \leq (\tilde{\kappa}, \tilde{\rho}),\ |\tilde{\kappa}| + \tilde{\rho} = |\boldsymbol{k}| + l - j\}$$

$$\leq \binom{M+|k|+l-j}{M}$$

holds. The lemma follows immediately from the above inequality and this completes the proof.

Lemma 7.5. *For a given $K \in N$ and a given $(k, l) \in D_+$ let $\mathscr{B}(K; k, l)$ be the set of multi-indices given by* (5.5). *Then we have*

$$\sum_{\omega=((j_i,k_i,l_i))\in\mathscr{B}(K;k,l)} \binom{K}{j_1 \cdots j_r} \leq A_3(|k|+l)^{(M+1)(K-1)}$$

with some positive constant A_3 dependent only on K and M.

Proof. Put

$$\beta_{k,l} = \sum_{((j_i,k_i,l_i))\in\mathscr{B}(K;k,l)} \binom{K}{j_1 \cdots j_r}$$

and set

$$G(X, Y) = \sum_{(k,l)\in D} X^k Y^l$$

with $X=(X_1, \cdots, X_M)$. Then, since it holds

$$(G(X, Y))^K = \sum_{(k,l)\in D} \beta_{k,l} X^k Y^l$$

and that

$$G(X, Y) \ll \sum_{(k,l)\in Z_+^M \times Z_+} X^k Y^l,$$

it follows

$$\beta_{k,l} \leq \frac{1}{k!l!}(\partial/\partial X)^k (\partial/\partial Y)^l \left\{\left(\prod_{i=1}^M \frac{1}{1-X_i}\right)\left(\frac{1}{1-Y}\right)\right\}^K \Bigg|_{(X,Y)=(0,0)}$$

$$\leq \left\{\prod_{i=1}^M \binom{K+k_i-1}{K-1}\right\}\binom{K+l-1}{K-1} \quad \text{for} \quad k=(k_1, \cdots, k_M).$$

We can obtain the lemma easily from the above inequality and this completes the proof.

For any $\omega=((j_i, k_i, l_i))_{1\leq i\leq r} \in \mathscr{B}(K; k, l)$ set

$$\gamma(\omega) = \sum_{i=1}^r j_i(|k_i|+l_i)g_1(|k_i|+l_i) + b(|k|+l)^{1/2} - m\log(|k|+l)$$

with a contract $0 \cdot g_1(0) = 0$ and put

$$\Gamma(\omega)=\binom{K}{j_1 \cdots j_r}\exp(\gamma(\omega)). \tag{7.16}$$

LEMMA 7.6. *Let $K\leq p$, let $\Gamma(\omega)$ be denoted by* (7.16) *and let b satisfy* (7.13). *Then, for any $(\boldsymbol{k}, l)\in \boldsymbol{D}_+$ we can find a positive constant A_4 dependent only on M, K and m such that*

$$\sum_{\omega\in \mathscr{B}_2(K;\boldsymbol{k},l)}\Gamma(\omega)\leq A_4\exp\left(-\frac{1}{5}b\right) \tag{7.17}$$

with a convention $\sum_{\mathscr{B}_2}=0$ if $\mathscr{B}_2=\phi$ and that

$$\sum_{\omega\in \mathscr{B}(K;\boldsymbol{k},l)}\Gamma(\omega)\leq A_4+K. \tag{7.18}$$

PROOF. Throughout the proof we denote C_i, $i=1, 2, \cdots$, some positive constants which depend only on K, M and m. For any $\omega=((j_i, \boldsymbol{k}_i, l_i))_{1\leq i\leq r}\in\mathscr{B}(K;\ \boldsymbol{k}, l)$ put

$$\begin{cases} k_i=|\boldsymbol{k}_i|+l_i; \quad i=1, \cdots, r, \\ k=|\boldsymbol{k}|+l. \end{cases}$$

Then we have by definition $k_1\leq k_2\leq \cdots \leq k_r$,

$$\sum_{i=1}^{r} j_i k_i=k$$

and

$$\gamma(\omega)=\sum_{i=1}^{r} j_i k_i g_1(k_i)+bk^{1/2}-m\log k.$$

Set

$$\begin{cases} \Delta=\sum_{\omega\in \mathscr{B}_2(K;\boldsymbol{k},l)}\Gamma(\omega), \\ \Delta_1=\sum_{\omega\in \mathscr{B}_{2,1}}\Gamma(\omega) \end{cases}$$

with $\mathscr{B}_{2,1}=\{\omega\in\mathscr{B}_2;\ k_r/k>1/2\}$. Let us estimate Δ_1 and $\Delta-\Delta_1$, separately.

