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The 2 dimensional cohomology group of moduli space of instantons

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§ 1. Introduction and statement of results

We shall denote by M_k' and M_k the moduli space and the framed moduli space of $SU(2)$ instantons with instanton number $-C_2=k$ respectively. Precisely, M_k' and M_k are quotients of the space of self-dual $SU(2)$ connections on the standard 4-sphere S^4 with $-C_2=k$ devided by the gauge group and the restricted gauge group (consisting automorphisms which are the identity on the base point $\infty\in S^4$) respectively. The followings are known concerning M_k' and M_k.

(1) [10] M_k' and M_k are path connected.

(2) [9] $\pi_1(M_k')=\begin{cases} \boldsymbol{Z}_2 & k:\text{even} \\ 0 & k:\text{odd}\end{cases}$

$\pi_1(M_k)=\boldsymbol{Z}_2$.

(3) [1] M_1' is diffeomorphic to $\{x\in\boldsymbol{R}^5;\ |x|<1\}$.

(4) [2] M_2' has the same homotopy type as $G_2(\boldsymbol{R}^5)$ where $G_2(\boldsymbol{R}^5)$ is the Grassmann manifold of 2 planes in $\boldsymbol{R}^5$. Besides the cohomology groups of M_2' and M_2 with $\boldsymbol{Z}$ and $\boldsymbol{Z}_2$ coefficients are given in [7, 8].

(5) [6] The Euler characteristic of M_k equals to the number of positive divisors of k.

In this note we shall prove the following Theorem.

THEOREM A. *For $k\geqq 3$ and p a prime number which satisfies $p>k$, $H^2(M_k;\boldsymbol{Z}_p)=0$ holds.*

COROLLARY. *For $k\geqq 3$ and p a prime number which satisfies $p>k$, $H^2(M_k';\boldsymbol{Z}_p)=0$ holds.*

In fact it is well known that there is a principal bundle

$$SO(3)\longrightarrow M_k\longrightarrow M_k'.$$

Hence Corollary follows immediately from Theorem A.

REMARK. From (3) and (4) we have $H^2(M_k'; Z_p)=0$ and $H^2(M_k; Z_p)=0$ for $k=1, 2$ and $p\geqq 3$.

This paper is organized as follows. In § 2, 3 and 4 we study the Donaldson's description of M_k and auxiliary spaces for computing $H^2(M_k; Z_p)$. (The proof of Proposition 2 in § 2 is deferred to § 7.) In § 5, 6 we prove Theorem A by using the steps given in § 2, 3 and 4. In § 7 we shall prove Proposition 2.

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§ 2. Donaldson's description

Let $M(i, j)$ be the set of complex matrices of size $i\times j$. We write $M(i)$ for $M(i, i)$. We define P by the set of matrices $(A, B, a, b)\in M(k)\times M(k)\times M(2, k)\times M(k, 2)$ satisfying the following conditions.

(i) $[A, B]+ba=0$.

(ii) For all λ, $\mu\in C$,

$$\operatorname{rank}\begin{pmatrix} A+\lambda \\ B+\mu \\ a \end{pmatrix}=k \quad \text{and} \quad \operatorname{rank}(A+\lambda \ \ B+\mu \ \ b)=k.$$

Then the following Proposition holds.

PROPOSITION 1 ([5]). *M_k is the quotient of P by the action of $GL(k, C)$*

$$\left.\begin{array}{l} A\to RAR^{-1} \\ B\to RBR^{-1} \\ a\to aR^{-1} \\ b\to Rb \end{array}\right\} \quad R\in GL(k, C).$$

Clearly we can interpret P as follows.

$P=\{(A, B, a, b)\in M(k)\times M(k)\times M(2, k)\times M(k, 2)$; The following (i), (ii) and (iii) hold.$\}$

(i) $[A, B]+ba=0$.

(ii) A and B have no common eigenvector h which satisfies $ah=0$.

(iii) ${}^tA$ and ${}^tB$ have no common eigenvector h' which satisfies ${}^tbh'=0$ (^{t}=transpose).

Let Q be the set of matrices $(A, B, a, b)\in M(k)\times M(k)\times M(2, k)\times M(k, 2)$ satisfying the following conditions.

(i) $[A, B]+ba=0$.
(ii) A and B have no common eigenvectors.
(iii) ${}^tA$ and ${}^tB$ have no common eigenvectors.
(iv) $\operatorname{rank} a=2$ and $\operatorname{rank} b=2$.
Clearly $Q\subset P$ holds.
We shall prove the following Proposition in §7.

