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The coendomorphism bialgebra of an algebra

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Introduction

If A is a finite dimensional algebra over a field k, the functor $C\mapsto A\otimes C$ from k-algebras to k-algebras has a left adjoint functor, which we denote by $a(A, -)$. Namely there is a bijection

$$\mathrm{Hom}_{k\text{-alg}}(B, A\otimes C)\cong\mathrm{Hom}_{k\text{-alg}}(a(A, B), C)$$

for k-algebras B, C. The algebra $a(A, A)$ has a natural structure of a bialgebra and coacts on the algebra A through the adjunction map $A\rightarrow A\otimes a(A, A)$. It is characterized as a universal bialgebra which coacts on the algebra A.

This construction is in the same spirit as Manin's one, which produces quadratic bialgebras based on the pair of adjoint functors $\circ$ and $\underline{\mathrm{hom}}$ on the category of quadratic algebras [9]. The algebra $a(A, B)$ is also regarded as a dual object of Sweedler's universal measuring coalgebra $M(B, A)$ [12].

In this paper we mainly study $a(A, B)$-modules, $a(A, A)$-module algebras, and $a(A, A)$-comodules. One of our theorems is that if $\dim A>1$, the category of right $a(A, A)$-comodules and the category of chain complexes of k-modules are equivalent as monoidal categories.

In Section 1 we give the construction of $a(A, B)$. We discuss its relations to other universal constructions of algebras and coalgebras. Some variations and generalizations of $a(A, B)$ are mentioned.

In Section 2 we state various interpretations of $a(A, B)$-modules and $a(A, A)$-module algebras. For example, a right action of $a(A, B)$ on a k-module V is equivalent to a right action of B on the left A-module $A\otimes V$ making $A\otimes V$ an $A\otimes B^{\mathrm{op}}$-module. A right action of $a(A, A)$ on a k-algebra R is equivalent to an algebra structure on $A\otimes R$ such that $(a\otimes 1)(a'\otimes 1)=aa'\otimes 1$, $(1\otimes r)(1\otimes r')=1\otimes rr'$, $(a\otimes 1)(1\otimes r)=a\otimes r$ for $a, a'\in A$, $r, r'\in R$.

In Section 3 we first give a tensor product decomposition of $a(A, B)$ and then show that $\mathrm{Ext}^i_{a(A,B)}(V, W)\cong\mathrm{Ext}^i_{A\otimes B^{\mathrm{op}}}(A\otimes V, A\otimes W)$ for $i\geq 2$ and right $a(A, B)$-modules V, W. When $A=k\times k$, this reduces to a special case of Dicks' result on the homology of coproducts [4].

In Section 4 we prove the previously mentioned theorem about $a(A, A)$-comodules and chain complexes. In fact this is a rather immediate consequence of the observation that the dual algebra $a(A, A)^{\vee}$ is isomorphic to the endomorphism algebra of the complex

$$k \to A \;\substack{\to\\ \leftarrow\\ \to}\; A\otimes A \;\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}\; A\otimes A\otimes A \;\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}\; \cdots$$

with the arrows induced by the algebra structure of A, and Dold and Kan's theorem that the normalization functor yields an equivalence between simplicial k-modules and chain complexes of k-modules.

In Section 5 we study action of $a(A, A)$ on full matrix algebras, using the above interpretation of $a(A, A)$-module algebras. It is proved that every action of $a(A, A)$ on the algebra $M_n(k)$ comes from a representation $a(B, A)\to M_n(k)$ for some algebra B. The contents of Sections 3, 4, 5 are independent.

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Notation and conventions

We work over a fixed field k. Hom_k, $\otimes_k$ are written as Hom, $\otimes$. For a k-module V we write $V^{\vee}=\mathrm{Hom}(V, k)$. For a k-algebra A, $A\text{-}\mathcal{M}$ (resp. $\mathcal{M}\text{-}A$) denotes the category of left (resp. right) A-modules. For k-algebras A and B, an (A, B)-module means a k-module having a left A-action and a right B-action commuting with each other. The category of (A, B)-modules is denoted by $A\text{-}\mathcal{M}\text{-}B$. For an (A, A)-module M the tensor algebra $\bigoplus_{n\geq 0}M^{\otimes_A n}=A\oplus M\oplus(M\otimes_A M)\oplus\cdots$ is denoted by $T_A(M)$. When $A=k$, we write $T_A(M)=T(M)$. For a coalgebra C, $C\text{-}\mathcal{C}om$ (resp. $\mathcal{C}om\text{-}C$) denotes the category of left (resp. right) C-comodules. If A is a k-algebra, we write $\bar{A}=A/k1$ and the class of $a\in A$ in $\bar{A}$ is denoted by $\bar{a}$.

1. The construction of $a(A, B)$

Let A, B be k-algebras and assume $\dim A<\infty$. Define a k-algebra $a(A, B)$ to be the quotient of the tensor algebra $T(A^{\vee}\otimes B)$ by the ideal generated by the elements

$$\begin{aligned}&\xi(1)-\xi\otimes 1\\&\xi\otimes y_1y_2-\sum_i(\xi_{1i}\otimes y_1)\otimes(\xi_{2i}\otimes y_2)\end{aligned}$$

for $\xi\in A^\vee$, $y_1, y_2\in B$, where $\sum_i \xi_{1i}\otimes\xi_{2i}$ is the image of ξ under the map $A^\vee\to A^\vee\otimes A^\vee$, the dual of multiplication of A. We denote by (ξ, y) the class of $\xi\otimes y$ in $a(A, B)$ for $\xi\in A^\vee$, $y\in B$.

Define a map $\sigma_{AB}: B\to A\otimes a(A, B)$ by

$$\sigma_{AB}(y)=\sum_i x_i\otimes(\xi_i, y)$$

for $y\in B$, where $\{x_i\}$, $\{\xi_i\}$ are bases of $A, A^\vee$ respectively such that $\xi_i(x_j)=\delta_{ij}$. Then σ_{AB} is an algebra homomorphism.

THEOREM 1.1. *For any k-algebra C we have a bijection*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{alg}}(a(A, B), C)&\cong\mathrm{Hom}_{\mathrm{alg}}(B, A\otimes C)\\ f&\longleftrightarrow(1\otimes f)\circ\sigma_{AB}.\end{aligned}$$

Proof is obvious.

Let A, B, C be k-algebras such that $\dim A<\infty$, $\dim B<\infty$. By Theorem 1.1 there is a unique algebra map

$$\Delta_{ABC}: a(A, C)\longrightarrow a(A, B)\otimes a(B, C)$$

such that the diagram

$$\begin{tikzcd} C \arrow[r, "\sigma_{BC}"] \arrow[d, "\sigma_{AC}"'] & B\otimes a(B, C) \arrow[d, "\sigma_{AB}\otimes 1"] \\ A\otimes a(A, C) \arrow[r, "1\otimes\Delta_{ABC}"'] & A\otimes a(A, B)\otimes a(B, C) \end{tikzcd}$$

commutes. Also there is a unique algebra map

$$\epsilon_A: a(A, A)\longrightarrow k$$

such that the diagram

$$\begin{tikzcd} A \arrow[r, "\sigma_{AA}"] \arrow[dr, equal] & A\otimes a(A, A) \arrow[d, "1\otimes\epsilon_A"] \\ & A \end{tikzcd}$$

commutes. On the generators these maps are given by

$$\Delta_{ABC}(\xi, z) = \sum_j (\xi, y_j) \otimes (\eta_j, z)$$
$$\epsilon_A(\xi, x) = \xi(x)$$

for $\xi \in A^\vee$, $z \in C$, $x \in A$ with $\{y_j\}$, $\{\eta_j\}$ a pair of dual bases of B and $B^\vee$. The maps Δ_{ABC}, ϵ_A satisfy the conditions of coassociativity and counit. In particular $a(A, A)$ becomes a bialgebra with comultiplication Δ_{AAA} and counit ϵ_A. The map $\sigma_{AA}: A \to A \otimes a(A, A)$ makes A a right $a(A, A)$-comodule algebra, i.e., a monoid object of $\mathcal{C}om\text{-}a(A, A)$.

Example 1.2. Let us describe $a(A,\ )$ for two dimensional algebras A.

(i) Let $A = k \times k$. For a k-algebra B, the algebra $a(k \times k, B)$ is isomorphic to the coproduct $B * B$ of two copies of B. The universal map $B \to A \otimes (B * B) \cong (B * B) \times (B * B)$ is given by the two canonical injections $B \rightrightarrows B * B$. If we write $e = (1, 0) \in A$ and $\sigma_{AA}(e) = e \otimes e_1 + (1-e) \otimes e_2$, then $a(A, A)$ is generated by e_1, e_2 with relations $e_1^2 = e_1$, $e_2^2 = e_2$, and its coalgebra structure is given by

$$\Delta(e_1) = e_1 \otimes e_1 + (1-e_1) \otimes (1-e_2)$$
$$\Delta(e_2) = (1-e_2) \otimes (1-e_1) + e_2 \otimes e_2$$
$$\epsilon(e_1) = \epsilon(e_2) = 1.$$

The author computed the Grothendieck ring of $a(A, A)$ in [15].

(ii) Let $A = k1 \oplus kt$ with $t^2 = 0$. For a k-algebra B, let Ω be the kernel of the multiplication map $B \otimes B \to B$ and $\delta: B \to \Omega$ the map $b \mapsto 1 \otimes b - b \otimes 1$. Then $a(A, B)$ is isomorphic to the tensor algebra $T_B(\Omega)$ of the B-bimodule Ω, and σ_{AB} is identified with the map $B \to A \otimes T_B(\Omega)$ taking b to $1 \otimes b + t \otimes \delta(b)$. If we write $\sigma_{AA}(t) = 1 \otimes x + t \otimes y$, then $a(A, A)$ is generated by x, y with relations $x^2 = 0$, $xy + yx = 0$. The coalgebra structure of $a(A, A)$ is given by

$$\Delta(x) = 1 \otimes x + x \otimes y, \quad \Delta(y) = y \otimes y$$
$$\epsilon(x) = 0, \quad \epsilon(y) = 1.$$

The Hopf algebra $a(A, A)[y^{-1}]$ was studied by Pareigis in connection with the category of complexes [10]. See also Remark 4.7.

The content of the rest of this section is not used in later sections.

REMARK 1.3. The coalgebra $a(A, B)^\circ$ in the dual space $a(A, B)^\vee$ is isomorphic to the universal measuring coalgebra $M(B, A)$ in the terminology of Sweedler [12]. Indeed, for any coalgebra C there are natural bijections

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{coalg}}(C, M(B, A))\\ &\cong \mathrm{Hom}_{\mathrm{alg}}(B, \mathrm{Hom}(C, A))\\ &\cong \mathrm{Hom}_{\mathrm{alg}}(B, A\otimes C^{\vee})\\ &\cong \mathrm{Hom}_{\mathrm{alg}}(a(A, B), C^{\vee})\\ &\cong \mathrm{Hom}_{\mathrm{coalg}}(C, a(A, B)^{\circ})\end{aligned}$$

hence $M(B, A)\cong a(A, B)^{\circ}$ as coalgebras. Takeuchi pointed out to me that $a(A, B)^{\vee}$ is isomorphic to his universal measuring topological coalgebra $\mathrm{Mes}(B, A)$ as topological coalgebras [14]. This is seen similarly from the bijection

$$\mathrm{Hom}_{\mathrm{conti.coalg}}(C, \mathrm{Mes}(B, A))\cong \mathrm{Hom}_{\mathrm{alg}}(B, \mathrm{Hom}_{\mathrm{conti}}(C, A))$$

for any topological coalgebra C.

REMARK 1.4. When $A=M_n(k)$ the full matrix algebra, $a(A, B)$ is the universal coefficient ring $\mathfrak{w}_n(B)$ for $n\times n$ matrix representations of B defined by Bergman [1].

REMARK 1.5. The adjoint between $a(A, -)$ and $A\otimes(-)$ is a special case of the following one. Let $\mathcal{B}, \mathcal{C}$ be monoidal categories and let $\lambda:\mathcal{B}\to\mathcal{C}$, $\rho:\mathcal{C}\to\mathcal{B}$ be functors such that λ is left adjoint to ρ and ρ has a structure of a monoidal functor. See Section 4 (a) for the definitions of monoidal categories and functors. We denote by $\mathcal{B}_m$, $\mathcal{C}_m$ the categories of monoid objects of $\mathcal{B}$, $\mathcal{C}$. The monoidal functor ρ induces a functor $\rho_m:\mathcal{C}_m\to\mathcal{B}_m$. Now suppose that $\mathcal{C}$ has inductive limits and the tensor product of $\mathcal{C}$ commutes with inductive limits. Then ρ_m has a left adjoint functor $\lambda^m:\mathcal{B}_m\to\mathcal{C}_m$. Theorem 1.1 is the case where $\mathcal{B}=\mathcal{C}=k\text{-}\mathcal{M}$, $\lambda=A^{\vee}\otimes(-)$, and $\rho=A\otimes(-)$ with monoidal structure induced by the algebra structure of A. The construction of λ^m is similar to that of $a(A, -)$.