The estimate of Δ_1. Let $\omega=((j_i, \boldsymbol{k}_i, l_i))_{1\leq i\leq r}\in\mathscr{B}_{2,1}$. Then, since

$$k/2<k_r\leq k-1,$$

we can see

$$j_r=1,\ \sum_{i=1}^{r-1} j_i k_i=k-k_r \quad\text{and}\quad 1\leq k_{r-1}\leq k-k_r.$$

Put

$$\rho=\sum_{i=1}^{r-1}\frac{j_i(k_i)^2}{k-k_r}.$$

Then, as it holds

$$1\leq\rho\leq k_{r-1}\sum_{i=1}^{r-1}\frac{j_ik_i}{k-k_r}$$
$$\leq k-k_r,$$

we can see from Lemma 7.2 (2)

$$\sum_{i=1}^{r-1}j_ik_ig_1(k_i)=(k-k_r)\sum_{i=1}^{r-1}\frac{j_ik_i}{k-k_r}g_1(k_i) \tag{7.19}$$
$$\leq(k-k_r)g_1(\rho)$$
$$\leq(k-k_r)g_1(k-k_r)$$
$$\leq-b(1-(2m/be))(k-k_r)^{1/2}$$

with Napia's number e. On the other hand, since $k_r/k>1/2$ implies $k_r^{-1}<(k-k_r)^{-1}$, it holds that

$$k_rg_1(k_r)+bk^{1/2}-m\log k\leq-bk_r^{1/2}+bk^{1/2} \tag{7.20}$$
$$\leq(b/2)\int_{k_r}^{k}x^{-1/2}dx$$
$$\leq(b/2)(k-k_r)k_r^{-1/2}$$
$$\leq(b/2)(k-k_r)^{1/2}.$$

Hence we have by (7.19) and (7.20)

$$\gamma(\omega)\leq-b(1-\delta_1)(k-k_r)^{1/2}\quad\text{for }\omega\in\mathscr{B}_{2,1}, \tag{7.21}$$

where we can see from (7.13)

$$\delta_1=(2m/be)+1/2<3/4. \tag{7.22}$$

Now, as we have noting that $j_r=1$ and using Lemma 7.4 and Lemma 7.5

$$\sum_{\substack{\omega\in\mathscr{B}_{2,1}\\ k_r=\kappa}}\begin{pmatrix}K\\ j_1\ \cdots\ j_r\end{pmatrix}\leq K\sum_{\substack{(k_r,l_r)\leq(k,l)\\ |k_r|+l_r=\kappa}}\left\{\sum_{\omega\in\mathscr{B}(K-1;k-k_r,l-l_r)}\begin{pmatrix}K-1\\ j_1\ \cdots\ j_{r-1}\end{pmatrix}\right\}$$
$$\leq C_3(k-\kappa)^{(M+1)(K-1)}\quad\text{for }k/2\leq\kappa\leq k-1,$$

it follows from (7.21), (7.22) and Lemma 7.3

$$\Delta_1=\sum_{k/2\leq\kappa\leq k-1}\ \sum_{\substack{\omega\in\mathscr{B}_{2,1}\\ k_r=\kappa}}\Gamma(\omega)$$

$$\leq C_3 \sum_{k/2\leq\kappa\leq k-1} (k-\kappa)^{(M+1)(K-1)} \exp(-b(1-\delta_1)(k-\kappa)^{1/2})$$

$$\leq C_3\left\{\int_1^\infty x^{(M+1)(K-1)} \exp(-b(1-\delta_1)x^{1/2})dx+\exp(-b(1-\delta_1))\right\}$$

$$\leq C_4 \exp(-(1/4)b).$$

The estimate of $\Delta-\Delta_1$. Set for any $\omega=((j_i, k_i, l_i))_{1\leq i\leq r}\in\mathcal{B}_2\setminus\mathcal{B}_{2,1}$

$$\tilde{\kappa}=\sum_{i=1}^r \frac{j_i(k_i)^2}{k}.$$

Since it holds

$$\tilde{\kappa}\leq k_r,$$

we have using Lemma 7.2 (2)

$$\begin{aligned}\gamma(\omega)&\leq k(g_1(\tilde{\kappa})-g_1(k))\\ &\leq k(g_1(k_r)-g_1(k))\\ &\leq -(1/2)k^{-1/2}b(1-\delta_2)(k-k_r)\end{aligned}$$

with

$$\delta_2=(4m/b)e^{-3/2}<1/5.$$

Therefore it follows from Lemma 7.4 and Lemma 7.3

$$\Delta-\Delta_1\leq \sum_{\kappa\leq k/2} \sum_{\substack{\omega\in\mathcal{B}_2\\ k_r=\kappa}} \Gamma(\omega)$$

$$\leq C_5 \sum_{\kappa\leq k/2} k^{(M+1)(K-1)} \exp(-(2/5)k^{-1/2}b(k-\kappa))$$

$$\leq C_5 k^{(M+1)(K-1)}\left[\int_{k/2}^\infty \{\exp(-(2/5)k^{-1/2}bx)\}dx+\exp(-(1/5)bk^{1/2})\right]$$

$$\leq C_5 k^{(M+1)(K-1)}\left[(k/2)\int_1^\infty \{\exp(-(1/5)k^{1/2}bt)\}dt+\exp(-(1/5)bk^{1/2})\right]$$

$$\leq C_6 k^{(M+1)(K-1)+(1/2)} \exp(-(1/5)bk^{1/2})$$

$$\leq C_6 \exp(-(1/5)b).$$

Consequently, we have

$$\begin{aligned}\Delta&=\Delta_1+(\Delta-\Delta_1)\\ &\leq C_7 \exp(-(1/5)b)\end{aligned}$$

and hence (7.17) is obtained. Next, as we can see

$$\sum_{\omega\in\mathcal{B}(K;k,l)} \Gamma(\omega)=\Delta+K,$$

we have (7.18) trivially and this completes the proof.

