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On the l-adic representations attached to some absolutely simple abelian varieties of type II

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§ 1. Introduction

Let K be a number field and let $\bar{K}$ be an algebraic closure of K. Let A be an abelian variety over K of dimension $n\geq 1$, and let $G=\mathrm{Gal}(\bar{K}/K)$. For each prime number l, let $V_l(A)$ be the $\boldsymbol{Q}_l$-adic Tate module of A. One general problem is to study the image of the l-adic representation $V_l(A)$ of G. Let $\mathcal{G}_l$ be the l-adic Lie algebra of the image of this l-adic representation. Fix an embedding $\sigma: K\rightarrow \boldsymbol{C}$, and denote by $A(\boldsymbol{C})$ the abelian variety $A\times_{K,\sigma}\boldsymbol{C}$. In [7], Mumford and Tate conjectured that $\mathcal{G}_l=m\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l$, where m is the Lie algebra of the Mumford-Tate group $MT(A(\boldsymbol{C}))$ (cf. [4], § 3).

As is now well known, in general $\mathcal{G}_l$ is contained in $m\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l$ (cf. [4]). Various results toward this general problem have been obtained for some classes of abelian varieties. Moreover, some important general result on l-adic representations attached to abelian varieties have been established. Especially, one has the following results:

(i) the rank of $\mathcal{G}_l$ is independent of l (Serre [13], Zarhin [18]).

(ii) $\mathcal{G}_l$ is algebraic and contains the homotheties (Bogomolov [1]).

(iii) $\mathcal{G}_l$ is reductive and $\mathrm{End}_{\mathcal{G}_l}(V_l(A))=\mathrm{End}_K(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l$ (Faltings [5]).

These make the determination of $\mathcal{G}_l$ possible for some other classes of abelian varieties. For example, when $d=\dim A$ is odd and $\mathrm{End}_{\bar{K}}(A)=\boldsymbol{Z}$, Serre has proved that $\mathcal{G}_l=m\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l\simeq sp(2d,\boldsymbol{Q}_l)\oplus\boldsymbol{Q}_l\cdot\mathrm{id.}$, where id. is the $2d\times 2d$ identity matrix (cf. [11]).

The purpose of this article is to determine $\mathcal{G}_l$ for the following type II absolutely simple abelian varieties: $\dim A=2d$, where $d=1,2$ or an odd number, and $\mathrm{End}_{\bar{K}}(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=D$ is an indefinite quaternion algebra over $\boldsymbol{Q}$. It follows that we have the equality $\mathcal{G}_l=m\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l$. Accordingly, the well-known conjectures (Hodge, Tate) on algebraic cycles (cf. [9], [17]) are true for such abelian varieties.

The key idea is to use theorem A in [3], then the situation can be treated as the case where $\mathrm{End}_{\bar{K}}(A)=\boldsymbol{Z}$ with $\dim A$ is 2 or odd, and hence Serre's method applies.

Finally, I would like to thank the referee for many useful suggestions.

§ 2. Preliminaries

In this section, we recall some results which will be used later. For most, we simply give a reference to the proof. Nevertheless, proofs for some variants of known results are provided.

2.1 Abelian varieties over finite fields

Let F_q be a finite field of q elements, where $q=p^a$ for some prime number p and some positive integer a.

We first recall two theorems of Tate on abelian varieties defined over finite fields.

THEOREM 2.1.A (Tate). *Let A be an abelian variety of dimension g defined over F_q. Let π be the Frobenius endomorphism of A relative to F_q and $p(t)$ its characteristic polynomial. One has the following statements:*

(a) *The algebra $F=\boldsymbol{Q}[\pi]$ is the center of the semisimple algebra $E=\mathrm{End}_{F_q}(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$, and $2g\leq[E:\boldsymbol{Q}]\leq(2g)^2$.*

(b) *The following statements are equivalent:*

(b1) $[E:\boldsymbol{Q}]=2g$

(b2) *$p(t)$ has no multiple roots*

(b3) $E=F$

(b4) *E is commutative*

(c) *The following statements are equivalent:*

(c1) $[E:\boldsymbol{Q}]=(2g)^2$

(c2) *$p(t)$ is a power of a linear polynomial*

(c3) $F=\boldsymbol{Q}$

(c4) *$E\simeq M(g,D_p)$, where D_p is the division quaternion algebra over $\boldsymbol{Q}$ which splits at all primes $l\neq p,\infty$.*

(c5) *A is F_q-isogenous to the g-th power of a supersingular elliptic curve, all of whose endomorphisms are defined over F_q.*

(d) *A is F_q-isogenous to a power of a F_q-simple abelian variety if and only if $p(t)$ is a power of a $\boldsymbol{Q}$-irreducible polynomial.*

When this is the case, E is a central simple algebra over F which splits at all finite primes w of F not dividing p, but does not split at any real prime of F.

PROOF. Cf. Tate [15], pp. 140, Theorem 2.

THEOREM 2.1.B (Tate). *Let A be F_q-simple, w be a place of F, and $\| \|_w$ be the normalized absolute value for w. If $\|\pi_A\|_w=q^{-i}$, then i is the invariant of the division algebra E at w. Explicitly, this is*

$$i=\begin{cases} 1/2 & , \quad \textit{if } w \textit{ is a complex place} \\ 1/2 & , \quad \textit{if } w \textit{ is a real place} \\ 0 & , \quad \textit{if } w \nmid p \\ \dfrac{\mathrm{ord}_w(\pi_A)}{\mathrm{ord}_w(q)}\cdot[F_w : Q_p], & \quad \textit{if } w \mid p. \end{cases}$$

PROOF. Cf. [6], Theorem 8.

REMARK. Suppose $q=p$ and A is F_p-simple. Then E splits at all finite places of F (cf. Tate [16], pp. 352-02, Theorem 1). Thus, $E=F$ and $\mathrm{End}_{F_p}(A)$ is commutative. Under this situation, one can apply Theorem 2.1.A-(b).

2.2 Representations defined by minuscule weights

Let $\mathcal{G}$ be a semisimple Lie algebra over an algebraically closed field of characteristic 0, and $\mathcal{G}=\mathcal{G}_1\times\cdots\times\mathcal{G}_t$ be the decomposition of $\mathcal{G}$ into the product of its simple ideals.

For any faithful irreducible representation V of $\mathcal{G}$, V decomposes as a tensor product of irreducible representations V_i of $\mathcal{G}_i$. Since V is faithful, none of the V_i's is trivial. Moreover, if the representation V admits a non-degenerate invariant bilinear form, then so does each V_i.

Further, if V is symplectic (resp. orthogonal), then the number of factors V_i such that V_i is symplectic, is odd (resp. even). We say that the representation V is defined by *minuscule weights* if the highest weight of each V_i is a minuscule weight in the sense of Bourbaki [2], Ch. VIII, § 7.

