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Nonlinear eigenvalue problem associated with the generalized capillarity equation

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1. Introduction.

Consider an embedded surface S in $B\times \boldsymbol{R}^{+}$ whose mean curvature H at each point $(x, u)\in B\times \boldsymbol{R}^{+}$ is given by $-\frac{1}{2}ku^{q}$ and whose boundary ∂S is $\partial B\times\{0\}$. Here B denotes the unit disk in $\boldsymbol{R}^{2}$, $\boldsymbol{R}^{+}=\{x\in \boldsymbol{R}\mid x\geq 0\}$, $q\geq 1$ and $k>0$ are fixed constants, and the sign of the mean curvature is defined so as to be negative at the point of maximum height. The surface S is realized as a solution of the problem (referred to [P] in the sequel)

$$2H+ku^{q}=0 \qquad \text{in } B, \tag{1}$$

$$u\geq 0 \qquad \text{in } B,\ u=0 \text{ on } \partial B. \tag{2}$$

The purpose of the present paper is to show the condition for existence and nonexistence of nontrivial solutions of [P] in terms of the parameter k.

As is well known, the equation (1) with $q=1$ describes the free surface in a capillary tube under negative gravitational field. In this case k is called the capillarity constant, given by $k=\rho g/\sigma$, where ρ denotes the density difference across the free surface, g the gravitational acceleration, σ the surface tension. P. Concus and R. Finn, in their celebrated paper [2], studied extensively the radial solution of this equation and obtained many interesting properties. One of the most striking results is that the solution is generally a multi-valued function and has an oscillatory trajectory; for more physical backgrounds and other informations see the elegant monograph of R. Finn [3].

Recently, from the viewpoint of the analogue of Laplace operator, many authors have investigated the general equation $nH+f(u)=0$ (see, for instance, [1] [5] [6]). Especially, F. V. Atkinson, L. A. Peletier and J. Serrin studied the generalized ground state solution; that is, a positive radial solution which is zero at infinity and generally multi-valued.

In order to overcome the difficulty of multi-valued functions, Concus and Finn introduce the arc length s along the trajectory as independent variable so as to transform (1) into the system of first order differential equations, while Atkinson et al. regard u as independent variable and discuss the single-valued function $r=r(u)$.

We now return to our problem. First we see from H. C. Wente [7] that our problem [P] has only the radial solution, possibly multi-valued. Following the idea of [1] we think of u as independent variable and discuss the single-valued function $r=r(u)$. The problem [P] then means the ordinary differential equation

$$\frac{r_{uu}}{(1+r_u^2)^{3/2}}-\frac{1}{r}\frac{1}{(1+r_u^2)^{1/2}}+ku^q=0, \tag{3}$$

with

$$r(u)>0 \qquad \text{for } 0<u<u_0\equiv u(0) \tag{4}$$

$$r(u_0)=0, \qquad \lim_{u\to u_0} r_u=-\infty \tag{5}$$

$$\lim_{u\to 0} r(u)=1. \tag{6}$$

Note that the function $u(r)$ satisfies either

$$\frac{1}{r}\frac{d}{dr}\frac{ru_r}{(1+u_r^2)^{1/2}}+ku^q=0 \tag{7}$$

or

$$\frac{1}{r}\frac{d}{dr}\frac{ru_r}{(1+u_r^2)^{1/2}}-ku^q=0 \tag{8}$$

according as $u_r<0$ or $u_r>0$, respectively.

We now state our main theorem.

THEOREM. (A) *Let* $q=1$. *Then there exist positive constants* $k_1<k_2$ *such that* [P] *has only the trivial solution for* $k\notin(k_1, k_2)$.

(B) *Let* $q>1$. *Then there exist positive constants* $k_3<k_4$ *depending only on* q *such that*:

(i) *for* $k<k_3$ [P] *has only the trivial solution.*

(ii) *for any* $k>k_4$ *there exists a nontrivial solution of* [P], *which can be represented as a single-valued function* $u=u(r)$.