Let $V_i(X, Y)$, $i=1, 2, \cdots, K$, and $\prod_{i=1}^{K} V_i(X, Y)$ be formal power series given by (5.1) and (5.2) with (5.3), respectively, such that

$$v_{0,0}^i \ll A_1\phi_0(s); \qquad i=1, \cdots, K \tag{7.23}$$

with A_1 and $\phi_0(s)$ given by (7.4) and (7.2) for some $L>0$, respectively. Then let us find a majorant estimate on $\hat{v}_{k,l}$ and ${}^{\sigma}\mathcal{C}_2(\prod V_i(X, Y); \boldsymbol{k}, l)$, $(\boldsymbol{k}, l) \in \boldsymbol{D}_+$, under the condition that it holds

$$(\tilde{M}_{k,l}) \qquad v_{k,l}^i \ll B\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|k|}(t)$$
$$\text{for any } i=1, \cdots, K \text{ and any } (\boldsymbol{k}, l) \in \boldsymbol{D}_+$$

for some positive constants $a, b, c \geq 1, R$ and B independent on i and $(\boldsymbol{k}, l)$ with c and R used in $\phi_{|k|}(t)$ given by (7.5) with (7.1).

Lemma 7.7. *Let $V_i(X, Y)$, $i=1, \cdots, K$, and $\prod_{i=1}^{K} V_i(X, Y)$ with $K \leq p$ be formal power series given by* (5.1) *and* (5.2) *with* (5.3), *respectively, satisfying* (7.23). *Then for a given $(\boldsymbol{k}, l) \in \boldsymbol{D}_+$ we have*:

(1) *If $v_{\kappa,\rho}^i$ satisfies $(\tilde{M}_{\kappa,\rho})$ for any $(\kappa, \rho)<(\boldsymbol{k}, l)$ with positive constants a, b satisfying* (7.12) *and* (7.13), *then we have*

$$\begin{aligned}{}^{\sigma}\mathcal{C}_2\Big(\prod_{i=1}^{K} V_i(X, Y);\ \boldsymbol{k}, l\Big) \ll A_4B^K \exp(-b/5)\{\exp(a(|\boldsymbol{k}|+l)\\ -b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|k|}(t).\end{aligned} \tag{7.24}$$

(2) *Let $v_{\kappa,\rho}^i$ satisfy $(\tilde{M}_{\kappa,\rho})$ for any $(\kappa, \rho) \leq (\boldsymbol{k}, l)$ with a, b satisfying the same conditions as in* (1). *Then we have*

$$\hat{v}_{k,l} \ll A_5\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|k|}(t), \tag{7.25}$$

with $A_5=(A_4+K)B^K$.

Proof. (1) Since it follows from the assumption

$$\begin{aligned}{}^{\sigma}\mathcal{C}_2\Big(\prod_{i=1}^{K} V_i(X, Y);\ \boldsymbol{k}, l\Big) \ll \sum_{((j_i,k_i,l_i)) \in \mathcal{B}_2(K;k,l)} B^K \binom{K}{j_1 \cdots j_r} \times \\ \times\Big[\exp\Big\{\sum_{i=1}^{r} j_i(a(|\boldsymbol{k}_i|+l_i)-b(|\boldsymbol{k}_i|+l_i)^{1/2} \\ +m\log(|\boldsymbol{k}_i|+l_i))\Big\}\Big]\phi_{|k|}(t)\end{aligned}$$

$$\ll B^{K}\Big(\sum_{\omega\in\mathscr{B}_2(K;k,l)}\Gamma(\omega)\Big)\{\exp(a(|\boldsymbol{k}|+l)$$
$$-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}(t),$$

we have (7.24) using (7.17).

(2) We can obtain (7.25) by (7.17), because we have

$$\hat{v}_{k,l}\ll B^{K}\Big(\sum_{\omega\in\mathscr{B}(K;k,l)}\Gamma(\omega)\Big)\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}(t),$$

and this completes the proof.

Now let us say simply that "$(M_{k,l})$ holds for $u_{k,l}$" if $u_{k,l}$ satisfies $(M_{k,l})$ for the prescribed a, b, c and R independent on $(\boldsymbol{k}, l)$. Let u be a given formal functional series of the form (4.10) for which it holds

$$u_{0,0}\ll A_1\phi_0(s). \tag{7.26}$$

LEMMA 7.8. *Let u be an arbitrary formal functional series of the form* (4.10) *satisfying* (7.26) *and let positive constants $a, b, c\geq 1$ and R satisfy* (7.12), (7.13) *and* (7.6). *Then we can find a positive constant A_6 which does not depend on a, b, c, R, $(\boldsymbol{k}, l)\in\boldsymbol{D}$ and $\alpha\in\boldsymbol{Z}_+^n$, $|\alpha|\leq m$, such that the following properties hold for any $(\boldsymbol{k}, l)\in\boldsymbol{D}_+$ and α.*