It is known (loc. cit., Proposition 8) that the minuscule weights of simple Lie algebras occur only among the fundamental weights. For the complete list of minuscule weights of simple Lie algebras, we refer to Bourbaki [2], Ch. VIII, pp. 129.

The following results follows easily from the nice information provided by Bourbaki [2], Ch. VIII, § 13, Table 1, 2 of pp. 213–214.

LEMMA (2.2.1). *Let $\mathcal{G}$ be a simple Lie algebra of type A_l ($l \geq 2$), and let V be an irreducible representation of $\mathcal{G}$ defined by minuscule weights. If V is symplectic, then* $\dim V$ *is divisible by* 4.

PROOF. From Table 1 and 2 of Bourbaki [2], pp. 213–214; one knows that the fundamental weight w_r ($1 \leq r \leq l$) defines a symplectic representation if and only if r is odd and $2r = l+1$. Let $r = 2k+1$ for some integer $k \geq 0$. Then the dimension of the irreducible representation with highest weight w_r is $\binom{4k+2}{2k+1}$.

It is easy to check that $\operatorname{ord}_2\binom{4k}{2k} \geq 1$. Hence $\operatorname{ord}_2\binom{4k+2}{2k+1} \geq 2$.

LEMMA (2.2.2). *Let V be a faithful, irreducible, symplectic representation of a semisimple Lie algebra $\mathcal{G}$ over an algebraically closed field of characteristic* 0. *If* $\dim V = 2d$ *with d odd, and V is defined by minuscule weights, then $\mathcal{G}$ is isomorphic to the simple Lie algebra of type C_d and V is the standard symplectic representation of C_d.*

PROOF. This follows immediately from the list of minuscule weights, the dimensions of their associated irreducible representations, and Lemma (2.2.1).

2.3 Theorem of Sen (cf. [10], § 6; [12], § 1)

Let K be a complete field of characteristic 0 with respect to a discrete valuation, whose residue field k is algebraically closed and of characteristic $p > 0$. Let C be the completion of an algebraic closure $\bar{K}$ of K. The Galois group H_K of $\bar{K}$ over K acts continuously on $\bar{K}$. This action extends continuously to C.

Let V be a Hodge-Tate module over K (cf. [12], § 1), and H_V be the algebraic envelope of H, where H is the image of H_K in $\operatorname{Aut}(V)$. Let $h_V : G_{m/C} \to GL_{V/C}$ be the one-parameter subgroup of GL_V over C defined by the Hodge-Tate decomposition of $V \underset{Q_p}{\otimes} C$(loc. cit.).

THEOREM (Sen). (a) *H is open in $H_V(Q_p)$. Equivalently,* $\operatorname{Lie}(H) = \operatorname{Lie}(H_V)$, *so that* $\operatorname{Lie}(H)$ *is algebraic.*

(b) *The connected component of the identity H_V° of H_V is the smallest algebraic subgroup of GL_V, defined over Q_p, which after extension of scalars to C contains the image of h_V.*

2.4 Theorem of Serre

Let M be a connected reductive algebraic group over a field E of characteristic 0. Let E' be a finite Galois extension of E such that M is E'-split. Choose a maximal torus $\underline{T}$ of $M_{/E'}$ and a Borel subgroup $\underline{B}$ of $M_{/E'}$ containing $\underline{T}$. Then, we have the associated system $(X, Y, R, \alpha\mapsto\alpha, B)$, where $X=\mathrm{Hom}(\underline{T}, G_{m/E'})$, $Y=\mathrm{Hom}(G_{m/E'}, \underline{T})$, R is the root system of $M_{/E'}$ relative to $\underline{T}$, B is the basis of R associated to $\underline{B}$.

Assume that we are given the following data along with M:

(a) a linear representation V of M defined over E.

(b) an algebraically closed field C containing E'.

(c) an one-parameter subgroup $h_M: G_{m/C}\to M_{/C}$ of $M_{/C}$ defined over C.

THEOREM (Serre [12, § 3]). *Suppose that the triple (V, C, h_M) satisfies the following conditions*:

(i) *V is faithful.*

(ii) *If N is a normal algebraic subgroup of M, defined over E, such that $N_{/C}$ contains* $\mathrm{Im}\, h_M$, *then $N=M$.*

(iii) *The action of $G_{m/C}$ on $V_C=C\bigotimes_E V$ defined by h_M is of weights* 0 *and* 1.

Then, the representation V is defined by minuscule weights.

§ 3. Review of l-adic representations attached to abelian varieties over number fields

Throughout this section, we shall adopt the following notations

$\boldsymbol{R}$: the field of real numbers	l: a prime number
$\boldsymbol{C}$: the field of complex numbers	$\boldsymbol{Q}_l$: the field of l-adic numbers
K: a number field	$\bar{\boldsymbol{Q}}_l$: an algebraic closure of $\boldsymbol{Q}_l$
$\bar{K}$: an algebraic closure of K	C_l: the completion of $\bar{\boldsymbol{Q}}_l$
G: $\mathrm{Gal}(\bar{K}/K)$	S_l: $\{v\in\Sigma_K \mid v\mid l\}$
G_m: the multiplicative group GL_1	
Σ_K: the set of all finite places of K	
χ_l: $G\to\boldsymbol{Z}_l^*$ the cyclotomic character	

In the following, A will be an abelian variety defined over K of dimension $n\geq1$. We denote by V_l the $\boldsymbol{Q}_l$-adic Tate module $T_l(A)\bigotimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l$ of A. Let $\rho_l: G\to\mathrm{Aut}(V_l)$ be the associated l-adic representation. The group $G_l=\mathrm{Im}(\rho_l)$ is a closed subgroup of $\mathrm{Aut}(V_l)$, hence an l-adic Lie group. Let $\mathcal{G}_l$ be the Lie algebra of G_l. Then $\mathcal{G}_l$ is easily seen to be invariant under finite extensions of the number field K.

Let G_V be the algebraic envelope of G_l. By replacing K by a finite extension, we may assume that $\mathrm{End}_{\bar{K}}(A)=\mathrm{End}_K(A)$. Let $V_l(\mu)$ be the 1-dimensional $\boldsymbol{Q}_l$-vector space $T_l(\mu)\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l$, where $T_l(\mu)=\varprojlim_n \mu_{l^n}$ and μ_{l^n} is the group of l^n-th roots of unity. Fix a K-polarization on A once and for all. We denote by $\psi: V_l\times V_l\to V_l(\mu)$ the induced Riemann form on V_l, and by $'$ the corresponding Rosati involution on $\mathrm{End}_K(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ (cf. [8], § 20).