PROPOSITION 2. *Q is a Zariski open set of P and the complex codimension of P-Q in P equals k-1.*

Let M be the set of matrices $(A, B)\in M(k)\times M(k)$ satisfying the following conditions.
(i) $\operatorname{rank}[A, B]=2$.
(ii) A and B have no common eigenvectors.
(iii) ${}^tA$ and ${}^tB$ have no common eigenvectors.
We define $\pi: Q\rightarrow M$ by $\pi(A, B, a, b)=(A, B)$ and we define the action of $GL(2, \boldsymbol{C})$ on Q by $(A, B, a, b)g=(A, B, g^{-1}a, bg)$ where g is an element of $GL(2, \boldsymbol{C})$.

PROPOSITION 3. *$\pi: Q\rightarrow M$ is a principal bundle with structure group $GL(2, \boldsymbol{C})$.*

PROOF. We define L and X as follows.

$$L=\{(a, b)\in M(2, k)\times M(k, 2);\ \operatorname{rank} a=2 \text{ and } \operatorname{rank} b=2\}$$
$$X=\{C\in M(k);\ \operatorname{rank} C=2\}.$$

We define the action of $GL(2, \boldsymbol{C})$ on L by $(a, b)g=(g^{-1}a, bg)$. Then we can easily prove the following Assertion.

ASSERTION. *The map $p: L\rightarrow X$ defined by $(a, b)\mapsto ba$ gives a principal bundle with structure group $GL(2, \boldsymbol{C})$.*

Now if we pull back $p: L\rightarrow X$ by the map $M\rightarrow X$ defined by $(A, B)\rightarrow -[A, B]$ we see that the total space of the induced bundle equals to Q. Hence Proposition 3 follows.

§3. The topology of M

For $A\in M(k)$ we denote by $\gamma_A(s)$ the characteristic polynomial of A. We define N to be

$$N=\{(A, B) \in M;\ \gamma_A(s)=0 \text{ has no common root.}\}.$$

Clearly N is the complement of a complex hypersurface in M.

We shall study the topology of N. We define the action of $GL(k, \boldsymbol{C})$ on N by $R(A, B)=(RAR^{-1}, RBR^{-1})$ where R is an element of $GL(k, \boldsymbol{C})$.

Let $\boldsymbol{C}^*$ be the group of non-trivial complex numbers and we embed $\boldsymbol{C}^*$ in $GL(k, \boldsymbol{C})$ by diagonal. We see that the action of $GL(k, \boldsymbol{C})$ on N naturally defines the action of $GL(k, \boldsymbol{C})/\boldsymbol{C}^*$ on N.

PROPOSITION 4. *The action of $GL(k, \boldsymbol{C})/\boldsymbol{C}^*$ on N is free. Therefore we have following principal bundle.*

$$GL(k, \boldsymbol{C})/\boldsymbol{C}^* \to N \to N/GL(k, \boldsymbol{C}).$$

PROOF. Let (A, B) be an element of N. Assume $(RAR^{-1}, RBR^{-1})=(A, B)$ for some $R \in GL(k, \boldsymbol{C})$. As $\gamma_A(s)=0$ has no common root, we diagonalize A as follows.

$$R'AR'^{-1}=\operatorname{diag}(a_1, \cdots, a_k) \qquad \text{for some } R' \in GL(k, \boldsymbol{C}).$$

Then we see the following relation.

$$\begin{aligned}&((R'RR'^{-1})\operatorname{diag}(a_1, \cdots, a_k)(R'RR'^{-1})^{-1},\ (R'RR'^{-1})(R'BR'^{-1})(R'RR'^{-1})^{-1})\\ &=((\operatorname{diag}(a_1, \cdots, a_k),\ R'BR'^{-1}).\end{aligned}$$

We shall admit the following Lemma for a moment.

LEMMA 5. *Let (A', B') be an element of N where $A'=\operatorname{diag}(a_1, \cdots, a_k)$ is a diagonal matrix. If $(\tilde{R}A'\tilde{R}^{-1}, \tilde{R}B'\tilde{R}^{-1})=(A', B')$ for some $\tilde{R} \in GL(k, \boldsymbol{C})$ then $\tilde{R} \in \boldsymbol{C}^*$ holds.*

Now by Lemma 5 we conclude that $R'RR'^{-1}$ is an element of $\boldsymbol{C}^*$. This implies $R \in \boldsymbol{C}^*$ and the proof of Proposition 4 completes.

PROOF OF LEMMA 5. We state two Assertions of which the first one is well known.

ASSERTION 1. *$\tilde{R}$ is in the following form.*

$$\tilde{R}=\operatorname{diag}(z_1, \cdots, z_k) \qquad \textit{for some } z_i \in \boldsymbol{C}^*$$

Note that $\operatorname{rank}[A', B']=2$. By the conjugate action of the symmetric group of k elements, we can assume that the first and second columns of $[A', B']$ are linearly independent. We write the $i \times j$ element of B' by $b'_{i,j}$. Using the defining relation (ii) for M we can easily see the following.