(1) *Assume that $(M_{\kappa,\rho})$ holds for any $(\boldsymbol{0}, 0)<(\boldsymbol{\kappa},\rho)\leq(\boldsymbol{k}, l)$. Then it holds*

$$u_{k,l}^{\alpha} \text{ and } S_{k,l}^{|\alpha|}\ll A_6c^{|\alpha|}\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+|\alpha|\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}(t). \tag{7.27}$$

(2) *If $(M_{\kappa,\rho})$ holds for any $(\boldsymbol{0}, 0)<(\boldsymbol{\kappa},\rho)<(\boldsymbol{k}, l)$, then we have*

$$R_{k,l}^{\alpha} \text{ and } \tilde{S}_{k,l}^{|\alpha|}\ll A_6c^{|\alpha|}\{\exp(a(|\boldsymbol{k}|+l-1)-b(|\boldsymbol{k}|+l-1)^{1/2}+|\alpha|\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}(t). \tag{7.28}$$

(3) *Assume $D\sigma\equiv 0$ and that $(M_{\kappa,\rho})$ holds for any $(\boldsymbol{\kappa},\rho)\leq(\boldsymbol{k}-\boldsymbol{f}_1, l-1)$. Then it holds*

$$Q_{k,l}^{\alpha}\ll A_6c^{|\alpha|}\{\exp(a(|\boldsymbol{k}|+l-2)-b(|\boldsymbol{k}|+l-2)^{1/2}+|\alpha|\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}(t).$$

(4) *Assume $D\sigma\equiv 0$ and assume $(\boldsymbol{k}, l)\geq(\boldsymbol{f}_1, 1)$ and that $(M_{\kappa,\rho})$ holds for any $(\boldsymbol{\kappa},\rho)<(\boldsymbol{k}-\boldsymbol{f}_1, l-1)$. Then we have*

$$\tilde{R}_{k,l}^{\alpha}\ll A_6c^{|\alpha|}\{\exp(a(|\boldsymbol{k}|+l-3)-b(|\boldsymbol{k}|+l-3)^{1/2}+|\alpha|\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}(t).$$

Proof. In this proof we suppose C_i, $i=1, 2, \cdots$, are positive constants independent on a, b, c, R and $(\boldsymbol{k}, l)\in \boldsymbol{D}$. Fix an arbitrary $(\boldsymbol{k}, l)\in \boldsymbol{D}_+$ and let us estimate each term of the right hand side of (4.12). Putting

$$(\kappa, \rho)=(\boldsymbol{k}-(|\alpha|-j)\boldsymbol{f}_1, l-|\alpha|+i),$$

we can see, first, from Lemma 4.4 and (7.3) that

$$N_{i,j}^{\alpha,\beta}(\sigma+\boldsymbol{d}\cdot\kappa, |\kappa|-\rho, \phi)\ll C_1(|\boldsymbol{k}|+l)^{|\beta|}(\phi_0)^{3|\alpha|}.$$

Next, as we may assume $|\kappa|+|\alpha|-|\beta|\leq|\boldsymbol{k}|$ from (4.5), we have applying $(M_{\kappa,\rho})$ to $u_{\kappa,\rho}$

$$D^{\alpha-\beta}u_{\kappa,\rho}\ll C_2 c^{|\alpha|-|\beta|}|\boldsymbol{k}|^{|\alpha|-|\beta|}\{\exp(a(|\kappa|+\rho)-b(|\kappa|+\rho)^{1/2})\}\phi_{|k|}.$$

Consequently, we can see

$$N_{i,j}^{\alpha,\beta}(\alpha+\boldsymbol{d}\cdot\kappa, |\kappa|-\rho, \phi)D^{\alpha-\beta}u_{\kappa,\rho}\ll C_3 c^{|\alpha|-|\beta|}\{\exp(a(|\kappa|+\rho) -b(|\kappa|+\rho)^{1/2}+|\alpha|\log(|\boldsymbol{k}|+l))\}\phi_{|k|}. \tag{7.29}$$

(1) Since (7.29) holds by the assumption for any term of the right-hand side of (4.12) and since

$$a(|\kappa|+\rho)-b(|\kappa|+\rho)^{1/2}\leq a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}$$

holds from Lemma 7.2 (1), we can see that $u_{k,l}^{\alpha}$ is a sum of at most $(|\alpha|+1)^{n+2}$ terms which are dominated by

$$C_3 c^{|\alpha|}\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+|\alpha|\log(|\boldsymbol{k}|+l))\}\phi_{|k|}.$$

Hence we have (7.27).

(2) As $R_{k,l}^{\alpha}$ is obtained by taking off all terms containing $u_{k,l}$ from the right hand side of (4.12), it is a sum of at most $(|\alpha|+1)^{n+2}$ such terms as

$$N_{i,j}^{\alpha,\beta}(\sigma+\boldsymbol{d}\cdot\kappa, |\kappa|-\rho, \phi)D^{\alpha-\beta}u_{\kappa,\rho};\qquad (\kappa, \rho)<(\boldsymbol{k}, l).$$

Therefore we have (7.28) easily using (7.29).

(3) or (4) can be shown similarly as (2), because $Q_{k,l}^{\alpha}$ or $\tilde{R}_{k,l}^{\alpha}$ is obtained by taking off all terms containing $u_{k,\rho}$, $\max(0, l-|\alpha|)\leq\rho\leq l$, or $u_{\kappa,\rho}$, $(\kappa, \rho)\not<(\boldsymbol{k}-\boldsymbol{f}_1, l-1)$, and their derivatives from the right hand side of (4.12), respectively. This completes the proof.

