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On the Anosov diffeomorphisms corresponding to geodesic flows on negatively curved closed surfaces

By Norikazu HASHIGUCHI

§ 1. Introduction.

In this paper, we study Birkhoff's sections for the geodesic flows of closed orientable surfaces with the constant negative curvature.

In 1917, Birkhoff defined a surface of section for a flow in his topological study of Laglange's equations of motion ([B]). This surface of section is called Birkhoff's section. In [F], Fried constructed explicitly Birkhoff's section for the geodesic flows of negatively curved surfaces, and he showed the way of reconstruction of the geodesic flows from "first return maps" for these sections. Recently in [G], Ghys proved that these maps are topologically semi-conjugate to hyperbolic toral automorphisms, and he calculated the traces of these matrices. In this paper, the author determines the conjugacy classes of these matrices and constructs the geodesic flows concretely.

The author would like to thank Professor T. Tsuboi for his patient advice and continuous encouragement.

§ 2. Birkhoff's section.

Let Σ_g be a closed orientable surface of genus g ($g \geqq 2$). We consider a Riemannian metric with the constant negative curvature -1 on it. (We have the same results as far as the metric has the constant negative curvature -1. So we use the symmetric one.) Let F_t denote the geodesic flow on the unit tangent vector bundle $T_1\Sigma_g$ of Σ_g. Fried defined Birkhoff's section for F_t as follows. Let $g_1, g_2, \cdots, g_{2g+2}$ be the simple closed geodesics shown in Figure 1. These geodesics divide Σ_g into four $2g+2$ gons P_1, P_2, P_3, P_4 where P_1 and P_2 are named so that they intersect at only $2g+2$ vertices. Then for $i=1, 2$, we choose a family C_i of convex smooth simple closed curves which fills the interior of P_i with one singularity o_i deleted (Figure 2). Let $S \subset T_1\Sigma_g$ be the closure of the set of the unit vectors tangent to the curves belonging

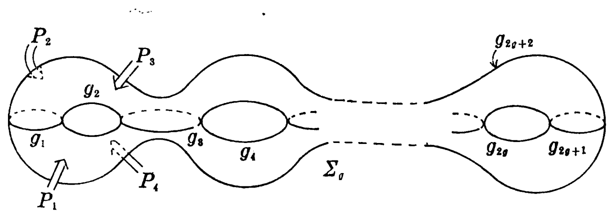

Figure 1

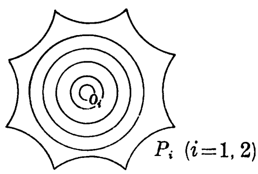

Figure 2

to C_i. Ghys showed that S has the following properties.
1. S is a smooth orientable surface with boundaries which consist of $4g+4$ closed orbits of F_t.
2. The interior of S is transverse to F_t, and the first return map

$$F: \mathrm{Int}(S) \longrightarrow \mathrm{Int}(S)$$

extends to a diffeomorphism $\bar{F}$ of S.
3. The Euler characteristic of S is $-(4g+4)$.

From the facts 1 and 3, we see that S is diffeomorphic to a torus with $4g+4$ open disks deleted.

We look at the section S more closely. Put $S_i = S \cap p_0^{-1}(P_i)$ $(i=1,2)$ where $p_0: T_1\Sigma_g \to \Sigma_g$ is the bundle projection. S_i is an annulus which contains the fibre over the singularity of C_i as a center circle. Now we consider the universal covering D of Σ_g (which is the Poincaré disk), and its unit tangent vector bundle T_1D. We use a natural trivialization t_0 of T_1D given by