The part (A) of this theorem is already known (see [3] [5]). But our proof of (B), in its special case, also applies to that of (A) and seems to give new proof. See Propositions below.

To prove the theorem we set $u=k^{-1/(1+q)}\bar{u}$, $r=k^{-1/(1+q)}\bar{r}$, and use the letter u and r instead of $\bar{u}$ and $\bar{r}$ for simplicity. Then $r(u)$ satisfies (3) with $k=1$ (respectively, $u(r)$, (7) or (8) with $k=1$). The parameter k appears only in the condition (6): $\lim_{u\to 0} r(u)=k^{1/(1+q)}$. After the above transformation we solve the equation (3) with (4) (5) as an initial value problem (IVP) with the shooting parameter u_0. It follows from [1] that for any u_0 a solution $r(u; u_0)$ of IVP exists in the interval $0<u\leq u_0$. Denoting the first zero by R, that is, $R=\lim_{u\to 0} r(u; u_0)$, we want to compare R with $k^{1/(1+q)}$. Hereafter we only investigate this IVP and write $r(u)$ instead of $r(u; u_0)$ for simplicity. We also discuss the function $u(r)$, the inverse of $r(u)$, which is possibly multi-valued.

In the following two sections we prove two propositions below. It is easy to see that these propositions imply the theorem.

PROPOSITION 1. *When* $u_0^{1+q}<2(1+q)$ *the inverse function* $u(r)$ *is a single-valued function for* $0\leq r<R$, *and we have the estimate*

$$C_1(q)u_0^{-(q-1)/2}<R<C_2(q)u_0^{-(q-1)/2}$$

for some constants $C_1(q)<C_2(q)$.

PROPOSITION 2. *When* $u_0^{1+q}>8q$ *the inverse function* $u(r)$ *is necessarily a multi-valued function and there exist positive constants* $\delta_1(q)<\delta_2(q)$ *independent of* u_0 *such that*

$$\delta_1(q)<R<\delta_2(q).$$

Note that when $q>1$ Proposition 1 implies $R\to\infty$ as $u_0\to 0$ and R is bounded for u_0^{1+q} in the left vicinity of $2(1+q)$. Thus the continuity with respect to u_0 implies the existence of the solution mentioned in the theorem.

2. Proof of Proposition 1.

First we want to prove that the inverse function $u(r)$ is single-valued for $0\leq r<R$.

At least for small values of r it is evident that $u(r)$ is single-valued and monotone decreasing. Integrating (7) we find

$$-\frac{ru_r}{(1+u_r^2)^{1/2}}=\int_0^r \rho u(\rho)^q d\rho \geq \frac{1}{2}r^2 u(r)^q,$$

and so,

$$\frac{1}{r}\frac{u_r}{(1+u_r^2)^{1/2}} \leq -\frac{1}{2}u(r)^q \tag{9}$$

as long as $u(r)$ is single-valued.

Let us write (7) in the form

$$\frac{u_{rr}}{(1+u_r^2)^{3/2}}+\frac{1}{r}\frac{u_r}{(1+u_r^2)^{1/2}}+u^q=0. \tag{10}$$

Using (9) we find

$$\frac{u_{rr}}{(1+u_r^2)^{3/2}}+\frac{1}{2}u^q \leq 0.$$

Multiplying u_r (<0) and integrating from 0 to r, we obtain

$$-\frac{1}{(1+u_r^2)^{1/2}}+1<-\frac{1}{2(1+q)}(u(r)^{1+q}-u_0^{1+q}).$$

Thus under our assumption $u_0^{1+q}<2(1+q)$ we have

$$(1+u_r(r)^2)^{1/2}<\frac{1}{1-u_0^{1+q}/2(1+q)},$$

from which we conclude that u can be continued as a single-valued function of r for $0 \leq r<R$.