Lemma 7.9. *Under the same notations and assumptions as in Lemma*

7.8, *we can find a positive constant* A_7 *which does not depend on* a, b, c, R *and* $(\boldsymbol{k}, l) \in \boldsymbol{D}_+$ *such that following estimates hold for any* $(\boldsymbol{k}, l) \in \boldsymbol{D}_+$.

(1) *If* $(M_{\kappa,\rho})$ *holds for any* $(\kappa, \rho) \leq (\boldsymbol{k}, l)$, *then we have*

$$u_{k,l}^{\mu} \ll A_7 c^{m|\mu|}\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}. \tag{7.30}$$

(2) *If* $|\boldsymbol{k}|+l \geq 2$ *and if* $(M_{\kappa,\rho})$ *holds for any* $(\kappa, \rho) < (\boldsymbol{k}, l)$, *then we have*

$$R_{k,l}^{\mu} \ll A_7 c^{m|\mu|}(e^{(-a+b)}+e^{-b/5})\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2} + m\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}. \tag{7.31}$$

(3) *Under the same conditions as in* (2) *and the condition* $D\sigma \equiv 0$, *we have*

$$\tilde{R}_{k+f_1,l+1}^{\mu} \ll A_7 c^{m|\mu|}(e^{(-a+b)}+e^{-b/5})\{\exp(a(|\boldsymbol{k}|+l) - b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|+1}. \tag{7.32}$$

PROOF. (1) Since (7.27) holds from Lemma 7.8 (1), we can apply Lemma 7.7 (2) to the formal power series $((U^{(\alpha)}(X, Y)))^{\mu}$ given by (5.6). Therefore it holds

$$\begin{aligned} u_{k,l}^{\mu} = {}^{\sigma}\mathcal{C}(((U^{(\alpha)}))^{\mu};\ \boldsymbol{k}, l) &\ll (A_4+|\mu|)A_6^{|\mu|}c^{m|\mu|} \\ &\times\{\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|} \end{aligned}$$

and we have (7.30).

(2) Applying Lemma 7.8 (2) or Lemma 7.7 (1) to $R_{k,l}^{\alpha}$ or ${}^{\sigma}\mathcal{C}_2(((U^{(\alpha)}))^{\mu};\ \boldsymbol{k}, l)$ of the right hand side of (5.10), respectively, we have

$$\begin{aligned} R_{k,l}^{\mu} \ll C_1\{&c^m\exp(a(|\boldsymbol{k}|+l-1)-b(|\boldsymbol{k}|+l-1)^{1/2}+m\log(|\boldsymbol{k}|+l)) \\ &+c^{m|\mu|}e^{-b/5}\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l))\}\phi_{|\boldsymbol{k}|}, \end{aligned}$$

which implies (7.31).

(3) Apply Lemma 7.8 (4), (2), Lemma 7.7 (1) or (2) to the first, second, third or fourth term of the right hand side of (5.14). Then we have

$$\begin{aligned} \tilde{R}_{k+f_1,l+1}^{\mu} \ll C_2\{&c^m\exp(a(|\boldsymbol{k}|+l-1)-b(|\boldsymbol{k}|+l-1)^{1/2}+m\log(|\boldsymbol{k}|+l+2)) \\ &+c^{m|\mu|}e^{-b/5}\exp(a(|\boldsymbol{k}|+l)-b(|\boldsymbol{k}|+l)^{1/2}+m\log(|\boldsymbol{k}|+l)) \\ &+c^{m|\mu|}\exp(a(|\boldsymbol{k}|+l-1)-b(|\boldsymbol{k}|+l-1)^{1/2}+m\log(|\boldsymbol{k}|+l-1))\}\phi_{|\boldsymbol{k}|+1} \\ &\text{for } (\boldsymbol{k}, l) \in \boldsymbol{D}_+. \end{aligned}$$

Consequently, we have (7.32) and this completes the proof.

§ 8. Proof of the estimate $(M_{k,l})$ for the formal solution.

In this section we will show that every coefficient $u_{k,l}$, $(k, l)\in D_+$, of the formal solution obtained in § 6 satisfies $(M_{k,l})$ for appropriate positive constants $a, b, c\geq 1$ and R independent on (k, l) under an additional condition in Case B. We suppose for each three cases that C_i, $i=1, 2, \cdots$, denotes a positive constant which does not depend on a, b, c and $(k, l)\in D_+$. Let b satisfy (7.13) and put

$$a=(6/5)b. \tag{8.1}$$

Case A: Give any $u_{0,0}\in\mathcal{O}^1(\Omega)$ and determine every $u_{k,l}$, $(k, l)\in D_+$, using (6.9), inductively. Let us show $(M_{k,l})$ with $c=1$ for any $(k, l)\in D_+$. Owing to Cond. II for Case A, we may assume that there exists a positive constant C_1 independent on k such that