$$t_0: T_1D \ni v \longmapsto (p(v),\ e(v)) \in D \times S^1,$$

where $p: T_1D \to D$ is the bundle projection and $e(v)$ $(\in S^1 = \partial D)$ is the end point of the geodesic starting at $p(v)$ in the direction of v. We give ∂D the counterclockwise orientation. We fix the lifts of P_i to D as are shown in Figure 3. Then D is divided into infinitely many $2g+2$ gons which are identified with P_i under the covering transformations on D. Let C_{i+} and C_{i-} denote the curves of C_i with the counterclockwise orientation and the clockwise orientation, respectively. We identify S_i with a lift $S_i \subset T_1D$, and parametrize S_i as follows. Let $\delta_{i\pm}$ be the boundary component of S_i corresponding to $C_{i\pm}$. $\delta_{i\pm}$ is divided into $4g+4$ edges $v_{i\pm}^1, h_{i\pm}^1, \cdots, v_{i\pm}^{2g+2}, h_{i\pm}^{2g+2}$ where $h_{i\pm}^j$ corresponds to a half of the geodesic g_k. We obtain S by identifying v_{1+}'s with v_{2-}'s and v_{2+}'s with v_{1-}'s. So $\{h_{i\pm}^j\}_{i=1,2,j=1,2,\cdots,2g+2}$ form the boundary of S.

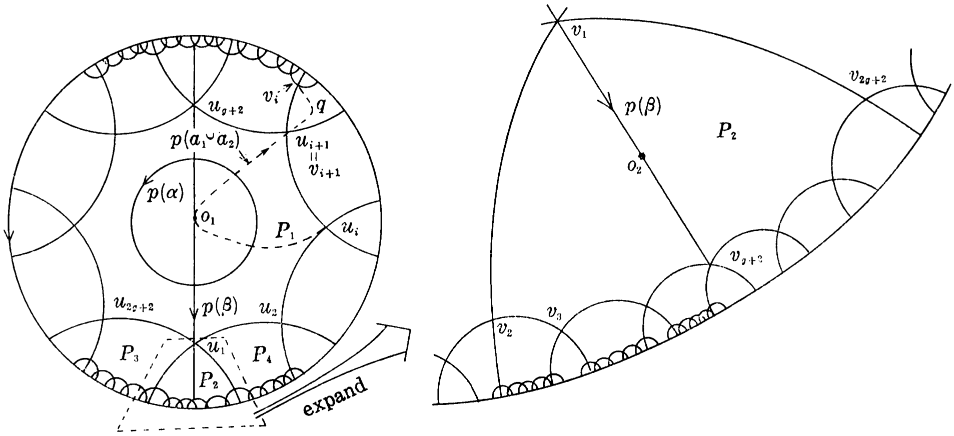

Figure 3

Now we define the maps $m_i: S_i \to S^1$ to be the restrictions of e to S_i. $\mathrm{Int}(S_i)$ is foliated by open intervals $m_i^{-1}(a)$ $(a \in S^1)$. So $\mathrm{Int}(S_i)$ is identified with $(0,1)\times S^1$ such that $(0,1)\times\{a\}$ $(a\in S^1)$ is a leaf and $\{b\}\times S^1$ $(b\in(0,1),\ b\neq 1/2)$ is the lift of an oriented curve of C_{i+} when $0<b<1/2$, of C_{i-} when $1/2<b<1$. Let $k_i: \mathrm{Int}(S_i)\to(0,1)$ be a projection with respect to this trivialization. By contracting $h^j_{i\pm}$ to a point, we define $\hat{S}_i$, which is also an annulus. $\hat{S}_i$ is identified with $[0,1]\times S^1$ by $\hat{k}_i\times\hat{m}_i$, where $\hat{k}_i: \hat{S}_i\to[0,1]$ is the extention of k_i and $\hat{m}_i: \hat{S}_i\to S^1$ is induced by m_i. ($\hat{m}_i$ can be defined because m_i maps each $h^j_{i\pm}$ to a point.)