ASSERTION 2. $b'_{1,2}\neq 0$, $b'_{2,1}\neq 0$, $(b'_{i,1}, b'_{i,2})\neq(0,0)$ *for* $3\leqq i\leqq k$ *hold.*

Now since $\tilde{R}B'\tilde{R}^{-1}=B'$, the first and second columns of $\tilde{R}B'-B'\tilde{R}$ are zero. By using Assertion 1 and 2, we easily see that $z_i=z_j$ for all i, j. This completes the proof of Lemma 5.

Let S be the set of matrices $(A, B)\in M(k)\times M(k)$ satisfying the following conditions.

(i) A is a diagonal matrix with no common diagonal elements.

(ii) $\operatorname{rank}[A, B]=2$.

(iii) Let $\tilde{B}$ be the matrix made from B by changing all diagonal elements to zero. Then for each row and column of $\tilde{B}$ there exists a non-trivial element.

We embed $(\boldsymbol{C}^*)^k$, the k times direct product of $\boldsymbol{C}^*$, in $GL(k, \boldsymbol{C})$ by diagonal. We define the action of $(\boldsymbol{C}^*)^k$ on S as follows.

$$Z(A, B)=(A, ZBZ^{-1})$$

where Z is an element of $(\boldsymbol{C}^*)^k$. Note that the action of $(\boldsymbol{C}^*)^k$ on S naturally defines the action of $(\boldsymbol{C}^*)^k/\boldsymbol{C}^*$ on S.

LEMMA 6. *The action of* $(\boldsymbol{C}^*)^k/\boldsymbol{C}^*$ *on* S *is free. Therefore we have the following principal bundle.*

$$(\boldsymbol{C}^*)^k/\boldsymbol{C}^* \longrightarrow S \longrightarrow S/(\boldsymbol{C}^*)^k.$$

This Lemma is proved in the same way as Proposition 4.

Let Σ_k be the symmetric group of k elements. We embed Σ_k in $GL(k, \boldsymbol{C})$ in a usual manner. We define the action of Σ_k on $S/(\boldsymbol{C}^*)^k$ by the following way.

$$\sigma(A, B)=(\sigma A\sigma^{-1}, \sigma B\sigma^{-1})$$

where σ is an element of Σ_k and (A, B) is an element of $S/(\boldsymbol{C}^*)^k$.

PROPOSITION 7. *The action of* Σ_k *on* $S/(\boldsymbol{C}^*)^k$ *is free. Therefore we have the following covering space.*

$$\Sigma_k \longrightarrow S/(\boldsymbol{C}^*)^k \longrightarrow S/(\boldsymbol{C}^*)^k/\Sigma_k.$$

PROOF. Let (A, B) be an element of $S/(\boldsymbol{C}^*)^k$ and let σ be an element of Σ_k. We assume $\sigma(A, B)=(A, B)$ in $S/(C^*)^k$. Then clearly $\sigma A\sigma^{-1}=A$ holds. As A has no common diagonal elements, this implies $\sigma=1$. This completes the proof of Proposition 7.

Now by Proposition 4 we have the following principal bundle.

$$GL(k, C)/C^* \longrightarrow N \longrightarrow N/GL(k, C).$$

By Proposition 7 we have the following covering space.

$$\Sigma_k \longrightarrow S/(C^*)^k \longrightarrow S/(C^*)^k/\Sigma_k.$$

PROPOSITION 8. *$N/GL(k, C)$ is homeomorphic to $S/(C^*)^k/\Sigma_k$.*

PROOF. We define $\alpha: S/(C^*)^k/\Sigma_k \to N/GL(k, C)$ by $\alpha(A, B)=(A, B)$ where (A, B) on the left hand is an element of $S/(C^*)^k/\Sigma_k$ and (A, B) on the right hand is an element of $N/GL(k, C)$.

We define $\beta: N/GL(k, C) \to S/(C^*)^k/\Sigma_k$ as follows. Let (A, B) be an element of $N/GL(k, C)$. As $\gamma_A(s)$ has no common root, there is an element R of $GL(k, C)$ such that $RAR^{-1}=\operatorname{diag}(a_1, \cdots, a_k)$. We define $\beta(A, B)$ by $\beta(A, B)=(RAR^{-1}, RBR^{-1})$. We can easily prove $\alpha\beta=1$ and $\beta\alpha=1$. This proves Proposition 8.

We shall study $S/(C^*)^k$ more carefully. We define Ω and W as follows.

$\Omega=\{A \in M(k);\ A$ is a diagonal matrix with no common diagonal elements$\}$.

$W=\{C \in M(k);$ all diagonal elements of C are zero$\}$.