3.1 Some fundamental properties

Since the Galois action on V_l is equivariant with respect to ψ (cf. [8], § 20), we have $\psi(\sigma v, \sigma w)=\chi_l(\sigma)\psi(v, w)$, for all $\sigma\in G$; $v, w\in V_l$. Thus, $G_V\subset GSp(V_l, \psi)$, the group of symplectic similitudes with respect to the alternating form ψ. On the other hand, we have $\psi(ev, w)=\psi(v, e'w)$ for all $v, w\in V_l$, $e\in \mathrm{End}_K(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$.

By a theorem of Faltings ([5], § 5, Satz 3), $\mathcal{G}_l$ is reductive. Let $\mathcal{G}_l=[\mathcal{G}_l, \mathcal{G}_l]\oplus\mathcal{C}_l$, where $[\mathcal{G}_l, \mathcal{G}_l]$ is the derived subalgebra of $\mathcal{G}_l$, and $\mathcal{C}_l$ is the center of $\mathcal{G}_l$. Let F be the center of the semisimple algebra $\mathrm{End}_K(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$. Since the commutant of $\mathcal{G}_l$ in $\mathrm{End}(V_l)$ is $\mathrm{End}_K(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}_l$ ([5], § 5, Satz 4), $\mathcal{C}_l$ is contained in $F_l=F\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l$. Let $A\sim\prod A_j^{m_j}$ be the decomposition of A into product of simple abelian varieties up to isogeny. Then, according to Albert's classification of involutional division algebras (cf. [8], § 21, Theorem 2), we have $F=F_R\times F_I$, where F_R is a product of totally real fields, and F_I is a product of CM-fields.

To fix notation, let $\boldsymbol{Q}_l\cdot\mathrm{id}$ be the algebra of homotheties in the general linear Lie algebra $gl(V_l)$.

PROPOSITION (3.1.1). *$\mathcal{C}_l$ is contained in* $(F_I\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l)+\boldsymbol{Q}_l\cdot\mathrm{id}$.

PROOF. Let $\Theta: G_l\to\boldsymbol{Z}_l^*$ be the determinant map. The Lie algebra $\mathcal{G}_l^\circ$ of $\ker\Theta$ is an ideal of $\mathcal{G}_l$ containing $[\mathcal{G}_l, \mathcal{G}_l]$ and of codimension 1 in $\mathcal{G}_l$. In particular, $\mathcal{G}_l^\circ$ is again reductive. Let $\mathcal{G}_l^\circ=[\mathcal{G}_l^\circ, \mathcal{G}_l^\circ]\oplus\mathcal{C}_0$, where $\mathcal{C}_0$ is the center of $\mathcal{G}_l^\circ$.

It is clear that $[\mathcal{G}_l, \mathcal{G}_l]=[\mathcal{G}_l^\circ, \mathcal{G}_l^\circ]$ and $\mathcal{C}_0$ is a subspace of $\mathcal{C}_l$ of codimension 1. By the theorem of Bogomolov ([1], Corollary 1), we have $\mathcal{C}_l=\mathcal{C}_0+\boldsymbol{Q}_l\cdot\mathrm{id}$.

Since $\mathcal{C}_0$ is contained in the symplectic Lie algebra $sp(V_l, \psi)$, we have $\psi(cv, w)+\psi(v, cw)=0$, for all v, w in V_l, c in $\mathcal{C}_0$. Therefore, $c'+c=0$. The Rosati involution is trivial on F_R. Hence $\mathcal{C}_0 \subset F_I \underset{Q}{\otimes} Q_l$.

COROLLARY (3.1.2). *If* $A \sim \prod A_j^{m_j}$, *and none of the* A_j*'s is of type* (IV), *then* $\mathcal{G}_l=[\mathcal{G}_l, \mathcal{G}_l] \oplus Q_l \cdot \mathrm{id}$.

Let S_V be the connected component of the identity of $G_V \cap SL_{V_l}$. Then S_V is a connected semisimple algebraic group defined over Q_l. If G_V is connected, then the commutator group $[G_V, G_V]$ is closed and connected. In particular, $S_V=[G_V, G_V]$. Further, if A has no factor (up to isogeny) of type (IV), we have $G_V=S_V \cdot G_m$. In fact, after replacing the base field K by a finite extension, one may assume that G_V is connected (cf. 3.3).

3.2 The Hodge-Tate module V_l

For each $v \in S_l$, let $\bar{K}_v$ be the algebraic closure of K_v in C_l. As a $\mathrm{Gal}(\bar{K}_v/K_v)$-module, it is well-known that V_l is a Hodge-Tate module of weights 0 and 1, each of them with multiplicity $\dim A$ (cf. [12], pp. 157, Raynaud-Tate theorem). We denote by $V_l \underset{Q_l}{\otimes} C_l = V_{C_l}(0) \oplus V_{C_l}(1)$ the Hodge-Tate decomposition. For each $v \in S_l$, let $\bar{v}$ be an extension of v to $\bar{K}$. Then the local Galois group $\mathrm{Gal}(\bar{K}_v/K_v)$ can be identified with the decomposition group $D_{\bar{v}}$ for $\bar{v}$ in $\mathrm{Gal}(\bar{K}/K)$. Let $I_{\bar{v}}$ be the inertia subgroup of $D_{\bar{v}}$. Then the algebraic envelope of $\rho_l(I_{\bar{v}})$ is an algebraic subgroup of G_V. By the theorem of Sen (§ 2.3), the one-parameter subgroup h_V of GL_{V/C_l} defined by

$$h_V(c)(x)=\begin{cases} x, & \text{if } x \in V_{C_l}(0) \\ cx, & \text{if } x \in V_{C_l}(1) \end{cases}$$

maps G_{m/C_l} into the algebraic envelope of $\rho_l(I_{\bar{v}})$ over C_l. Thus, h_V is an one-parameter subgroup of G_V defined over C_l.

Recall that a connected algebraic group defined over Q_l is called almost Q_l-simple if it has no proper infinite normal algebraic subgroup defined over Q_l.

By Corollary (3.1.2), if A has no factor (up to isogeny) of type (IV), then $\mathcal{G}_l=[\mathcal{G}_l, \mathcal{G}_l] \oplus Q_l \cdot \mathrm{id}$. In this situation, combining the theorems of Serre and Sen, we have the following:

PROPOSITION (3.2). *If* S_V *is an almost* Q_l*-simple algebraic group, then the representation* V_l *of* $[\mathcal{G}_l, \mathcal{G}_l]$ *is defined by minuscule weights.*

PROOF. Consider the triple (V_l, C_l, h_V) associated to G_V. Since S_V is almost $\boldsymbol{Q}_l$-simple, the condition (ii) of Serre's theorem holds. By Sen's theorem, h_V satisfies condition (iii). Therefore, the faithful representation V_l of $[\mathcal{G}_l, \mathcal{G}_l]$ is defined by minuscule weights.