We obtain 2-dimensional torus $\hat{S}$ by contracting each boundary component of S to a point. The diffeomorphism $\bar{F}$ induces a homeomorphism $\hat{F}$ of $\hat{S}$. Ghys showed that:

THEOREM A. *$\hat{F}$ is conjugate to a hyperbolic toral automorphism. That is, there is a matrix $A_g\in SL(2,\boldsymbol{Z})$ with $|\mathrm{trace}(A_g)|>2$, and a homeomorphism $H: T^2=\boldsymbol{R}^2/\boldsymbol{Z}^2\to\hat{S}$ such that*

$$\hat{A}_g=H^{-1}\circ\hat{F}\circ H.$$

(Here $\hat{A}_g$ is the diffeomorphism of T^2 induced by A_g.)

In this paper, the author determines the conjugacy class of A_g.

THEOREM B. *Under some basis of $\hat{S}$, A_g is written as follows:*

$$A_g=\begin{pmatrix} 2g^2-1 & 2g(g-1) \\ 2g(g+1) & 2g^2-1 \end{pmatrix}=\left(-\begin{pmatrix} g & g-1 \\ g+1 & g \end{pmatrix}\right)^2.$$

In § 3, we find the basis of $\hat{S}$, and calculate entries of A_g.

§ 3. Proof of the Theorem B.

To compute the entries of A_g, we find a nice basis of T^2 and examine the action of $\hat{A}_g$. The basis $\langle\hat{\alpha}, \hat{\beta}\rangle$ of T^2 are two simple closed curves representing generators of the fundamental group $\pi_1(T^2)$ of T^2.

To begin with, we find two simple closed curves of S which give rise to the basis of $\hat{S}$. Choose an oriented simple closed curve belonging to C_{1+} which is near the singularity o_1. The lift of this curve to $T_1\Sigma_g$ is the simple closed curve α in S. β is the lift of the simple closed oriented curve which is a geodesic through u_{g+2}, o_1, $u_1=v_1$, o_2, $v_{g+2}=u_{g+2}$ in $P_1\cup P_2$ (Where u_i and v_i are vertices of P_1 and P_2 respectively. They are named counterclockwise as in Figure 3). This geodesic intersects the curves of C_i transversely. This geodesic has two lifts in S. Now we take the one such that the part over (u_{g+2}, o_1) is in k_1^{-1} $(0, 1/2)$ and not in k_1^{-1} $(1/2, 1)$.

LEMMA. *The two simple closed curves $\hat{\alpha}$ and $\hat{\beta}$ in $\hat{S}$ which are the image of α and β form the basis of $\hat{S}$.*

PROOF. Because α and β intersect transversely only once in $\mathrm{Int}(S)$, so do $\hat{\alpha}$ and $\hat{\beta}$ in $\hat{S}$. Then they represent the basis of $\pi_1(\hat{S})$. ||

To obtain $\hat{S}$, we identified $v_{1\pm}$'s with $v_{2\mp}$'s, and contract each boundary component of this surface S to a point. So we can get $\hat{S}$ by pasting $\hat{S}_1$ and $\hat{S}_2$ on their edges $\hat{\delta}_{i\pm}$ which correspond to $\delta_{i\pm}$. The restriction of $\hat{m}_i$ to $\hat{\delta}_{i\pm}$ is an orientation preserving homeomorphism. Identifying $\hat{\delta}_{i\pm}$ with $\partial D=S^1$ by this map, the pasting maps between $\hat{\delta}_{1+}$ and $\hat{\delta}_{2-}$, $\hat{\delta}_{2+}$ and $\hat{\delta}_{1-}$ are homeomorphisms of ∂D. We also obtain a map $k:\hat{S}\to S^1=[0, 2]/0\sim 2$ such that

$$\begin{cases} \hat{k}_1(x) & \text{for } x\in\hat{S}_1 \\ \hat{k}_2(x)+1 & \text{for } x\in\hat{S}_2. \end{cases}$$

The curve α is contained in $\mathrm{Int}(S_i)$ and $m_1|\alpha$ is an orientation preserving homeomorphism because of the convexity of elements of C_i. The curve β starts from δ_{1+}, and goes across $p^{-1}(o_1)$, $\delta_{1-}=\delta_{2+}$, $p^{-1}(o_2)$ and $\delta_{2-}=\delta_{1+}$ in

S. $p(\beta)$ intersects elements of C_i transversely, so $\hat{\beta}$ intersects $k^{-1}(a)$ ($a \in S^1=[0,2]/0\sim 2$) transversely. $k|\hat{\beta}$ is also an orientation preserving homeomorphism. From this fact, we see the following fact;

If γ is a closed curve in $\hat{S}$, then the fact that γ represents

$$a[\hat{\alpha}]+b[\hat{\beta}] \quad a, b \in \boldsymbol{Z}$$

in $\pi_1(\hat{S})$ is the same as follows.