REMARK 1.6. Here is a graded version of $a(\ ,\ )$. Let $A=\bigoplus_{n\geq 0}A_n$ be a graded algebra such that $\dim A_n<\infty$ for all n. Consider the functor

$$A\otimes(-):\{k\text{-algebras}\}\longrightarrow\{\text{graded } k\text{-algebras}\}$$

where for a k-algebra C the grading of the algebra $A\otimes C$ is given by $(A\otimes C)_n=A_n\otimes C$ for $n\geq 0$. Using Remark 1.5, we see that $A\otimes(-)$ has a left adjoint functor $a'(A, -)$. On the other hand, for a graded algebra $B=\bigoplus_{n\geq 0}B_n$, define a graded algebra $A\circ B$ as $A\circ B=\bigoplus_{n\geq 0}(A_n\otimes B_n)$ with multiplication given by $(a\otimes b)(a'\otimes b')=aa'\otimes bb'$ for $a\in A_n$, $b\in B_n$, $a'\in A_{n'}$, $b'\in B_{n'}$. The functor

$$A\circ(-):\{\text{graded } k\text{-algebras}\}\longrightarrow\{\text{graded } k\text{-algebras}\}$$

also has a left adjoint $\underline{\hom}(A, -)$. We have $\underline{\hom}(A, B)\cong a'(A, B)$ as algebras. If A, B are quadratic algebras in the sense of Manin, then $A\circ B$ and $\underline{\hom}(A, B)$ are also quadratic algebras and coincide with Manin's ones [9]. As another example, if $A=k[x_1, \cdots, x_n]$ the polynomial algebra and $B=k[T]/(T^h)$ for n, $h\geq 0$ with the natural gradings, then $\underline{\hom}(A, B)$ is Roby's h-exterior algebra on n variables [11].

REMARK 1.7. The construction of the bialgebra $a(A, A)$ has a generalization for $\times_R$-bialgebras in the sense of [13]. Let R be a k-algebra and A an R-ring, i.e., a k-algebra equipped with an algebra map $R\rightarrow A$. Takeuchi defined the functor

$$(-)\times_R A:\{R\otimes R^{\mathrm{op}}\text{-rings}\}\longrightarrow\{R\text{-rings}\}$$

generalizing Sweedler's construction. Assume that A is finitely generated projective as a left R-module. Using Remark 1.5 in an appropriate way, we see that $(-)\times_R A$ has a left adjoint functor, which we denote by $a_R(A, -)$. Then $a_R(A, A)$ becomes a $\times_R$-bialgebra.

2. $a(A, B)$-modules and $a(A, A)$-module algebras

PROPOSITION 2.1. *Let A, B be k-algebras with* $\dim A<\infty$. *Let V be a k-module. There are one to one correspondences among the following objects.*

(i) *A right $a(A, B)$-module structure on V.*

(ii) *A k-algebra map $B\rightarrow A\otimes\mathrm{End}(V)^{\mathrm{op}}$.*

(iii) *An (A, B)-module structure on $A\otimes V$ such that A acts on $A\otimes V$ by left multiplication.*

(iv) *A linear map $f: V\otimes B\rightarrow A\otimes V$ such that the diagrams*

$$\begin{tikzcd}
V\otimes B\otimes B \arrow[r, "f\otimes 1"] \arrow[d, "1\otimes\mu_B"'] & A\otimes V\otimes B \arrow[r, "1\otimes f"] & A\otimes A\otimes V \arrow[d, "\mu_A\otimes 1"] \\
V\otimes B \arrow[rr, "f"'] & & A\otimes V
\end{tikzcd}$$

$$\begin{tikzcd} & V \arrow[dl, "1\otimes\iota_B"'] \arrow[dr, "\iota_A\otimes 1"] & \\ V\otimes B \arrow[rr, "f"'] & & A\otimes V \end{tikzcd}$$

are commutative, with $\mu_A : A\otimes A\to A$, $\mu_B : B\otimes B\to B$ *the multiplications,* $\iota_A : k\to A$, $\iota_B : k\to B$ *the unit maps.*

PROOF. (i)$\leftrightarrow$(ii): This is given by the bijection of Theorem 1.1 with $C=\mathrm{End}(V)^{\mathrm{op}}$. (ii)$\leftrightarrow$(iii): As $\dim A<\infty$, we have $A\otimes\mathrm{End}(V)^{\mathrm{op}}\cong \mathrm{End}_A(A\otimes V)^{\mathrm{op}}$ naturally. Thus an algebra map $B\to A\otimes\mathrm{End}(V)^{\mathrm{op}}$ defines a right action of B on $A\otimes V$ which commutes with left multiplication by A. (iii)$\leftrightarrow$(iv): Given an (A, B)-module structure on $A\otimes V$ as in (iii), we set $f(v\otimes b)=(1\otimes v)b$ for $v\in V$, $b\in B$.

For a right $a(A, B)$-module V, we call the (A, B)-module $A\otimes V$ in (iii) the extended bimodule of V and the map f in (iv) the transition map for V. We have similar interpretations of a left $a(A, B)$-module. We associate with a left $a(A, B)$-module V a (B, A)-module $V\otimes A$ and a linear map $f: B\otimes V\to V\otimes A$.

The correspondences (i)$\leftrightarrow$(iii)$\leftrightarrow$(iv) of Proposition 2.1 preserve tensor products. Let A, B, C be k-algebras such that $\dim A<\infty$, $\dim B<\infty$. For a right $a(A, B)$-module V and a right $a(B, C)$-module W, $V\otimes W$ becomes a right $a(A, C)$-module through the map $\Delta_{ABC} : a(A, C)\to a(A, B)\otimes a(B, C)$. If ${}_AM_B, {}_BN_C$ are the extended bimodules of V, W and $f: V\otimes B\to A\otimes V$, $g: W\otimes C\to B\otimes W$ are the transition maps for V, W respectively, then the extended bimodule of $V\otimes W$ is isomorphic to ${}_AM\otimes_B N_C$ and the transition map for $V\otimes W$ is the composite $(f\otimes 1)\circ(1\otimes g)$: $V\otimes W\otimes C\to V\otimes B\otimes W\to A\otimes V\otimes W$.

We regard k as a right $a(A, A)$-module through the counit $\epsilon_A : a(A, A)\to k$. This module k has the extended bimodule ${}_AA_A$ on which A acts by multiplication on both sides, and has the transition map $\mathrm{id}: A\to A$.

Let V be a left $a(A, B)$-module having the extended bimodule ${}_BM_A$ and the transition map $f: B\otimes V\to V\otimes A$. Then the extended bimodule of the right $a(A, B)$-module $V^\vee$ is isomorphic to $\mathrm{Hom}_A(M, A)\in A\text{-}\mathcal{M}\text{-}B$, and the transition map $g: V^\vee\otimes B\to A\otimes V^\vee$ for $V^\vee$ is related to f by the commutative diagram

$$\begin{array}{ccc} V^\vee\otimes B\otimes V & \xrightarrow{g\otimes 1} & A\otimes V^\vee\otimes V \\ {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle 1\otimes\pi} \\ V^\vee\otimes V\otimes A & \xrightarrow[\pi\otimes 1]{} & A \end{array}$$

with $\pi: V^\vee\otimes V\to k$ the canonical pairing.

We say that the bialgebra $a(A, A)$ acts on an algebra R on the right if R is a right $a(A, A)$-module algebra in the terminology of Sweedler [12]. The one to one correspondences of Proposition 2.1 induce those for monoid objects as follows.

PROPOSITION 2.2. *Let A be a k-algebra with* $\dim A<\infty$. *For a k-algebra R there are one to one correspondences among the following objects.*

(i) *A right action of $a(A, A)$ on the algebra R.*

(ii) *A linear map $f: R\otimes A\to A\otimes R$ such that the diagrams*

$$\begin{array}{ccccc} R\otimes A\otimes A & \xrightarrow{f\otimes 1} & A\otimes R\otimes A & \xrightarrow{1\otimes f} & A\otimes A\otimes R \\ {\scriptstyle 1\otimes\mu_A}\downarrow & & & & \downarrow{\scriptstyle \mu_A\otimes 1} \\ R\otimes A & & \xrightarrow[f]{\qquad\qquad} & & A\otimes R \end{array}$$

$$\begin{array}{ccc} & R & \\ {\scriptstyle 1\otimes\iota_A}\swarrow & & \searrow{\scriptstyle \iota_A\otimes 1} \\ R\otimes A & \xrightarrow[f]{} & A\otimes R \end{array}$$

$$\begin{array}{ccccc} R\otimes R\otimes A & \xrightarrow{1\otimes f} & R\otimes A\otimes R & \xrightarrow{f\otimes 1} & A\otimes R\otimes R \\ {\scriptstyle \mu_R\otimes 1}\downarrow & & & & \downarrow{\scriptstyle 1\otimes\mu_R} \\ R\otimes A & & \xrightarrow[f]{\qquad\qquad} & & A\otimes R \end{array}$$

$$\begin{array}{ccc} & A & \\ {\scriptstyle \iota_R\otimes 1}\swarrow & & \searrow{\scriptstyle 1\otimes\iota_R} \\ R\otimes A & \xrightarrow[f]{} & A\otimes R \end{array}$$

are commutative, where $\mu_A, \mu_R, \iota_A, \iota_R$ denote the multiplications and the unit maps of A, R.

(iii) *An algebra structure on the k-module $A\otimes R$ whose multiplica-*

tion satisfy the identities

$$(a\otimes 1)(a'\otimes 1)=aa'\otimes 1$$
$$(1\otimes r)(1\otimes r')=1\otimes rr'$$
$$(a\otimes 1)(1\otimes r)=a\otimes r$$

for $a, a'\in A, r, r'\in R$.

PROOF. (i)$\leftrightarrow$(ii): f is the transition map of the right $a(A, A)$-module R. (ii)$\leftrightarrow$(iii): $f(r\otimes a)=(1\otimes r)(a\otimes 1)$ for $r\in R$, $a\in A$.

REMARK 2.3. We add one more interpretation of $a(A, B)$-modules. If $U\in\mathcal{M}\text{-}A$ and $V\in\mathcal{M}\text{-}a(A, B)$, then $U\otimes V$ becomes a right B-module through the canonical map $\sigma_{AB}: B\to A\otimes a(A, B)$. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}\text{-}A\times\mathcal{M}\text{-}a(A, B) & \xrightarrow{\otimes} & \mathcal{M}\text{-}B \\ \downarrow & & \downarrow \\ \mathcal{M}\text{-}k\times\mathcal{M}\text{-}k & \xrightarrow[\otimes]{} & \mathcal{M}\text{-}k \end{array}$$

where the vertical arrows are the forgetful functors. The category $\mathcal{M}\text{-}a(A, B)$ over $\mathcal{M}\text{-}k$ is a universal one such that the functor $\otimes:\mathcal{M}\text{-}k\times\mathcal{M}\text{-}k\to\mathcal{M}\text{-}k$ lifts to a functor $\mathcal{M}\text{-}A\times\mathcal{M}\text{-}a(A, B)\to\mathcal{M}\text{-}B$ in the above way.

3. Ext of $a(A, B)$-modules

The main result of this section is Theorem 3.4, which expresses Ext of $a(A, B)$-modules in terms of Ext of $A\otimes B^{\mathrm{op}}$-modules. Throughout this section we fix k-algebras A, B such that $0<\dim A<\infty$ and set $\Lambda=a(A, B)$. We use the notation $\bar{A}, \bar{B}, \bar{a}, \bar{b}$ introduced at the beginning of the paper. We always identify $\bar{A}^{\vee}=\{\alpha\in A^{\vee}\mid\alpha(1)=0\}$.

There is a well-defined linear map $\bar{A}^{\vee}\otimes\bar{B}\to\Lambda:\alpha\otimes\bar{b}\mapsto(\alpha, b)$ and this extends to a k-algebra map $f: T(\bar{A}^{\vee}\otimes\bar{B})\to\Lambda$.

THEOREM 3.1. *Take* $\xi_0\in A^{\vee}$ *such that* $\xi_0(1)=1$. *Then we have isomorphisms of k-modules*

$$T(\bar{A}^{\vee}\otimes\bar{B})\otimes B\xrightarrow{\sim}\Lambda: u\otimes b\mapsto f(u)(\xi_0, b)$$
$$B\otimes T(\bar{A}^{\vee}\otimes\bar{B})\xrightarrow{\sim}\Lambda: b\otimes u\mapsto(\xi_0, b)f(u)$$

where $b \in B$, $u \in T(\bar{A}^\vee \otimes \bar{B})$.