We regard the projections onto the first factors, $\Omega\times M(k) \to \Omega$ and $\Omega\times W \to \Omega$, as trivial complex vector bundles. We define the vector bundle homomorphism $f: \Omega\times M(k) \to \Omega\times W$ by $f(A, B)=(A, [A, B])$.

LEMMA 9. *f is a surjective vector bundle homomorphism.*

PROOF. As $\dim_C M(k)=k^2$ and $\dim_C W=k^2-k$, all we need to prove is that for each $A\in\Omega$ the complex dimension of the vector space $\{B\in M(k);\ [A, B]=0\}$ equals to k. But this is proved easily.

We define T by

$$T=\{C\in W;\ \text{the following (i) and (ii) hold}\}.$$

(i) $\operatorname{rank} C=2$.

(ii) For each row and column of C, there exists a non-trivial element.

Then we easily see the following Proposition.

PROPOSITION 10. *$f^{-1}(\Omega\times T)=S$ holds. Hence by Lemma 9, $f|S:S\to\Omega\times T$ is an affine bundle whose fiber is an affine space of complex dimension k.*

We define the action of $(C^*)^k$ on $\Omega\times T$ by $Z(A,C)=(A,ZCZ^{-1})$ where Z is an element of $(C^*)^k$. Note that this action naturally defines the action of $(C^*)^k/C^*$ on $\Omega\times T$.

LEMMA 11. *The action of $(C^*)^k/C^*$ on $\Omega\times T$ is free. Therefore we have the following principal bundle.*

$$(C^*)^k/C^*\longrightarrow\Omega\times T\longrightarrow(\Omega\times T)/(C^*)^k.$$

This Lemma is proved in the same way as Lemma 5.

Note that $(\Omega\times T)/(C^*)^k=\Omega\times T/(C^*)^k$.

Now Lemma 6 and Lemma 11, we have two principal bundles

$$(C^*)^k/C^*\longrightarrow S\longrightarrow S/(C^*)^k \quad\text{and}\quad (C^*)^k/C^*\longrightarrow\Omega\times T\longrightarrow\Omega\times T/(C^*)^k.$$

It is easily proved that $f|S:S\to\Omega\times T$ naturally defines $\tilde{f}:S/(C^*)^k\to\Omega\times T/(C^*)^k$. We define the action of Σ_k on $S/(C^*)^k$ as before and we define the action of Σ_k on $\Omega\times T/(C^*)^k$ by $\sigma(A,C)=(\sigma A\sigma^{-1},\sigma C\sigma^{-1})$ where σ is an element of Σ_k and (A,C) is an element of $\Omega\times T/(C^*)^k$. Then we see that $\tilde{f}$ is Σ_k equivariant.

As $f|S$ is homotopy equivalence by Proposition 10, we easily see that $\tilde{f}$ is also homotopy equivalence. Therefore we have the following Proposition.

PROPOSITION 12. *$\tilde{f}^*:H^q(\Omega\times T/(C^*)^k;K)\to H^q(S/(C^*)^k;K)$ is isomorphism for all q and for any K a field. Moreover $\tilde{f}^*$ is Σ_k equivariant.*

§ 4. Some additional facts

In order to complete the steps of proving Theorem A, we need two more Lemmas.

LEMMA 13. *In Proposition 1, the action of $GL(k,C)$ on P is free.*

PROOF. Let (A,B,a,b) be an element of P and let R be an element of $GL(k,C)$. Suppose that $(RAR^{-1},RBR^{-1},aR^{-1},Rb)=(A,B,a,b)$. Then we must show $R=1$. But from the beginning we may assume R is in the Jordan normal form.

Then we must show the followings.

(1) Each eigenvalue of R equals to 1.

(2) The size of each Jordan cell of R equals to 1.

We shall show (1). (2) is proved similarly. We suppose that R has an eigenvalue not equal to 1. Then we may assume that R is in the following Jordan normal form.

$$R=J(\lambda_1)\oplus\cdots\oplus J(\lambda_l) \quad \lambda_i\neq\lambda_j \text{ for } i\neq j \text{ and } \lambda_1\neq 1$$

where $J(\lambda_i)$ is the direct sum of Jordan cells corresponding to the eigenvalue λ_i. We write A, B, a, b in the submatrices of appropriate size as follows.

$$A=\begin{pmatrix} A_{1,1} & \cdots & A_{1,l} \\ \vdots & & \vdots \\ A_{l,1} & \cdots & A_{l,l} \end{pmatrix} \qquad B=\begin{pmatrix} B_{1,1} & \cdots & B_{1,l} \\ \vdots & & \vdots \\ B_{l,1} & \cdots & B_{l,l} \end{pmatrix}$$

$$a=(a_1, \cdots, a_l) \qquad b=\begin{pmatrix} b_1 \\ \vdots \\ b_l \end{pmatrix}.$$

Note that $AR=RA$ implies $A_{i,j}=0$ for $i\neq j$. Hence we can write A as follows.

$$A=\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_l \end{pmatrix}.$$

Similarly we can write B as follows.

$$B=\begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_l \end{pmatrix}.$$

ASSERTION. $a_1=0$.