3.3 The rank of G_V

Let S be the set of all places of K where A has bad reduction. As is well known, ρ_l is unramified outside of $S\cup S_l$. For each $v\in\Sigma_K-(S\cup S_l)$, let F_v be the conjugacy class of the Frobenius element $F_{\bar{v}}$, where $\bar{v}$ is any extension of v to $\bar{K}$. As stressed by Taniyama, the characteristic polynomial of $\rho_l(F_{\bar{v}})$ coincides with the characteristic polynomial of the Frobenius endomorphism π_v of the reduction A_v of A at v.

Let $H_{\bar{v}}$ be the Zariski closure of the set $\{\rho_l(F_{\bar{v}})^n \mid n\in\boldsymbol{Z}\}$. As $\rho_l(F_{\bar{v}})$ is semisimple, $H_{\bar{v}}\subset G_V$ is an algebraic group of multiplicative type (not necessarily connected). The character group of $H_{\bar{v}}$ is the subgroup Γ_v of $\bar{\boldsymbol{Q}}_l^*$ generated by the eigenvalues of $\rho_l(F_{\bar{v}})$ (or the Frobenius endomorphism π_v).

For the rest of this paper, we shall assume that $A(K)$ contains all the l-division points (if $l\neq 2$) or 4-division points (if $l=2$). Then, the eigenvalues of each element in G_l are congruent to 1 modulo l or 4 respectively. Recall the following elementary fact: Suppose $\alpha\in\bar{\boldsymbol{Q}}_l^*$ is a root of unity and $\alpha-1$ is divisible by l for $l\neq 2$ (resp. 4 for $l=2$). Then $\alpha=1$. Thus, for each $v\in\Sigma_K-(S\cup S_l)$, the subgroup Γ_v of $\bar{\boldsymbol{Q}}_l^*$ is torsion-free. In particular, each $H_{\bar{v}}$ is a torus. By the Cebotarev's density theorem, G_V is generated by the family of its subtori $\{H_{\bar{v}} \mid v\in\Sigma_K-(S\cup S_l)\}$. Hence G_V is connected.

In general, $\operatorname{rank}\Gamma_v\leq\operatorname{rank}G_V$. Furthermore, one has the following nice result due to Serre.

THEOREM (Serre). *The set* $\{v\in\Sigma_K-(S\cup S_l) \mid \operatorname{rank}\Gamma_v=\operatorname{rank}G_V\}$ *is of density* 1.

PROOF. Cf. [13].

3.4 The Frobenius trace

Let t_v be the trace of $\rho_l(F_v)$. The following result is useful for our later discussion.

LEMMA (3.4). *After replacing K by some finite extension, the set* $\{v\in\Sigma_K-(S_A\cup S_l) \mid Nv=p_v$ *is a prime number and* $p_v\nmid t_v\}$ *is of positive* *density.*

PROOF. Let $K'=K(A[l^n])$ be the subfield of $\bar{K}$ generated by K and the l^n-torsion points of A. Denote by $G_{K'}$ the Galois group of $\bar{K}$ over K'.

Recall that $\underline{P}=\{v\in\Sigma_K-(S_A\cup S_l)\mid Nv=p_v$ is a prime number$\}$ is of positive density. On the other hand, by the definition of ρ_l, one sees easily that $\rho_l|_{G_{K'}}$ is trivial modulo l^n. So, for each $v\in\underline{P}$, one has $t_v\equiv 2\dim A \pmod{l^n}$ and $|t_v|<2\dim A\cdot\sqrt{p_v}$.

Let n be chosen to be such that $l^n>2\dim A$. Suppose $t_v=p_v\cdot m$ for some integer m. Then $|t_v|=p_v|m|<2\dim A\cdot p_v^{1/2}$. Hence $|m|<2\dim A\cdot p_v^{-1/2}$. If $p_v>(2\dim A)^2$, then m$=0$. This contradicts $t_v\equiv 2\dim A \pmod{l^n}$. Our assertion follows.

§ 4. Determine $\mathcal{G}_l$ for some absolutely simple abelian varieties of type II

Throughout this section, let A be a type II absolutely simple abelian variety as in § 1. Namely, $\dim A=2d$ $(d\geq 1)$ and its endomorphism algebra D is an indefinite quaternion algebra over $\boldsymbol{Q}$.

By Theorem A of [3], one has the following result.

Suppose l is a prime at which D splits. Then there exists a G-submodule W_l of V_l such that the following hold:

(i) $\dim W_l=2d$.

(ii) The Galois module V_l is isomorphic, over $\boldsymbol{Q}_l$, to the direct sum of two copies of W_l.

(iii) $\psi_{|W_l\times W_l}$ is a non-degenerate S_V-equivariant alternating form.

Thus, the representation W_l of the semisimple Lie algebra $[\mathcal{G}_l,\mathcal{G}_l]$ is faithful, symplectic, and absolutely irreducible of dimension $2d$ over $\boldsymbol{Q}_l$.

Now, let $v\in\Sigma_K-S_l$ be a place of K where A has good reduction. We retain the same notations as in §§ 2 and 3. Let $P_{v,l}(T)\in\boldsymbol{Z}[T]$ be the characteristic polynomial of $\rho_l(F_v)$. Then $P_{v,l}(T)$ equals the characteristic polynomial of the Frobenius endomorphism π_v of the reduction A_v of A at v. In our situation, we have $P_{v,l}(T)=f(T)^2$, where

$$f(T)=\det(1-T\cdot\rho_l(F_v)\mid W_l),\ f(T)\in\boldsymbol{Z}[T],\ \text{and degree } f(T)=2d.$$

For $d=1$, $[\mathcal{G}_l,\mathcal{G}_l]$ is of rank 1. Hence $[\mathcal{G}_l,\mathcal{G}_l]\simeq sl(2,\boldsymbol{Q}_l)$, and we have $\mathcal{G}_l\simeq sl(2,\boldsymbol{Q}_l)\oplus\boldsymbol{Q}_l\cdot\mathrm{id}$. For $d=2$, $[\mathcal{G}_l,\mathcal{G}_l]$ is either of rank 1 or 2. If $[\mathcal{G}_l,\mathcal{G}_l]$

is of rank 2, then it must be one of the following Lie types: $A_1 \times A_1$, A_2, C_2, or G_2. In this situation, $[\mathcal{G}_l, \mathcal{G}_l]$ admits a faithful, symplectic, and absolutely irreducible representation of dimension 4. By the Weyl's character formula, it is easy to eliminate the cases $A_1 \times A_1$, A_2, and G_2. Thus, we have either $[\mathcal{G}_l, \mathcal{G}_l] \simeq sl(2, \boldsymbol{Q}_l)$ or $[\mathcal{G}_l, \mathcal{G}_l] \simeq sp(4, \boldsymbol{Q}_l)$. Furthermore, we have the following result, which is a simple variant of Zarhin ([19], Theorem 0.3.1):

LEMMA (4.1). *If* $\mathrm{Lie}(S_V(\boldsymbol{Q}_l)) \simeq sl(2, \boldsymbol{Q}_l)$, *then* $d=1$.