1. The image $k_\sharp(\gamma)$ winds $S^1=[0,2]/0\sim 2$ b times. (S^1 has the natural orientation.)
2. We can choose a closed curve γ' in $\hat{S}_1$ which represents $[\gamma]-b[\hat{\beta}]$. Then a is the winding number of the image $\hat{m}_{1\sharp}(\gamma')$ with respect to $S^1=\partial D$.

(Here we identify $\pi_1(\hat{S})$ with a free homotopy class $[S^1, \hat{S}]$.)

Using the above fact, we can calculate how $\hat{F}_*[\hat{\alpha}]$ and $\hat{F}_*[\hat{\beta}]$ are represented by $[\hat{\alpha}]$ and $[\hat{\beta}]$ in $\pi_1(\hat{S})$. We obtain

$$\hat{F}_*[\hat{\alpha}]=(2g^2-1)[\hat{\alpha}]+2g(g+1)[\hat{\beta}]$$
$$\hat{F}_*[\hat{\beta}]=2g(g-1)[\hat{\beta}]+(2g^2-1)[\hat{\beta}] \quad \text{in } \pi_1(\hat{S}).$$

(When we calculate the winding number, $p(\alpha)$ and $p(\beta)$ in D are considered (Figure 3). Moving $p(\alpha)$ and $p(\beta)$ along geodesics, we observe through which vertices the images pass in $2g+2$ gons corresponding to P_1 and P_2. This observation gives us the winding numbers.)

So Theorem B is proved.

REMARK. Ghys also showed that the trace of A_g is $4g^2-2$. But the conjugacy classes of $SL(2, \boldsymbol{Z})$ are not determined by traces ([S-F]).

§ 4. Reconstruction of F_t.

In this section, we reconstruct F_t from A_g. Fried constructed transitive Anosov flows on closed 3-manifolds from some pseudo-Anosov maps. We use his method.

Let $x_1, x_2, \cdots, x_{4g+4} \in T^2$ be the fixed points of $\hat{A}_g : T^2 \to T^2$, T_0 be $T^2 \setminus \{x_1, x_2, \cdots, x_{4g+4}\}$ and T be the natural compactification of T_0 obtained by adding boundary circles. $\hat{A}_g | T_0$ induces a homeomorphism $\bar{A}_g : T \to T$ which maps each boundary circle onto itself. We consider the suspension flow $\phi_t^* : M^* \to M^*$ where M^* is the mapping torus of $\bar{A}_g$. The boundary of M^* consists of $4g+4$ tori $x_1^*, x_2^*, \cdots, x_{4g+4}^*$ where x_i^* corresponds to x_i. x_i^* is given a fixed system of coordinate of $H_1(x_i^*)$ as follows.

a) The first generator, meridian m, of $H_1(x_i^*)$ is one of the closed orbits of $\phi_t^*|x_i^*$.

b) The second generator, longitude l, of $H_1(x_i^*)$ is the boundary of fibre of the mapping torus $M^* \to S^1$ with the clockwise orientation (Figure 4).

x_i^* is foliated by circles which are transverse to all orbits of $\varphi_t^*|x_i^*$ and represent $m+l$ in $H_1(x_i^*)$. Collapsing each leaf of the circle foliation on x_i^*, we obtain a new flow $\phi_t : M \to M$ which is topologically conjugate to the geodesic flow $F_t : T_1\Sigma_g \to T_1\Sigma_g$.