In fact $aR=a$ implies $a_1J(\lambda_1)=a_1$. But since $\lambda_1\neq 1$, we must have $a_1=0$.

Now $[A, B]+ba=0$ implies $[A_1, B_1]+b_1a_1=0$. But since $a_1=0$, we must have $A_1B_1=B_1A_1$. This implies that A_1 and B_1 have a common eigenvector. But the condition

$$\operatorname{rank}\begin{pmatrix} A+\lambda \\ B+\mu \\ a \end{pmatrix}=k \qquad \text{for all } \lambda, \mu\in \boldsymbol{C}$$

and Assertion imply that A_1 and B_1 have no common eigenvector. This is a contradiction. Therefore each eigenvalue of R equals to 1. This completes the proof of (1).

Recall that in § 2, N is the complement of a complex hypersurface in M.

LEMMA 14. *The complex hypersurface $M-N$ of M is irreducible.*

This Lemma follows from the following fact. Let $A=(a_{i,j})$ be an element of $M(k)$. We denote the discriminant of $\gamma_A(s)$ by $h(A)$. Clearly $h(A)$ is a polynomial of k^2 variables $a_{i,j}$ for $1\leqq i, j\leqq k$. Then we see that $h(A)$ is an irreducible polynomial. (The fact that $h(A)$ is an irreducible polynomial follows from the following fact. Let U be $U=\{A\in M(k); h(A)=0\}$ and U_{sing} denotes the singular points of U. By considering the Jordan normal form, the dimension counting of the isotropy subgroup shows that the following space U' is a dense subspace of $U-U_{\text{sing}}$.

$$U'=\left\{A\in M(k);\ A \text{ has a Jordan normal form } \begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{pmatrix}\oplus(\lambda_2)\oplus\cdots\oplus(\lambda_{k-1}).\right\}$$

where $\lambda_i\neq\lambda_j$ for $i\neq j$.
Clearly U' is connected. Hence $U-U_{\text{sing}}$ is connected.)

We constructed the following principal bundle in Proposition 3.

$$GL(2, C)\longrightarrow Q\stackrel{\pi}{\longrightarrow}M.$$

Let $Q|N$ be $\pi^{-1}(N)$. Then we have the following principal bundle.

$$GL(2, C)\longrightarrow Q|N\longrightarrow N.$$

We see that $Q|N$ is a $GL(k, C)$ invariant subspace of P. (cf. Proposition 1). Hence by using Lemma 13, we have the following principal bundle.

$$GL(k, C)\longrightarrow Q|N\longrightarrow Q|N/GL(k, C).$$

By using Proposition 2 and Lemma 13, we see that $Q/GL(k, C)$ is a Zariski open set of $M_k=P/GL(k, C)$ and the complex codimension of $M_k-Q/GL(k, C)$ in M_k is $k-1$. By using Lemma 13 and Lemma 14, we see that $Q|N/GL(k, C)$ is the complement of an irreducible complex hypersurface in $Q/GL(k, C)$.

We summarize the facts stated in § 2, 3, 4 as follows.

1∘ $Q/GL(k, C)$ is a Zariski open set of $M_k=P/GL(k, C)$.

2∘ $Q/GL(k, C)-Q|N/GL(k, C)$ is an irreducible hypersurface.

3∘ $GL(k, \boldsymbol{C}) \to Q|N \to Q|N/GL(k, \boldsymbol{C})$.
4∘ $GL(2, \boldsymbol{C}) \to Q|N \to N$.
5∘ $GL(k, \boldsymbol{C})/\boldsymbol{C}^* \to N \to N/GL(k, \boldsymbol{C})$.
6∘ $N/GL(k, \boldsymbol{C})$ is homeomorphic to $S/(\boldsymbol{C}^*)^k/\Sigma_k$.
7∘ $\Sigma_k \to S/(\boldsymbol{C}^*)^k \to S/(\boldsymbol{C}^*)^k/\Sigma_k$.
8∘ For any K a field, $H^*(\Omega \times T/(\boldsymbol{C}^*)^k; K)$ is isomorphic to $H^*(S/(\boldsymbol{C}^*)^k; K)$ as representation spaces of Σ_k.

§ 5. The cohomology of Ω and $T/(\boldsymbol{C}^*)^k$

In this section we shall compute the cohomology of Ω and $T/(\boldsymbol{C}^*)^k$. As for $H^*(\Omega; \boldsymbol{Z}_p)$ the following Proposition is known.