$$-\frac{1}{s_\pi(\sigma(z), d(z)\cdot k, z, D\phi(z))(u_{0,0})^{|\pi|-1}}\ll C_1|k|^{-m}\phi_0(s);\ k\in Z_+^M\setminus\{0\} \tag{8.2}$$

holds in some subdomain Ω_1. First, find a sufficiently large $b_1>0$ so that it holds

$$u_{f_i,0}\ll\{\exp(b_1/5)\}\phi_1(s);\qquad i=1, \cdots, M.$$

Then we have $(M_{f_i,0})$, $i=1, \cdots, M$, if we take

$$b>b_1. \tag{8.3}$$

Next, let $(k, l)\in D_+$ and $|(k, l)|\geq 2$. Assume that $(M_{\kappa,\rho})$ holds for any $(0, 0)<(\kappa, \rho)<(k, l)$. Then applying Lemma 7.9 (1) or (2) to $u^\mu_{k-h_\mu,l-|h_\mu|}$ or $R^\mu_{k,l}$ in the right hand side of (6.7), respectively, we have noting that $|h_\mu|\geq 1$

$$R_{k,l}\ll C_2e^{-b/5}\{\exp((6/5)b(|k|+l)-b(|k|+l)^{1/2}+m\log(|k|+l))\}\phi_{|k|}(s). \tag{8.4}$$

Now (8.2) and (8.4) enable us to make a majorant estimate of the right-hand side of (6.9) and then we have

$$u_{k,l}\ll C_3e^{-b/5}\{\exp((6/5)b(|k|+l)-b(|k|+l)^{1/2}+m\log(|k|+l))\}\phi_{|k|}(s) \tag{8.5}$$

for some constant C_3. Therefore, if we choose b so large that it satisfies (7.13), (8.3) and

$$C_3\exp(-b/5)<1,$$

we have $(M_{k,l})$ for any $(\boldsymbol{k}, l) \in \boldsymbol{D}_+$.

Case B: Take and fix any regular hypersurface

$$T=\{\phi(z)=0\} \cap \Omega$$

in Ω for which $\eta=D\phi(0)$ satisfies Cond. II for Case B, and give any $v_{k,l} \in \mathcal{O}(T)$, $(\boldsymbol{k}, l) \in \boldsymbol{D}$, with $v_{0,0}(0) \neq 0$. Then we can determine all $u_{k,l}$, $(\boldsymbol{k}, l) \in \boldsymbol{D}$, inductively, solving initial data problem (6.15) and (6.16). By virtue of Lemma 3.1 we may assume

$$\phi(z)=z_1$$

and hence that $\eta=\eta_1=(1, 0, \cdots, 0)$. Moreover, we impose the following assumption on $v_{k,l}$, $(\boldsymbol{k}, l) \in \boldsymbol{D}_+$.

ASSUMPTION B. We can find a positive constant $\tilde{a}$ independent on $(\boldsymbol{k}, l)$ such that

$$v_{k,l} \ll \{\exp(\tilde{a}(|\boldsymbol{k}|+l))\}\phi_{|k|}(s')$$

with $s'=z_2+\cdots+z_n$ and $\phi_{|k|}(s')=(L-s')^{-|k|}$ holds for any $(\boldsymbol{k}, l) \in \boldsymbol{D}_+$.

Now we may assume that it holds

$$u_{0,0} \text{ and } (u_{0,0})^{-1} \ll A_1\phi_0(s), \tag{8.6}$$

$$-\{t_\pi(\sigma, \boldsymbol{d}\cdot\boldsymbol{k}, z, D\phi(z), \eta_1)\}^{-1} \ll C_1|\boldsymbol{k}|^{-m+1}\phi_0(s), \tag{8.7}$$

$$t_\pi(\sigma, \boldsymbol{d}\cdot\boldsymbol{k}, z, D\phi(z), \eta_i) \ll C_1|\boldsymbol{k}|^{m-1}\phi_0(s); \quad i=2, \cdots, n, \tag{8.8}$$

$$r_\pi(\sigma, \boldsymbol{d}\cdot\boldsymbol{k}, z, D\phi(z), D(\log u_{0,0}), D^\gamma\phi(z), u_{0,0}) \ll C_1|\boldsymbol{k}|^m\phi_0(s); \quad \boldsymbol{k}>0 \tag{8.9}$$

with $\eta_i=(0, \cdots, \overset{\overset{i}{\vee}}{1}, \cdots, 0)$ retaking A_1 and L if necessary, because (8.7) follows from Cond. II for Case B. First, take a $b_2>0$ so large that

$$u_{f_i,0} \ll \{\exp(b_2/5)\}\phi_1(t); \qquad i=1, \cdots, M$$

holds for any $c \geq 1$. Then we have $(M_{f_i,0})$, $i=1, \cdots, M$, for any $c \geq 1$, if we take

$$b>b_2. \tag{8.10}$$

Next, let $|(\boldsymbol{k}, l)| \geq 2$ and assume that we can find some $b>0$ and $c \geq 1$ for which $(M_{\kappa,\rho})$ holds for any $(\kappa, \rho)<(\boldsymbol{k}, l)$. Then we can apply Lemma 7.9

to each term of the right hand side of (6.14) and have

$$\tilde{R}_{k+f_1,l+1} \ll C_2 c^{mp}\, e^{-b/5}\{\exp((6/5)b(|k|+l)-b(|k|+l)^{1/2}+m\log(|k|+l))\}\phi_{|k|+1}(t).$$