PROOF. We may assume $B \neq 0$. Take k-bases $\{x_i\}_{i\in I}$ of A and $\{y_l\}_{l\in L}$ of B such that $0 \in I$, $x_0=1$, $0 \in L$, $y_0=1$ and $\xi_0(x_i)=0$ for $i \in I-\{0\}$. Let $\{\xi_i\}_{i\in I}$ be the basis of $A^\vee$ dual to $\{x_i\}_{i\in I}$. For the first isomorphism we must show that the monomials

$$(\xi_{i_1}, y_{l_1})\cdots(\xi_{i_t}, y_{l_t}),\quad (\xi_{i_1}, y_{l_1})\cdots(\xi_{i_t}, y_{l_t})(\xi_0, y_{l_{t+1}})$$

for $i_1, \cdots, i_t \in I-\{0\}$, $l_1, \cdots, l_{t+1} \in L-\{0\}$, $t \geq 0$ from a k-basis of Λ. Write

$$x_i x_j = \sum_{k\in I} c_{ij}^k x_k$$
$$y_l y_m = \sum_{n\in L} d_{lm}^n y_n$$

with $c_{ij}^k, d_{lm}^n \in k$. Then $c_{0j}^k = \delta_{jk}$. The defining relations of Λ with respect to the generators (ξ_i, y_l) for $i \in I$, $l \in L$ can be written as

$$(\xi_0, y_l)(\xi_k, y_m) = -\sum_{\substack{i,j\in I\\ i\neq 0}} c_{ij}^k(\xi_i, y_l)(\xi_j, y_m) + \sum_{n\in L} d_{lm}^n(\xi_k, y_n)$$
$$(\xi_k, y_0) = \xi_k(1)$$

for $k \in I, l, m \in L$. It follows that the above monomials span Λ. The linear independence of them is assured by the diamond lemma [2], once we check that the ambiguities $(\xi_0, y_l)(\xi_0, y_m)(\xi_k, y_n)$, $(\xi_0, y_l)(\xi_k, y_0)$, $(\xi_0, y_0)(\xi_k, y_m)$ are resolved. This is left to the reader. See the argument in [1, p. 62]. The second isomorphism is proved similarly.

We shall give another proof in Remark 4.18.

COROLLARY 3.2. *If $B \to B'$ is an injective algebra map, then the induced map $a(A, B) \to a(A, B')$ is injective. If $A' \to A$ is a surjective algebra map, then the induced map $a(A, B) \to a(A', B)$ is injective.*

Proof is immediate.

Hereafter we write $R = \operatorname{Im} f$ and $\sigma = \sigma_{AB} : B \to A \otimes \Lambda$ the canonical map.

COROLLARY 3.3. *The map*

$$t : B \otimes A \otimes R \longrightarrow A \otimes \Lambda$$
$$b \otimes a \otimes r \longmapsto \sigma(b)(a \otimes r)$$

is an isomorphism of k-modules.

PROOF. Let $\{x_i\}$, $\{\xi_i\}$ be as in the above proof. Then

$$t(1\otimes a\otimes r)=a\otimes r$$
$$t(b\otimes a\otimes r)=\sum_i x_i a\otimes(\xi_i, b)r$$
$$\equiv a\otimes(\xi_0, b)r \mod A\otimes R.$$

By the theorem the map $\bar{B}\otimes R\to\Lambda/R$ taking $\bar{b}\otimes r$ to $(\xi_0, b)r \bmod R$ is an isomorphism. Hence t is an isomorphism.

We use the following notation for bimodules. If $X, Y\in A\text{-}\mathcal{M}\text{-}B$, $\mathrm{Ext}^i_{A,B}(X, Y)$ means $\mathrm{Ext}^i_{A\otimes B^{\mathrm{op}}}(X, Y)$ with X, Y viewed as left $A\otimes B^{\mathrm{op}}$-modules. If $X\in A\text{-}\mathcal{M}\text{-}B$, $Y\in B\text{-}\mathcal{M}\text{-}A$, $X\otimes_{B,A}Y$ means $X\otimes_{B\otimes A^{\mathrm{op}}}Y$ with X viewed as a right $B\otimes A^{\mathrm{op}}$-module, Y as a left $B\otimes A^{\mathrm{op}}$-module.

Let $q:\mathcal{M}\text{-}\Lambda\to A\text{-}\mathcal{M}\text{-}B$ be the functor taking a Λ-module V to the extended bimodule $A\otimes V$. As q is exact, it induces the maps

$$q_*:\mathrm{Ext}^i_\Lambda(V, W)\longrightarrow\mathrm{Ext}^i_{A,B}(A\otimes V, A\otimes W)$$

for $V, W\in\mathcal{M}\text{-}\Lambda$, $i\geq 0$.

THEOREM 3.4. *For any right Λ-modules V and W we have an exact sequence*

$$0\longrightarrow\mathrm{Hom}_\Lambda(V, W)\xrightarrow[q_*]{}\mathrm{Hom}_{A,B}(A\otimes V, A\otimes W)\xrightarrow[r]{}\mathrm{Hom}(V, \bar{A}\otimes W)$$
$$\longrightarrow\mathrm{Ext}^1_\Lambda(V, W)\xrightarrow[q_*]{}\mathrm{Ext}^1_{A,B}(A\otimes V, A\otimes W)\longrightarrow 0$$

and isomorphisms

$$q_*:\mathrm{Ext}^i_\Lambda(V, W)\xrightarrow{\sim}\mathrm{Ext}^i_{A,B}(A\otimes V, A\otimes W)$$

for $i\geq 2$. The map r is induced by the maps $v\mapsto 1\otimes v$ and $a\otimes w\mapsto\bar{a}\otimes w$ for $v\in V$, $w\in W$, $a\in A$.

LEMMA 3.5. (i) *The functor q has a left adjoint*

$$p: A\text{-}\mathcal{M}\text{-}B\longrightarrow\mathcal{M}\text{-}\Lambda$$
$$M\longmapsto(A^\vee\otimes_A M)\otimes_{B,A}(A\otimes\Lambda)$$

where we regard $A^\vee$ as an (A, A)-module naturally, $A^\vee\otimes_A M$ as an (A, B)-module by $a\cdot(\alpha\otimes m)\cdot b=a\alpha\otimes mb$ for $a\in A$, $b\in B$, $\alpha\in A^\vee$, $m\in M$, and $A\otimes\Lambda$ as a $(B, A\otimes\Lambda)$-module by $b\cdot(a\otimes\lambda)\cdot(a'\otimes\lambda')=\sigma(b)(aa'\otimes\lambda\lambda')$ for $a, a'\in A$, $b\in B$, $\lambda, \lambda'\in\Lambda$. For $V\in\mathcal{M}\text{-}\Lambda$, the morphism of adjunction

$$e:(A^\vee\otimes V)\otimes_{B,A}(A\otimes\Lambda)\cong(A^\vee\otimes_A A\otimes V)\otimes_{B,A}(A\otimes\Lambda)\to V$$

is given by $\alpha\otimes v\otimes a\otimes\lambda\mapsto\alpha(a)v\lambda$.

(ii) *Let $M \in A\text{-}\mathcal{M}\text{-}B$ and $V \in \mathcal{M}\text{-}\Lambda$ and suppose that M is projective as an A-module. Then the adjointness of p and q gives rise to isomorphisms*

$$\operatorname{Ext}^i_\Lambda(pM, V) \cong \operatorname{Ext}^i_{A,B}(M, qV)$$

for $i \geq 0$.

PROOF. (i) For $M \in A\text{-}\mathcal{M}\text{-}B$ and $V \in \mathcal{M}\text{-}\Lambda$ we have

$$pM \cong ((A \otimes \Lambda) \otimes_A A^\vee) \otimes_{A,B} M \cong (A^\vee \otimes \Lambda) \otimes_{A,B} M$$

and

$$\begin{aligned} \operatorname{Hom}_\Lambda((A^\vee \otimes \Lambda) \otimes_{A,B} M, V) &\cong \operatorname{Hom}_{A,B}(M, \operatorname{Hom}_\Lambda(A^\vee \otimes \Lambda, V)) \\ &\cong \operatorname{Hom}_{A,B}(M, A \otimes V) \end{aligned}$$

naturally. Thus p is a left adjoint of q. The proof of the second statement is left to the reader.

(ii) By Corollary 3.3 the (B, A)-module $A \otimes \Lambda$ is free on $1 \otimes R$. So $pL \cong A^\vee \otimes_A L \otimes R$ for any $L \in A\text{-}\mathcal{M}\text{-}B$. Take a projective resolution $F. \to M$ in $A\text{-}\mathcal{M}\text{-}B$. As M is A-projective, this resolution splits in $A\text{-}\mathcal{M}$, so $pF. \to pM$ is also a resolution. Since q is exact, p preserves projectives. Thus

$$\begin{aligned} \operatorname{Ext}^i_\Lambda(pM, V) &\cong H^i \operatorname{Hom}_\Lambda(pF., V) \\ &\cong H^i \operatorname{Hom}_{A,B}(F., qV) \cong \operatorname{Ext}^i_{A,B}(M, qV) \end{aligned}$$

as required.

PROPOSITION 3.6. *For $V \in \mathcal{M}\text{-}\Lambda$ there is an exact sequence in $\mathcal{M}\text{-}\Lambda$*

$$0 \longrightarrow \bar{A}^\vee \otimes V \otimes \Lambda \xrightarrow{j} (A^\vee \otimes V) \otimes_{B,A} (A \otimes \Lambda) \xrightarrow{e} V \longrightarrow 0$$

where $j(\alpha \otimes v \otimes \lambda) = \alpha \otimes v \otimes 1 \otimes \lambda$.

PROOF. By Corollary 3.3 the map

$$\begin{aligned} s : A^\vee \otimes V \otimes R &\longrightarrow (A^\vee \otimes V) \otimes_{B,A} (A \otimes \Lambda) \\ \alpha \otimes v \otimes r &\longmapsto \alpha \otimes v \otimes 1 \otimes r \end{aligned}$$

is an isomorphism. Put $j' = s^{-1} \circ j$, $e' = e \circ s$. Let $\iota^\vee : A^\vee \to k$ be the map $\alpha \mapsto \alpha(1)$ and $m : V \otimes R \to V$ the map $v \otimes r \mapsto vr$. Then we have a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& \bar{A}^\vee\otimes V\otimes R \arrow[r, equal] \arrow[d, "\text{inclu}"'] & \bar{A}^\vee\otimes V\otimes R \arrow[d, "\text{inclu}"'] & & \\
0 \arrow[r] & \bar{A}^\vee\otimes V\otimes \Lambda \arrow[r, "j'"] \arrow[d] & A^\vee\otimes V\otimes R \arrow[r, "e'"] \arrow[d, "\iota^\vee\otimes 1\otimes 1"'] & V \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bar{A}^\vee\otimes V\otimes \bar{B}\otimes R \arrow[r, "h"'] \arrow[d] & V\otimes R \arrow[r, "m"'] \arrow[d] & V \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd}$$

where the first column is the exact sequence induced by the isomorphism $\bar{B}\otimes R\to\Lambda/R : \bar{b}\otimes r\mapsto(\xi_0, b)r$ with $\xi_0\in A^\vee$ as in Theorem 3.1, the second column is also exact, and h is a map induced by j'. We show

$$h(\alpha\otimes v\otimes\bar{b}\otimes r)=v(\alpha, b)\otimes r-v\otimes(\alpha, b)r$$

for $\alpha\in\bar{A}^\vee$, $v\in V$, $b\in B$, $r\in R$. Then the bottom row will be exact because R is a tensor algebra of $\bar{A}^\vee\otimes\bar{B}$, and so the middle row will be also. Let $\{1=x_0, x_1, \cdots\}$, $\{\xi_0, \xi_1, \cdots\}$ be bases of A and $A^\vee$ respectively which are dual to each other. Then

$$\begin{aligned}
&\alpha\otimes v\otimes 1\otimes(\xi_0, b)r\\
&=\alpha\otimes v\otimes\sigma(b)(1\otimes r)-\sum_{i\neq 0}\alpha\otimes v\otimes x_i\otimes(\xi_i, b)r\\
&=(\alpha\otimes v)b\otimes 1\otimes r-\sum_{i\neq 0}x_i\alpha\otimes v\otimes 1\otimes(\xi_i, b)r
\end{aligned}$$

in $(A^\vee\otimes V)\otimes_{B,A}(A\otimes\Lambda)$. Hence

$$\begin{aligned}
&h(\alpha\otimes v\otimes\bar{b}\otimes r)\\
&=(\iota^\vee\otimes 1\otimes 1)j'(\alpha\otimes v\otimes(\xi_0, b)r)\\
&=(\iota^\vee\otimes 1\otimes 1)\Big((\alpha\otimes v)b\otimes r-\sum_{i\neq 0}x_i\alpha\otimes v\otimes(\xi_i, b)r\Big)\\
&=v(\alpha, b)\otimes r-\sum_{i\neq 0}\alpha(x_i)v\otimes(\xi_i, b)r\\
&=v(\alpha, b)\otimes r-v\otimes(\alpha, b)r,
\end{aligned}$$

as required.