PROPOSITION 15 ([4]). *For p a prime number which satisfies $p>k$, $H^1(\Omega; \boldsymbol{Z}_p) = V$ and $H^2(\Omega; \boldsymbol{Z}_p) = 0$ hold where V is a $\boldsymbol{Z}_p$-representation space of Σ_k such that V^{Σ_k}, the set of fixed points, equals to $\boldsymbol{Z}_p$.*

Next we shall compute $H^q(T/(\boldsymbol{C}^*)^k; \boldsymbol{Z}_p)$ for $q=0, 1, 2$.

PROPOSITION 16. *For $k \geqq 3$ and p a prime number which satisfies $p>k$*

$$H^q(T/(\boldsymbol{C}^*)^k; \boldsymbol{Z}_p) = \begin{cases} \boldsymbol{Z}_p & q=0 \\ 0 & q=1, 2 \end{cases} \qquad \textit{hold.}$$

PROOF. Recall that in § 2 we constructed the following principal bundle $GL(2, \boldsymbol{C}) \to L \xrightarrow{p} X$. Let $L|T$ be $p^{-1}(T)$. We define Ψ by the set of matrices $a = (a_{i,j}) \in M(2, k)$ satisfying the following conditions.

(i) $\begin{pmatrix} a_{1,i} \\ a_{2,i} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all $1 \leqq i \leqq k$.

(ii) $\operatorname{rank} a = 2$.

We define $q: \Psi \times (\boldsymbol{C}^*)^k \to L|T$ as follows. Let $(a, \operatorname{diag}(\zeta_1, \cdots, \zeta_k))$ be an element of $\Psi \times (\boldsymbol{C}^*)^k$ then $q(a, \operatorname{diag}(\zeta_1, \cdots, \zeta_k))$ is defined by

$$\left(a, \operatorname{diag}(\zeta_1, \cdots, \zeta_k) \begin{pmatrix} a_{2,1} & -a_{1,1} \\ \vdots & \vdots \\ a_{2,k} & -a_{1,k} \end{pmatrix} \right).$$

ASSERTION 1. *q is a homeomorphism.*

This Assertion is proved by direct computations.

Recall that we defined the action of $GL(2, \boldsymbol{C})$ on L by $(a, b)g=(g^{-1}a, bg)$ where (a, b) is an element of L and g is an element of $GL(2, \boldsymbol{C})$.

ASSERTION 2. *Under the homeomorphism q, the action of $GL(2, \boldsymbol{C})$ on $\Psi\times(\boldsymbol{C}^*)^k$ is as follows.*

$$(a, \operatorname{diag}(\zeta_1, \cdots, \zeta_k))g=(g^{-1}a, (\det g)\operatorname{diag}(\zeta_1, \cdots, \zeta_k)).$$

This Assertion is also proved by direct computations.

Recall that we defined the action of $(\boldsymbol{C}^*)^k$ on T by conjugation. We lift this action to $L|T$ as follows. $Z(a, b)=(aZ^{-1}, Zb)$ where Z is an element of $(\boldsymbol{C}^*)^k$ and (a, b) is an element of $L|T$. Then we see the following Assertion.

ASSERTION 3. *Under the homeomorphism q, the action of $(\boldsymbol{C}^*)^k$ on $\Psi\times(\boldsymbol{C}^*)^k$ is as follows.*

$$Z(a, \operatorname{diag}(\zeta_1, \cdots, \zeta_k))=(aZ^{-1}, Z^2\operatorname{diag}(\zeta_1, \cdots, \zeta_k)).$$

Now by Assertion 2 and 3, we have the left action of $(\boldsymbol{C}^*)^k$ on $\Psi\times(\boldsymbol{C}^*)^k$ and the right action of $GL(2, \boldsymbol{C})$ on $\Psi\times(\boldsymbol{C}^*)^k$. But we can easily prove that these actions commute. Therefore we can define the following double coset.

$$(\boldsymbol{C}^*)^k\backslash\Psi\times(\boldsymbol{C}^*)^k/GL(2, \boldsymbol{C}).$$

We shall study $(\boldsymbol{C}^*)^k\backslash\Psi\times(\boldsymbol{C}^*)^k$. We embed $\boldsymbol{Z}_2^k$, the k times direct product of $\boldsymbol{Z}_2=\{\pm 1\}$, in $GL(k, \boldsymbol{C})$ by diagonal. We define the action of $\boldsymbol{Z}_2^k$ on Ψ by right multiplication.

We define $r:(\boldsymbol{C}^*)^k\backslash\Psi\times(\boldsymbol{C}^*)^k\to\Psi/\boldsymbol{Z}_2^k$ by

$$r(a, \operatorname{diag}(\zeta_1, \cdots, \zeta_k))=a\operatorname{diag}(\sqrt{\zeta_1}, \cdots, \sqrt{\zeta_k})).$$

Then we see the following Assertion.