Now let us transform the initial data problem (6.16) in the form

$$(8.11)\quad \begin{cases} D_1 u_{k,l} = -\dfrac{1}{t_\pi(\sigma, \boldsymbol{d}\cdot \boldsymbol{k}, z, D\phi, \eta_1)}\Big\{\Big(\sum_{i=2}^{n} t_\pi(\sigma, \boldsymbol{d}\cdot\boldsymbol{k}, z, D\phi, \eta_i)D_i \\ \qquad + r_\pi(\sigma, \boldsymbol{d}\cdot\boldsymbol{k}, z, D\phi, D(\log u_{0,0}),\ D^r\phi, u_{0,0})\Big)u_{k,l} + \dfrac{\tilde{R}_{k+f_1,l+1}}{(u_{0,0})^{|\pi|-1}}\Big\} \\ u_{k,l}(z') = v_{k,l}(z'). \end{cases}$$

Applying (8.6)∼(8.9) to the right hand side of the above equation, we have the following majorant differential inequality:

$$(8.12)\quad \begin{aligned} &D_1 U_{k,l} \gg (C_1)^2(\phi_0(s))^2\Big(\sum_{i=2}^{n} D_i + |k|\Big)U_{k,l} + C_1C_2A_1^{|\pi|-1}c^{mp}e^{-b/5} \\ &\qquad \times (\phi_0)^{|\pi|}\{\exp((6/5)b(|k|+l)-b(|k|+l)^{1/2}+m\log(|k|+l) \\ &\qquad\qquad -(m-1)\log|k|)\}\phi_{|k|+1}(t) \\ &U_{k,l}(z') \gg \{\exp(\tilde{a}(|k|+l))\}\phi_{|k|}(s'). \end{aligned}$$

Let us find a solution of (8.12) of the form

$$U_{k,l} = \{\exp((6/5)b(|k|+l)-b(|k|+l)^{1/2})\}\phi_{|k|}(t)$$

for which we can see from (7.10) that (8.12) is equivalent to

$$(8.13)\quad \begin{cases} c\phi_{|k|+1}(t) \gg (C_1)^2(\phi_0)^2((n-1)\phi_{|k|+1}(t)+\phi_{|k|}(t)) \\ \qquad + C_1C_2A_1^{|\pi|-1}(\phi_0)^{|\pi|}c^{mp}\Big(\dfrac{|k|+l}{|k|}\Big)^m e^{-b/5}\phi_{|k|+1}(t) \\ \exp((6/5)b(|k|+l)-b(|k|+l)^{1/2}) \geq \exp(\tilde{a}(|k|+l)). \end{cases}$$

Then it can be seen from (7.8), (7.9) and (7.6) that (8.13) follows sufficiently if the inequality

$$(8.14)\quad n(C_1)^2(2/L)^2c^{-1} + 2^m C_1C_2A_1^{|\pi|-1}(2/L)^{|\pi|}c^{mp-1}e^{-b/5} < 1$$

holds. First, put

$$c = \frac{8nC_1^2}{L^2}.$$

Next, choose $b>0$ so large that

$$2^m C_1 C_2 A^{|\pi|-1}(2/L)^{|\pi|} c^{mp-1} e^{-b/5} < 1/2 \quad \text{and} \quad b/5 > \tilde{a}$$

holds in addition to (7.13) and (8.10). Then we have

$$u_{k,l} \ll U_{k,l}$$

and hence we can obtain $(M_{k,l})$ for any $(k, l) \in D_+$.

Case C: Since Cond. II for Case C enables us to find a positive constant C_1 satisfying

$$-\frac{1}{s_{\sigma_r}(z, d\cdot k, D\phi(z))} \ll C_1 |k|^{-m} \phi_0(s); \qquad |k| \geq 1$$

and since we have the same majorant inequality as (8.5) with $R_{k,l}$ given by (6.18), we can obtain $(M_{k,l})$ in the same way as Case A for $(k, l) \in D_+$ with sufficiently large b and $c=1$. Consequently we have the following theorem.

THEOREM 2. *Let $\sigma \in \mathcal{O}(\Omega)$, $\omega \in C^+$ and $\pi^* \subset \mathcal{L}$ satisfy the condition in Case* A, *Case* B *or Case* C *and let $S=\{\phi(z)=0\}$ with $\phi \in \mathcal{O}^0(\Omega)$ satisfy Condition* I *and Condition* II *corresponding to each case. Moreover, in Case* B, *let $T=\{\psi(z)=0\} \cap \Omega$ with $\psi \in \mathcal{O}^0(\Omega)$ be an arbitrary hypersurface such that $\eta = D\psi(0)$ satisfies properties of Condition* II, *and assume that $v_{0,0} \in \mathcal{O}^1(T)$ and that $v_{k,l}$, $(k, l) \in D_+$, satisfies Assumption* B. *Let $u_{0,0}$ be given by an arbitrary element of $\mathcal{O}^1(\Omega)$, the solution of* (6.15) *or* (6.19) *in Case* A, *Case* B *or Case* C, *respectively. Then we can find positive constants a, b and c so that $(M_{k,l})$ holds for any $u_{k,l}$, $(k, l) \in D_+$, which satisfies* (6.9), (6.16) *or* (6.20) *in Case* A, *Case* B *or Case* C, *respectively.*

§ 9. Convergence of the formal solution.