PROOF OF THEOREM 3.4. For $V, W\in\mathcal{M}\text{-}\Lambda$ and $i\geq 0$ we have a factorization

$$q_* : \mathrm{Ext}^i_\Lambda(V, W)\to\mathrm{Ext}^i_\Lambda(pqV, W)\xrightarrow{\sim}\mathrm{Ext}^i_{A,B}(qV, qW)$$

where the left arrow is the map induced by the adjunction map $e : pqV\to V$

and the isomorphism is that of Lemma 3.5 (ii). Therefore the theorem follows by applying $\mathrm{Ext}_{\Lambda}(\ ,W)$ to the exact sequence of Proposition 3.6. The description of the map r is left to the reader.

REMARK 3.7. Let $V, W \in \mathcal{M}\text{-}\Lambda$. Since $A\otimes V$ is a free A-module, there are natural isomorphisms

$$\mathrm{Ext}^i_{A,B}(A\otimes V, A\otimes W)\cong \mathrm{Ext}^i_{B,B}(B, \mathrm{Hom}_A(A\otimes V, A\otimes W))$$

for all $i\geq 0$.

REMARK 3.8. We have an explicit free resolution of $V\in\mathcal{M}\text{-}\Lambda$

$$\cdots\longrightarrow V\otimes A^\vee\otimes B^{\otimes n}\otimes\Lambda\xrightarrow{d_n}\cdots\xrightarrow{d_2}V\otimes A^\vee\otimes B\otimes\Lambda\xrightarrow{d_1}V\otimes\Lambda\xrightarrow{d_0}V\longrightarrow 0$$

with differentials given by

$$\begin{aligned}
&d_0(v\otimes\lambda)=v\lambda\\
&d_1(v\otimes\alpha\otimes b_1\otimes\lambda)=v(\alpha, b_1)\otimes\lambda-v\otimes(\alpha, b_1)\lambda\\
&d_n(v\otimes\alpha\otimes b_1\otimes\cdots\otimes b_n\otimes\lambda)\\
&\quad=\sum_i v(\alpha_{1i}, b_1)\otimes\alpha_{2i}\otimes b_2\otimes\cdots\otimes b_n\otimes\lambda\\
&\qquad+\sum_{p=1}^{n-1}(-1)^p v\otimes\alpha\otimes b_1\otimes\cdots\otimes b_pb_{p+1}\otimes\cdots\otimes b_n\otimes\lambda\\
&\qquad+(-1)^n\sum_i v\otimes\alpha_{1i}\otimes b_1\otimes\cdots\otimes b_{n-1}\otimes(\alpha_{2i}, b_n)\lambda
\end{aligned}$$

for $v\in V$, $\lambda\in\Lambda$, $\alpha\in A^\vee$, $b_1,\cdots,b_n\in B$, $n>1$, where $\mu^\vee(\alpha)=\sum_i\alpha_{1i}\otimes\alpha_{2i}$ with $\mu^\vee: A^\vee\to A^\vee\otimes A^\vee$ the dual of multiplication of A. Comparing this with the standard resolution of the B-module $A\otimes V$ [3, p. 174], we can derive Theorem 3.4.

REMARK 3.9. Bergman gave a different basis of $\mathfrak{w}_n(B)=a(M_n(k), B)$ [1, Section 9]. He also proved that

$$\text{r.gl.dim}\,\mathfrak{w}_n(B)=\begin{cases}\text{r.gl.dim}\,B & \text{if r.gl.dim}\,B>0\\ 0\text{ or }1 & \text{if r.gl.dim}\,B=0\end{cases}$$

where r.gl.dim means the right global dimension [1, Theorem 7.3]. One can deduce this from Theorem 3.4.

REMARK 3.10. When $A=k\times k$, $a(A, B)$ is isomorphic to the coproduct $B*B$ by Example 1.2 (i), and Theorem 3.4 reduces to a special case of Dicks' result on the homology of colimits [4, Theorem 6]. Write $\Lambda=B_1*B_2$

with $B_1=B_2=B$. The exact sequence of Proposition 3.6 for a right Λ-module V is identified with the sequence

$$0 \longrightarrow V\otimes\Lambda \xrightarrow{f} V\otimes_{B_1}\Lambda\oplus V\otimes_{B_2}\Lambda \xrightarrow{g} V \longrightarrow 0$$

where $f(v\otimes\lambda)=(v\otimes\lambda, -v\otimes\lambda)$, $g(v\otimes\lambda, v'\otimes\lambda')=v\lambda+v'\lambda'$. This is a special case of the Mayer-Vietories sequence [4, Section 4 (16)].

4. $a(A, A)$-comodules and chain complexes

This section consists of subsections (a)-(f). In (a) and (b) we introduce the terminology and notation about monoidal categories and chain complexes. In (c) we define for an algebra A a functor $(-)\otimes_c Q_A$ from {chain complexes of k-modules} to {right $a(A, A)$-comodules}. Our main result Theorem 4.4 states that this functor preserves tensor products and is an equivalence if $\dim A>1$. In (d) we review standard facts about the relation between chain complexes and simplicial complexes. Theorem 4.4 is proved in (e). In (f) we give an additional result in which the algebra $a(A, B)$ is expressed in two ways by differential graded algebras $Q_A[1]$, $Q_B[1]$ and by differential graded coalgebras Q_A, Q_B.

(a) Monoidal categories

Our main reference is Eilenberg and Kelly [7]. A monoidal category is a category $\mathcal{A}$ equipped with a functor $\otimes:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$, an object $I\in\mathcal{A}$ and isomorphisms $a_{XYZ}:(X\otimes Y)\otimes Z\cong X\otimes(Y\otimes Z)$, $l_X:I\otimes X\cong X$, $r_X:X\otimes I\cong X$ for $X, Y, Z\in\mathcal{A}$ such that a_{XYZ}, l_X, r_X are natural in X, Y, Z and satisfy certain coherence conditions. We call $\otimes$ the tensor product, I the unit object, a_{XYZ} the associativity isomorphism and l_X, r_X the unit isomorphisms. For example, if Λ is a bialgebra over then field k, then $\mathcal{M}$-Λ and $\mathcal{C}om$-Λ become monoidal categories with tensor product $\otimes_k$, unit object k and the obvious associativity and unit isomorphisms.

Suppose given monoidal categories $\mathcal{A}$ and $\mathcal{B}$, whose tensor products and unit objects are denoted commonly by $\otimes$ and I. A monoidal functor from $\mathcal{A}$ to $\mathcal{B}$ is a functor $\phi:\mathcal{A}\to\mathcal{B}$ equipped with natural morphisms $\mu_{X,Y}:\phi(X)\otimes\phi(Y)\to\phi(X\otimes Y)$ for $X, Y\in\mathcal{A}$ and a morphism $\iota:I\to\phi(I)$ which are compatible with the associativity and unit isomorphisms of $\mathcal{A}$ and $\mathcal{B}$. When $\mu_{X,Y}$ for all $X, Y\in\mathcal{A}$ and ι are isomorphisms, we say that ϕ is a strictly monoidal functor. Dually, a comonoidal functor from $\mathcal{A}$ to $\mathcal{B}$ is a functor $\phi:\mathcal{A}\to\mathcal{B}$ equipped with natural morphisms $\Delta_{X,Y}:\phi(X\otimes Y)\to\phi(X)\otimes\phi(Y)$ for $X, Y\in\mathcal{A}$ and a morphism $\epsilon:\phi(I)\to I$ satisfying similar conditions. A strictly comonoidal functor is defined similarly and is essentially the same thing as a strictly monoidal functor. If $\phi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{B} \to \mathcal{C}$ are (co)monoidal functors, the composite $\psi \circ \phi : \mathcal{A} \to \mathcal{C}$ has a natural structure of a (co)monoidal functor. A strictly (co)monoidal functor is called a monoidal equivalence if it is an equivalence of categories, that is, fully faithful and dense.

(b) Chain and cochain complexes

By a chain (resp. cochain) complex we mean a complex of the form

$$F_0 \xleftarrow{d} F_1 \xleftarrow{d} \cdots \qquad (\text{resp. } F_0 \xrightarrow{d} F_1 \xrightarrow{d} \cdots \qquad).$$

Let $\mathcal{C}_-$ (resp. $\mathcal{C}_+$) be the category of chain (resp. cochain) complexes of k-modules. For $F \in \mathcal{C}_\pm$ and $i \geq 0$, define $F[i] \in \mathcal{C}_\pm$ by $F[i]_n = F_{i+n}$ for $n \geq 0$ with differentials a part of those of F. Hom-sets in the categories $\mathcal{C}_-$, $\mathcal{C}_+$ are denoted by $\mathrm{Hom}_c(\ ,\ )$.

$\mathcal{C}_-$, $\mathcal{C}_+$ become monoidal categories in the standard way, which we review below. For $F, G \in \mathcal{C}_\pm$, we define $F \otimes G \in \mathcal{C}_\pm$ by

$$(F \otimes G)_n = \bigoplus_{p+q=n} F_p \otimes G_q$$
$$d(x \otimes y) = dx \otimes y + (-1)^p x \otimes dy$$

for $n \geq 0$, $x \in F_p$, $y \in G_q$. Define $k[0] \in \mathcal{C}_\pm$ by $k[0]_0 = k$, $k[0]_n = 0$ for $n > 0$. Then $\mathcal{C}_\pm$ is a monoidal category with tensor product $\otimes$, unit object $k[0]$ and the obvious associativity and unit isomorphisms.

For $F \in \mathcal{C}_\pm$ we define $F^\vee \in \mathcal{C}_\mp$ by $(F^\vee)_n = F_n^\vee$, $(d\varphi)(x) = \varphi(dx)$ for $n \geq 0$, $\varphi \in F_n^\vee$, $x \in F_{n\mp 1}$.

For $F \in \mathcal{C}_-$ and $G \in \mathcal{C}_+$ we define $F \otimes_c G \in k\text{-}\mathcal{M}$ to be the quotient of $\bigoplus_{n \geq 0} F_n \otimes G_n$ by the subspace spanned by the elements $dx \otimes y - x \otimes dy$ for $x \in F_{n+1}$, $y \in G_n$, $n \geq 0$. For $x \in F_n$, $y \in G_n$, the class of $x \otimes y$ in $F \otimes_c G$ is denoted by $x \otimes_c y$. The functor $\otimes_c : \mathcal{C}_- \times \mathcal{C}_+ \to k\text{-}\mathcal{M}$ is endowed with a structure of a comonoidal functor by the maps

$$(F \otimes F') \otimes_c (G \otimes G') \longrightarrow (F \otimes_c G) \otimes (F' \otimes_c G')$$
$$(x \otimes x') \otimes_c (y \otimes y') \longmapsto \begin{cases} (x \otimes_c y) \otimes (x' \otimes_c y') & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

for $F, F' \in \mathcal{C}_-$, $G, G' \in \mathcal{C}_+$, $x \in F_n$, $x' \in F'_{n'}$, $y \in G_m$, $y' \in G'_{m'}$, $n + n' = m + m'$ and the map

$$k[0] \otimes_c k[0] \longrightarrow k : 1 \otimes 1 \longmapsto 1.$$

(c) The construction of a monoidal equivalence

Let A be a k-algebra. Let Ω be the kernel of the multiplication map $A\otimes A\to A$ and $\delta: A\to\Omega$ the map $a\mapsto 1\otimes a-a\otimes 1$. The tensor algebra $T_A(\Omega)$ of the A-bimodule Ω has the grading such that $T_A(\Omega)_n=\Omega^{\otimes_A n}$. Define a linear map $d: T_A(\Omega)\to T_A(\Omega)$ of degree 1 by the formula

$$d(a_0\delta(a_1)\otimes_A\cdots\otimes_A\delta(a_n))=\delta(a_0)\otimes_A\delta(a_1)\otimes_A\cdots\otimes_A\delta(a_n)$$

for $a_0, \cdots, a_n\in A$, $n\geq 0$. Then

$$\begin{aligned}&d(\omega\otimes_A\theta)=d\omega\otimes_A\theta+(-1)^p\omega\otimes_A d\theta \qquad \text{for } \omega\in\Omega^{\otimes_A p},\ \theta\in\Omega^{\otimes_A q}\\ &d^2=0.\end{aligned}$$

Namely $(T_A(\Omega), d)$ is a monoid object of $\mathcal{C}_+$. Let Q_A be the cochain complex

$$k\longrightarrow A\longrightarrow\Omega\longrightarrow\Omega\otimes_A\Omega\longrightarrow\cdots$$

where the differential is the unit map in degree 0 and that of $T_A(\Omega)$ in degree ≥ 1.

For $F, G\in\mathcal{C}_-$ define a map

$$\mu_{F,G}: (F\otimes_c Q_A)\otimes(G\otimes_c Q_A)\longrightarrow(F\otimes G)\otimes_c Q_A$$

by the formulas

$$\begin{aligned}&\mu_{F,G}((x\otimes_c\omega)\otimes(y\otimes_c\theta))\\ &\quad=(x\otimes y)\otimes_c(\omega\otimes_A d\theta)+(x\otimes dy)\otimes_c(\omega\otimes_A\theta) && \text{if } p, q>0\\ &\mu_{F,G}((x\otimes_c\omega)\otimes(y\otimes_c 1))=(x\otimes y)\otimes_c\omega && \text{if } p>0, q=0\\ &\mu_{F,G}((x\otimes_c 1)\otimes(y\otimes_c\theta))=(x\otimes y)\otimes_c\theta && \text{if } p=0, q>0\\ &\mu_{F,G}((x\otimes_c 1)\otimes(y\otimes_c 1))=(x\otimes y)\otimes_c 1 && \text{if } p=q=0\end{aligned}$$

where $x\in F_p$, $y\in G_q$, $\omega\in\Omega^{\otimes_A(p-1)}$, $\theta\in\Omega^{\otimes_A(q-1)}$. Let

$$\iota: k\longrightarrow k[0]\otimes_c Q$$

be the map $1\mapsto 1\otimes_c 1$.