In order to obtain a lower bound of R we compare $u(r)$ with the lower hemisphere g whose mean curvature is $-\frac{1}{2}u_0^q$. Then, on the interval $0<r<2/u_0^q$, we have

$$g(r)=((2/u_0^q)^2-r^2)^{1/2}+u_0-2/u_0^q$$

and

$$\frac{1}{r}\frac{d}{dr}\frac{rg_r}{(1+g_r^2)^{1/2}}<\frac{1}{r}\frac{d}{dr}\frac{ru_r}{(1+u_r^2)^{1/2}}$$

with the initial conditions

$$g(0)=u(0)=u_0, \qquad g_r(0)=u_r(0)=0.$$

An integration now yields

$$g(r)<u(r)$$

for $0<r\leq 2/u_0^q$. This implies the desired bound:

$$R>C_1(q)u_0^{-(q-1)/2}.$$

Finally we wish to give an upper bound of R. Let us use the notation $R_q=u^{-1}(\bar{q}u_0)$ where we set $\bar{q}=q/(1+q)$. The lower bound of R_q easily follows as in the case of R. Indeed we have

$$R_q>g^{-1}(\bar{q}u_0)=(4/(1+q)u_0^{q-1}-(u_0/(1+q))^2)^{1/2}\geq 2u_0^{(1-q)/2}/(1+q). \tag{11}$$

On the other hand,

$$-u_r(r)>-\frac{u_r(r)}{(1+u_r(r)^2)^{1/2}}=\frac{1}{r}\int_0^r \rho u(\rho)^q d\rho \tag{12}$$

$$>\frac{1}{r}\int_0^a \rho u(\rho)^q d\rho$$

where $0<a\leq r$. An integration from a to R_q yields

$$R_q<a\exp\left(\frac{2qu_0/(1+q)}{a^2(\bar{q}u_0)^q}\right)<C_4(q)u_0^{-(q-1)/2}. \tag{13}$$

We also integrate (12) from a to R, obtaining

$$R<a\exp\left(\frac{2}{a^2u(a)^{q-1}}\right)<\infty.$$

Placing $a=R_q$ in this inequality and using (11) (13), we deduce

$$R<C_2(q)u_0^{-(q-1)/2}.$$

This completes the proof of proposition.

3. Proof of Proposition 2.

Let us begin with:

LEMMA 1. *When $u_0^{1+q}>8q$ there exist a first critical point (r_1, u_1) on the graph $r(u)$, with*

$$2/u_0^q<r_1<4/u_0^q, \qquad u_0-4/u_0^q<u_1<u_0-2/u_0^q.$$

Moreover $r_{uu}(u_0)<-u_0^q/4$.

This lemma is the special case of Theorem 2.1 in [1]. Hence we omit the proof.

Before proving the rest of Proposition 2, we collect some properties of the graph of solution which are needed later. Proofs of these results can be found in [1] or [3].

Properties of solutions.
(P1) *The solution* $r(u)$ *has at most a finite number of minima and maxima over the interval* $[0, u_0]$. *The trajectory is bounded above by the curve* $ru^q=2$.
(P2) *Let* $\{u_{2m}\}$, $u_{2(m+1)}<u_{2m}$, $m=1, 2, \cdots$, *denote the sequence of minima of* $r(u)$ *when we trace it from the point* $(0, u_0)$ *and let*

$$u_{2m-1}=\sup\{u>u_{2m} \mid r_u>0 \text{ on the interval } (u_{2m}, u)\},$$

then we have

$$r_{2m-1}u_{2m-1}^q>1, \quad r_{2m}u_{2m}^q<1, \quad m\geq 1$$

and

$$r_{2m-1}<r_{2m+1}, \quad r_{2m}<r_{2(m+1)}, \quad m\geq 1.$$

(P3) *We have an upper bound of* r_k *independent of* k. *Indeed the inequality*

$$r_k^3-\frac{3\sqrt{3}}{q}r_k^{(2q-1)/q}-(\sqrt{3}/q)^{3q/(1+q)}<0,$$

holds, from which we conclude in particular $r_k<3$.
(P4) *There hold* $-\infty<u_r(0^+)<0$ *and* $R=\lim_{u\to 0}r(u)<\infty$.