Now we can prove the following theorem.

THEOREM 3. *Under the same conditions and notations as in Theorem* 2, *let $u(z)$ be any formal solution of the form* (4.10) *whose coefficient $u_{k,l}$, $(k, l) \in D$, is given in the same way as in Theorem* 2 *in each case. Then $u(z)$ is a genuine solution of the equation* (1.5) *which has regular singularities of exponent σ with spiral exponent ω, interpreting $\omega=0$ in Case* C.

PROOF. We shall prove this theorem only in Case A because the other two cases can be treated similarly. As it follows from the condition in Case A (1) that there exists a positive constant c_1 such that

$$\mathcal{R}e(\omega d_j(z))>c_1; \qquad z\in\Omega,\ j=1,\cdots,M,$$

we can see the nonnegative value

$$c_2=\sup_{k\in Z_+^M\setminus\{0\}}\left|\frac{\mathcal{I}m(\omega d\cdot k)}{\mathcal{R}e(\omega d\cdot k)}\right|$$

is finite. Let $\psi(z)$ satisfy

$$\phi(z)=(\psi(z))^{\omega}.$$

Let K be an arbitrary positive constant and put

$$\delta_1=\mathrm{e}^{-2c_2K}.$$

Then, since it holds

$$\begin{aligned}|\phi^{d\cdot k}|&=|\psi^{\omega d\cdot k}|\\&=\exp\{(\mathcal{R}e(\omega d\cdot k))\log|\psi|-(\mathcal{I}m(\omega d\cdot k))\arg\psi\},\end{aligned}$$

we have

$$\begin{aligned}(9.1)\qquad |\phi^{d\cdot k}|&=\exp\left\{(\mathcal{R}e(\omega d\cdot k))(\log|\psi|)\left(1-\frac{\mathcal{I}m(\omega d\cdot k)}{\mathcal{R}e(\omega d\cdot k)}\frac{\arg\psi}{\log|\psi|}\right)\right\}\\&\le|\psi|^{(1/2)c_1|k|}; \qquad k\in Z_+^M\setminus\{0\},\end{aligned}$$

if $|\psi|<\delta_1$ and $|\arg\psi|<K$. Choose a $\delta_2>0$ so small that it holds

$$(9.2)\qquad x^{(1/4)c_1}(|\log x|+K)<1; \qquad 0<x<\delta_2$$

and put

$$(9.3)\qquad \delta=\min(\delta_1,\delta_2,(\mathrm{e}^{-12b/5}R/2)^{4/c_1}).$$

Now take any $r>0$ so small that

$$(9.4)\qquad 0<r<R/2 \quad\text{and}\quad \mathcal{R}(\{|s|<r\}\setminus S)\subset\mathcal{R}(\Omega\setminus S)$$

with $|s|=|z_1|+\cdots+|z_n|$ holds and let

$$\Omega_{r,K,\delta}=\mathcal{R}(\{|s|<r\}\setminus S)\cap\{|\arg\psi|<K\}\cap\{|\psi|<\delta\}$$

be a subdomain of $\mathcal{R}(\Omega\setminus S)$. Since, owing to Theorem 2, every $u_{k,l}$,

$(k, l) \in D_+$, satisfies $(M_{k,l})$ with a, b satisfying (7.13) and (8.1), $c=1$ and any $0<R<L/2$, we have using (9.4), (9.1) and (9.2)

$$(9.5) \qquad |\sum_{l=0}^{|k|} u_{k,l}\phi^{d\cdot k}(\log\phi)^l| \leq \sum_{l=0}^{|k|}\{\exp((6/5)b(|k|+l)-b(|k|+l)^{1/2})\} \times|\phi_{|k|}(s)||\phi^{d\cdot k}||(\log\phi)^l|$$
$$\leq \sum_{l=0}^{|k|} e^{(12b/5)|k|}(R-r)^{-|k|}|\phi|^{(c_1/2)|k|}(|\log\phi|+K)^l$$
$$\leq C_1 e^{(12b/5)|k|}(R/2)^{-|k|}|\phi|^{(c_1/4)|k|} \qquad \text{on } \Omega_{r,K,\delta}$$

with some positive constant C_1 independent on $|k|$. As it holds from (9.3)

$$(9.6) \qquad (2/R)e^{12b/5}|\phi|^{c_1/4}<1 \qquad \text{for } |\phi|<\delta,$$

we can see that the formal solution $u(z)$ is absolutely convergent and is a holomorphic solution of (1.5) on $\Omega_{r,K,\delta}$. Further, it is clear from the same estimates as (9.5) and (9.6) that $u(z)$ has regular singularities of exponent $\sigma(z)$ with spiral exponent ω on S, if we put

$$F_0(z)=u_{0,0}(z),$$
$$F_1(z)=\sum_{(k,l)\in D_+} u_{k,l}(z)(\phi(z))^{d(z)\cdot k}(\log\phi(z))^{l^*}.$$

This completes the proof.

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