PROOF. Note that $W_l \underset{\boldsymbol{Q}_l}{\otimes} \bar{\boldsymbol{Q}}_l$ is an irreducible $2d$-dimensional symplectic representation of $sl(2, \bar{\boldsymbol{Q}}_l)$. So $W_l \underset{\boldsymbol{Q}_l}{\otimes} \bar{\boldsymbol{Q}}_l$ must be the $(2d-1)^{\text{th}}$ symmetric power of the fundamental representation of $sl(2, \bar{\boldsymbol{Q}}_l)$.

By the theorem of Serre in § 3.3, we can choose a place $v \in \Sigma_K - S_l$ such that (i) A has good reduction A_v at v (ii) $\mathrm{rank}(\Gamma_v) = \mathrm{rank}(G_V) = 2$. Let $Nv = q = p^m$, and let $\bar{\boldsymbol{Q}}$ be the algebraic closure of $\boldsymbol{Q}$ in $\bar{\boldsymbol{Q}}_l$. Put $u = q^{-1}\rho_l(F_{\bar{v}})^2$. Then $\det u = 1$. Since S_V is of finite index in $G_V \cap SL_{V_l}$, after replacing u by a power of u, we may assume that $u \in S_V(\boldsymbol{Q}_l)$. Let $\tilde{f}(T) = \det(1 - T \cdot u | W_l)$. Then $\tilde{f}(T) = q^{-2d} h(qT)$, where $h(T) = \det(1 - T \cdot \rho_l(F_{\bar{v}})^2 | W_l)$. Since π_v^2 is the Frobenius endomorphism of $A_v \underset{\boldsymbol{F}_q}{\otimes} \boldsymbol{F}_{q^2}$, we have $h(T) \in \boldsymbol{Z}[T]$. Consequently, $\tilde{f}(T) \in \boldsymbol{Q}[T]$.

By our hypothesis, there exists $\alpha \in \bar{\boldsymbol{Q}}^*$ which is not a root of unity, such that $\Delta = \{\alpha^{2d-1}, \alpha^{2d-3}, \cdots, \alpha, \alpha^{-1}, \alpha^{-3}, \cdots, (\alpha^{-1})^{2d-1}\}$ is the complete set of roots of $\tilde{f}(T)$. Since $\tilde{f}(T) \in \boldsymbol{Q}[T]$, Δ is a $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$-set. In fact, we have the following:

SUBLEMMA. *For each* $i, i=1, 3, \cdots, 2d-1$, *the subset* $\{\alpha^i, \alpha^{-i}\}$ *of* Δ *is* $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$*-stable.*

PROOF. Let α^j be a conjugate of α over $\boldsymbol{Q}$. Then $\alpha^{j^2}, \alpha^{j^3}, \cdots$ are conjugates of α over $\boldsymbol{Q}$. Since α is an algebraic number which is not a root of unity, the set $\{j, j^2, j^3, \cdots\}$ must be finite. Therefore, $j=1$ or -1. This implies that $\{\alpha^i, \alpha^{-i}\}$ is $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$-stable.

Now, by the preceding sublemma, $A_v \underset{\boldsymbol{F}_q}{\otimes} \boldsymbol{F}_{q^2}$ is $\boldsymbol{F}_{q^2}$-isogenous to a product of elliptic curves over $\boldsymbol{F}_{q^2}$ (cf. [19], Lemma 1.0.3). Since $\mathrm{rank}(\Gamma_v) = 2$, these elliptic curves must be ordinary. Let w be a place of $\boldsymbol{Q}(\alpha)$ dividing p. By the theorem of Tate (Theorem 2.1.B), we have

$$\mathrm{ord}_w(q\alpha^{2i-1})/\mathrm{ord}_w(q^2) = 0 \quad \text{or} \quad 1 \quad (i=1, \cdots, d).$$

Thus, $\mathrm{ord}_w(\alpha^{2i-1})/\mathrm{ord}_w(q^2)=1/2$ or $-1/2$ $(i=1, \cdots, d)$. In particular, $\mathrm{ord}_w(\alpha^{2d-1})=\pm\mathrm{ord}_w(\alpha)$. If $\mathrm{ord}_w(\alpha)=0$, then α is a root of unity. Hence we have $d=1$.

COROLLARY (4.2). *If* $d=2$, *then* $\mathcal{G}_l \simeq sp(4, \boldsymbol{Q}_l)\oplus \boldsymbol{Q}_l\cdot\mathrm{id}$.

PROOF. This follows immediately from Lemma (4.1).

For the remainder of this section, we assume that d is odd and that $d\geq 3$. The key step in determining $\mathcal{G}_l$ is the following result:

THEOREM (4.3). *The connected semisimple algebraic group S_V is almost $\boldsymbol{Q}_l$-simple.*

To prove this theorem, it suffices to show that the Lie algebra $\mathrm{Lie}(S_V)=[\mathcal{G}_l, \mathcal{G}_l]$ is simple. In order to apply classical representation theory (over an algebraically closed field of characteristic 0), we extend the base field from $\boldsymbol{Q}_l$ to C_l. Let $W=W_l \otimes_{\boldsymbol{Q}_l} C_l$; $S=S_{V/C_l}$; and $\mathcal{G}=\mathrm{Lie}(S)$. Then W is a faithful, symplectic, irreducible representation of S and $\mathcal{G}$ (cf. [3], Theorem A).

Suppose that $\mathcal{G}$ is not simple. We may assume that $\mathcal{G}=\mathcal{G}_1\times\mathcal{G}_2$ is the product of a simple Lie algebra $\mathcal{G}_1$ and a semisimple Lie algebra $\mathcal{G}_2$, such that $W=W_1\otimes W_2$, where W_1 (resp. W_2) is a faithful, symplectic (resp. orthogonal), irreducible representation of $\mathcal{G}_1$ (resp. $\mathcal{G}_2$). Let $\dim W_1=2s$, and $\dim W_2=t$. Then $st=d$, and s, t are odd numbers. Let $\tilde{S}, \tilde{S}_1, \tilde{S}_2$ be the connected, simply connected algebraic groups with Lie algebras $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ respectively. Thus, we have $\tilde{S}=\tilde{S}_1\times\tilde{S}_2$. The representations W, W_1, W_2 of $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ integrate to representations ρ, ρ_1, ρ_2 of $\tilde{S}, \tilde{S}_1, \tilde{S}_2$ on W, W_1, W_2 respectively. Further, we have $\rho=\rho_1\otimes\rho_2$ on $\tilde{S}=\tilde{S}_1\times\tilde{S}_2$.