We denote the last minimum by (r_i, u_i) and last maximum by (r_a, u_a). Notice that by (P4) the solution is monotone decreasing on the interval $[0, u_i)$, and hence the inverse $u(r)$ satisfies (7) for $r_i\leq r<R$.

LEMMA 2. *When* $u_i>\pi^{1/(1+q)}$ *there holds the inequality*

$$C_5(q)<R<C_6(q)$$

for some constants $C_5(q)<C_6(q)$.

PROOF. Let us put $r_c = r(\pi^{1/(1+q)})$. Notice that r_c is well defined.

We give a lower bound of R first. If $r_c \geq \pi^{-q/(1+q)}$ then $R \geq \pi^{-q/(1+q)}$. Whence the case $0 < r_c < \pi^{-q/(1+q)}$ needs to be considered.

Let an ellipse of major axis $2\pi^{-q/(1+q)}$ with focal points at $(r_c, \pi^{1/(1+q)})$ and $(2\pi^{-q/(1+q)} - r_c, \pi^{1/(1+q)})$ roll rigidly downward on a u-axis without slipping. Let $\hat{v}(r)$ denote the curve swept out by the focal point $(r_c, \pi^{1/(1+q)})$. The curve is called a Delaunay arc and $\hat{v}(r)$ satisfies

$$\frac{1}{r}\frac{d}{dr}\frac{r\hat{v}_r}{(1+\hat{v}_r^2)^{1/2}} + \pi^{q/(1+q)} = 0$$

on $r_c < r < 2\pi^{-q/(1+q)} - r_c$, with

$$\hat{v}(r_c) = \pi^{1/(1+q)}, \qquad \hat{v}_r(r_c) = -\infty.$$

For more informations about a Delaunay arc, see Section 4.7 in [3]. We compare $\hat{v}(r)$ with $u(r)$, obtaining

$$\hat{v}(r) < u(r)$$

for $r_c < r \leq 2\pi^{-q/(1+q)} - r_c$, on integration. Since $\pi^{1/(1+q)} - 2\pi^{-q/(1+q)} \times \mathrm{E}(1 - r_c\pi^{1/(1+q)}) \geq 0$ we find

$$R > (r_c + 2\pi^{-q/(1+q)} - r_c)/2 = \pi^{-q/(1+q)}.$$

Here $\mathrm{E}(k)$ denotes the complete elliptic integral of the second kind and we note that $2\pi^{-q/(1+q)}\mathrm{E}(1 - r_c\pi^{q/(1+q)})$ is the half circumference of the ellipse we considered.

Next let us give an upper bound of R. An integration of (7) from r_i to r yields

$$\begin{aligned}
\frac{ru_r}{(1+u_r^2)^{1/2}} + r_i &= -\int_{r_i}^{r} \rho u(\rho)^q d\rho \\
&\leq -\int_{r_i}^{a} \rho u(\rho)^q d\rho \\
&= -\frac{1}{2}(a^2 u(a)^q - r_i^2 u_i^q) + \frac{q}{2}\int_{r_i}^{a} \rho^2 u(\rho)^{q-1} u_r(\rho) d\rho \\
&\leq -\frac{1}{2}a^2 u(a)^q + \frac{1}{2}r_i
\end{aligned}$$

where $r_i \leq a < r < R$. It follows that

$$-u_r(r) > -\frac{u_r}{(1+u_r^2)^{1/2}} \geq \frac{1}{r}\frac{1}{2}a^2u(a)^q. \tag{14}$$

On the other hand, let $R_0 \equiv u^{-1}(\pi^{1/(1+q)}/2)$ and comparing $\hat{v}(r)$ with $u(r)$ as before, we obtain

$$R_0 > \pi^{-q/(1+q)}. \tag{15}$$

Observing that (14) and (15) correspond to (12) and (11) respectively, we can give the rest of the proof on the same line as that of Proposition 1. So we omit the details. □