Now we assume $\dim A<\infty$. Then we have the bialgebra $a(A, A)$. We regard $\mathcal{C}om\text{-}a(A, A)$ as a monoidal category in the usual way. The right coaction of $a(A, A)$ on A makes Q_A a cochain complex in $\mathcal{C}om\text{-}a(A, A)$ so that the functor $(-)\otimes_c Q_A: \mathcal{C}_-\to k\text{-}\mathcal{M}$ takes values in $\mathcal{C}om\text{-}a(A, A)$. The maps $\mu_{F,G}$ and ι are $a(A, A)$-homomorphisms. Our main result is the following.

THEOREM 4.4. *If* $1<\dim A<\infty$, *then the functor*

$$(-)\otimes_c Q_A : \mathcal{C}_- \longrightarrow \mathcal{C}om\text{-}a(A, A)$$

equipped with the maps $\mu_{-,-}, \iota$ *is a monoidal equivalence.*

As an easy consequence, we have

COROLLARY 4.5. *Suppose* $1<\dim A<\infty$. *The* $a(A, A)$*-comodules* $\bar{A}^{\otimes n}$, $A\otimes\bar{A}^{\otimes n}$ *for* $n\geq 0$ *furnish a complete list of indecomposable objects in* $\mathcal{C}om\text{-}a(A, A)$. *The Grothendieck ring of the category of finite dimensional* $a(A, A)$*-comodules with respect to* $\oplus$ *and* $\otimes$ *is a commutative ring generated by the classes* $[A]$ *and* $[\bar{A}]$ *with relation* $[A]^2=(1+[\bar{A}])[A]$.

REMARK 4.6. Taking the dual, we see that if $1<\dim A<\infty$, then $\mathcal{C}_+$ and $a(A, A)$-$\mathcal{C}om$ are monoidally equivalent.

REMARK 4.7. Let $A=k1\oplus kt$ with $t^2=0$. Set $Q=Q_A$, $v_0=1\in Q_0$, $v_n=t\delta(t)\otimes_A\cdots\otimes_A\delta(t)\in Q_n$ for $n\geq 1$. Then $Q_n=d(Q_{n-1})\oplus kv_n$ and

$$\mu_{F,G}((x\otimes_c v_p)\otimes(y\otimes_c v_q))=(x\otimes y)\otimes v_{p+q}$$
$$\iota(1)=1\otimes_c v_0.$$

It follows that the monoidal functor $(-)\otimes_c Q$ is isomorphic to the obvious functor $\sigma: F\mapsto\oplus_n F_n$. Therefore we have a commutative diagram of strictly monoidal functors

$$\begin{tikzcd} \mathcal{C}_- \arrow[rr, "(-)\otimes_c Q"] \arrow[dr, "\sigma"'] & & \mathcal{C}om\text{-}a(A, A) \arrow[dl, "\omega"] \\ & k\text{-}\mathcal{M} & \end{tikzcd}$$

where ω is the forgetful functor. In view of Example 1.2 (ii) this is compatible with Pareigis' result that the category of unbounded complexes of k-modules is equivalent to $\mathcal{C}om\text{-}a(A, A)[y^{-1}]$ as monoidal categories over k-$\mathcal{M}$ [10].

(d) The normalization

Define a category S as follows. Objects of S are the ordered sets $\underline{n}=\{1<2<\cdots<n\}$ for $n=0, 1, 2, \cdots$, where $\underline{0}=\emptyset$, and morphisms of S are order preserving maps. Define a functor $+: S\times S\to S$ as follows. For $n, m\geq 0$ we set $\underline{n}+\underline{m}=\underline{n+m}$, and for morphisms $\alpha:\underline{n}\to\underline{n}'$, $\beta:\underline{m}\to\underline{m}'$, we define $\alpha+\beta:\underline{n+m}\to\underline{n'+m'}$ to be the map taking i to $\alpha(i)$ for $1\leq i\leq n$ and $n+j$ to $n'+\beta(j)$ for $1\leq j\leq m$. Then S together with $+$ as the tensor product and $\underline{0}$ as the unit object is a monoidal category.

Let $\mathcal{S}_-$, $\mathcal{S}_+$ be the categories of the functors $S^{\mathrm{op}}\to k\text{-}\mathcal{M}$, $S\to k\text{-}\mathcal{M}$ respectively. Usually a simplicial k-module means a functor $(S-\{\underline{0}\})^{\mathrm{op}}\to k\text{-}\mathcal{M}$ (see [8] for example), but most properties of the category of simplicial k-modules hold also for $\mathcal{S}_-$. Hom-sets in the categories $\mathcal{S}_-$, $\mathcal{S}_+$ are denoted by $\mathrm{Hom}_s(\ ,\ )$. For each $n\geq 0$ we define $h_n\in\mathcal{S}_-$ to be the functor taking $\underline{m}$ to $k[\mathrm{Hom}_S(\underline{m},\underline{n})]$, the free k-module on the set $\mathrm{Hom}_S(\underline{m},\underline{n})$. For $X\in\mathcal{S}_\pm$ and we define $X^\vee\in\mathcal{S}_\mp$ by $X^\vee(\underline{n})=X(\underline{n})^\vee$ for objects $\underline{n}$ and $X^\vee(\alpha)=X(\alpha)^\vee$ for morphisms α.

The monoidal structure of S extends to $\mathcal{S}_-$, $\mathcal{S}_+$ in the following way. If $X, Y\in\mathcal{S}_\pm$, we define $X\otimes Y\in\mathcal{S}_\pm$ by

$$(X\otimes Y)(\underline{n})=\bigoplus_{n=p+q} X(\underline{p})\otimes Y(\underline{q})$$
$$(X\otimes Y)(\alpha)=\bigoplus_{\alpha=\beta+\gamma} X(\beta)\otimes Y(\gamma)$$

for objects $\underline{n}$ and morphisms α in S. Then $\mathcal{S}_-$, $\mathcal{S}_+$ become monoidal categories with tensor products $\otimes$ and unit objects h_0, $h_0^\vee$ respectively and the obvious associativity and unit isomorphisms.

For $X\in\mathcal{S}_-$ and $Y\in\mathcal{S}_+$ we define $X\otimes_s Y$ to be the quotient of $\bigoplus_{n\geq 0}X(\underline{n})\otimes Y(\underline{n})$ by the subspace spanned by the elements $X(\alpha)(x)\otimes y-x\otimes Y(\alpha)(y)$ for $x\in X(\underline{n})$, $y\in Y(\underline{m})$ and morphisms $\alpha:\underline{m}\to\underline{n}$ in S. For $x\in X(\underline{n})$, $y\in Y(\underline{n})$ the class of $x\otimes y$ in $X\otimes_s Y$ is denoted by $x\otimes_s y$. Similarly to $\otimes_c$, the functor $\otimes_s:\mathcal{S}_-\times\mathcal{S}_+\to k\text{-}\mathcal{M}$ has a natural structure of a comonoidal functor.

For each $\underline{n}\in S$ and $i\in\underline{n}$, let $\partial^i:\underline{n-1}\to\underline{n}$ be the order preserving injection such that $i\notin\mathrm{Im}\,\partial^i$, and $\sigma^i:\underline{n+1}\to\underline{n}$ the order preserving surjection such that $\sigma^i(i)=\sigma^i(i+1)$. We write $X(\partial^i)=d_i$, $X(\sigma^i)=s_i$ for $X\in\mathcal{S}_\pm$.

The functors $N'_-:\mathcal{S}_-\to\mathcal{C}_-$, $N'_+:\mathcal{S}_+\to\mathcal{C}_+$ are defined by

$$(N'_-X)_n=\bigcap_{i=2}^{n}\mathrm{Ker}(d_i: X(\underline{n})\to X(\underline{n-1}))$$
$$(N'_+Y)_n=Y(\underline{n})/\sum_{i=2}^{n}\mathrm{Im}(d_i:\ Y(\underline{n-1})\to Y(\underline{n}))$$

for $X\in\mathcal{S}_-$, $Y\in\mathcal{S}_+$, $n\geq 0$ and the differentials of N'_-X, N'_+Y are induced by the maps $d_1: X(\underline{n})\to X(\underline{n-1})$, $d_1: Y(\underline{n-1})\to Y(\underline{n})$. There are also functors $N_-:\mathcal{S}_-\to\mathcal{C}_-$, $N_+:\mathcal{S}_+\to\mathcal{C}_+$ defined by

$$(N_-X)_n=X(\underline{n})/\sum_{i=1}^{n-1}\mathrm{Im}(s_i: X(\underline{n-1})\to X(\underline{n}))$$

$$(N_+Y)_n = \bigcap_{i=1}^{n-1} \mathrm{Ker}(s_i : Y(\underline{n}) \to Y(\underline{n-1}))$$

for $X \in \mathcal{S}_-$, $Y \in \mathcal{S}_+$, $n \geq 0$ and the differentials of N_-X, N_+Y are induced by the maps

$$\sum_{i=1}^{n}(-1)^{i-1}d_i : X(\underline{n}) \longrightarrow X(\underline{n-1})$$
$$\sum_{i=1}^{n}(-1)^{i-1}d_i : Y(\underline{n-1}) \longrightarrow Y(\underline{n}).$$

(4.8) The natural maps $(N'_-X)_n \hookrightarrow X(\underline{n}) \twoheadrightarrow (N_-X)_n$, $(N_+Y)_n \hookrightarrow Y(\underline{n}) \twoheadrightarrow (N'_+Y)_n$ for $n \geq 0$ yield isomorphisms of functors $N'_- \xrightarrow{\sim} N_-$, $N_+ \xrightarrow{\sim} N'_+$.

(4.9) The functors N_-, N'_-, N_+, N'_+ are equivalences of categories.

These are easily derived from the results of Dold [6, Theorem 1.9, Corollary 1.12] and Kan which state the analogous facts about simplicial k-modules.

We shall next make $N_\pm$, $N'_\pm$ strictly (co)monoidal functors. Let

$$N_-X \otimes N_-Y \longrightarrow N_-(X \otimes Y)$$
$$k[0] \longrightarrow N_-h_0$$

be the isomorphisms in $\mathcal{C}_-$ induced by the natural injections $X(\underline{p}) \otimes Y(\underline{q}) \to (X \otimes Y)(\underline{p+q})$ for $X, Y \in \mathcal{S}_-$ and the natural map $k \xrightarrow{\sim} h_0(\underline{0})$ respectively. Then N_- together with these isomorphisms is a strictly monoidal functor. Dually the natural isomorphisms

$$N_+(X \otimes Y) \longrightarrow N_+X \otimes N_+Y$$
$$N_+(h_0^\vee) \longrightarrow k[0]$$

in $\mathcal{C}_+$ give N_+ a structure of a strictly comonoidal functor. By the transport of these (co)monoidal structures through the isomorphisms of (4.8), N'_-, N'_+ become also strictly (co)monoidal functors.

(4.10) We have an isomorphism

$$\otimes_c \circ (N_- \times N_+) \cong \otimes_s \cong \otimes_c \circ (N'_- \times N'_+)$$

of comonoidal functors $\mathcal{S}_- \times \mathcal{S}_+ \to k\text{-}\mathcal{M}$ defined by the maps

$$N_-X \otimes_c N_+Y \longrightarrow X \otimes_s Y : \bar{x} \otimes_c y \longmapsto x \otimes_s y$$
$$N'_-X \otimes_c N'_+Y \longrightarrow X \otimes_s Y : x' \otimes_c \bar{y}' \longmapsto x' \otimes_s y'$$

for $X \in \mathcal{S}_-$, $Y \in \mathcal{S}_+$, $x \in X(\underline{n})$, $y \in (N_+Y)_n$, $x' \in (N'_-X)_n$, $y' \in Y(\underline{n})$, $n \geq 0$, where the bar means the residue class.

This follows from the fully faithfulness of N_-, N'_- and the monoidal adjointness of $\otimes_s$ and Hom_s, $\otimes_c$ and Hom_c.