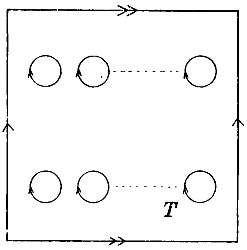

Figure 4

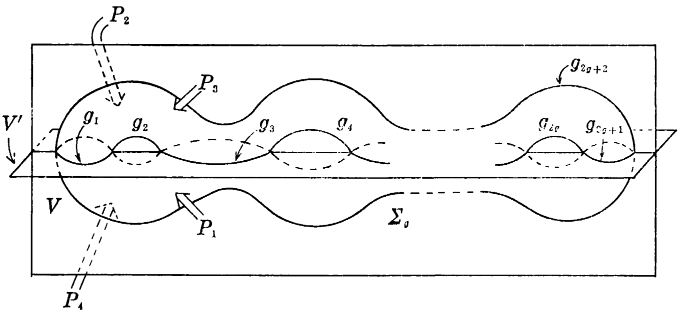

Figure 5

REMARK. From a topological point of view, M is obtained from the mapping torus of $\hat{A}_g$ by (1, 1)-Dehn surgeries.

To begin with, we choose $x_1, x_2, \cdots, x_{4g+4}$ from the fixed points of $\hat{A}_g$ ($\hat{A}_g$ has $4g^2-4$ fixed points). For this purpose, it is useful to know why A_g is written as the square of $B_g = -\begin{pmatrix} g & g-1 \\ g+1 & g \end{pmatrix}$. Let V be a plane which includes $g_2, g_4, \cdots, g_{2g+2}$ (Figure 5). So V divides Σ_g into $P_1 \cup P_3$ and $P_2 \cup P_4$. Consider the reflection $t : \Sigma_g \to \Sigma_g$ with respect to V. We may assume $t(P_1) = P_4$ and $t(P_2) = P_3$. We define the family of simple closed curves in P_3 and P_4 as $C_3 = t(C_2)$ and $C_4 = t(C_1)$ respectively. For $i = 3, 4$, as in the case of C_1 and C_2, we get sections $S_i \subset T_1\Sigma_g$ for F_t over P_i. $S' = S_3 \cup S_4$ is also Birkhoff's section for F_t. $S \cup S'$ is a closed surface and $F : S \cup S' \to S \cup S'$ is the "first return map" for F_t. Because of the convexity of elements of C_i, $F(S) = S'$ and $F(S') = S$. Clearly, $F = F|S' \circ F|S$. t induces $\mathcal{T} : T_1\Sigma_g \to T_1\Sigma_g$. Then $\mathcal{T} \circ \mathcal{T} = \mathrm{id}_{T_1\Sigma_g}$ and $\mathcal{T}(S) = S'$. Because

$F_t \circ \mathcal{T} = \mathcal{T} \circ F_t$, we can see the following.

LEMMA. *F and $\mathcal{T}$ commute i.e. $F \circ \mathcal{T} = \mathcal{T} \circ F$.*

By this lemma, we get

$$F|S' = \mathcal{T}|S' \circ F|S \circ (\mathcal{T}^{-1})|S' = \mathcal{T}|S' \circ F|S \circ \mathcal{T}|S'.$$

So $F = F|S' \circ F|S = \mathcal{T}|S' \circ F|S \circ \mathcal{T}|S' \circ F|S = (\mathcal{T}|S' \circ F|S)^2$.

We obtain 2-dimensional torus $\hat{S}'$ from S' in the same way as we get $\hat{S}$. Let $\hat{\mathcal{T}}|\hat{S}'$ and $\hat{F}|\hat{S}$ be the map induced by $\mathcal{T}|S'$ and $F|S$, respectively. Then $\hat{F} = (\hat{\mathcal{T}}|\hat{S}' \circ \hat{F}|\hat{S})^2$. In order to see that $H^{-1} \circ \hat{\mathcal{T}}|\hat{S}' \circ \hat{F}|\hat{S} \circ H = \hat{B}_g$, we need the next proposition.