LEMMA (4.4). *Under the preceding situation,* $\mathrm{Ker}(\rho)=\mathrm{Ker}(\rho_1)\times\mathrm{Ker}(\rho_2)$ *in* $\tilde{S}=\tilde{S}_1\times\tilde{S}_2$.

PROOF. Let Z_1 (resp. Z_2) be the center of $\tilde{S}_1$ (resp. $\tilde{S}_2$). Then $Z=Z_1\times Z_2$ is the center of $\tilde{S}$. Since $\mathrm{Lie}(\rho)$, $\mathrm{Lie}(\rho_1)$, $\mathrm{Lie}(\rho_2)$ are faithful, the kernels of ρ, ρ_1, ρ_2, lie in Z, Z_1, Z_2 respectively. The restriction of ρ (resp. ρ_1, ρ_2) to Z (resp. Z_1, Z_2) is a character χ (resp. χ_1, χ_2). We have

$$\chi(z_1, z_2)=\chi_1(z_1)\chi_2(z_2) \qquad \text{for all} \quad z_1\in Z_1,\ z_2\in Z_2.$$

Because ρ_1 (resp. ρ_2) is a symplectic (resp. an orthogonal) irreducible representation of $\tilde{S}_1$ (resp. $\tilde{S}_2$), we have $\rho_1(Z_1)\subset Z(Sp(W_1))=\{\pm 1\}$, and $\rho_2(Z_2)\subset Z(So(W_2))=\{1\}$ (since dim W_2 is odd). Hence χ_2 is trivial. Thus, $\chi_1(z_1)\chi_2(z_2)=1$ if and only if $\chi_1(z_1)=\chi_2(z_2)=1$.

Now, let $\rho: S\rightarrow GL(W)$ be the faithful, symplectic, irreducible representation of S, which was obtained (by extension of scalars) from the original l-adic representation $\rho_l: S_V\rightarrow GL(W_l)$ over $\boldsymbol{Q}_l$.

LEMMA (4.5). *If $\mathcal{G}=\mathrm{Lie}(S)$ is not simple, then we can decompose S as $S_1\times S_2$, where S_1 is a connected, almost simple algebraic group; and S_2 is a nontrivial, connected semisimple algebraic group. Moreover, $\rho=\rho_1\otimes\rho_2$ on $W=W_1\otimes W_2$, where ρ_1 (resp. ρ_2) is a faithful, symplectic (resp. orthogonal), irreducible representation of S_1 (resp. S_2).*

PROOF. In the situation of Lemma (4.4), we have $S=\tilde{S}/\mathrm{Ker}(\rho)$. If we take $S_i=\tilde{S}_i/\mathrm{ker}(\rho_i)$, $i=1,2$; then our assertion follows immediately.

LEMMA (4.6). *If $\mathcal{G}=\mathrm{Lie}(S)$ is not simple, then there exist integers $s\geq 1$, $t\geq 3$ with the following properties:*

(i) *$st=d$, and hence s, t are odd integers.*

(ii) *$\mathcal{G}\simeq sp(2s, C_l)\times\mathcal{H}$, where $sp(2s, C_l)$ is the simple Lie algebra of type C_s, and $\mathcal{H}$ is a nontrivial semisimple Lie algebra.*

(iii) *The representation W of $\mathcal{G}$ is the tensor product $W_1\otimes W_2$, where W_1 is the standard representation of $sp(2s, C_l)$, and W_2 is an irreducible orthogonal representation of $\mathcal{H}$ of dimension t (odd).*

PROOF. Suppose $\mathcal{G}=\mathrm{Lie}(S)$ in not simple. By Lemma (4.5), we can write $S=S_1\times S_2$ (S_1 is almost simple), and $W=W_1\otimes W_2$. It suffices to show that $\mathrm{Lie}(S_1)\simeq sp(2s, C_l)$, where $2s=\dim W_1$.

Note that W has Hodge-Tate decomposition $W=W_{C_l}(0)\oplus W_{C_l}(1)$ with $\dim W_{C_l}(0)=\dim W_{C_l}(1)=d$ (cf. [3], Lemma (3.2) and Theorem A). By Sen's local theory, one has an one-parameter subgroup $h: G_m\rightarrow G_m\cdot S$ defined as follows:

$$h(c)(x)=\begin{cases} x, & \text{if } x\in W_{C_l}(0)\\ cx, & \text{if } x\in W_{C_l}(1).\end{cases}$$

Let $\phi: G_m\rightarrow GL(W)$ be the group homomorphism defined by

$$\phi(c)(x)=\begin{cases} cx, & \text{if } x\in W_{C_l}(0)\\ c^{-1}x, & \text{if } x\in W_{C_l}(1).\end{cases}$$

The image Φ of ϕ is contained in $(G_m\cdot\mathrm{Im}\, h)\cap SL(W)$, and hence is a subgroup of $S=S_1\times S_2$. Let p_i be the projection of S onto S_i, where $i=1,2$. Then $\phi_i=p_i\phi$ is an one-parameter subgroup of S_i, where $i=1,2$. We have the following result:

SUBLEMMA. *The G_m-action on W_1 defined by ϕ_1 is of weights -1 and* 1.

PROOF. Since the S-module W is isomorphic to the $S_1\times S_2$-module $W_1\otimes W_2$, we have $\phi(c)(w_1\otimes w_2)=\phi_1(c)(w_1)\otimes\phi_2(c)(w_2)$, for all $c\in G_m$, $w_1\in W_1$, $w_2\in W_2$. W_2 is an orthogonal representation of S_2 of odd dimension. Hence 1 is an eigenvalue of $\phi_2(c)\in GL(W_2)$, for all $c\in G_m$. By the definition of ϕ, both of c and c^{-1} are the eigenvalues (with certain multiplicities) of $\phi_1(c)\in GL(W_1)$. Our assertion follows.

Now, consider the connected reductive algebraic subgroup $M=G_m\cdot S_1$ of $GL(W_1)$, where $G_m\subset GL(W_1)$ is the group of homotheties of W_1. W_1 is a faithful representation of M. On the other hand, by the preceding sublemma, $G_m\cdot\mathrm{Im}\,\phi_1\subset M/_{C_l}$ contains an 1-dimensional torus of weights 0 and 1. Thus, there exists an one-parameter subgroup ψ of M, such that the G_m-action on W_1 which defined by ψ is of weights 0 and 1.

Since S_1 is almost simple and $M=G_m\cdot S_1$, the triple (W_1, C_l, ψ) attached to M satisfies the conditions (i), (ii), (iii), of the theorem of Serre (cf. § 2.4). Therefore, W_1 is defined by minuscule weights. Moreover, W_1 is a faithful, symplectic, irreducible representation of dimension $2s$, where s is odd. By passing to Lie algebras, our assertions follow from Lemma (2.2.2). This completes the proof of Lemma (4.6).