Fix an integer $m \geq 0$. Let S_m be the full subcategory of S consisting of the objects $\underline{0}, \underline{1}, \cdots, \underline{m}$ and $\mathcal{S}_{-m}, \mathcal{S}_{+m}$ the categories of functors $S_m^{\mathrm{op}} \to k\text{-}\mathcal{M}$, $S_m \to k\text{-}\mathcal{M}$ respectively. Let $\mathcal{C}_{-m}$ (resp. $\mathcal{C}_{+m}$) be the category of complexes

$$F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_m \qquad (\text{resp. } F_0 \longrightarrow F_1 \longrightarrow \cdots \longrightarrow F_m)$$

of k-modules. Hom-sets in $\mathcal{S}_{\pm m}$, $\mathcal{C}_{\pm m}$ are written as $\mathrm{Hom}_s(\ ,\ )$, $\mathrm{Hom}_c(\ ,\ )$ respectively. We can define functors $N'_\pm : \mathcal{S}_{\pm m} \to \mathcal{C}_{\pm m}$ similarly to $N'_\pm$: $\mathcal{S}_\pm \to \mathcal{C}_\pm$ and these are also equivalences. There is also a functor $\otimes_c$: $\mathcal{C}_{-m} \times \mathcal{C}_{+m} \to k\text{-}\mathcal{M}$ defined analogously to $\otimes_c : \mathcal{C}_- \times \mathcal{C}_+ \to k\text{-}\mathcal{M}$.

(e) Proof of Theorem 4.4

The proof consists of three parts. First we define a cochain complex Q'_A for an algebra A and show that $(-)\otimes_c Q'_A : \mathcal{C}_- \to \mathcal{C}om\text{-}a(A, A)$ is an equivalence if $1 < \dim A < \infty$. Secondly we define on Q'_A a comonoid structure, which makes $(-)\otimes_c Q'_A : \mathcal{C}_- \to k\text{-}\mathcal{M}$ a comonoidal functor. We prove that this is strictly comonoidal. Finally we give an isomorphism $Q_A \cong Q'_A$, and show that through the induced isomorphism $(-)\otimes_c Q_A \cong (-)\otimes_c Q'_A$ the monoidal structure of the left side defined in (c) and the one of the right side defined here coincide, which finishes the proof of Theorem 4.4. The reason why we adopt Q_A instead of Q'_A in the statement of the theorem is that the monoidal structure maps of $(-)\otimes_c Q_A$ have a simpler form.

Let A be a k-algebra. Define $P = P_A \in \mathcal{S}_+$ as follows. For $\underline{n} \in S$ we set $P(\underline{n}) = A^{\otimes n}$. For a morphism $\alpha : \underline{n} \to \underline{m}$ in S we define $P(\alpha) : A^{\otimes n} \to A^{\otimes m}$ by

$$P(\alpha)(a_1 \otimes \cdots \otimes a_n) = b_1 \otimes \cdots \otimes b_m, \quad b_j = \prod_{\alpha(i)=j} a_i$$

for $a_1, \cdots, a_n \in A$, where we multiply a_i in the increasing order of i, and the empty product means the unit element 1_A of A.

Define $Q' = Q'_A \in \mathcal{C}_+$ as follows.

$$\begin{aligned}
&(Q_A)_0 = k \\
&(Q_A)_n = A \otimes \bar{A}^{\otimes(n-1)} \qquad n \geq 1 \\
&d(1) = 1_A \\
&d(a_1 \otimes \bar{a}_2 \otimes \cdots \otimes \bar{a}_n) = 1_A \otimes \bar{a}_1 \otimes \bar{a}_2 \otimes \cdots \otimes \bar{a}_n \qquad n \geq 1,
\end{aligned}$$

where the bar notation introduced at the beginning of the paper is used. The complex Q'_A is exact if $A\neq 0$. The natural surjections $A^{\otimes n}\to A\otimes \bar{A}^{\otimes(n-1)}$ induce an isomorphism $N'_+P\xrightarrow{\sim}Q'$ in $\mathcal{C}_+$.

Now assume $\dim A<\infty$. Let B be a k-algebra. We have isomorphisms of k-modules

$$a(A, B)\cong P_A^\vee\otimes_s P_B\cong Q_A'^\vee\otimes_c Q'_B. \tag{4.11}$$

Indeed, the first isomorphism is given by the correspondence

$$(\xi_1, y_1)\cdots(\xi_n, y_n)\longleftrightarrow(\xi_1\otimes\cdots\otimes\xi_n)\otimes_s(y_1\otimes\cdots\otimes y_n)$$

for $\xi_1, \cdots, \xi_n\in A^\vee$, $y_1, \cdots, y_n\in B$, $n\geq 0$. That this map is a well-defined isomorphism follows from the construction of $a(A, B)$ in Section 1 and the definition of $\otimes_s$. Use the fact that every morphism of S is a composite of ∂^i, σ^i in the notation of (d). The second isomorphism is induced by the second isomorphism of (4.10) as

$$P_A^\vee\otimes_s P_B\cong N'_-(P_A^\vee)\otimes_c N'_+P_B\cong(N'_+P_A)^\vee\otimes_c N'_+P_B\cong Q_A'^\vee\otimes_c Q'_B.$$

Taking the dual of (4.11), we have

$$a(A, B)^\vee\cong\mathrm{Hom}_s(P_B, P_A)\cong\mathrm{Hom}_c(Q'_B, Q'_A) \tag{4.12}$$

as k-modules. The first isomorphism is the restriction of the natural map

$$T(A^\vee\otimes B)^\vee\xrightarrow{\sim}\Big(\bigoplus_n(A^{\otimes n})^\vee\otimes B^{\otimes n}\Big)^\vee\xrightarrow{\sim}\prod_n \mathrm{Hom}(B^{\otimes n}, A^{\otimes n})$$

and the second one is induced by the functor N'_+. Thus, if $A=B$, (4.12) are isomorphisms among the dual algebra $a(A, A)^\vee$ of the coalgebra $a(A, A)$ and the endomorphism algebras $\mathrm{End}_s P_A$, $\mathrm{End}_c Q'_A$.

Since $P=P_A$ can be viewed as a functor $S\to\mathcal{C}om\text{-}a(A, A)$, $Q'=Q'_A\cong N'_+P$ is a cochain complex in $\mathcal{C}om\text{-}a(A, A)$. Thus $(-)\otimes_c Q':\mathcal{C}_-\to k\text{-}\mathcal{M}$ is a functor into $\mathcal{C}om\text{-}a(A, A)$.

(4.13) If $\dim A>1$, then the functor $(-)\otimes_c Q':\mathcal{C}_-\to\mathcal{C}om\text{-}a(A, A)$ is an equivalence.

PROOF. For each integer $m\geq 0$, let $a(A, A)_m$ be the subspace of $a(A, A)$ spanned by the monomials of (ξ, x) of degree $\leq m$ with $\xi\in A^\vee$, $x\in A$. Then $a(A, A)_m$ is a subcoalgebra of $a(A, A)$. We use the notation in (d). Let $P_{(m)}\in\mathcal{S}_{+m}$ be the restriction of P to S_m, $Q'_{(m)}\in\mathcal{C}_{+m}$ the truncation of

Q' in degree $\leq m$. We claim that there is a commutative diagram of k-algebras

$$\begin{array}{ccccc} a(A, A)^{\vee} & \cong & \mathrm{End}_s P & \cong & \mathrm{End}_c Q' \\ \downarrow & & \downarrow & & \downarrow \\ a(A, A)^{\vee}_m & \cong & \mathrm{End}_s P_{(m)} & \cong & \mathrm{End}_c Q'_{(m)} \end{array}$$

where the top row is (4.12) with $B=A$ and the vertical arrows are the restriction maps. In fact, the fully faithful functor $N'_+ : \mathcal{S}_{+m} \to \mathcal{C}_{+m}$ and the natural isomorphism $N'_+(P)_{(m)} \cong N'_+(P_{(m)})$ induce an algebra isomorphism $\mathrm{End}_s P_{(m)} \cong \mathrm{End}_c Q'_{(m)}$ making the right square commutative. The isomorphism $a(A, A)^{\vee} \cong \mathrm{End}_s P$ induces an injection $a(A, A)^{\vee}_m \to \mathrm{End}_s P_{(m)}$ such that the left square of the diagram commutes. Since the complex Q is exact, the restriction map $\mathrm{End}_c Q' \to \mathrm{End}_c Q'_{(m)}$ is surjective. Hence the map $a(A, A)^{\vee}_m \to \mathrm{End}_s P_{(m)}$ is bijective.

The functor $(-)\otimes_c Q' : \mathcal{C}_- \to \mathcal{C}om\text{-}a(A, A)$ restricts to the functor $(-)\otimes_c Q'_{(m)} : \mathcal{C}_{-m} \to \mathcal{C}om\text{-}a(A, A)_m$. Since $(-)\otimes_c Q'$ is an inductive limit of $(-)\otimes_c Q'_{(m)}$ as $m \to +\infty$, it is enough to show that $(-)\otimes_c Q'_{(m)}$ is an equivalence for each m. This functor is identified with the canonical functor $(-)\otimes_c Q'_{(m)} : \mathcal{C}_{-m} \to \mathrm{End}_c Q'_{(m)}\text{-}\mathcal{M}$ through the algebra isomorphism $a(A, A)^{\vee}_m \cong \mathrm{End}_c Q'_{(m)}$. Since $\bar{A} \neq 0$, $Q'_{(m)}$ is a projective generator in $\mathcal{C}_{+m}$. Therefore $(-)\otimes_c Q'_{(m)} : \mathcal{C}_{-m} \to \mathrm{End}_c Q'_{(m)}\text{-}\mathcal{M}$ is an equivalence by the Morita theorem. Thus (4.13) is proved.

From now on $\dim A$ is arbitary again. We make $P \in \mathcal{S}_+$, $Q' \in \mathcal{C}_+$ comonoid objects. Let $\Delta : P \to P \otimes P$, $\epsilon : P \to h_0^{\vee}$ be the morphisms whose components $\Delta_{p,q} : P(\underline{p+q}) \to P(\underline{p}) \otimes P(\underline{q})$, $\epsilon_0 : P(\underline{0}) \to k$ are the identity maps. Then (P, Δ, ϵ) is a comonoid object of $\mathcal{S}_+$. Since $N'_+ : \mathcal{S}_+ \to \mathcal{C}_+$ is a comonoidal functor, $N'_+ P$ becomes a comonoid object of $\mathcal{C}_+$ and so is Q' through the isomorphism $N'_+ P \cong Q'$.

For $X, Y \in \mathcal{S}_-$, $F, G \in \mathcal{C}_-$, let

$$(X \otimes Y)\otimes_s P \longrightarrow (X \otimes Y)\otimes_s (P \otimes P) \longrightarrow (X \otimes_s P) \otimes (Y \otimes_s P)$$
$$h_0 \otimes_s P \longrightarrow h_0 \otimes_s h_0^{\vee} \longrightarrow k$$
$$(F \otimes G)\otimes_c Q' \longrightarrow (F \otimes G)\otimes_c (Q' \otimes Q') \longrightarrow (F \otimes_c Q') \otimes (G \otimes_c Q')$$
$$k[0]\otimes_c Q' \longrightarrow k[0]\otimes_c k[0] \longrightarrow k$$

be the maps induced by the comonoidal structures of $\otimes_s$, $\otimes_c$ defined in (d), (b) and the comonoid structures of P, Q'. With these maps, $(-)\otimes_s P : \mathcal{S}_- \to k\text{-}\mathcal{M}$, $(-)\otimes_c Q' : \mathcal{C}_- \to k\text{-}\mathcal{M}$ become comonoidal functors.

(4.14) $(-)\otimes_s P$, $(-)\otimes_c Q'$ are strictly comonoidal functors.

PROOF. The comultiplication maps $(X\otimes Y)\otimes_s P\to(X\otimes_s P)\otimes(Y\otimes_s P)$ for $X=h_p$, $Y=h_q$ and the counit map $h_0\otimes_s P\to k$ are isomorphic because through the natural isomorphisms $h_p\otimes h_q\cong h_{p+q}$, $h_n\otimes_s P\cong P(\underline{n})$, they are identified with the maps $\Delta_{p,q}: P(\underline{p+q})\to P(\underline{p})\otimes P(\underline{q})$, $\epsilon_0: P(\underline{0})\to k$ which are the identity maps. Since $\otimes_s$, $\otimes$ commute with inductive limits, the comultiplication maps are isomorphic for all $X, Y\in\mathcal{S}_-$. Thus $(-)\otimes_s P$ is a strictly comonoidal functor.

By (4.10) we have an isomorphism $N'_-(-)\otimes_c Q'\cong(-)\otimes_s P$ of comonoidal functors on $\mathcal{S}_-$. Since N'_- is a monoidal equivalence, it follows that $(-)\otimes_c Q'$ is also a strictly comonoidal functor. This proves (4.14).

We must relate $(-)\otimes_c Q'$ and $(-)\otimes_c Q$ of (c). We first claim that the natural injections $\Omega^{\otimes_A n}\hookrightarrow(A\otimes A)^{\otimes_A n}\cong A^{\otimes(n+1)}$ induce an isomorphism $Q\xrightarrow{\sim}N_+P$ in $\mathcal{C}_+$. Indeed, by the definition of N_+ and the fact that Ω is a direct summand of $A\otimes A$ as a one-sided A-module, we see that the induced map $Q\to N_+P$ is a bijection. We must show that this is a cochain map. We already have the cochain isomorphisms $N_+P\xrightarrow{\sim}N'_+P$ of (4.8) and $N'_+P\xrightarrow{\sim}Q'$. So it is enough to show that the composite $f: Q\to N_+P\xrightarrow{\sim}N'_+P\xrightarrow{\sim}Q'$ is a cochain map. Clearly $f(1)=1$ in degree 0, and an easy induction on n shows

$$f(a_1\delta(a_2)\otimes_A\cdots\otimes_A\delta(a_n))=a_1\otimes\bar{a}_2\otimes\cdots\otimes\bar{a}_n$$

for $n\geq 1$. Hence f is a cochain map. This proves the claim.