PROPOSITION. *$h : T^2 \to T^2$ is a hyperbolic toral automorphism induced by $\tilde{h} \in SL(2, Z)$, and $f : T^2 \to T^2$ is an orientation preserving C^1 diffeomorphism. If $f^2 = h^2$, then $f = \pm h$.*

PROOF. Let $\mathcal{F}^s(\mathcal{F}^u)$ be the (un)stable foliation for h and $\mathcal{F}'^\sigma$ $(\sigma = s, u)$ be the foliation induced from $\mathcal{F}^\sigma$ by f i.e. $\mathcal{F}'^\sigma = f^{-1}(\mathcal{F}^\sigma)$. Because $f^2 = h^2$ preserves $\mathcal{F}'^\sigma$, we have the following lemma.

LEMMA. *f preserves $\mathcal{F}^\sigma$ $(\sigma = s, u)$ i.e. $\mathcal{F}'^\sigma = \mathcal{F}^\sigma$.*

λ and $\dfrac{1}{\lambda}$ $(|\lambda| > 1)$ denote the eigenvalue of $\tilde{h}$. Since $\mathcal{F}^\sigma$ is ergodic in T^2, there exists the unique transversal measure μ^σ up to positive multiples. $f^*\mu^\sigma = \mu^{*\sigma}$ is also the transversal measure of $\mathcal{F}^\sigma$. So $\mu^{*\sigma} = c^\sigma \mu^\sigma$ where c^σ is a positive constant. Clearly $(h^2)^*\mu^s = \left(\dfrac{1}{\lambda}\right)^2 \mu^s$ and $(h^2)^*\mu^u = \lambda^2 \mu^u$. These imply $(c^s)^2 = \left(\dfrac{1}{\lambda}\right)^2$ and $(c^u)^2 = \lambda^2$. So $c^s = |1/\lambda|$ and $c^u = |\lambda|$.

Let $\pi : R^2 \to T^2$ be the orientation preserving universal covering. Put $\pi^*\mu^\sigma = \tilde{\mu}^\sigma$. Then $\pi^*\mu^{*\sigma} = c^\sigma \tilde{\mu}^\sigma$. Let $\tilde{\mathcal{F}}^\sigma$ be the lift of $\mathcal{F}^\sigma$ and $\tilde{f} : R^2 \to R^2$ be the lift of f.

Now we transform the coordinate of R^2 as follows;

We take $\mathcal{F}^u_0$ as the x-axis and $\mathcal{F}^s_0$ as the y-axis where $\mathcal{F}^\sigma_0$ is the leaf of $\tilde{\mathcal{F}}^\sigma$ containing the origin. $\mathcal{F}^u_0$ (resp. $\mathcal{F}^s_0$) is measured by $\tilde{\mu}^s$ (resp. $\tilde{\mu}^u$). (i.e. we take the coordinate of which basis are the eigenvectors of $\tilde{h}$.)

In this coordinate system of R^2, we can see that $\tilde{f}$ is affine. So

$\tilde{f}=\pm\begin{pmatrix} \tilde{h} & & a \\ & & b \\ 0 & 0 & 1\end{pmatrix}$ $a, b\in \boldsymbol{R}$. We can take the universal covering $\pi:\boldsymbol{R}^2\to T^2$ such that $a=b=0$. Hence $\tilde{f}=\pm\tilde{h}$. This implies $f=\pm h$. ||

Now we fix the universal covering $\pi:\boldsymbol{R}^2\to T^2$ which satisfies that $H(\pi(0,0))$ is the fixed point of $\hat{F}$ corresponding to $+g_2$. (Each g_i corresponds to two oriented closed geodesics $\pm g_i$.) Then, from this proposition,

$$H^{-1}\circ\hat{\mathcal{T}}|\hat{S}'\circ\hat{F}|\hat{S}\circ H=\pm\hat{B}_g \text{ where } \pm\hat{B}_g \text{ is induced by } \pm B_g.$$