Now, we are ready to prove Theorem (4.3). The method essentially follows the line of Serre's method in [11].

PROOF. Suppose that S_V is not almost $\boldsymbol{Q}_l$-simple. In the notations of Lemma (4.6), the Lie algebra $\mathcal{G}_l\underset{\boldsymbol{Q}_l}{\otimes}C_l$ is isomorphic to

$$(sp(2s, C_l)\oplus C_l\cdot\mathrm{id})\times\mathcal{H}.$$

Further, the representation $W_l\underset{\boldsymbol{Q}_l}{\otimes}C_l$ of $\mathcal{G}_l\underset{\boldsymbol{Q}_l}{\otimes}C_l$ is the tensor product $W_1\otimes W_2$, where W_1 is the standard representation of $sp(2s, C_l)$, and W_2 is an irreducible orthogonal representation of $\mathcal{H}$ of dimension t. $st=d$, and hence s, t are odd numbers.

The following steps lead to a contradiction:

(1) By the theorem of Serre in § 3.3 and Lemma (3.4), we can choose

a place $v \in \Sigma_K - (S \cup S_l)$, such that we have the following properties:

(a) A has good reduction A_v at v.
(b) v is of degree 1, and the prime number $Nv = p$ is large enough such that the conclusion of Lemma (3.4) holds.
(c) $\operatorname{rank}(\Gamma_v) = \operatorname{rank}(G_V) = (s+1) + \operatorname{rank}(\mathcal{H})$.

(2) The characteristic polynomial $P_{v,l}(T)$ of π_v equals $f(T)^2$, where $f(T) = \det(1 - T \cdot \rho_l(F_v) \mid W_l)$, $f(T) \in \boldsymbol{Z}[T]$, and $\deg f = 2d$. Because W_2 is an orthogonal representation of $\mathcal{H}$ of odd dimension, 0 is a weight for W_2. Let Δ be the set (counting multiplicities) of all eigenvalues of $\rho_l(F_v)|_{W_l} \in GL(W_l)$. By Lemma (4.6), Δ can be expressed as the set (counting multiplicities) $\Phi \cdot \Psi$ of pairwise products of

$\Phi = \{\lambda\alpha_1, \lambda\alpha_1^{-1}, \cdots, \lambda\alpha_s, \lambda\alpha_s^{-1}\}$ (corresponds to W_1) and
$\Psi = \{\beta_1, \cdots, \beta_{(t-1)/2}, 1, \beta_1^{-1}, \cdots, \beta_{(t-1)/2}^{-1}\}$ (corresponds to W_2).

Here λ, α_i $(1 \leq i \leq s)$, β_j $\left(1 \leq j \leq \frac{t-1}{2}\right)$, are in C_l^*. Let $\bar{Q}$ be the algebraic closure of Q in C_l. Since $f(T) \in \boldsymbol{Z}[T]$, Φ and Ψ are contained in $\bar{Q}^*$. Note that Φ is a subset of Δ. In other words, Φ consists of eigenvalues of π_v.

(3) Let Γ_1 (resp. Γ_2) be the multiplicative subgroup of $\bar{Q}^*$ generated by Φ (resp. Ψ). Since $\Phi \subset \Delta$, Γ_1 is a free abelian subgroup of Γ_v. Consequently, Γ_2 is also a free abelian subgroup of Γ_v. Recall that W_1 and W_2 are faithful (cf. Lemma (4.5)), and the weights of W_1 and W_2 are linearly independent over $\boldsymbol{Q}$. By property (c) of (1), we have the following properties:

(a) $\operatorname{rank}(\Gamma_1) = \operatorname{rank}(G_m \cdot Sp_{2s}) = s+1$. In particular, all the elements in Φ are distinct.
(b) $\operatorname{rank}(\Gamma_2) = \operatorname{rank}(\mathcal{H}) \geq 1$.
(c) $\Gamma_v = \Gamma_1 \oplus \Gamma_2$ in $\bar{Q}^*$.

(4) Note that Δ is a $\operatorname{Gal}(\bar{Q}/Q)$-set. For each $\gamma \in \Delta$, let $T_\gamma : \Delta \to \bar{Q}^*$ be the map defined by $T_\gamma(\mu) = \gamma^2 \mu^{-1}$ for all $\mu \in \Delta$. We have the following result:

LEMMA (4.7). *The cardinality* (*counting multiplicities*) *of* $T_\gamma(\Delta) \cap \Delta$ *is at least* t *if and only if* $\gamma \in \Phi$.

PROOF. Let $x \in \Phi$. For each $\beta \in \Psi$, we have $T_x(x\beta) = x\beta^{-1} \in \Delta$. Hence the cardinality of $T_x(\Delta) \cap \Delta$ is at least t. Conversely, suppose that the cardinality of $T_{x\beta}(\Delta) \cap \Delta$ is at least t for some $x \in \Phi$ and $\beta \in \Psi$. If $T_{x\beta}(x'\beta') = (x\beta)^2(x'\beta')^{-1} \in \Delta$ for some $x' \in \Phi - \{x\}$ and $\beta' \in \Psi$, then the condition $(x\beta)^2(x'\beta')^{-1} \in \Delta$ yields a nontrivial relation between certain elements of Γ_1 and Γ_2. This contradicts property (a), (b), (c), of (3). Thus, we must have $(x\beta)^2(x\beta')^{-1} \in \Delta$ for all $\beta' \in \Psi$. This shows that $\beta^2 \cdot \Psi \subset \Psi$. Ψ is a finite set, hence β is a root of unity. Since Γ_2 is a free abelian group, we have $\beta = 1$.

(5) Recall that the characteristic polynomial $P_{v,l}(T)$ of π_v equals $(f(T))^2$.

Let $q_1(T) = \prod_{x \in \Phi} (T - x)$. Then we have the following:

LEMMA (4.8). (a) *$q_1(T)$ is a monic polynomial over $\boldsymbol{Z}$.*
(b) $\sum_{x \in \Phi} x \in \boldsymbol{Z}$.

PROOF. For each $\sigma \in \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, $\gamma \in \Delta$, it is clear that $T_\gamma(\Delta) \cap \Delta$ and $T_{\sigma(\gamma)}(\Delta) \cap \Delta$ (counting multiplicities) have the same cardinalities. By Lemma (4.7), Φ is stable under the action of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. (a) and (b) follow immediately.

LEMMA (4.9). *Suppose that $p \gg 0$. Then we have the following*:
(a) $\sum_{x \in \Phi} x \not\equiv 0 \pmod p$.
(b) *For any prime $\mathfrak{P}$ of $\bar{\boldsymbol{Q}}$ lying over p, there exists $x \in \Phi$ such that* $\mathrm{ord}_{\mathfrak{P}}(x) = 0$.