Through this isomorphism $Q\cong N_+P$ we transport to Q the comonoidal structure of N_+P induced by the comonoidal structures of N_+ and P. Let $\Delta: Q\to Q\otimes Q$, $\epsilon: Q\to k[0]$ be the comultiplication, counit respectively. Then ϵ is the identity in degree 0 and the components $\Delta_{p,q}: Q_{p+q}\to Q_p\otimes Q_q$ of Δ are given by the formulas

$$\begin{aligned}&\Delta_{p,q}(\omega\otimes_A\delta(a)\otimes_A\theta)=\omega\otimes a\theta-\omega a\otimes\theta\\ &\Delta_{p,0}(\omega)=\omega\otimes 1\\ &\Delta_{0,q}(\theta)=1\otimes\theta\\ &\Delta_{0,0}(1)=1\otimes 1\end{aligned}\tag{4.15}$$

where $\omega\in\Omega^{\otimes_A(p-1)}$, $\theta\in\Omega^{\otimes_A(q-1)}$, $a\in A$, $p, q>0$.

Similarly to $(-)\otimes_c Q'$, the comonoid structure of Q makes the functor $(-)\otimes_c Q: \mathcal{C}_-\to k\text{-}\mathcal{M}$ a comonoidal functor. Let

$$\Delta_{F,G}:(F\otimes G)\otimes_c Q\longrightarrow(F\otimes_c Q)\otimes(G\otimes_c Q)$$
$$\epsilon: k[0]\otimes_c Q\longrightarrow k$$

be the structure maps. The isomorphism $f:Q\tilde{\to}Q'$ of comonoid objects induces the isomorphism $(-)\otimes_c Q\tilde{\to}(-)\otimes_c Q'$ of strictly comonoidal functors.

(4.16) The maps $\mu_{F,G}$ and ι defined in (c) are inverse to $\Delta_{F,G}$ and ϵ respectively.

PROOF. This is clear for ι and ϵ. Let $F, G\in\mathcal{C}_-$. Since $\Delta_{F,G}$ is an isomorphism, it is enough to show $\mu_{F,G}\Delta_{F,G}=\mathrm{id}$. If $p, q>0$, $x\in F_p$, $y\in G_q$, $\omega\in\Omega^{\otimes_A(p-1)}$, $\theta\in\Omega^{\otimes_A(q-1)}$, $a\in A$, then

$$\begin{aligned}&\mu_{F,G}\Delta_{F,G}((x\otimes y)\otimes_c(\omega\otimes_A\delta(a)\otimes_A\theta))\\&=\mu_{F,G}((x\otimes_c\omega)\otimes(y\otimes_c a\theta)-(x\otimes_c\omega a)\otimes(y\otimes_c\theta))\\&=(x\otimes y)\otimes_c(\omega\otimes_A d(a\theta))+(x\otimes dy)\otimes_c(\omega\otimes_A a\theta)\\&\qquad-(x\otimes y)\otimes_c(\omega a\otimes_A d\theta)-(x\otimes dy)\otimes_c(\omega a\otimes_A\theta)\\&=(x\otimes y)\otimes_c(\omega\otimes_A\delta(a)\otimes_A\theta).\end{aligned}$$

The remaining cases are trivial.

By (4.13), (4.14) and (4.16) the proof of Theorem 4.4 is completed.

REMARK 4.17. $\mathrm{Hom}_s(P_B, P_A)$ in (4.12) is the underlying k-module of Takeuchi's universal measuring topological coalgebra $\mathrm{Mes}(B, A)$ [14]. Therefore the isomorphism $a(A, B)^\vee\cong\mathrm{Hom}_s(P_B, P_A)$ of (4.12) follows also from Remark 1.3.

REMARK 4.18. We can deduce Theorem 3.1 from the isomorphism $a(A, B)\cong Q_A'^\vee\otimes_c Q_B'$ of (4.11). Indeed, take $\xi_0\in A^\vee$ such that $\xi_0(1)=1$. The complex $Q_A'^\vee$ is exact and the subspace $\bigoplus_{n\geq 0}\xi_0\otimes(\bar{A}^\vee)^{\otimes n}$ of $Q_A'^\vee$ is a complement of $d(Q_A'^\vee)$. Therefore we have an isomorphism

$$\bigoplus_{n\geq 0}\xi_0\otimes(\bar{A}^\vee)^{\otimes n}\otimes B\otimes\bar{B}^{\otimes n}\cong Q_A'^\vee\otimes_c Q_B'\cong a(A, B),$$

which takes $\xi_0\otimes\xi_1\otimes\cdots\otimes\xi_n\otimes y_0\otimes\bar{y}_1\otimes\cdots\otimes\bar{y}_n$ to $(\xi_0, y_0)(\xi_1, y_1)\cdots(\xi_n, y_n)$ for $\xi_1,\cdots,\xi_n\in\bar{A}^\vee$, $y_0,\cdots,y_n\in B$.

REMARK 4.19. We have defined for a k-algebra A the strictly monoidal functor $(-)\otimes_c Q_A:\mathcal{C}_-\to k\text{-}\mathcal{M}$. It is also true that every strictly monoidal faithful functor $\mathcal{C}_- \to k\text{-}\mathcal{M}$ is isomorphic to $(-)\otimes_c Q_A$ for a k-algebra A with $\dim A > 1$.

(f) Two presentations of $a(A, B)$ by differential graded (co)algebras

Let A be a k-algebra. We made $Q_A[1] = T_A(\Omega)$ a monoid object of $\mathcal{C}_+$ in (c) and Q_A a comonoid object of $\mathcal{C}_+$ in (e). We assume $\dim A < \infty$. In this subsection we prove two propositions below.

PROPOSITION 4.20. *The comonoidal functor* $Q_A[1]^\vee \otimes_c (-) : \mathcal{C}_+ \to k\text{-}\mathcal{M}$ *induced by the monoid structure of* $Q_A[1]$ *is strict.*

Admit this for a moment. Let B be a k-algebra. Then the k-modules $Q_A^\vee \otimes_c Q_B$, $Q_A[1]^\vee \otimes_c Q_B[1]$, $\mathrm{Hom}_c(Q_A, Q_B[1])$ become k-algebras by the comonoid structure of Q_A and the monoid structures of $Q_B[1]$, $(-)\otimes_c Q_B$, $Q_A[1]^\vee \otimes_c (-)$.

PROPOSITION 4.21. *We have algebra isomorphisms*

$$a(A, B) \cong Q_A^\vee \otimes_c Q_B \cong Q_A[1]^\vee \otimes_c Q_B[1]$$

and an algebra injection

$$a(A, B) \longrightarrow \mathrm{Hom}_c(Q_A, Q_B[1]).$$

Before the proofs we introduce some notation. Let $F, G \in \mathcal{C}_+$. Denote by $\mathrm{Hom}_c^0(F, G)$ the set of morphisms $u : F \to G$ such that $\dim u(F) < \infty$. Define

$$t(F, G) : F^\vee \otimes_c G \longrightarrow \mathrm{Hom}_c(F, G[1])$$
$$t'(F, G) : F[1]^\vee \otimes_c G \longrightarrow \mathrm{Hom}_c(F, G)$$

by the formulas

$$t(F, G)(\xi \otimes y)(x) = \xi(x)dy + \xi(dx)y$$
$$t'(F, G)(\xi' \otimes y)(x) = \xi'(x)dy + \xi'(dx)y$$

for $x \in F_n$, $y \in G_m$, $\xi \in F_m^\vee$, $\xi' \in (F[1]^\vee)_m = F_{m+1}^\vee$. We note that if F is a projective object and $\dim F_n < \infty$ for all n, then $t(F, G)$, $t'(F, G)$ are injections onto the subspaces $\mathrm{Hom}_c^0(F, G[1])$, $\mathrm{Hom}_c^0(F, G)$ respectively.

Write $Q = Q_A$. Let

$$\Delta : Q \longrightarrow Q \otimes Q \qquad \epsilon : Q \longrightarrow k[0]$$
$$\mu : Q[1] \otimes Q[1] \longrightarrow Q[1] \qquad \iota : k[0] \longrightarrow Q[1]$$

be the (co)multiplication and the (co)unit. Let

$$\Delta_{p,q}: Q_{p+q} \longrightarrow Q_p \otimes Q_q \qquad \epsilon_0: Q_0 \longrightarrow k$$
$$\mu_{p,q}: Q_{p+1} \otimes Q_{q+1} \longrightarrow Q_{p+q+1} \qquad \iota_0: k \longrightarrow Q_1$$

be the components of Δ, ϵ, μ, ι. Using the description of $\Delta_{p,q}$ in (4.15), one can verify that the following relations hold.

(i) $\mu_{p-1,q-1}\Delta_{p,q}=0$ for p, $q \geq 1$
(ii) $\mu_{p-1,q}(1\otimes d)\Delta_{p,q}=\mathrm{id}$ for $p\geq 1$, $q\geq 0$
(iii) $\mu_{p,q-1}(d\otimes 1)\Delta_{p,q}=(-1)^p\mathrm{id}$ for $p\geq 0$, $q\geq 1$
(iv) $\mu_{p,q}(d\otimes d)\Delta_{p,q}=d$ for $p, q\geq 0$
(v) $\iota_0\epsilon_0=d$.

These relations imply that for any $F, G\in \mathcal{C}_+$ the following diagrams are commutative.

(4.22)

$$\begin{tikzcd}
(F^\vee\otimes_c Q)\otimes(G^\vee\otimes_c Q) \arrow[r, "t(F,Q)\otimes t(G,Q)"] & \mathrm{Hom}_c(F, Q[1])\otimes\mathrm{Hom}_c(G, Q[1]) \arrow[dd, "\mu_*"] \\
(F^\vee\otimes G^\vee)\otimes_c Q \arrow[u, "\Delta_*"] \arrow[d, "\mathrm{can}"'] & \\
(F\otimes G)^\vee\otimes_c Q \arrow[r, "t(F\otimes G, Q)"'] & \mathrm{Hom}_c(F\otimes G, Q[1])
\end{tikzcd}$$

$$\begin{tikzcd}
k[0]\otimes_c Q \arrow[r, "\epsilon_*"] \arrow[d, equal] & k \arrow[d, "\iota_*"] \\
k[0]^\vee\otimes_c Q \arrow[r, "t(k[0], Q)"'] & \mathrm{Hom}_c(k[0], Q[1])
\end{tikzcd}$$

$$\begin{tikzcd}
(Q[1]^\vee\otimes_c F)\otimes(Q[1]^\vee\otimes_c G) \arrow[r, "t'(Q,F)\otimes t'(Q,G)"] & \mathrm{Hom}_c(Q, F)\otimes\mathrm{Hom}_c(Q, G) \arrow[d, "\Delta^*"] \\
Q[1]^\vee\otimes_c(F\otimes G) \arrow[u, "\mu^*"] \arrow[r, "t'(Q, F\otimes G)"'] & \mathrm{Hom}_c(Q, F\otimes G)
\end{tikzcd}$$

$$\begin{tikzcd}
& k \arrow[dr, "\epsilon^*"] & \\
Q[1]^\vee\otimes_c k[0] \arrow[ur, "\iota^*"] \arrow[rr, "t'(Q, k[0])"'] & & \mathrm{Hom}_c(Q, k[0])
\end{tikzcd}$$

Here Δ^*, ϵ^*, $\cdots$ are the maps induced by Δ, ϵ, $\cdots$. The proof is straightforward.

PROOF OF PROPOSITION 4.20. We have to show that μ^*, ι^* in the lower half of the diagrams (4.22) are bijections. This is clear for ι^*. As for μ^* we may assume $\dim F$, $\dim G<\infty$. Then the rows of the third diagram are bijections. Since $(-)\otimes_c Q$ is strictly comonoidal and $\mathrm{Hom}_c(Q, (-)^\vee)\cong((-)\otimes_c Q)^\vee$, Δ^* is a bijection. Hence μ^* is a bijection.

PROOF OF PROPOSITION 4.21. We first show $a(A, B)\cong Q_A^\vee\otimes_c Q_B$ as algebras. Since $(-)\otimes_s P_B : \mathcal{S}_-\to k\text{-}\mathcal{M}$ is a monoidal functor and $P_A\in\mathcal{S}_+$ is a comonoid object, $P_A^\vee\otimes_s P_B$ becomes a k-algebra. As in (4.11) we have

$$a(A, B)\cong P_A^\vee\otimes_s P_B\cong N_-(P_A^\vee)\otimes_c N_+P_B\cong Q_A^\vee\otimes_c Q_B$$

and each isomorphism preserves algebra structure.