ASSERTION 4. *r is well defined and homeomorphism. And under r, the action of $GL(2, \boldsymbol{C})$ on $\Psi/\boldsymbol{Z}_2^k$ is given by $a.g=(\sqrt{\det g})g^{-1}a$ where a is an element of $\Psi/\boldsymbol{Z}_2^k$ and g is an element of $GL(2, \boldsymbol{C})$.*

As $L|T/GL(2, \boldsymbol{C})=T$, $T/(\boldsymbol{C}^*)^k=\Psi/\boldsymbol{Z}_2^k/GL(2, \boldsymbol{C})$ holds by Assertion 1 and 4. Hence to prove Proposition 16, all we need is to show that

$$H^q(\Psi/\boldsymbol{Z}_2^k/GL(2, \boldsymbol{C});\ \boldsymbol{Z}_p)=\begin{cases}\boldsymbol{Z}_p & q=0\\ 0 & q=1, 2.\end{cases}$$

We embed C^* in $GL(2, C)$ by diagonal. Then we see that the action of $GL(2, C)$ on Ψ/Z_2^k (cf. Assertion 4) naturally defines the action of $GL(2, C)/C^*$ on Ψ/Z_2^k. Then we readily see the following Assertion.

ASSERTION 5. *The action of $GL(2, C)/C^*$ on Ψ/Z_2^k is free. Therefore we have the following principal bundle.*

$$GL(2, C)/C^* \longrightarrow \Psi/Z_2^k \longrightarrow \Psi/Z_2^k/GL(2, C).$$

ASSERTION 6. $H^q(\Psi/Z_2^k; Z_p) = H^q(\Psi/Z_2^k/GL(2, C); Z_p)$ *for* $q=0, 1, 2$.

In fact we can easily prove the following.

$$H^q(GL(2, C)/C^*; Z_p) = \begin{cases} Z_p & q=0 \\ 0 & q=1, 2. \end{cases}$$

Then by using the Serre spectral sequence of the principal bundle in Assertion 5, we easily prove Assertion 6.

By Assertion 6, all we need to prove is the following fact.

$$H^q(\Psi/Z_2^k; Z_p) = \begin{cases} Z_p & q=0 \\ 0 & q=1, 2. \end{cases}$$

ASSERTION 7. $H^q(\Psi; Z_p) = \begin{cases} Z_p & q=0 \\ 0 & q=1, 2. \end{cases}$

PROOF. Let Ψ' be the set of matrices $a=(a_{i,j})$ satisfying only the condition (i) of Ψ. Then we see that Ψ is a Zariski open set of Ψ' and the complex codimension of $\Psi' - \Psi$ in Ψ' is $k-1$. Recall that we assumed $k \geqq 3$. Now the general position argument shows that $H^q(\Psi; Z_p) = H^q(\Psi'; Z_p)$ for $q=0, 1, 2$.

Note that Ψ' has the same homotopy type as $(S^3)^k$, the k times direct product of S^3. Hence $H^q(\Psi'; Z_p) = \begin{cases} Z_p & q=0 \\ 0 & q=1, 2. \end{cases}$

Note that the action of Z_2^k on Ψ is free. Therefore we have the following covering space. $Z_2^k \to \Psi \to \Psi/Z_2^k$. The following Lemma is well known.

LEMMA 17 ([3]). *Let $\tilde{Y} \to Y$ be a finite Galois covering space with covering transformation group Φ. Let p a prime number. If the order of Φ is not divided by p, then $H^*(Y; Z_p) = H^*(\tilde{Y}; Z_p)^\Phi$ holds.*

Now by using Assertion 7 and Lemma 17, we see that

$$H^q(\Psi/Z_2^k;\, Z_p)=\begin{cases} Z_p & q=0 \\ 0 & q=1,2. \end{cases}$$

This completes the proof of Proposion 16.

§ 6. Proof of Theorem A

In this section, we shall prove Theorem A by using the steps given at the end of § 4.

ASSERTION 1. *In step* 8∘ *at the end of* § 4, $H^1(S/(C^*)^k;\, Z_p)=V$ *and* $H^2(S/(C^*)^k;\, Z_p)=0$.

Assertion 1 follows from Proposition 15, 16 and the Künneth Theorem.

ASSERTION 2. *In step* 7∘, $H^1(S/(C^*)^k/\Sigma_k;\, Z_p)=Z_p$ *and* $H^2(S/(C^*)^k/\Sigma_k;\, Z_p)=0$.

Assertion 2 follows from Lemma 17 and the fact that $V^{\Sigma_k}=Z_p$.
Note that $H^1(N/GL(k, C);\, Z_p)=Z_p$ and $H^2(N/GL(k, C);\, Z_p)=0$ by step 6∘.