LEMMA 3. *When $u_i < \pi^{1/(1+q)}$ we have*

$$r_i < R < r_i \cosh(\pi^{1/(1+q)}/r_i).$$

PROOF. By (P4) it is clear that $r_i < R$. To prove the opposite side we compare $u(r)$ with the minimal surface

$$w(r) = -r_i \cosh^{-1}(r/r_i) + u_i. \quad \square$$

LEMMA 4. *When $u_i \leq \pi^{1/(1+q)}$ we have positive constants δ, C such that if $0 < r_i \leq \delta$ then $u_a \leq C$.*

PROOF. Consider a Delaunay arc $v(r)$ which is determined by the conditions

$$\begin{aligned} a &= 1/(u_i+\tau)^q \\ a-c &= r_i \\ \tau &= 2a\mathrm{E}(c/a) \end{aligned} \tag{16}$$

with its lower vertical point at (r_i, u_i), where $2a$ is the length of major axis and $2c$ is the distance between focal points. We find

$$\frac{1}{r}\frac{d}{dr}\frac{rv_r}{(1+v_r^2)^{1/2}} - (u_i+\tau)^q = 0$$

on $r_i < r < 2/(u_i+\tau)^q - r_i$, with

$$v(r_i) = u_i, \qquad v_r(r_i) = \infty.$$

Suppose that such $v(r)$ exists, then, comparing it with $u(r)$, we have

$$u(r) < v(r)$$

for $r_i<r\leq 2/(u_i+\tau)^q-r_i$; in particular, $r_a>2/(u_i+\tau)^q-r_i$.

We discuss the criterion of the existence of this Delaunay arc. Interpreting (16) as the determining equation of τ, we introduce the function

$$f(\tau)=(u_i+\tau)^q\tau-2\mathrm{E}(1-r_i(u_i+\tau)^q).$$

In order that $f(\tau)$ is well defined, there must hold

$$r_i^{-1/q}-u_i\geq\tau. \tag{17}$$

If we have, in addition to this,

$$f(r_i^{-1/q}-u_i)=(r_i^{-1/q}-u_i)/r_i-\pi>0. \tag{18}$$

Then, in view of $f(0)<0$, the intermediate value theorem yields the solution τ of the equation $f(\tau)=0$. Under our assumption that u_i is bounded above, we can find a positive constant δ such that both (17) and (18) hold on the interval $0<r_i<\delta$.

On the other hand,

$$\tau(u_i+\tau)^q=2\mathrm{E}(1-r_i(u_i+\tau)^q)<\pi,$$

and so,

$$(u_i+\tau)^q<C^q/2$$

for some constant C. This implies, on the interval $0<r_i<\delta$,

$$r_a>2/(u_i+\tau)^q-r_i>1/(u_i+\tau)^q>2/C^q,$$

from which it finally follows that

$$u_a<(2/r_a)^{1/q}<C. \quad \square$$

Proof of Proposition 2 concluded.

First we see from Lemma 2 that only the case $u_i<\pi^{1/(1+q)}$ remains to be considered.

If $r_i>\delta$ then by Lemma 3 the conclusion follows. We intend to show that if $0<r_i\leq\delta$ then in fact $r_i>\varepsilon$ for some positive constant ε. By Lemma 3 the result follows even in this case.

In order to do it, we introduce the functional

$$F(u)=\frac{r(u)}{(1+r_u(u)^2)^{1/2}}-\frac{1}{2}u^q r(u)^2.$$

Since $F_u<0$ and $F(u_a)>0$, we have

$$r_i>F(u_i)\geq F(u_a)+\varepsilon(u_a)>\varepsilon(u_a),$$

where

$$\varepsilon(u_a)=\int_{u_i}^{u_a}\frac{1}{2q}u^{q-1}r(u)^2du.$$

By Lemma 4, u_a is bounded above in our present case. This bound of u_a yields the desired constant ε. This completes the proof. □

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