Let $r : Q_A[1]^\vee\otimes_c Q_B[1]\to Q_A^\vee\otimes_c Q_B$ be the map $\xi\otimes_c y\mapsto\xi\otimes_c y$. We have a commutative diagram

$$\begin{tikzcd}
Q_A[1]^\vee\otimes_c Q_B[1] \arrow[rr, "r"] \arrow[dr, "t'(Q_A, Q_B[1])"'] & & Q_A^\vee\otimes_c Q_B \arrow[dl, "t(Q_A, Q_B)"] \\
& \mathrm{Hom}_c(Q_A, Q_B[1]). &
\end{tikzcd}$$

By the lower half of the diagrams (4.22) with $F=G=Q_B[1]$, $t'(Q_A, Q_B[1])$ is an algebra map. By the upper half of the diagrams (4.22) with $F=G=Q_A$ and Q replaced by Q_B, $t(Q_A, Q_B)$ is an algebra map. Moreover $t(Q_A, Q_B)$, $t'(Q_A, Q_B[1])$ are injections onto $\mathrm{Hom}_c^0(Q_A, Q_B[1])$. Therefore r is an algebra isomorphism. This proves the proposition.

5. Action of $a(A, A)$ on full matrix algebras

We fix a finite dimensional k-algebra A throughout. Let B be a finite dimensional k-algebra and V a left $a(B, A)$-module such that $0<\dim V<\infty$. As in Proposition 2.1, V corresponds to an algebra map $p : A\to B\otimes\mathrm{End}\, V$ and a linear map $f : A\otimes V\to V\otimes B$. Define

$$p_1 : A\otimes\mathrm{End}\, V\to B\otimes\mathrm{End}\, V$$

by $p_1(a\otimes x)=p(a)(1\otimes x)$. Also let $g : V^\vee\otimes A\to B\otimes V^\vee$ be the transition map for the right $a(B, A)$-module $V^\vee$.

PROPOSITION 5.1. (i) *p_1 is bijective if and only if f is bijective.*

(ii) *Suppose f is bijective. Then the algebra* End V *admits a right action of $a(A, A)$ such that p_1 is an algebra isomorphism, where the algebra structure of $A \otimes \mathrm{End}\, V$ comes from the action of $a(A, A)$ on* End V *as in Proposition* 2.2 (iii) *and $B \otimes \mathrm{End}\, V$ is the standard tensor product algebra. The transition map for* End V *corresponds to the map* $(f^{-1} \otimes 1) \circ (1 \otimes g) : V \otimes V^{\vee} \otimes A \to V \otimes B \otimes V^{\vee} \to A \otimes V \otimes V^{\vee}$ *through the canonical isomorphism* $\mathrm{End}\, V \cong V \otimes V^{\vee}$.

PROOF. (i) We have a commutative diagram

$$\begin{array}{ccc} A \otimes V \otimes V^{\vee} & \xrightarrow{f \otimes 1} & V \otimes B \otimes V^{\vee} \\ \wr \Vert & & \wr \Vert \\ A \otimes \mathrm{End}\, V & \xrightarrow[p_1]{} & B \otimes \mathrm{End}\, V \end{array}$$

where the isomorphisms are the canonical ones. Since $V^{\vee} \neq 0$, (i) follows.

(ii) The first assertion is clear from (i) and Proposition 2.2. Let $p_2 : \mathrm{End}\, V \otimes A \to B \otimes \mathrm{End}\, V$ be the map $x \otimes a \mapsto (1 \otimes x) p(a)$. Using the relation of g and f given in Section 2, we see that the diagram

$$\begin{array}{ccc} V \otimes V^{\vee} \otimes A & \xrightarrow{1 \otimes g} & V \otimes B \otimes V^{\vee} \\ \wr \Vert & & \wr \Vert \\ \mathrm{End}\, V \otimes A & \xrightarrow[p_2]{} & B \otimes \mathrm{End}\, V \end{array}$$

is commutative. The transition map for the right $a(A, A)$-module End V is $p_1^{-1} \circ p_2$. By the above two diagrams, $p_1^{-1} \circ p_2$ is isomorphic to the map $(f \otimes 1)^{-1} \circ (1 \otimes g)$. This proves (ii).

PROPOSITION 5.2. *Let V be a k-module such that $0 < \dim V < \infty$. Then every right action of $a(A, A)$ on the algebra* End V *is obtained from an algebra B and a left $a(B, A)$-module structure on V having the invertible transition map as in Proposition* 5.1 (ii).

PROOF. Suppose given a right action of $a(A, A)$ on the algebra End V. It makes $A \otimes \mathrm{End}\, V$ an algebra by the correspondence (i)$\leftrightarrow$(iii) of Proposition 2.2. Let $i_1 : A \to A \otimes \mathrm{End}\, V$, $i_2 : \mathrm{End}\, V \to A \otimes \mathrm{End}\, V$ be the natural injections. Let B be the centralizer of $\mathrm{Im}\, i_2$ in $A \otimes \mathrm{End}\, V$. Then we have an algebra isomorphism $p_1 : A \otimes \mathrm{End}\, V \xrightarrow{\sim} B \otimes \mathrm{End}\, V$, where the right side is the usual tensor product of algebras. The algebra map

$p=p_1\circ i_1 : A\to B\otimes\mathrm{End}\ V$ makes V a left $a(B, A)$-module. By Proposition 5.1 (i) the transition map for V is invertible. This proves the proposition.

We next consider morphisms between $a(A, A)$-algebras of the form End V. Let V be a left $a(B, A)$-module and W a left $a(C, A)$-module such that $0<\dim V$, $\dim W<\infty$ and their transition maps are invertible. Then End V, End W become right $a(A, A)$-module algebras.

PROPOSITION 5.3. (i) *For a left $a(C, B)$-module U and an $a(C, A)$-isomorphism $l : U\otimes V\xrightarrow{\sim} W$, define an algebra map $j_{U,l}$: End $V\to$ End W by $j_{U,l}(x)=l\circ(1\otimes x)\circ l^{-1}$. Then $j_{U,l}$ is $a(A, A)$-linear.*

(ii) *For left $a(C, B)$-modules U, U' and $a(C, A)$-isomorphisms $l : U\otimes V\xrightarrow{\sim} W$, $l' : U'\otimes V\xrightarrow{\sim} W$, we have $j_{U,l}=j_{U',l'}$ if and only if there is an $a(C, B)$-isomorphism $m : U\xrightarrow{\sim} U'$ such that $l=l'\circ(m\otimes 1)$.*

(iii) *Every $a(A, A)$-module algebra map* End $V\to$ End W *is of the form $j_{U,l}$ for some pair U, l as in* (i).

PROOF. Let $p : A\to B\otimes\mathrm{End}\ V$, $p_1 : A\otimes\mathrm{End}\ V\to B\otimes\mathrm{End}\ V$ be the maps defined at the beginning of this section, and $q : A\to C\otimes\mathrm{End}\ W$, $q_1 : A\otimes\mathrm{End}\ W\to C\otimes\mathrm{End}\ W$ the similar maps for the $a(C, A)$-module W. The actions of $a(A, A)$ on End V, End W make $A\otimes\mathrm{End}\ V$, $A\otimes\mathrm{End}\ W$ algebras and p_1, q_1 are algebra isomorphisms.

(i) Let $r : B\to C\otimes\mathrm{End}\ U$ be the algebra map corresponding to the $a(C, B)$-module U. Then we have a commutative diagram

$$\begin{tikzcd} A\otimes\mathrm{End}\ V \arrow[rr, "1\otimes j_{U,l}"] \arrow[d, "p_1"'] & & A\otimes\mathrm{End}\ W \arrow[d, "q_1"] \\ B\otimes\mathrm{End}\ V \arrow[r, "r\otimes 1"'] & C\otimes\mathrm{End}\ U\otimes\mathrm{End}\ V \arrow[r, "1\otimes I_l"'] & C\otimes\mathrm{End}\ W, \end{tikzcd}$$

where I_l is the map $x\otimes y\mapsto l\circ(x\otimes y)\circ l^{-1}$ for $x\in\mathrm{End}\ U$, $y\in\mathrm{End}\ V$. Hence $1\otimes j_{U,l}$ is an algebra map, so $j_{U,l}$ is an $a(A, A)$-module algebra map.

(ii) This is left to the reader.

(iii) Let j : End $V\to$ End W be a map of $a(A, A)$-module algebras. Then there is a k-module U and an isomorphism $l : U\otimes V\xrightarrow{\sim} W$ such that $j=j_{U,l}$. We must show that U has a structure of a left $a(C, B)$-module and l is $a(C, A)$-linear. Set $s=q_1\circ(1\otimes j)\circ p_1^{-1} : B\otimes\mathrm{End}\ V\to C\otimes\mathrm{End}\ W$. Then s is an algebra map and the diagram

$$\begin{array}{ccc} \text{End } V & \xrightarrow{j} & \text{End } W \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle i_2} \\ B\otimes\text{End } V & \xrightarrow[s]{} & C\otimes\text{End } W \end{array}$$

is commutative, where i_2 are the natural injections. Considering the centralizer of $\mathrm{Im}(s\circ i_2)$ in $C\otimes\mathrm{End}\,W$, we see that there is an algebra map $r: B\to C\otimes\mathrm{End}\,U$ such that $s=(1\otimes I_l)\circ(r\otimes 1)$. The map r makes U a left $a(C, B)$-module, and by $q=s\circ p$, l is $a(C, A)$-linear. This proves (iii).

We finally consider a smash product. Let V be a left $a(B, A)$-module such that $0<\dim V<\infty$ and having the invertible transition map. The bialgebra $a(A, A)$ acts on the algebra End V on the right as in Proposition 5.1 (ii), so we can form the semi-direct product algebra $a(A, A)\otimes$ End V, which is also called the smash product in Sweedler [12].

PROPOSITION 5.4. *There is an algebra isomorphism*

$$a(A, A)\otimes\mathrm{End}\ V\cong a(A, B)\otimes\mathrm{End}\ V$$

where the right side is the usual tensor product of algebras.

PROOF. Set $\Lambda=a(A, A)$. For $X\in\mathcal{C}om\text{-}\Lambda$ and $M\in\mathcal{M}\text{-}\Lambda$, define $f_{X,M}$: $M\otimes X\to X\otimes M$ to be the composite

$$M\otimes X\xrightarrow{1\otimes c}M\otimes X\otimes\Lambda\xrightarrow{t\otimes 1}X\otimes M\otimes\Lambda\xrightarrow{1\otimes a}X\otimes M$$

with $c: X\to X\otimes\Lambda$ the coaction, $a: M\otimes\Lambda\to M$ the action, $t: M\otimes X\to X\otimes M$ the map $m\otimes x\mapsto x\otimes m$.

If R is a monoid object of $\mathcal{M}\text{-}\Lambda$, i.e., a right Λ-module algebra, the map $f_{X,R}$ defines an (R, R)-module structure on $X\otimes R$ in the same manner as in Proposition 2.1 and the functor $\phi: \mathcal{C}om\text{-}\Lambda\to R\text{-}\mathcal{M}\text{-}R$ taking X to the bimodule $X\otimes R$ preserves monoidal structure. Therefore ϕ takes a monoid object of $\mathcal{C}om\text{-}\Lambda$, i.e., a right Λ-comodule algebra, to a monoid object of $R\text{-}\mathcal{M}\text{-}R$, i.e., an R-ring. This is called a generalized smash product (see [5]). The smash product $\Lambda\otimes R$ is nothing but the R-ring $\phi(\Lambda)$, where Λ is viewed as a right Λ-comodule algebra naturally.

Now let $R=\mathrm{End}\ V$. Then there is an equivalence of monoidal categories $\zeta: R\text{-}\mathcal{M}\text{-}R\to k\text{-}\mathcal{M}$ given by $\zeta(X)=\{x\in X\mid rx=xr \text{ for all } r\in R\}$, and we have $X\cong R\otimes\zeta(X)$ naturally for $X\in R\text{-}\mathcal{M}\text{-}R$. We must show $\zeta\phi(\Lambda)\cong a(A, B)$ as monoid objects of $k\text{-}\mathcal{M}$. We shall show more generally that

$\zeta\phi(a(C, A))\cong a(C, B)$ as monoid objects for any algebra C with $\dim C<\infty$, where $a(C, A)$ is viewed as a right Λ-comodule algebra naturally. By Proposition 5.1 we know $\zeta\phi(A)\cong B$ as monoid objects. Also $\zeta\circ\phi: \mathcal{C}om\text{-}\Lambda\rightarrow k\text{-}\mathcal{M}$ commutes with inductive limits and $\otimes$. It then follows from the construction of $a(C, -)$ in Section 1 that $\zeta\phi(a(C, A))\cong a(C, B)$ as k-algebras. This finishes the proof.

REMARK 5.5. The general theory of the action of Hopf algebras on Azumaya algebras has been developed by Doi and Takeuchi [5]. Some of our result might be deduced from their theory.

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