ASSERTION 3. *In step* 5∘, $H^1(N;\, Z_p)=Z_p$ *and* $H^2(N;\, Z_p)=0$.

In fact we can easily prove that $H^q(GL(k, C)/C^*;\, Z_p)=0$ for $q=1, 2$. Hence Assertion 3 follows from the Serre spectral sequence of the principal bundle in step 5∘.

ASSERTION 4. *In step* 4∘, $H^1(Q|N;\, Z_p)=Z_p\oplus Z_p$ *and* $H^2(Q|N;\, Z_p)=Z_p$.

Assertion 4 follows from the Serre spectral sequence of the principal bundle in step 4∘.

ASSERTION 5. *In step* 3∘, $H^1(Q|N/GL(k, C);\, Z_p)=Z_p$ *or* $Z_p\oplus Z_p$. *If the first occurs then* $H^2(Q|N/GL(k, C);\, Z_p)=0$.

Assertion 5 also follows from the Serre spectral sequence in step 3∘.

ASSERTION 6. *In step* 2∘, $H^2(Q/GL(k, C),\ Q|N/GL(k, C);\, Z_p)=Z_p$.

In fact the Poincaré duality theorem shows that

$$H^2(Q/GL(k,C),\ Q|N/GL(k,C); Z_p)$$
$$=H^c_{\dim_R Q/GL(k,C)-2}(Q/GL(k,C)-Q|N/GL(k,C); Z_p)$$

where H^c_* is the homology with compact support. As $Q/GL(k,C)-Q|N/GL(k,C)$ is an irreducible complex hypersurface in $Q/GL(k,C)$, Assertion 6 follows.

Note that in step 1∘, the general position argument shows that $H^q(Q/GL(k,C); Z_p)=H^q(M_k; Z_p)$ for $q=0,1,2$. Note also that as $\pi_1(M_k)=Z_2$ [9], $H^1(M_k; Z_p)=0$.

Now by using these results, the cohomology long exact sequence of $(Q/GL(k,C),\ Q|N/GL(k,C))$ shows that $H^1(Q|N/GL(k,C); Z_p)=Z_p$. Therefore $H^2(Q/GL(k,C); Z_p)=H^2(M_k; Z_p)=0$. This completes the proof of Theorem A.

§ 7. Proof of Proposition 2

In this section we shall prove Proposition 2.

PROPOSITION 2. *Q is a Zariski open set of P and the complex codimension of $P-Q$ in P equals $k-1$.*

PROOF. We define the set of matrices F, F_0 and F_1 as follows.

$$F=\{(A,B,a,b)\in M(k)\times M(k)\times M(2,k)\times M(k,2);\ [A,B]+ba=0\}$$
$$F_0=\{(A,B,a,b)\in F;\ \operatorname{rank} a=2,\ \operatorname{rank} b=2\}$$
$$F_1=\{(A,B,a,b)\in F;\ \text{either } \operatorname{rank} a<2 \text{ or } b<2\}.$$

Clearly $P\subset F=F_0\amalg F_1$ holds. We easily see the following Assertion.

ASSERTION. $P\cap F_0=Q$.

Now we see $P-Q=P\cap F_1$ by Assertion. Next we shall compute the complex codimension of $P\cap F_1$ in P. We define F_1' and F_1'' as follows.

$$F_1'=\{(A,B,a,b)\in F_1;\ \operatorname{rank} a<2\}$$
$$F_1''=\{(A,B,a,b)\in F_1;\ \operatorname{rank} b<2\}.$$

All we need is to compute the complex codimension of $P\cap F_1'$ in P and $P\cap F_1''$ in P. We compute the former. The latter is computed similarly. We define P_1 and $\tilde{P}_1$ as follows.

$P_1=\{(A,B,a,b)\in P;\ A$ is a diagonal matrix with no common diagonal elements$\}$.

$$\tilde{P}_1=\{(A, B, a, b)\in P_1;\ \operatorname{rank} a<2\}.$$

Note that generically an element of P is equivalent to an element of P_1 by the $GL(k, C)$ action. (cf. Proposition 1). By the same argument as Proposition 4, 7 and 8, the codimension of $\tilde{P}_1$ in P_1 equals to the codimension of $P\cap F_1'$ in P. In P_1, if we fix A, a and the first column of b then generically the second column of b and the off-diagonals of B are determined automatically. Hence we see $\dim_C P_1=5k$. (Note that the condition (ii), in Proposition 1 is an open condition). Similarly we see $\dim_C \tilde{P}_1=4k+1$. Therefore the codimension of $\tilde{P}_1$ in P_1 equals to $k-1$. This completes the proof of Proposition 2.

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