$\hat{\mathcal{T}}|\hat{S}'\circ\hat{F}|\hat{S}$ fixes $2g+2$ points corresponding to closed geodesics $\pm g_2$, $\pm g_4$, $\pm g_6, \cdots, \pm g_{2g+2}$. And

$$\#\,\mathrm{Fix}(\hat{B}_g)=|\mathrm{Trace}(B_g)-2|=2g+2,$$
$$\#\,\mathrm{Fix}(-\hat{B}_g)=|\mathrm{Trace}(-B_g)-2|=2g-2.$$

So $H^{-1}\circ\hat{\mathcal{T}}|\hat{S}'\circ\hat{F}|\hat{S}\circ H=\hat{B}_g$.

Let V' be a plane which includes $g_1, g_3, g_5, \cdots, g_{2g+1}$ (Figure 5) and $t':\Sigma_g\to\Sigma_g$ be the reflection with respect to V'. In the same way used in the case of V, we get $\mathcal{T}':T_1\Sigma_g\to T_1\Sigma_g$. $\hat{\mathcal{T}}'|\hat{S}'$ is induced by $\mathcal{T}'|S'$.

We also see that the lift of $H^{-1}\circ\hat{\mathcal{T}}'|\hat{S}'\circ\hat{F}|\hat{S}\circ H$ with respect to π is the affine map. So

$$B_g(a,b)=\begin{pmatrix} B_g & & a \\ & & b \\ 0 & 0 & 1\end{pmatrix} \qquad (a, b\in(0,1),\ (a,b)\neq(0,0)).$$

$(\hat{\mathcal{T}}'|\hat{S}'\circ\hat{F}|\hat{S})^2=\hat{F}$, so $(H^{-1}\circ\hat{\mathcal{T}}'|\hat{S}'\circ\hat{F}|\hat{S}\circ H)^2=\hat{A}_g$. Hence

$$(B_g(a,b))^2=\begin{pmatrix} A_g & & B_g\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix} \\ 0 & 0 & 1\end{pmatrix}$$

and A_g induce the same diffeomorphisms of T^2. Therefore,

$$B_g\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}a\\b\end{pmatrix}\equiv\begin{pmatrix}0\\0\end{pmatrix} \mod \boldsymbol{Z}^2 \quad \text{i.e. } \pi\begin{pmatrix}a\\b\end{pmatrix}\in\mathrm{Fix}(-\hat{B}_g).$$

$\mathrm{Fix}(\hat{B}_g)$ and $\mathrm{Fix}(-\hat{B}_g)$ are as follows;

$$\mathrm{Fix}(-\hat{B}_g)=\left\{\pi\left(\frac{m}{g-1},0\right),\ \pi\left(\frac{m}{g-1},\frac{1}{2}\right);\ m=0,1,2,\cdots,g-2\right\} \quad (g \text{ odd})$$

$$=\left\{\pi\left(\frac{m}{g-1},0\right),\ \pi\left(\frac{2m+1}{2(g-1)},\frac{1}{2}\right);\ m=0,1,2,\cdots,g-2\right\}\quad (g\text{ even})$$

$$\mathrm{Fix}(\hat{B}_g)=\left\{\pi\left(\frac{m}{g+1},0\right),\ \pi\left(\frac{m}{g+1},\frac{1}{2}\right);\ m=0,1,2,\cdots,g\right\}\quad (g\text{ odd})$$

$$=\left\{\pi\left(\frac{m}{g+1},0\right),\ \pi\left(\frac{2m+1}{2(g+1)},\frac{1}{2}\right);\ m=0,1,2,\cdots,g\right\}\quad (g\text{ even}).$$

$\mathrm{Fix}(\hat{B}_g)$ corresponds to the oriented closed geodesics $\pm g_{2m}$. Each $\pm g_{2m}$ is mapped to the reversely oriented closed geodesics $\mp g_{2m}$ by $\mathcal{T}'|S'\circ F|S$. So $-\hat{B}_g(a,b)(\mathrm{Fix}(\hat{B}_g))=\mathrm{Fix}(\hat{B}_g)$, especially

$$\pi\left(-B_g(a,b)\begin{pmatrix}0\\0\end{pmatrix}\right)=\pi\begin{pmatrix}a\\b\end{pmatrix}\in\mathrm{Fix}(\hat{B}_g).$$