PROOF. Suppose that $\sum_{x \in \Phi} x = 0$. Then $\mathrm{Tr}(\rho_l(F_v)) = 2\sum_{\beta \in \Psi}\left(\sum_{x \in \Phi} x\beta\right) = 0$. This contradicts our choice of the place v (cf. step (1), condition (b)). So we may assume that $\sum_{x \in \Phi} x \neq 0$. Our assertion (a) follows easily from Riemann hypothesis. Now, suppose that $\mathfrak{P}|p$ and $\mathrm{ord}_{\mathfrak{P}}(x) > 0$ for all $x \in \Phi$. Then we have $\mathrm{ord}_{\mathfrak{P}}\left(\sum_{x \in \Phi} x\right) > 0$; hence $\sum_{x \in \Phi} x$ is divisible by p, which contradicts assertion (a).

(6) Now, for any prime $\mathfrak{P}$ of $\bar{\boldsymbol{Q}}$ lying over p, let $x_{\mathfrak{P}}$ be an element of Φ such that $\mathrm{ord}_{\mathfrak{P}}(x_{\mathfrak{P}}) = 0$. For any β in Ψ, we have

$$\mathrm{ord}_{\mathfrak{P}}(x_{\mathfrak{P}}) = 1/2 \quad (\mathrm{ord}_{\mathfrak{P}}(x_{\mathfrak{P}} \cdot \beta) + \mathrm{ord}_{\mathfrak{P}}(x_{\mathfrak{P}} \cdot \beta^{-1})) = 0.$$

Thus, we have $\mathrm{ord}_{\mathfrak{P}}(\beta^2) = 0$ for all $\mathfrak{P}|p$. By Theorem 2.1.B, we conclude that β is a root of unity. Hence $\beta=1$ for all $\beta\in\Psi$, and $\mathrm{rank}(\mathcal{H})=0$. This contradicts our hypothesis and completes the proof of Theorem (4.3).

To sum up, for primes l where D splits, we have the following main result:

THEOREM (4.10). *If $d=1,2$ or an odd number, then*

$$\mathcal{G}_l\simeq sp(2d,\boldsymbol{Q}_l)\oplus\boldsymbol{Q}_l\cdot\mathrm{id}.$$

PROOF. This follows from Corollary (4.2), Theorem (4.3), Proposition (3.2), and Lemma (2.2.2).

REMARK. For prime l where D doesn't split, we shall discuss the result in § 5 (cf. Theorem (5.2)).

§ 5. The main theorems

In this section, we let $\dim A=2d$ with $d=1,2$ or an odd number, and $\mathrm{End}_K(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=D$ an indefinite quaternion algebra over $\boldsymbol{Q}$. For simplicity, we call them abelian varieties of type (IQ).

As in § 3, let us fix a polarization on A and let ψ be the nondegenerate alternating form (associated to the chosen polarization) on $V=H_1(A(\boldsymbol{C}),\boldsymbol{Q})$. Denote by $M=MT(A(\boldsymbol{C}))$ the Mumford-Tate group of $A(\boldsymbol{C})$, and $H=\mathrm{Hod}(A(\boldsymbol{C}))=(M\cap SL_V)^{\circ}$ the Hodge group of $A(\boldsymbol{C})$. We have $M=H\cdot G_m$ and $\mathrm{End}_H(V)=\mathrm{End}_M(V)=D$.

Let $V_{\boldsymbol{C}}=V\otimes_{\boldsymbol{Q}}\boldsymbol{C}$ and $\psi_{\boldsymbol{C}}$ be the $\boldsymbol{C}$-linear extension of ψ to $V_{\boldsymbol{C}}\times V_{\boldsymbol{C}}$. As we have pointed out in [3], the $\boldsymbol{C}$-linear extension $'$ of the Rosati involution on $D\otimes_{\boldsymbol{Q}}\boldsymbol{C}\simeq M_2(\boldsymbol{C})$ is the usual transpose (up to conjugation). By the same arguments as Theorem A in [3], we have the following:

PROPOSITION (5.1). *There exists an $M_{/\boldsymbol{C}}$-submodule $W_{\boldsymbol{C}}\subset V_{\boldsymbol{C}}$ of dimension $2d$, such that* (a) *the representation $V_{\boldsymbol{C}}$ of $H_{/\boldsymbol{C}}$ is isomorphic to the direct sum of two copies of the representation $W_{\boldsymbol{C}}$ of $H_{/\boldsymbol{C}}$ over $\boldsymbol{C}$.* (b) *$\psi_{\boldsymbol{C}}|W_{\boldsymbol{C}}\times W_{\boldsymbol{C}}$ is a nondegenerate $H_{/\boldsymbol{C}}$-equivariant alternating form.*

PROOF. The same as Theorem A in [3]. □

On the other hand, it is well known that the representation V of M is defined by minuscule weights (cf. [9], [12]). By Lemma (2.2.2), we have $M_{/C} \cong GSp(W_C, \psi_C)$. Let $m = \mathrm{Lie}(M)$.

THEOREM (5.2). *$\mathcal{G}_l = m \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l$ for all abelian varieties of type (IQ).*

PROOF. Since $\mathcal{G}_l \subset m \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l$, it suffices to show that $\mathcal{G}_l \otimes_{\boldsymbol{Q}_l} C_l = (m \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l) \otimes_{\boldsymbol{Q}_l} C_l$. Recall that the rank of $\mathcal{G}_l$ is independent of l. For primes l where D splits, the equality has been established by the results in § 4. For primes l where D does not split, by extension of scalars to C_l, the same arguments as in the case where D splits, S_{V/C_l} is an irreducible subgroup of Sp_{2d} of rank d. We conclude that $\mathrm{Lie}(S_{V/C_l}) \simeq sp(2d, C_l)$. This completes the proof.

It is well known that a theorem of S. G. Tankeev (Theorem 5.1, [14]) and a result of Serre [12] imply the Hodge conjecture for simple abelian varieties of type (IQ) except when $\dim A = 4$.

For $\dim A = 4$, one sees that the Lie algebra $m \simeq sp(4, \boldsymbol{Q}) \oplus \boldsymbol{Q} \cdot \mathrm{id}$. By the same calculation of invariants as in [9], it is easy to check that the $(1,1)$-criterion holds for such abelian varieties $A(\boldsymbol{C})$. Thus we have the following:

COROLLARY (5.3). *The Hodge conjecture is true for $A(\boldsymbol{C})$, where A is an abelian variety of type (IQ).*

COROLLARY (5.4). *The Tate conjecture is true for abelian varieties of type (IQ).*

PROOF. After Faltings proved his theorems ([4], § 5, Satz 3, Satz 4), it is well known that if $\mathcal{G}_l = m \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l$, then the conjecture of Hodge, Tate on algebraic cycles (cf. [9], [17]) over $A(\boldsymbol{C})$, A respectively are equivalent.

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