Because $\pi\begin{pmatrix}a\\b\end{pmatrix}\in\mathrm{Fix}(-\hat{B}_g)$, $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1/2\\1/2\end{pmatrix}$ when g is even, $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1/2\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1/2\end{pmatrix}$ or $\begin{pmatrix}1/2\\1/2\end{pmatrix}$ when g is odd. In S_1, $+g_2\subset\delta_{1+}$ and $-g_2\subset\delta_{1-}$. So $+g_2$ is actually moved to the y-direction by $\mathcal{T}'|S'\circ F|S$. Then $\begin{pmatrix}1/2\\0\end{pmatrix}$ is not valid. Let a_1 be the oriented arc which connects $+g_2$ and $-g_2$ in S_1 such that the image $m_1(a_1)$ goes in the direction of the orientation of $S^1=\partial D$ and does not wind S^1 any times, and a_2 be the oriented arc which connects $-g_2$ and $+g_2$ in S_2 such that the image $m_2(a_2)$ goes in the same direction and does not wind S^1 any times. $a_1\cup a_2$ is an oriented loop in S when we choose edges of a_i properly. The image $e(a_1\cup a_2)$ winds $S^1=\partial D$ once because $p(a_1\cup a_2)$ moves in D such that

$$u_i\longrightarrow o_1\longrightarrow u_{i+1}=v_{i+1}\longrightarrow q\longrightarrow v_i\qquad (\text{for some } i)$$

where u_i and u_{i+1} are vertices of P_1, v_i and v_{i+1} are vertices of $2g+2$ gon which corresponds to P_2 by an isometry of D, and q is the singularity of this $2g+2$ gon (Figure 3).

Hence $+g_2$ is actually moved to the x-direction by $\mathcal{T}'|S'\circ F|S$. So $\begin{pmatrix}0\\1/2\end{pmatrix}$ is not valid too. Even if g is odd, $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1/2\\1/2\end{pmatrix}$.

$\{x_1,x_2,\cdots,x_{4g+4}\}\subset T^2$ correspond to $\{\pm g_1,\pm g_2,\cdots,\pm g_{2g+2}\}\subset S$. And $\{\pm g_1,\pm g_3,\cdots,\pm g_{2g+1}\}$ correspond to $\mathrm{Fix}(\hat{\mathcal{T}}'|\hat{S}'\circ\hat{F}|\hat{S})$, $\{\pm g_2,\pm g_4,\cdots,\pm g_{2g+2}\}$ correspond to $\mathrm{Fix}(\hat{\mathcal{T}}|\hat{S}'\circ\hat{F}|\hat{S})$.

So

$$\{x_1,x_2,\cdots,x_{4g+4}\}=\mathrm{Fix}(\hat{B}_g)\cup\mathrm{Fix}\left(\hat{B}_g\left(\frac{1}{2},\frac{1}{2}\right)\right)$$

$$=\left\{\pi\left(\frac{m}{2(g+1)}, 1\right);\ 1=0, \frac{1}{2}, \quad m=0, 1, 2, \cdots, 2g+1\right\}.$$

Therefore,

THEOREM C. *The geodesic flow F_t can be reconstructed as follows;*
1. *Make the suspension flow ϕ_t^* from $\hat{A}_g$.*
2. *Operate Fried's* (1, 1)*-Dehn surgeries around* $4g+4$ *closed orbits of ϕ_t^* which correspond to* $4g+4$ *fixed points of $\hat{A}_g$*

$$\left\{\pi\left(\frac{m}{2(g+1)}, l\right);\ l=0, \frac{1}{2}, \quad m=0, 1, 2, \cdots, 2g+1\right\}.$$

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan